Uniquely colorable mixed hypergraphs

Zsolt Tuza\textsuperscript{a,1}, Vitaly Voloshin\textsuperscript{b,*,2}, Huishan Zhou\textsuperscript{c}

\textsuperscript{a}Computer and Automation Institute, Hungarian Academy of Sciences, H-1111 Budapest, Kende u. 13-17, Hungary
\textsuperscript{b}Institute of Mathematics and Computer Science, Moldovan Academy of Sciences, str. Academiei, 5, Chişinău, MD-2028, Moldova
\textsuperscript{c}Department of Mathematics and Computer Science, Georgia State University, Atlanta, Georgia 30303-3083, USA

Received 15 June 1996; revised 15 January 2001; accepted 29 January 2001

Abstract

A mixed hypergraph consists of two families of edges: the \( C \)-edges and \( D \)-edges. In a coloring, every \( C \)-edge has at least two vertices of the same color, while every \( D \)-edge has at least two vertices colored differently. The largest and smallest possible numbers of colors in a coloring are termed the upper and lower chromatic number, \( \bar{\chi} \) and \( \chi \), respectively. A mixed hypergraph is called uniquely colorable if it has precisely one coloring apart from the permutation of colors. We begin a systematic study of uniquely colorable mixed hypergraphs.

In particular, we show that every colorable mixed hypergraph can be embedded into some uniquely colorable mixed hypergraph; we investigate the role of uniquely colorable subhypergraphs being separators, study recursive operations (orderings and subset contractions) and unique colorings, and prove that it is NP-hard to decide whether a mixed hypergraph is uniquely colorable.

We also discuss the weaker property where the mixed hypergraph has a unique coloring with \( \bar{\chi} \) colors and a unique coloring with \( \chi \) colors, where \( \bar{\chi} > \chi \). The class of these “weakly uniquely colorable” mixed hypergraphs contains all uniquely colorable graphs in the usual sense.

Keywords: Coloring; Mixed hypergraphs; Lower (upper) chromatic number; Unique colorability; Algorithmic complexity

\textsuperscript{*} Supported by COBASE.
\textsuperscript{*} Corresponding author.
\textit{E-mail address:} voloshin@math.mldnet.com (V. Voloshin).
\textsuperscript{1} Supported in part by the Hungarian Scientific Research Fund through Grants OTKA T-026575 and T-032969.
\textsuperscript{2} Partially supported by Volkswagen-Stiftung Project No. I = 69041, COBASE short- and long-term grants (Georgia State University and University of Illinois at Urbana-Champaign), and CNR (University of Catania); while revising the paper partially supported by DFG (TU-Dresden).
1. Introduction

In the classical theory of coloring for graphs and hypergraphs, one usual goal is to find a color assignment that satisfies prescribed properties (e.g., no monochromatic edge occurs), and minimizes the number of colors.

In the present work, we deal with the assumptions introduced in [32], where the (hyper)edges, as usual, must not be monochromatic, while some other subsets of the structure are supposed to contain at least one monochromatic pair of elements. Such constraints imply lower as well as upper bounds on the possible number of colors. This paper is the first step towards the solution of the problem [32, p. 43, Problem 4] to describe the structures in which the two bounds coincide. More precisely, we study the structures admitting exactly one coloring apart from the permutation of colors. These uniquely colorable mixed hypergraphs are just the generalizations of the usual complete graphs (cliques) from the point of view of coloring. In the simplest form they appeared first in [6] as complete mixed interval hypergraphs.

We also investigate those structures where the colorings with both the smallest and largest possible numbers of colors are unique. In this way, we obtain a generalization of the uniquely colorable graphs in the classical sense.

Introduced in [31,32], the theory of mixed hypergraphs grows rapidly and represents an area with many possible applications. As models they may be applied in list-free modeling of list-colorings of graphs [29], investigation of the coloring properties of block designs [23,21,22,24] (with potential applications in coding theory [5,8]), integer programming [29,20], in resource allocation, Data Base Management, parallel computing, and scheduling of systems of power supplies, in the study of heredity in populations with sexual reproduction [33], and some other topics where the problems have a combinatorial nature. Also, a variant of a canonical Ramsey problem on edge colorings studied in [18,14] can be formulated using colorings of mixed hypergraphs.

