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Note

On edge-disjoint pairs of matchings[☆]

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Abstract

For a graph G, consider the pairs of edge-disjoint matchings whose union consists of as many edges as possible. Let H be the largest matching among such pairs. Let M be a maximum matching of G. We show that $5/4$ is a tight upper bound for $|M|/|H|$.

Keywords: Matching; Maximum matching; Pair of edge-disjoint matchings

We consider finite, undirected graphs without multiple edges or loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The cardinality of a maximum matching of a graph G is denoted by $\nu(G)$.

For a graph G define $B_2(G)$ as follows:

$$B_2(G) \equiv \{(H, H') : H, H' \text{ are edge-disjoint matchings of } G\},$$

and set:

$$\lambda_2(G) \equiv \max\{|H| + |H'| : (H, H') \in B_2(G)\}.$$

Define:

$$\alpha_2(G) \equiv \max\{|H|, |H'| : (H, H') \in B_2(G) \text{ and } |H| + |H'| = \lambda_2(G)\},$$
$$M_2(G) \equiv \{(H, H') : (H, H') \in B_2(G), |H| + |H'| = \lambda_2(G), |H| = \alpha_2(G)\}.$$

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It is clear that $\alpha_2(G) \leq \nu(G)$ for all G. By Mkrtchyan's result [4], reformulated as in [2], if G is a matching covered tree then the inequality turns to an equality. Note that a graph is said to be matching covered (see [5]) if its every edge belongs to a maximum matching (not necessarily a perfect matching as it is usually defined, see e.g. [3]).

The aim of this paper is to obtain a tight upper bound for $\frac{\nu(G)}{\alpha_2(G)}$. We prove that $\frac{5}{4}$ is an upper bound for $\frac{\nu(G)}{\alpha_2(G)}$, and exhibit a family of graphs, which shows that $\frac{5}{4}$ cannot be replaced by any smaller constant. Terms and concepts that we do not define can be found in [1,3,7].

Let A and B be matchings of a graph G.

Definition. A path (cycle) $e_1, e_2, \ldots, e_l$ $(l \geq 1)$ is called A–B alternating if the edges with odd indices belong to $A \backslash B$ and the others to $B \backslash A$, or vice versa.

Definition. An A–B alternating path P is called maximal if there is no other A–B alternating path that contains P as a proper subpath.

The sets of A–B alternating cycles and maximal alternating paths are denoted by $C(A, B)$ and $P(A, B)$, respectively.

The set of the paths from $P(A, B)$ that have even (odd) length is denoted by $P_e(A, B)$ $(P_o(A, B))$.

The set of the paths from $P_o(A, B)$ starting from an edge of A (B) is denoted by $P_o^A(A, B)$ $(P_o^B(A, B))$.

Note that every edge $e \in A \cup B$ either belongs to $A \cap B$ or lies on a cycle from $C(A, B)$ or lies on a path from $P(A, B)$.

Moreover,

Property 1. (a) *if $F \in C(A, B) \cup P_e(A, B)$, then A and B have the same number of edges that belong to F,*
(b) *if $P \in P_o^A(A, B)$, then the difference between the numbers of edges that lie on P and belong to A and B is one.*

This property immediately implies:

Property 2. *If A and B are matchings of a graph G then*

$$|A| - |B| = |P_o^A(A, B)| - |P_o^B(A, B)|.$$

Berge's well-known theorem states that a matching M of a graph G is maximum if and only if G does not contain an M-augmenting path [1,3,7]. This theorem immediately implies:

Property 3. *If M is a maximum matching and H is a matching of a graph G then*

$$P_o^H(M, H) = \emptyset,$$

and therefore $|M| - |H| = |P_o^M(M, H)|$.

The proof of the following property is similar to the one of Property 3:

Property 4. *If $(H, H') \in M_2(G)$, then $P_o^{H'}(H, H') = \emptyset$.*

Let G be a graph. Over all $(H, H') \in M_2(G)$ and all maximum matchings M of G, consider the pairs $((H, H'), M)$ for which $|M \cap H|$ is maximized. Among these, choose a pair $((H, H'), M)$ such that $|M \cap H'|$ is maximized.

From now on H, H' and M are assumed to be chosen as described above. For this choice of H, H' and M, consider the paths from $P_o^M(M, H)$ and define M_A and H_A as the sets of edges lying on these paths that belong to M and H, respectively.

