Compact factors of countable state Markov shifts

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Abstract

We study continuous shift commuting maps from transitive countable state Markov shifts into compact subshifts. The closure of the image is a coded system. On the other hand, any coded system is the surjective image of some transitive Markov shift, which may be chosen locally compact by construction. These two results yield a formal analogy to “the transitive sofic systems are the subshift factors of transitive shifts of finite type”. Then we consider factor maps which have bounded coding length in some graph presentation (label maps). Now the image has to be synchronized, but not every synchronized system can be obtained in this way. We show various restrictions for a surjective label map to exist.

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1. Introduction

Let \( \mathbb{N}^\mathbb{Z} \) be endowed with the product topology of the discrete topology on \( \mathbb{N} \). Let \( \sigma : \mathbb{N}^\mathbb{Z} \to \mathbb{N}^\mathbb{Z} \) denote the left shift map, i.e. for \( x = (x_n)_{n \in \mathbb{Z}} \in \mathbb{N}^\mathbb{Z} \) let \( (\sigma x)_n := x_{n+1}, n \in \mathbb{Z} \). Let \( T \) be a subset of \( \mathbb{N}^\mathbb{Z} \). For \( x = (x_n)_{n \in \mathbb{Z}} \in T \) and \( -\infty < n \leq m < \infty \) let \( x[n,m] \) denote the block \( x_n, \ldots, x_m \) and let \( n[x_n,\ldots,x_m] \) denote the cylinder set \( \{ y \in T \mid y[n,m] = x[n,m] \} \). The induced topology on \( T \) is generated by the cylinder sets. If \( T \) is shiftinvariant, \( \sigma(T) = T \), then \( (T, \sigma|_T) \) is called a subshift. For brevity we shall use the symbol \( T \) for either the space or the shift map, which one is meant will be clear from context.

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In particular, we shall say that $T$ is a subshift of $\mathbb{N}^\mathbb{Z}$. Note that $T$ need not to be closed in general. For $n \in \mathbb{Z}$ let $x(n, \infty)$ denote the right infinite sequence of symbols $x_n, x_{n+1}, \ldots$ and $x(-\infty, n]$ the left infinite sequence of symbols $\ldots, x_{n-1}, x_n$. A right infinite sequence of symbols will also be called a right-infinite ray (analogously we define left-infinite rays). A block $w \in \mathbb{N}^N$, $k \geq 1$, is a block of $T$ (or a $T$-block) if there is a point $x$ in $T$ with $x[1,k] = w$. Let $|w|$ denote the length of a block $w$.

A **coded system** is a compact subshift $S$ which has a (finite or countable) list of $S$-blocks $w_1, w_2, \ldots$ such that $S$ coincides with the shift invariant closure of all bi-infinite concatenations of these blocks, i.e., every bi-infinite concatenation $y$ of blocks from the list $w_1, w_2, \ldots$ is a point in $S$ and for every $x \in S$ and every $n \in \mathbb{N}$ there is such a concatenation $y$ with $y[-n,n] = x[-n,n]$, [1]. Any such list of blocks is called a code for $S$. The definition implies that coded systems are transitive, that is for any two $S$-blocks $a$ and $c$ there is a block $b$ such that $abc$ is an $S$-block. If the code is finite, then $S$ is transitive sofic, since there are only finitely many follower sets. A **synchronized system** is a transitive compact subshift $S$ which has a “synchronizing” $S$-block $w$ which means whenever $uw$ and $w$ are $S$-blocks, then also $uww$ is an $S$-block. Synchronized systems are coded, since for any synchronizing block $w$ the set $\{wwx|x \in S\text{-block}\}$ is a code for $S$, [1]. Sofic systems have synchronizing blocks, thus all transitive sofic systems are synchronized [11].

A **Markov shift** $S$ is (by definition) conjugate to the set $S_G$ of bi-infinite walks on a countable directed graph $G$, with the left shift transformation acting on $S_G$. We call $S_G$ a **graph shift** or a **graph presentation** of $S$. The Markov shift $S$ is locally compact (l.c.) iff $G$ has finite in- and out-degree (at most finitely many in-coming and out-going edges at every vertex of $G$). The Markov shift is compact iff $G$ is a finite graph iff it is a shift of finite type (SFT). Transitivity of $S$ means irreducibility of $G$, that is for every pair of vertices $x$ and $\beta$ of $G$ there is a path from $x$ to $\beta$.

If $S$ is a compact Markov shift then any continuous shift commuting map $f : S \rightarrow S_N$ (here $S_N := \{1, \ldots, N\}^\mathbb{Z}$ denotes the full $N$-shift) can be recoded to a 1-block map (there is a graph shift $T$ and a conjugacy $\phi : S \rightarrow T$ such that $(f \phi^{-1}(x))_0$ is determined by the edge $x_0$ for all $x \in T$). In analogy for a locally compact Markov shift $S$ we define a **label map** to be a continuous shift commuting map $f : S \rightarrow S_N$ into some full shift $S_N$ with the property that there is a graph presentation $T$ for $S$ and a conjugacy $\phi : S \rightarrow T$ such that the conjugate map $f\phi^{-1}$ is a 1-block map, i.e., that $(f\phi^{-1}(x))_0$ is determined by the edge $x_0$ for all $x \in T$. Thus, in the compact setting any continuous map is a label map, but for non-compact l.c. Markov shifts we shall see that not every continuous map is a label map (Remark 2.5).

By [3] and the preceding discussions every coded system allows a label map with dense image. Motivated by this and the fact that every transitive sofic shift is a surjective factor of a transitive SFT [11, 13], two natural questions arise: Which compact subshifts are surjective continuous images of transitive Markov shifts? And which of these are even surjective images of transitive Markov shifts under label maps?