It is worth mentioning that a special case of the problem of finding the maximum number of colors in hypergraphs was first discussed in [1], where it was shown that the essence of many multi-user source coding problems is a statement about vertex colorings of hypergraphs, which assign to the vertices of every edge a certain percentage of different colors.

There are several classes of mixed hypergraphs that have been introduced recently; among them are the mixed co-perfect [32], interval [6], quasi-interval [27], uncolorable [32,29], pseudo-chordal [35], circular [34], planar [32] hypergraphs, mixed hypertrees [29,32], monostars [9], mixed hypergraphs derived from block designs [23,21,22,24], and some other classes. They represent the generalizations of graphs, hypergraphs and colorings from different points of view. However, the mixed hypergraphs having precisely one feasible partition of the vertex set deserve special attention. First, they are situated on the “edge” of colorability, being the last structure that can be colored along the way from empty to uncolorable mixed hypergraphs. Second, as cliques in graphs, they lead to important theoretical bounds and efficient practical algorithms for finding both chromatic numbers and optimal colorings. Third, having an unexpectedly general
nature, they show that cliques in graphs represent the “tip of the iceberg” in general combinatorial relations of set partitions. Fourth and finally, as in all mathematics, where there is no solution for some structures and there are many solutions for some others, it is always important to know the conditions ensuring that a solution is unique. Therefore, the aim of the paper is to begin a systematic study of uniquely colorable mixed hypergraphs.

Motivated by the first version of this paper, the following classes of uniquely colorable mixed hypergraphs have been characterized: those with \( \chi = n - 1 \) and \( \chi = n - 2 \) in [26]; mixed hypertrees in [25]; and circular mixed hypergraphs in [34]. Moreover, pseudo-chordal mixed hypergraphs as a generalization of chordal graphs have been introduced and described in [35].

The paper is organized as follows. The necessary definitions are introduced and some properties of the uniquely colorable structures are described in the remaining part of this section. Section 2 presents a method to embed an arbitrary mixed hypergraph into a (weakly, lower, and upper) uniquely colorable hypergraph. In Section 3, it is shown how the chromatic polynomial can be computed from that of smaller substructures when the hypergraph in question has a uniquely colorable separator. Section 4 deals with recursive operations preserving unique colorability. Finally, in Section 5 we apply some known results of the theory of computational complexity to this new concept of hypergraph colorings.

1.1. Definitions and notation

Unless otherwise stated, we use the standard terminology of [4], and its extension for mixed hypergraphs, introduced in [32].

A mixed hypergraph is a triple \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \), where \( X \) is the vertex set and each of \( \mathcal{C}, \mathcal{D} \) is a family of subsets of \( X \), termed \( \mathcal{C} \)-edges and \( \mathcal{D} \)-edges, respectively. A proper \( k \)-coloring of a mixed hypergraph is a mapping from the vertex set into a set of \( k \) colors so that each \( \mathcal{C} \)-edge has two vertices with a common color and each \( \mathcal{D} \)-edge has two vertices with distinct colors.

A mixed hypergraph is \( k \)-colorable if it has a proper coloring with at most \( k \) colors, and uncolorable if it admits no colorings.

Throughout the paper we consider colorable mixed hypergraphs. The chromatic polynomial \( P(\mathcal{H}, k) \) gives the number of different proper \( k \)-colorings of \( \mathcal{H} \), where two colorings \( c_1 \) and \( c_2 \) are counted to be different whenever there is at least one vertex \( v \) with \( c_1(v) \neq c_2(v) \). A strict \( k \)-coloring is a proper \( k \)-coloring using all the \( k \) colors. The minimum number of colors in a strict coloring of \( \mathcal{H} \) is its lower chromatic number \( \underline{\chi} \); the maximum number is its upper chromatic number \( \overline{\chi} \).