Lemma 1. $C(M, H) = P_e(M, H) = P_o^H(M, H) = \emptyset$.

Proof. Property 3 implies $P_o^H(M, H) = \emptyset$. Let us show that $C(M, H) = P_e(M, H) = \emptyset$. Suppose that there is $F_0 \in C(M, H) \cup P_e(M, H)$. Define:

$$M' \equiv [M \backslash E(F_0)] \cup [H \cap E(F_0)].$$

Consider the pair $((H, H'), M')$. Note that M' is a maximum matching, and

$$|H \cap M'| > |H \cap M|,$$

which contradicts $|H \cap M|$ being maximum. ■

Corollary 1. $M \cap H = M \backslash M_A = H \backslash H_A$.

Lemma 2. *Each edge of* $M_A \backslash H'$ *is adjacent to two edges of* H'.

Proof. Let e be an arbitrary edge from $M_A \backslash H'$. Note that $e \in M$, $e \notin H$, $e \notin H'$. Now, if e is not adjacent to an edge of H', then $H \cap (H' \cup \{e\}) = \emptyset$ and

$$|H| + |H' \cup \{e\}| > |H| + |H'| = \lambda_2(G),$$

which contradicts $(H, H') \in M_2(G)$.

On the other hand, if e is adjacent to only one edge $f \in H'$, then consider the pair (H, H''), where $H'' \equiv (H' \backslash \{f\}) \cup \{e\}$. Note that

$$H \cap H'' = \emptyset, \qquad |H''| = |H'|$$

and

$$|H'' \cap M| > |H' \cap M|,$$

which contradicts $|H' \cap M|$ being maximum. ■

Lemma 3. $C(M_A, H') = P_e(M_A, H') = P_o^{M_A}(M_A, H') = \emptyset$.

Proof. First of all, let us show that $C(M_A, H') = P_e(M_A, H') = \emptyset$. For the sake of contradiction, suppose that there is $F_0 \in C(M_A, H') \cup P_e(M_A, H')$. Define:

$$H'' \equiv [H' \backslash E(F_0)] \cup [M_A \cap E(F_0)].$$

Consider the pair of matchings (H, H''). Note that due to the definition of an alternating path or cycle we have $M_A \cap H = \emptyset$, therefore

$$H \cap H'' = \emptyset,$$
$$|H| + |H''| = |H| + |H'| = \lambda_2(G)$$

(see (a) of Property 1).

Thus $(H, H'') \in M_2(G)$ and

$$|H'' \cap M| > |H' \cap M|,$$

which contradicts $|H' \cap M|$ being maximum.

On the other hand, the end-edges of a path from $P_o^{M_A}(M_A, H')$ are from M_A and are adjacent to only one edge from H' contradicting Lemma 2. Therefore $P_o^{M_A}(M_A, H') = \emptyset$. ■

Lemma 4. $|H'| = |P_o^{H'}(M_A, H')| + |H_A| + \nu(G) - \alpha_2(G)$.

Proof. Due to Property 2

$$|H'| - |M_A| = |P_o^{H'}(M_A, H')| - |P_o^{M_A}(M_A, H')|,$$

and due to (b) of Property 1 and Property 3

$$|M_A| - |H_A| = |P_o^{M}(M, H)| = |M| - |H| = \nu(G) - \alpha_2(G).$$

By Lemma 3, $P_o^{M_A}(M_A, H') = \emptyset$, therefore

$$|H'| = |P_o^{H'}(M_A, H')| + |M_A| = |P_o^{H'}(M_A, H')| + |H_A| + \nu(G) - \alpha_2(G). \quad ■$$

Lemma 5. *Let $P \in P_o(M, H)$ and assume that $P = m_1, h_1, m_2, \ldots, h_{l-1}, m_l$, $l \geq 1, m_i \in M, 1 \leq i \leq l, h_j \in H, 1 \leq j \leq l-1$. Then $l \geq 3$ and $\{m_1, m_l\} \subseteq H'$.*