We shall give a complete answer to the first question. We show that every compact subshift $T$, for which a transitive Markov shift $S$ exists and a shift commuting con-
tinuous map \( f : S \to T \) with \( f(S) \) dense in \( T \), is coded (Lemma 2.2), in particular a surjective factor of a transitive Markov shift is coded. Then we show that for every coded system there is a locally compact transitive Markov shift and a surjective factor map onto the coded system (Theorem 2.4).

If for a transitive subshift \( T \) there is a Markov shift \( S \) and a label map \( f : S \to T \) such that \( f(S) \) is a residual subset of \( T \), then \( T \) is a synchronized system [3, Theorem 1.1]. Thus, for the second question, we know that every surjective image of a transitive Markov shift under a label map is a synchronized system. But we shall show that not every synchronized system can be obtained in this way. We give various restrictions, see Lemmas 3.1, 3.6, 3.8, and Examples 3.2, 3.7, 3.9.

Since we frequently use [3, Theorem 1.1], we recall the argument here.

**Theorem 1.1** (Fiebig and Fiebig [3, Theorem 1.1]). Let \( T \) be a compact subshift, \( S \) be a transitive Markov shift in graph presentation and \( f : S \to T \) a 1-block map such that \( f(S) \subset T \) is a residual set. Then \( T \) is synchronized.

**Proof.** Let \( V \) denote the vertex set of the graph of \( S \) and let \( B_x := \{ f(x) | x_0 \text{ starts in vertex } x \} \). Since \( \bigcup_{x \in V} B_x = f(S) \) is a residual subset of \( T \), not all \( B_x \) are nowhere dense. Thus there is an \( x \in V \) and a block \( m \) of length, say \( 2n + 1 \), such that \( B_x \) is dense in \( M := \{ y \in T | y[-n,n] = m \} \). Now let \( a, b \) be blocks such that \( am \) and \( mb \) are \( T \)-blocks. Since \( B_x \cap M \) is dense in \( M \), there are \( x, x' \in S \) such that \( x_0 \) and \( x'_0 \) start in \( x \), \( x(\infty,-n] \) ends with \( am \), and \( x'[-n,\infty) \) begins with \( mb \). Since \( S \) is in graph presentation, there is a point \( s \in S \) with \( s(-\infty,-1] = x(\infty,-1] \) and \( s[0,\infty) = x'[0,\infty) \). Since \( f \) is 1-block, \( f(s)(\infty,n] \) ends with \( am \) and \( f(s)(-n,\infty) \) begins with \( mb \). Thus \( amb \) is a \( T \)-block, which shows that \( m \) is a synchronizing block for \( T \).

Part of this work is from the second authors Habilitationsschrift [2].

2. The coded systems are the continuous images of transitive l.c. Markov shifts

In this section we consider surjective factor maps \( f : S \to T \), where \( S \) is a transitive Markov shift and \( T \) a compact subshift. We show that \( T \) is coded (Lemma 2.2). Then we show that in fact for any coded system \( T \) there is some transitive l.c. Markov shift \( S \) and a surjective continuous shift commuting map \( f : S \to T \) (Theorem 2.4). This is a formal analogy to “the transitive sofic systems are the continuous factors of transitive SFTs”.

First we point out that the transitivity condition on \( S \) is essential for selecting a proper subclass of the compact subshifts.

**Proposition 2.1.** There is a l.c. Markov shift \( S \) such that any compact metric dynamical system is a factor of \( S \).
Proof. Of course, $S$ will not be transitive. The following will be a dynamical extension of the Theorem of Alexandroff. Let $G_0$ be the graph with vertex set $\mathbb{Z}$ and two edges from vertex $n$ to $n+1$ for all $n \in \mathbb{Z}$. Let $S$ be the Markov shift defined by this graph. Let $(X, \sigma)$ be a compact metric dynamical system. Let $A$ be the subset of bi-infinite paths $x$ where the initial vertex of $x_0$ is 0 (thus the initial vertex of $x_i$ is $i$ for all $i$). Then $A$ is homeomorphic to $\{0,1\}^\mathbb{Z}$. By the Theorem of Alexandroff [10, Paragraph 41, Section VI, Corollary 3a] every compact metric HD-space is a continuous image of $\{0,1\}^\mathbb{Z}$. Thus there is a continuous onto map $g : A \to X$. Now extend this map to a map $f : S \to X$ as follows. For each $x \in S$ there is a unique $n \in \mathbb{Z}$ such that $S^n(x) \in A$, define $f(x) = \sigma^{-n}gS^n(x)$. Then $f$ is obviously onto and $fS = \sigma f$. We show continuity. Consider a convergent sequence of points $x^{(i)} \to x$ in $S$. Let $n$ such that $S^n(x) \in A$. There is an $i_0$ such that $S^n(x^{(i)}) \in A$ for all $i \geq i_0$, since $A$ is open. Thus $f(x^{(i)}) = \sigma^{-n}gS^n(x^{(i)})$, $i \geq i_0$ converges to $\sigma^{-n}gS^n(x) = f(x)$. \hfill \Box

In our notation, Proposition 2.1 of [1] says that a compact subshift $T$ is coded if and only if there is a transitive Markov shift $S$ in graph presentation and a 1-block map $f : S \to T$ with $f(S)$ dense in $T$. We shall extend this result in two ways. In Lemma 2.2 we show that $T$ is coded if there is a transitive Markov shift $S$ and a continuous shift commuting map $f : S \to T$ with $f(S)$ dense in $T$. Since not every continuous shift commuting map can be recoded to a 1-block map on a graph presentation this generalizes the sufficiency of the above mentioned result [1, Proposition 2.1]. On the other hand, in Theorem 2.4 we shall see that for a coded subshift $T$ there is always a surjective continuous shift commuting map from a transitive locally compact Markov shift $S$ onto $T$. The following lemma in particular implies that a surjective factor of a transitive Markov shift is always coded.