We use \( n \) to denote \( |X| \) for the mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \). For each \( k \), let \( r_k \) be the number of partitions of the vertex set into \( k \) nonempty subsets (color classes) such that the coloring constraint is satisfied on each \( \mathcal{C} \)- and \( \mathcal{D} \)-edge. Such partitions are feasible. The vector \( R(\mathcal{H}) = (r_1, \ldots, r_n) = (0, \ldots, 0, r_1, \ldots, r_2, 0, \ldots, 0) \) is the chromatic
spectrum of $\mathcal{H}$. As it was discovered recently in [15], the chromatic spectrum may contain gaps, i.e. zeros can occur between positive numbers.

Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph. The subhypergraph induced by $X' \subseteq X$, denoted $\mathcal{H}[X']$, is the mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ defined by setting $\mathcal{C}' = \{ C \in \mathcal{C} : C \subseteq X' \}$ and $\mathcal{D}' = \{ D \in \mathcal{D} : D \subseteq X' \}$. Moreover, the mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ is a mixed subhypergraph of $\mathcal{H}$ if $X' \subseteq X$, $\mathcal{C}' \subseteq \mathcal{C}$, and $\mathcal{D}' \subseteq \mathcal{D}$. For the latter, we use the notation $\mathcal{H}' \subseteq \mathcal{H}$. The terms “subhypergraph” and “induced subhypergraph” will always be applied in the context of mixed hypergraphs.

A mixed hypergraph $\mathcal{H} = (X, \emptyset, \mathcal{D})$ is called $\mathcal{D}$-hypergraph and is denoted by $\mathcal{H}_D$. Similarly, a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \emptyset)$ is called $\mathcal{C}$-hypergraph and denoted $\mathcal{H}_C$. Thus, the classic graphs and hypergraphs from [4] become in our terminology $\mathcal{D}$-graphs and $\mathcal{D}$-hypergraphs, respectively, or simply graphs and hypergraphs. Sometimes we use the terms “graph” or “edge” in the usual sense of graph theory, if we disregard colorings.

We say that a mixed hypergraph is reduced if no edge is a subset of any other edge of the same type, and moreover, the size of each $\mathcal{C}$-edge is at least 3, and the size of each $\mathcal{D}$-edge is at least 2. As it follows from the splitting-contraction algorithm [32], the coloring properties of an arbitrary mixed hypergraph can be derived from the respective reduced mixed hypergraph. Therefore, without loss of generality, throughout the paper we consider the reduced mixed hypergraphs, unless explicitly stated otherwise.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called connected if for any pair of vertices $x, y \in X$ there exists an alternating sequence $x = z_0S_0z_1S_1z_2 \ldots z_tS_tz_{t+1} = y$ of vertices and $\mathcal{C}$- and $\mathcal{D}$-edges with $z_0, z_1, \ldots, z_{t+1} \in X$ and $S_0, S_1, \ldots, S_t \in \mathcal{C} \cup \mathcal{D}$, satisfying $z_0 \in S_0$, $z_{t+1} \in S_t$, and $z_i \in S_{i-1} \cap S_i$ ($i = 1, 2, \ldots, t$). Otherwise, $\mathcal{H}$ is called disconnected. A connected induced subhypergraph maximal under inclusion is called a connected component.

**Definition 1.1.** A mixed hypergraph $\mathcal{H}$ is called uniquely colorable ($uc$ hypergraph or $uc$ for short) if it has precisely one strict coloring apart from permutations of colors.

Equivalently, $\mathcal{H}$ is $uc$ if it allows exactly one feasible partition of the vertex set $X$ into color classes. Let us agree that the expression “unique coloring” means in the sequel “unique partition” into the corresponding number of color classes. Next, in order to make the setting more general, we also introduce some weaker concepts:

**Definition 1.2.** A mixed hypergraph $\mathcal{H}$ is called

(a) lower uniquely colorable ($luc$ for short) if it has a unique coloring with $\chi(\mathcal{H})$ colors;

(b) upper uniquely colorable ($uuc$ for short) if it has a unique coloring with $\bar{\chi}(\mathcal{H})$ colors;

(c) weakly uniquely colorable ($wuc$ for short) if it is both upper and lower uniquely colorable.
Evidently, if \( H \) is a uc hypergraph, then \( \chi(H) = \overline{\chi}(H) = \chi \) and \( r_{\chi}(H) = 1 \), therefore, \( P(H, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - \chi + 1) = \lambda^{\chi} \). If we treat a classic graph \( G \) as a special type of mixed hypergraph with no \( C \)-edges, then \( G \) is uniquely colorable if and only if \( G \) is a complete graph. Therefore, the unique colorability defined in this paper, when restricted to graphs, is different from the usual unique colorability of graphs which refers only to the colorings with \( \chi(G) \) colors [13, pp. 48–49]. However, since every graph is a uuc hypergraph in our definition, one can see that the uniquely colorable graphs in the classic sense represent a special case of the weakly uniquely colorable mixed hypergraphs.