Proof. Let us show that $l \geq 3$. Note that if $l = 1$, then $P = m_1$, and clearly $m_1 \in H'$ as otherwise we could enlarge H by adding m_1 to it which contradicts $(H, H') \in M_2(G)$. Thus $m_1 \in H'$. Define

$$H_1 = H \cup \{m_1\}, H_1' = H' \setminus \{m_1\}.$$

Note that $|H_1| + |H_1'| = |H| + |H'| = \lambda_2(G)$ and $|H_1| > |H|$ which contradicts $(H, H') \in M_2(G)$. Thus $l \geq 2$. Let us show that $m_1 \in H'$. If $m_1 \notin H'$, then define

$$H_1 \equiv (H \backslash \{h_1\}) \cup \{m_1\}.$$

Note that

$$H_1 \cap H' = \emptyset, \qquad |H_1| = |H|,$$

and

$$|H_1 \cap M| > |H \cap M|$$

which contradicts $|H \cap M|$ being maximum. Thus $m_1 \in H'$. Similarly, one can show that $m_l \in H'$.

Due to Property 4, $P_o^{H'}(H, H') = \emptyset$, thus there is i, $1 \leq i \leq l$, such that $m_i \in M \backslash (H \cup H')$. Since $\{m_1, m_l\} \subseteq H'$, we have $l \geq 3$. ■

Corollary 2. $|H_A| \geq 2(\nu(G) - \alpha_2(G))$.

Proof. Due to Lemma 5, every path $P \in P_o(M, H)$ has length at least five, therefore it contains at least two edges from H. Due to Property 3, there are

$$|P_o(M, H)| = \left|P_o^M(M, H)\right| = \nu(G) - \alpha_2(G)$$

paths from $P_o(M, H)$, therefore

$$|H_A| \geq 2(\nu(G) - \alpha_2(G)). \quad ■$$

Corollary 3. *Every vertex lying on a path from $P(M, H) = P_o^M(M, H)$ is incident to an edge from H'.*

Proof. Suppose w is a vertex lying on a path from $P(M, H) = P_o^M(M, H)$ and assume that e is an edge from M_A incident to the vertex w. Clearly, if $e \in H'$, then the corollary is proved therefore we may assume that $e \notin H'$. Note that $e \in M_A \backslash H'$ therefore due to Lemma 2, e is adjacent to two edges from H'. Thus w is incident to an edge from H'. ■

Let Y denote the set of the paths from $P(H, H')$ starting from the end-edges of the paths from $P_o^M(M, H)$. Note that Y is well-defined since due to Lemma 5 these end-edges belong to H'. According to Property 4, $Y \subseteq P_e(H, H')$, thus the set of the last edges of the paths from Y is a subset of H. Let us denote it by H_Y.

Lemma 6. (a) $|Y| = 2(\nu(G) - \alpha_2(G))$ *and the length of the paths from Y is at least four,*
(b) $\left|P_o^{H'}(M_A, H')\right| \geq \nu(G) - \alpha_2(G)$.

Proof. (a) Due to Property 4, all end-edges of the paths from $P_o^M(M, H)$ lie on different paths from Y. Therefore $|Y| = 2|P_o^M(M, H)| = 2(\nu(G) - \alpha_2(G))$.

Since the edges from H_Y are adjacent to only one edge from H', we conclude that they do not lie on paths from $P_o^M(M, H)$ (Corollary 3). Thus, due to Corollary 1, $H_Y \subseteq M \cap H$. Furthermore, since the first two edges of a path from Y lie on a path from $P_o^M(M, H)$, and the last edge does not, we conclude that its length is at least four.

(b) From $H_Y \subseteq M \cap H$ we get

$$|M \cap H| \geq |H_Y| = |Y| = 2|P_o^M(M, H)| = 2(\nu(G) - \alpha_2(G)).$$

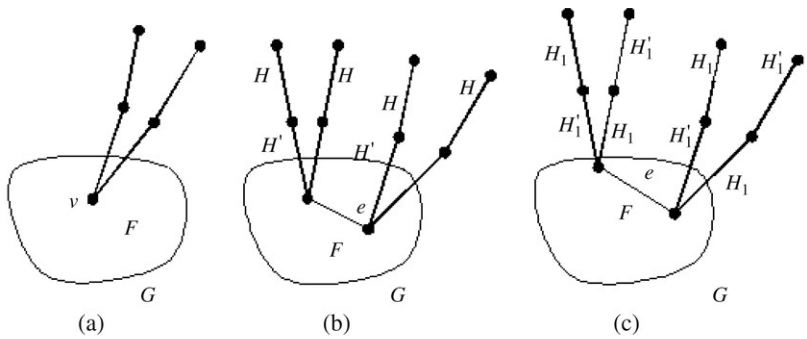

Fig. 1. Example achieving the bound of the theorem.