Lemma 2.2. Let $S$ be an transitive Markov shift and $f : S \to S_N$ a continuous shift commuting map into some full shift (the map $f$ has neither to be injective nor surjective). Then the closure $T = \text{cl}(f(S))$ of $f(S)$ in $S_N$ is coded.

Proof. We may assume that $S$ is given as a graph shift. Fix a vertex $z$ in this graph and a finite path $u$ from $z$ to itself. Let $q \in S$ be such that $q[0, \infty) = u^\infty$ and $q(-\infty, -1] = u^\infty$. Then $f(q)$ is a periodic point in $T$. Thus $f(q)[0, \infty) = p^\infty$ and $f(q)(-\infty, -1] = p^\infty$, for some $T$-block $p$ of length $|u|$. Since $f$ is continuous, there is $N$ such that $N$ is a multiple of $|u|$ and whenever $x[-N+1, N] = q[-N+1, N]$ then $f(x)[1, |u|] = p$.

Fix a $T$-block $w$. Since $f(S)$ is dense in $T$, there is an $x \in S$ with $f(x)[1, |w|] = w$. By continuity there is $n \in \mathbb{N}$ such that $f(y)[1, |w|] = w$ if $y \in S$ with $y[-n, |w| + n] = x[-n, |w| + n]$. Since $S$ is transitive and given in graph presentation, there is a point $z \in S$ with $z[-n, |w| + n] = x[-n, |w| + n]$, $z[r+1, \infty) = u^\infty$ and $z(-\infty, l-1] = u^\infty$ for some $r > |w| + n$, $l < -n$. By continuity there is $k > 2N$ such that $k$ is a multiple of $|u|$ and such that for $y \in S$ and $i \in \mathbb{Z}$ with $y[-k+i, i+k] = z[-k+i, i+k]$ it holds that $(fy)_i = (fz)_i$. Let $v_w := z[-2N-k+l, r+k+2N]$ and $C_w := (fz)[-2N-k+l, r+k+2N]$. Since $k$ and $N$ are multiples of $|u|$, the path $v_w$ is a loop at vertex $z$ beginning and
such that \(y \in S\) with \(y[-N+1,|C_w|+N] = q[1,N]v_wq[1,N]\) then \((f \cdot y)[1,|C_w|] = C_w\), by definition of \(k\) and \(N\).

We show that any bi-infinite concatenation of blocks from \(\{C_w|w\ a \ T\text{-block}\}\) gives a point in \(T\). Let \(w_i\) be a \(T\)-block, \(i \in \mathbb{Z}\), and let \(z \in S_N\) with \(z[0,\infty) = C_{w_q}C_{w_1}\) and \(z(-\infty,-1) = \cdots C_{w_{-2}}C_{w_{-1}}\). Let \(y \in S\) with \(y[0,\infty) = v_{w_q}v_{w_1}, \ldots, y(-\infty,-1) = \cdots v_{w_{-2}}v_{w_{-1}}\). Then \(f \cdot y \in T\) and by the above \(f \cdot y = z\). Thus \(z \in T\). Since the block \(w\) is a subblock of \(C_w\), these bi-infinite concatenations form a dense set in \(T\) and thus the set \(\{C_w|w\ a \ T\text{-block}\}\) is a code for \(T\). \(\Box\)

The following technical lemma will be used in the proof of the main Theorem 2.4.

**Lemma 2.3.** Let \(w_1, w_2, \ldots\) be a code for a coded system \(T\). Let \(p\) be the greatest common divisor (g.c.d.) of the lengths of the code words. Let \(C:= \{x \in T|\ for\ all\ n \geq 0\ there\ is\ an\ N_0\ such\ that\ for\ all\ N \geq N_0\ there\ is\ a\ point\ y \in T\ with\ y[-n,n] = x[-n,n] and y[-Np,Np - 1] is a concatenation of code words\}.

Then \(C\) is closed and for each point \(x \in T\) there is an \(0 \leq i < p\) such that \(T^i x \in C\).

**Proof.** We show that \(T-C\) is open. If \(x \in T-C\), then there is an \(n\) such that there is no \(N_0\) with the property that for all \(N \geq N_0\) there is a \(y \in T\) with \(y[-n,n] = x[-n,n]\) and \(y[-Np,Np - 1]\) is a concatenation of code words. But this is a property of \(x[-n,n]\), thus \(-n[x_{-n}, \ldots, x_n] \subset T-C\).

Now we show the second statement.

(*) By the choice of \(p\), there is a \(k \geq 0\) such that there is a concatenation of code words of length \(mp\) for each \(m \geq k\).

Let \(x \in T\) be given. For each \(n\) there is a point \(y(n)\) and numbers \(l(n) \leq -n, n \leq r(n)\) such that \(y(n)[-n,n] = x[-n,n]\) and \(y(n)[l(n), r(n)]\) is a concatenation of code words.

Write \(l(n)\) as \(-a_n p + i_n\) with \(a_n \geq 1, 0 \leq i_n < p\). Then \(r(n) = b_n p + i_n - 1\) for some \(b_n \geq 0\) since \(r(n) - l(n) + 1\) has to be divisible by \(p\).

Thus by (*), one can find for each large enough \(N\) a point \(y\) such that \(y[-Np + i_n, Np + i_n - 1]\) is a concatenation of code words and \(y[-n,n] = x[-n,n]\) (just add the suitable concatenations of code words to the left and right of \(y(n)[l(n), r(n)]\) and then extend somehow to a point in \(T\)). There is some \(0 \leq i < p\) such that \(i_n = i\) for infinitely many \(n\).

Thus, given any \(n\), there will be an \(m \geq n + i\) such that \(i_m = i\), thus for all large enough \(N\) there will be a point \(y\) such that \(y[-m,m] = x[-m,m]\) and \(y[-Np+i, Np + i - 1]\) is a concatenation of code words. But then \((T^i y)[-n,n] = (T^i x)[-n,n]\) and \((T^i y)[-Np,Np - 1]\) is a concatenation of code words, thus \(T^i x \in C\). \(\Box\)

**Theorem 2.4.** A compact subshift \(T\) is coded iff there is a transitive l.c. Markov shift \(S\) and a surjective continuous shift commuting map \(f:S \rightarrow T\).

**Remark 2.5.** In general \(f\) cannot be chosen to be a label map, since then the image system would have to be synchronized, Theorem 1.1. Thus if \(T\) is coded but not
synchronized, for example the Dyck shift ([3, Example 0.10, 5, Section 3]) and if \( f : S \to T \) is a surjective continuous shift commuting map then \( f \) cannot be a label map. Even if \( S \) is synchronized there are further restrictions for a label map to exist, see Section 3.

**Proof of the theorem.** Let \( T \) be a compact subshift, \( S \) a transitive l.c. Markov shift and \( f : S \to T \) a shift commuting continuous surjection. Then \( T \) is coded by Lemma 2.2.

Now let \( T \) be a coded system. We construct a transitive l.c. Markov shift \( S \), given by a graph \( G \), and a surjective factor map \( f \) from \( S \) onto \( T \), that is \( f : S \to T \) will be continuous, shift commuting and onto.

Let \( w_1, w_2, \ldots \) be a code for \( T \). Let \( p \) be the g.c.d. of the lengths of the code words. Let \( C \) be the set defined as in Lemma 2.3.

Since \( C \) is closed, it is a compact metric space. Let \( G_0 \) be the graph with vertex set \( \mathbb{Z} \) and two edges from vertex \( n \) to \( n+1 \) for all \( n \in \mathbb{Z} \). Let \( S_0 \) be the Markov shift defined by the graph \( G_0 \). Let \( A \) be the subset of bi-infinite paths \( x \) where the initial vertex of \( x_0 \) is 0 (thus the initial vertex of \( x_i \) is \( i \) for all \( i \)). Then \( A \) is homeomorphic to \( \{0,1\}^\mathbb{Z} \).

Thus, by the Theorem of Alexandroff [10, Paragraph 41, Section VI, Corollary 3a], there is a continuous surjection \( g : A \to C \).

**The graph \( G \) for the Markov shift \( S \).** We first need to choose a special subset of points in \( T \) and a sequence of integers \( n_0 < n_1 < n_2 < \cdots \).

Let \( n_0 = |w_1| \). Now assume \( n_{i-1} \) is given. Since \( C \) is compact, there are \( z^{i,1}, \ldots, z^{i,m(i)} \in C \) such that for every \( z \in C \) there is some \( 1 \leq m \leq m(i) \) with \( z[-n_{i-1},n_{i-1} - 1] = z^{i,m}[-n_{i-1},n_{i-1} - 1] \). We choose such a set \( \{z^{i,1}, \ldots, z^{i,m(i)}\} \) with minimal cardinality. By the definition of \( C \) and since \( A \) is compact, there is an \( n_i > n_{i-1} \) such that

- \( n_i \) is a multiple of \( |w_1| \),
- there are points \( y^{i,1}, \ldots, y^{i,m(i)} \in T \) such that for each \( 1 \leq m \leq m(i) \) it holds that
  \( y^{i,m}[-n_{i-1},n_{i-1} - 1] = z^{i,m}[-n_{i-1},n_{i-1} - 1] \) and \( y^{i,m}[-n_i,n_i - 1] \) is a concatenation of code words,
- for every point \( x \in A \) the block \( gx[-n_{i-1},n_{i-1} - 1] \) is determined by \( x[-n_i,n_i - 1] \).

Now we obtain the graph \( G \) from \( G_0 \) by simply adding for each \( n_i, i \geq 1 \), a new disjoint path of length \( |w_1| \) from the vertex \( n \) of \( G_0 \) to the vertex \( -n \) of \( G_0 \), more precisely:

- The vertex set of \( G \) is \( V := \mathbb{Z} \cup \bigcup_{i \geq 1} \{(i,j) | 0 \leq j < |w_1|\} \) and there
  - are two edges from vertex \( i \) to vertex \( i+1 \), called \( e_i \) and \( e_i' \), for each \( i \in \mathbb{Z} \),
  - is an edge \( e_{i,0} \) from vertex \( n_i \) to vertex \( (i,1) \), for each \( i \geq 1 \),
  - is an edge \( e_{i,j} \) from \( (i,j) \) to \( (i,j+1) \), \( 1 \leq j < |w_1| - 1 \), for each \( i \geq 1 \),
  - is an edge \( e_{i,|w_1|-1} \) from vertex \( (i,|w_1| - 1) \) to vertex \( -n_i \), for each \( i \geq 1 \).

Let \( S \) be the transitive l.c. Markov shift given by the graph \( G \).

**The factor map \( f \).** Write \( w_1 = s_0 s_1 \ldots s_{|w_1| - 1} \) where the \( s_i \) are symbols of \( T \) and let \( E_i := \{e_i, e_i'\} \), \( i \in \mathbb{Z} \).