On the other hand, every edge from H_Y is adjacent to an edge from $H' \backslash M$, which is an end-edge of a path from $P_o^{H'}(M_A, H')$, therefore

$$2(\nu(G) - \alpha_2(G)) \leq |M \cap H| \leq 2\left|P_o^{H'}(M_A, H')\right|$$

or

$$\nu(G) - \alpha_2(G) \leq \left|P_o^{H'}(M_A, H')\right|. \quad \blacksquare$$

Theorem. *For every graph G the inequality* $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$ *holds.*

Proof. Lemma 4, (b) of Lemma 6 and Corollary 2 imply

$$\alpha_2(G) \geq \left|H'\right| = \left|P_o^{H'}(M_A, H')\right| + |H_A| + \nu(G) - \alpha_2(G) \geq 4(\nu(G) - \alpha_2(G)).$$

Therefore, $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$. $\blacksquare$

Remark 1. We have given a proof of the theorem based on the structural Lemma 4, (b) of Lemma 6 and Corollary 2. It is not hard to see that the theorem can also be proved directly using only (a) of Lemma 6. As the length of every path from Y is at least four, there are at least two edges from H' lying on each path from Y, therefore

$$\alpha_2(G) \geq |H'| \geq 2|Y| = 4(\nu(G) - \alpha_2(G)).$$

Remark 2. There are infinitely many graphs G for which

$$\frac{\nu(G)}{\alpha_2(G)} = \frac{5}{4}.$$

In order to construct one, just take an arbitrary graph F containing a perfect matching. Attach to every vertex v of F two paths of length two, as it is shown on the Fig. 1(a):

Let G be the resulting graph. Note that:

$$\nu(G) = \frac{|V(F)|}{2} + 2\,|V(F)| = \frac{5\,|V(F)|}{2}.$$

Let us show that for every pair of disjoint matchings (H, H') satisfying $|H| + \left|H'\right| = \lambda_2(G)$ and $e \in E(F)$ we have $e \notin H \cup H'$. On the opposite assumption, consider an edge $e \in E(F)$ and a pair (H, H') with $|H| + \left|H'\right| = \lambda_2(G)$ and $e \in H \cup H'$. Note that without loss of generality, we may always assume that H and H' contain the edges shown on the Fig. 1(b).

Now consider a new pair of disjoint matchings (H_1, H_1') obtained from (H, H') by changing only the edges that are shown on the Fig. 1(b) as it is shown on the Fig. 1(c).

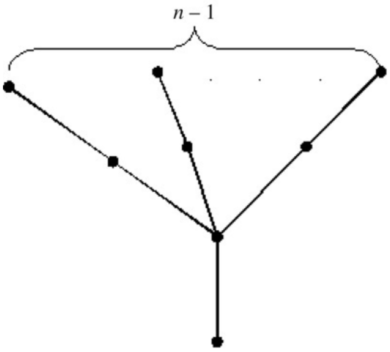

Fig. 2. An example of a graph G_n with $\frac{\nu(G_n)}{\lambda_2(G_n)-\alpha_2(G_n)} = n$.

Note that $|H_1| + \left|H_1'\right| = 1 + |H| + \left|H'\right| > \lambda_2(G)$, which contradicts the choice of (H, H'), therefore $e \notin H \cup H'$ and $\lambda_2(G) = 4\,|V(F)|$, $\alpha_2(G) = 2\,|V(F)|$, hence

$$\frac{\nu(G)}{\alpha_2(G)} = \frac{5}{4}.$$

See [6] for the characterization of these graphs.

Remark 3. In contrast with the bound $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$, it can be shown that for every positive integer $n \geq 2$ there is a graph G_n such that $\frac{\nu(G_n)}{\lambda_2(G_n)-\alpha_2(G_n)} = n$. Just consider the graph G_n shown on the Fig. 2.

Note that $\nu(G_n) = n$, $\lambda_2(G_n) = n + 1$, $\alpha_2(G) = n$ hence

$$\frac{\nu(G_n)}{\lambda_2(G_n) - \alpha_2(G_n)} = n.$$

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