For \( x \in S \) we define \( f(x)_0 \) as follows:

1. If \( x_0 = e_{i,j} \) for some \( i \geq 1 \), \( 0 \leq j < |w_1| \) then let \( f(x)_0 = s_j \).
2. If \( x_0 \in E_n \) for some \( n \in \mathbb{Z} \) then choose \( i \geq 1 \) minimal with \( -n_i \leq n \leq n_i - 1 \).
(2a) If there is some \(-n-n_1 \leq k \leq -n+n_1-1\) such that \(x_k \notin E_{n+k}\) then let \(f(x)_0 = s_j\) where \(j = n \mod |w_1|\).

(2b) If \(x_k \in E_{n+k}\) for all \(-n-n_1 \leq k \leq -n+n_1-1\) and \((x_{-n}-n_{-1} = e_{i_1} \mid w_1\) or \(x_{-n+n_i} = e_{i_0}\) then we can choose \(x' \in A\) with \(x'[-n_i,-n_i-1]\) and \(x_{-n+n_i} = e_{i_0}\).

(2c) If \(x_k \in E_{n+k}\) for all \(-n-n_{i+1} \leq k \leq -n+n_{i+1}-1\) then there is a unique \(1 \leq m \leq m(i)\) with \(g(x')[-n_{i+1},-n_{i+1}-1] = z^{i, m}[-n_{i+1},-n_{i+1}-1]\).

Let \(f(x)_0 = (y^{i, m})_n\).

We show that \(f\) is a continuous shift commuting surjection onto \(T\).

By definition \(f\) is shift commuting and continuous (for each edge \(e \in G\) there is some \(n \geq 0\) such that for \(x \in S\) with \(x_0 = e\), \(f(x)_0\) is determined by \(x[-n,n]\)). Assume we have shown that \(f(S) \subset T\). Then \(C \subset f(S) \subset T\), since for a point \(x \in A\) the case \(2c\) implies \(f(x) = g(x)\). Since \(f(S)\) is shift-invariant, it contains all translates of \(C\), thus by Lemma 2.3 \(f\) contains \(T\). Thus \(f\) is surjective factor map onto \(T\).

It remains to show that \(f(S) \subset T\).

Let \(x \in S\). If there is some \(n \in \mathbb{Z}\) such that \(S^n x \in A\) then \(f x = S^{-n} g(S^n x) \in S^{-n} C \subset T\).

If \(S^n x \notin A\) for all \(n \in \mathbb{Z}\), then let \(M := \{ r \in \mathbb{Z} \mid x_r \in \bigcup_{n \in \mathbb{Z}} E_n \text{ and } x_{r+1} \notin \bigcup_{n \in \mathbb{Z}} E_n \}\).

We will show that \(f(x)[r+1,s]\) is a concatenation of code words for any two consecutive numbers \(r < s\) in \(M\), that \(f(x)[-R,r]\) is a concatenation of code words if \(M\) has a minimum \(r\), and that \(f(x)[s+1, \infty)\) is a concatenation of code words if \(M\) has a maximum \(s\).

Thus assume that \(r < s\) are two consecutive numbers in \(M\). Then there is some \(i \geq 1\) such that \(x_r \in E_{n-1}\) and \(x_{r+1} = e_{i_0}\). Then \(x_{r+2} = e_{i_1}, \ldots, x_{r+|w_1|} = e_{i_0} x_{r+1}\) and thus \(f(x)[r+1,r+|w_1|] = w_1\) by definition of \(f\). Since \(s \in M\) there is some \(j \geq 1\) such that \(x_s \in E_{n-1}\) and \(x_{s+1} = e_{j_0}\). Since \(r < k < s\) implies \(k \notin M\), we have that \(x_{r+|w_1|}, \ldots, x_s\) is a shortest path from vertex \(-n_{i+1}\) to vertex \(n_j\) using only the edges from \(\bigcup_{n \in \mathbb{Z}} E_n\). By shifting, we may assume that \(s = n_j - 1\), then \(x[-n_{i+1}, n_j - 1]\) is a path from vertex \(-n_i\) to vertex \(n_j\) using only the edges from \(\bigcup_{n \in \mathbb{Z}} E_n\). Thus, we want to show that \(f x[-n_i, n_j - 1]\) is a concatenation of code words.

If \(i = j = 1\) then for each \(-n_i \leq k \leq n_1\) rule 2b determines \((f x)_k\) and thus for some \(1 \leq m \leq m(1)\) we have that \(f x[-n_i, n_i - 1] = y^{i, m}[-n_i, n_i - 1]\) which is a concatenation of codewords by choice of the point \(y^{i, m}\).

If \(i = j > 1\) then for each \(-n_i \leq k \leq n_i\) rule 2c applies, and thus \(f x[-n_i, n_i - 1] = g x'[-n_i, n_i - 1]\) for some \(x' \in A\) with \(x'[-n_i, n_i - 1] = x[-n_i, n_i - 1]\). For \(-n_i \leq k < -n_i\) and for \(-n_i \leq k < n_i\) rule 2b applies, and thus \(f x[-n_i, n_i - 1] = y^{i, m}[-n_i, n_i - 1]\) and \(f x[-n_i, n_i - 1] = y^{i, m}[-n_i, n_i - 1]\) where \(1 \leq m \leq m(i)\) is determined by the fact that \(g x'[-n_i, n_i - 1] = z^{i, m}[-n_i, n_i - 1]\) for any \(x' \in A\) with \(x'[-n_i, n_i - 1] = x[-n_i, n_i - 1]\) and \(y^{i, m}[-n_i, n_i - 1]\) we get \(f x[-n_i, n_i - 1] = y^{i, m}[-n_i, n_i - 1]\) which is a concatenation of code words by choice of \(y^{i, m}\).
If $i > j$, then as in the previous argument $f^x[-n_j, n_j - 1] = y^{i, m}[-n_j, n_j - 1]$ for some $1 \leq m \leq m(j)$, determined by $x[-n_j, -n_j - 1]$. For $-n_i \leq k < n_j$ rule 2a applies and thus $f^x[-n_i, -n_j - 1] = (w_1)^p$ where $p = (n_i - n_j)/|w_1|$, since $n_i$ and $n_j$ are both multiples of $|w_1|$. Thus $f^x[-n_i, n_j - 1] = (w_1)^p y^{i, m}[-n_j, n_j - 1]$, which is a concatenation of code words.

The case $i < j$ is symmetric.

If $M$ has a minimum $r$ then the arguments above for $i > j$ show that $f^x(-\infty, r] = (w_1)^\infty y^{i, m}[-n_j, n_j - 1]$ for some $1 \leq m \leq m(j)$. And similarly if $M$ has a maximum $s$ then $f^x(s + 1, \infty) = y^{i, m}[-n_i, n_i - 1](w_1)^\infty$. □

3. Factors of transitive l.c. Markov shifts via label maps

A compact subshift which is the surjective image of a transitive l.c. Markov shift under a label map has to be synchronized, Theorem 1.1. On the other hand, a given synchronized system does not have to be the image of a transitive l.c. Markov shift under a label map (although there will always be a continuous factor map, by Theorem 2.4). We show various restrictions which lead to three explicit examples. It remains an open problem to characterize the synchronized systems which admit a Markov shift cover with a surjective label map. At least we show that such synchronized systems do not have to be sofic (Example 3.10).

Recall that a (compact) subshift $R$ of a compact subshift $T$ is called an “isolated subsystem” if it can be obtained as $R = \bigcap_{i \in \mathbb{Z}} T^i O$ where $O$ is an open subset of $T$.

**Lemma 3.1.** If $T$ is a synchronized system and a 1-block factor of some transitive l.c. Markov shift $S$, then any isolated subsystem $R$ of $T$ has to have a synchronizing block (although it might be non-transitive).

**Proof.** Since $O$ is a countable union of cylinder sets and $R \subset O$, by compactness of $R$ there is a finite union of cylinder sets $A$ with $R \subset A \subset O$. Shift invariance of $R$ implies $R \subset \bigcap_{i \in \mathbb{Z}} T^i A$. Thus the subsystem $R$ is given by excluding a finite list of blocks. Recode $T$ and its Markov cover $S$ to a higher block system such that the blocks excluded for $R$ become symbols. Then $R$ is the image of the graph that remains after deleting all the edges labeled with the forbidden symbols. Thus $R$ is the image of a countable graph by a label map. Thus there is a synchronizing block in $R$ by Theorem 1.1. □

**Example 3.2.** A synchronized system $T$ with an isolated subsystem $M$ which does not have synchronizing blocks. In particular, $T$ does not have a surjective Markov cover via a label map, by Lemma 3.1.

Let $M \subset \{0, 1\}^\mathbb{Z}$ be a subshift without synchronizing blocks (for example the Morse-shift or the Dyck shift) and fix $m = (m_n)_{n \in \mathbb{Z}} \in M$. Consider the following labeled graph. The vertex set is $\mathbb{Z}$ and
• for each \( n \in \mathbb{Z} \) there is an edge from \( n \) to \( n + 1 \) labeled \( m_n \),
• for each \( n \geq 0 \) there is an edge from \( n \) to \( -n \) labeled 2.

Let \( T \subset \{0,1,2\}^\mathbb{Z} \) be the compact subshift which is the closure of all bi-infinite sequences of labels that can be read off in this labeled graph. Then the block 22 is a synchronizing block for \( T \), since the only path of length 2 which is labeled 22 begins and ends at vertex 0. The isolated subsystem \( M \) using as symbols only 0 and 1 has no synchronizing block. \( \square \)

We need some preparations for the next example. For a subshift \( T \) and \( n \in \mathbb{N} \) let \( B_n(T) \) denote the set of \( T \)-blocks of length \( n \) and let \( B(T) = \bigcup_{n \in \mathbb{N}} B_n(T) \). Let \( h(w) := \limsup 1/\log \text{card}(v \in B_n(T) \mid vw \in B(T)) \).

**Observation 3.3.** Let \( T \) be a synchronized system. Then \( h(w) = h(w') \) for all synchronizing \( T \)-blocks \( w \) and \( w' \).

**Proof.** Since \( T \) is transitive, for \( w,w' \in B(T) \) there is a \( T \)-block \( u \) such that \( wuw' \in B(T) \). If \( w' \) is synchronizing, then \( wuw'v \in B(T) \) iff \( w'v \in B(T) \). Thus \( h(w) \geq h(w') \). By symmetry \( h(w) = h(w') \). \( \square \)

**Definition 3.4.** Let \( T \) be synchronized. We say that \( T \) “obtains its entropy at the synchronizing blocks” if there is a synchronizing \( T \)-block \( w \) such that \( h(w) = h_{\text{top}}(T) \).

The non-synchronizing subshift \( NS(T) \) of a synchronized system \( T \) is the set of all points which never see a synchronizing block. It is a compact subshift, but not transitive in general. The synchronized subshift \( SYN(T) \), which is an open subshift, is the complement of \( NS(T) \) in \( T \). Since \( SYN(T) \) is an open subshift of \( T \), it is a transitive locally compact subshift and, by definition, every point in \( SYN(T) \) sees a synchronizing point. The subshift \( SYN(T) \) is actually a transitive l.c. Markov shift. We give a sketch of the proof, a complete argument can be found in [2, Lemma 7.1.10, 6, Lemma 1.10], where it was shown that a locally compact subshift \( R \) such that every point sees a synchronizing block is conjugate to a locally compact Markov shift \( S_G \) in graph presentation. Now we indicate the argument.

For each point \( x \in R \) choose \( k(x) \geq 0 \) minimal such that \( V(x) := \{ y \in R \mid y[-k(x),k(x)] = x[-k(x),k(x)] \} \) is compact and such that \( x[-k(x),k(x)] \) is a synchronizing block. By the minimality of \( k(x) \), the sets \( V(x) \) form a countable partition of \( R \) into compact open sets.

Define a graph \( G \) with vertex set \( V := \{ V(x) \mid x \in R \} \) and edge set \( E := \{ (V_0,V_1) \mid V_0 \cap R^{-1} V_1 \neq \emptyset, V_0,V_1 \in V \} \), where \( (V_0,V_1) \in E \) is a directed edge from vertex \( V_0 \) to vertex \( V_1 \). Let \( S_G \subset E^\mathbb{Z} \) be the Markov shift defined by \( G \). Then define \( g:S_G \to R \) by \( gs := \bigcap_{n \in \mathbb{Z}} R^{-n}V_s \) for \( s = (V_n)_{n \in \mathbb{Z}} \in S_G \). Now one can check that \( g \) is well defined and a conjugacy.

For transitive locally compact Markov shifts \( S \) the Gurevic entropy of \( S \), \( h_G(S) \), is defined to be the topological entropy of the 1-point compactification of \( S \) [7, 8]. Then
Lemma 3.5. Let $T$ be a synchronized system. Then
\[
h_{\top}(T) = \max \{h_{G}(\text{SYN}(T)), h_{\top}(\text{NS}(T))\}.
\]

Proof. “$\geq$” holds, by the variational principle for the Gurevic entropy [7, 8]. Let $\mu$ be an ergodic measure on $T$. If $\mu(\text{NS}(T)) = 1$ then $h_{\top}(\text{NS}(T)) \geq h_{\mu}(\text{NS}(T)) = h_{\mu}(T)$. If $\mu(\text{NS}(T)) < 1$ then by ergodicity $\mu(\text{SYN}(T)) = 1$ and thus by the variational principle for Gurevic entropy $h_{G}(\text{SYN}(T)) \geq h_{\mu}(\text{SYN}(T)) = h_{\mu}(T)$. Thus, max$\{h_{G}(\text{SYN}(T)), h_{\top}(\text{NS}(T))\} \geq h_{\mu}(T)$ and the lemma follows from the variational principle [12]. □

Lemma 3.6. Let $T$ be a synchronized system which does not obtain its entropy at the synchronizing blocks and where $\text{NS}(T)$ is a transitive SFT. Then $T$ cannot be a label factor of a transitive l.c. Markov shift.

Proof. We first show that the hypothesis implies that $\text{NS}(T)$ has full entropy. Let $\phi: \text{SYN}(T) \to S_{G}$ be a conjugacy from $\text{SYN}(T)$ to a transitive l.c. Markov shift $S_{G}$ in graph presentation [2, Lemma 7.1.10, 6, Lemma 1.10]. Let $w$ be a synchronizing $T$-block and fix $x \in \text{SYN}(T)$ and $k \leq k'$ such that $x[k, k'] = w$. Fix $n \geq k' - k$ such that $x'[n, \infty) = \phi x'[-n, n] \implies x'[k, k'] = w$. Let $p := \phi x'[-n, n]$. Let $z$ denote the initial vertex of $p$. Fix a path $p'$ such that $pp'$ is a loop at $z$ of length, say $M$. Let $q, q'$ be distinct loops at $z$ of the same length, say $L$. Let $y, y' \in S_{G}$ with $y = (pp'q)^{\infty}$, $y' = (pp'q')^{\infty}$, $y[-n, -n + M + L - 1] = pp'q$ and $y'[n, -n + M + L - 1] = pp'q'$. Then $\phi^{-1}y$ and $\phi^{-1}y'$ are distinct periodic points in $T$ of period $M + L$. Thus, $\phi^{-1}y[k, k + M + L - 1] \neq \phi^{-1}y'[k, k + M + L - 1]$ and $\phi^{-1}y[k, k'] = \phi^{-1}y'[k, k'] = w$. Thus for each $L \geq 1$ the number of loops in $G$ at $z$ of length $L$ is at most $\#\{v \in B(T)|wv \in B_{M+L}(T)\}$. Thus $h_{G}(\text{SYN}(T)) = h_{G}(S_{G}) \leq h(w) < h_{\top}(T)$, since $T$ does not obtain its entropy at synchronizing blocks. Thus by Lemma 3.5 $h_{\top}(\text{NS}(T)) = h_{\top}(T)$.

Now assume $T$ is a 1-block factor of a transitive l.c. Markov shift $S$ in graph presentation, say $f: S \to T$. Since $\text{NS}(T)$ is a SFT, it can be obtained from $T$ in excluding finitely many $T$-blocks $B$ of the same length $n$. Let $S' := \{s' \in (\mathbb{N}^{n})^{\mathbb{Z}} | s' = (s_{i}, s_{i+1}, \ldots, s_{i+n-1}), i \in \mathbb{Z} \text{ for some } s = (s_{i})_{i \in \mathbb{Z} \subseteq S} \}$ denote the $n$-block presentation of $S$. Then $S'$ is also in graph presentation (the edges being the $S$-paths of length $n$ and the vertices being the $S$-paths of length $n - 1$). There is a 1-block map from $S'$ onto $T$, namely $f': S' \to T$ where $(f's')_{i} = fs_{i}$ for $s = (s_{i})_{i \in \mathbb{Z} \subseteq S}$ with $s' = (s_{i}, s_{i+1}, \ldots, s_{i+n-1})$, $i \in \mathbb{Z}$. Now let $S''$ be the subgraph of $S'$ consisting of the edges $e' = (e_{0}, \ldots, e_{n-1})$ such that $(f'_{e_{0}}, \ldots, f'_{e_{n-1}}) \in B$. Then $f'(S'') = \text{NS}(T)$. By the proof of Theorem 1.1 there is some vertex $x$ in the graph of $S''$ and two $\text{NS}(T)$-blocks $a, b$ such that if $abc$ is a $\text{NS}(T)$-block, then there is a path in $S''$ leading to $x$ with label $a$, and a path starting from $x$ with label $bc$. Now fix a synchronizing $T$-block $m$ and a vertex $\beta$ in $S'$ such
that there is a path ending at $\beta$ with label $m$. Fix a path in $S'$ from $\beta$ to $\alpha$ with label $ua$ for some $T$-block $u$ and let $w = muab$. Then $w$ is a synchronizing $T$-block and for each block $c$ such that $abc$ is a $NS(T)$-block there is a path in $S'$ with label $wc$. Thus, since $NS(T)$ is a transitive SFT, $h_T(w) \geq h_{NS(T)}(ab) = h_{top}(NS(T)) = h_{top}(T)$, a contradiction. 

**Example 3.7.** A synchronized system $T$ which does not obtain its entropy at synchronizing blocks and whose non-synchronized subshift $NS(T)$ is a transitive SFT.

Fix $m = (m_n)_{n \in \mathbb{N}} \in \{2, 3\}^\mathbb{N}$ which sees every finite block from $\{2, 3\}^\mathbb{N}$. Define a labeled graph with vertex set $\mathbb{Z}$ and such that

- for each $n \geq 1$ there is an edge labeled $m_n$ from vertex $n$ to vertex $n + 1$,
- for each $n \geq 1$ there is an edge labeled 0 from vertex $n$ to vertex $-n$,
- there is an edge labeled 1 from vertex 0 to vertex 1,
- for each $n < 0$ there is an edge labeled 0 from vertex $n$ to vertex $n + 1$.

Let $T$ be the compact subshift which is the closure of all bi-infinite sequences of labels that can be read off in this graph. Then $T$ is synchronized and 1 is a synchronizing $T$-block. Since $\{2, 3\}^\mathbb{Z}$ is a subset of $T$, we have $h_{top}(T) \geq \log 2$. The number of paths in the graph of length $n$ starting in vertex 0 is $\leq 2^n/2$ and for each $T$-block $w$ of length $n$ beginning with symbol 1 there is a unique such path in the graph, thus $h(1) < \log 2$. Thus $h(1) < h_{top}(T)$ and $T$ does not obtain its entropy at synchronizing blocks. Furthermore, $NS(T) = \{2, 3\}^\mathbb{Z}$ which is a transitive SFT. Thus $T$ is not a label factor of a transitive l.c. Markovshift.

The third restriction is given by the following result from [3].

**Lemma 3.8.** If the synchronized system $T$ is almost Markov but the right Fischer cover is not a surjective cover (see [3] for definitions), then $T$ does not have a surjective label cover.

**Proof.** For almost Markov synchronized systems any cover has to factor through the right Fischer cover [3, Theorem 4.2]. Thus there cannot be a surjective label cover. 

**Example 3.9.** The $(1, \alpha, \beta)$-system $T$ is given by the closure of all bi-infinite sequences of labels that can be read off in the following labeled graph. The vertex set of the graph is $\mathbb{Z}$ and

- there is an edge labeled 1 from vertex $-1$ to vertex 0,
- for each $n \geq 0$ there is an edge labeled $\alpha$ from vertex $n$ to vertex $n + 1$,
- for each $n \leq -1$ there is an edge labeled $\beta$ from vertex $n - 1$ to vertex $n$,
- for each $n \geq 1$ there is an edge labeled $\beta$ from vertex $n$ to vertex $-n$.

This labeled graph is a 2-step-left resolving right Fischer cover for the $(1, \alpha, \beta)$-system $T$ and thus $T$ is almost Markov by [3, Theorem 4.2]. But this right Fischer cover is not surjective (for example it misses the point $\alpha^\infty$), thus by the above lemma the $(1, \alpha, \beta)$-system does not have a surjective Markov cover by a label map.
Finally, we show that in fact there are non-sofic synchronized systems admitting a surjective Markov cover by a label map.

**Example 3.10.** There are synchronized systems $T$ which are not sofic but have a surjective Markov cover by a label map.

Define a labeled graph as follows. The vertex set of the graph is $\mathbb{Z} \cup \{0'\}$ and
- there is an edge labeled 1 from vertex $-1$ to vertex 0,
- for each $n \geq 0$ there is an edge labeled $z$ from vertex $n$ to vertex $n+1$,
- for each $n \leq -1$ there is an edge labeled $\beta$ from vertex $n-1$ to vertex $n$,
- for each $n \geq 1$ there is an edge labeled $\beta$ from vertex $n$ to vertex $-n$,
- there is an edge labeled $\gamma$ from vertex 0 to vertex 0',
- there is an edge labeled $\delta$ from vertex 0' to vertex 0,
- there is an edge labeled $z$ from vertex 0' to vertex 0',
- there is an edge labeled $\beta$ from vertex 0' to vertex 0'.

Let the system $T$ be given by all bi-infinite sequences of labels that can be read off this labeled graph which is very similar to that in Example 3.9. Note that the set $T$ is already closed, i.e. the labeled graph yields a surjective Markov cover by a label map.

Then $T$ is not sofic since it has infinitely many follower sets, for example the follower sets of the blocks $1\xi^\omega$ are different for different $n$. The non-synchronized subshift is the closure of the orbit of $z^\omega \beta^\omega$.

**References**