It follows from the definitions that in mixed hypergraphs in each feasible partition of \( X \) into \( \chi \) color classes the union of any two color classes contains some \( D \)-edge. The combinatorially dual assertion (when minimum is inverted to maximum) states that in each feasible partition of \( X \) into \( \bar{\chi} \) color classes any further partition of any color class yields a \( C \)-edge which becomes improperly colored. Consequently, if \( H \) is uc, then both these properties hold.

2. Embedding into (weakly) UC hypergraphs

We show in this section that every mixed hypergraph having at least one coloring is a subhypergraph, and even an induced subhypergraph, of some (weakly) uniquely colorable mixed hypergraph.

**Definition 2.1.** In a mixed hypergraph \( H = (X, \mathcal{C}, \mathcal{D}) \) a sequence of vertices

\[ x = x_0, x_1, \ldots, x_k = y, \quad k \geq 1 \]

is called an \((x, y)\)-invertor if and only if \( x_i \neq x_{i+1} \) and \((x_i, x_{i+1}) \in \mathcal{D}\) for every \( i = 0, 1, \ldots, k - 1 \), and, moreover, the following implication holds:

\[ x_j \neq x_{j+1} \neq x_{j+2} \neq x_j \Rightarrow (x_j, x_{j+1}, x_{j+2}) \in \mathcal{C}, \quad j = 0, 1, \ldots, k - 2. \]

The \((x, y)\)-invertor is called odd or even if \( k \) is odd or even, respectively; and if \( k \geq 2 \), then \( x_1, \ldots, x_{k-1} \) are termed internal vertices.

In any proper coloring of an invertor, the colors alternate along consecutive vertices. If in a mixed hypergraph there exists an even \((x, y)\)-invertor, then in every coloring the vertices \( x \) and \( y \) have the same color. In contrast, if there exists an odd \((x, y)\)-invertor, then in every coloring the vertices \( x \) and \( y \) have different colors. Notice that both odd and even invertors are uc mixed subhypergraphs.

Now we embed the mixed hypergraph \( H \) into a mixed hypergraph \( H' \) such that \( H' \) has a unique coloring with \( \chi(H') \) colors and a unique coloring with \( \overline{\chi}(H') \) colors. More precisely, the following stronger assertion holds:

**Theorem 2.2.** Let \( H = (X, \mathcal{C}, \mathcal{D}) \) be a mixed hypergraph, \( \chi(H) \leq t \leq \overline{\chi}(H) \), \( \mathcal{P} = X_1 \cup \cdots \cup X_t \) and \( \mathcal{P}' = \bar{X}_1 \cup \cdots \cup \bar{X}_\bar{t} \) be two of its strict colorings such that
is a refinement of \( P \), i.e., every \( X_i \) is a subset of some \( X_j \). Then there exists a wuc hypergraph \( \mathcal{H}' = (X', \mathcal{C}', \mathcal{D}') \) with the following properties:

1. \( \mathcal{H} \) is an (induced) subhypergraph of \( \mathcal{H}' \),
2. \( \chi(\mathcal{H}') = t \), the strict \( t \)-coloring of \( X' \) is unique and when restricted to \( X \) it is given by \( P \),
3. \( \tilde{\chi}(\mathcal{H}') = \tilde{t} \), the strict \( \tilde{t} \)-coloring of \( X' \) is unique and when restricted to \( X \) it is given by \( \tilde{P} \).

**Proof.** Observe that the assertion will be verified if we ensure that the following two conditions are satisfied in building the wuc mixed hypergraph \( \mathcal{H}' \) in which \( \mathcal{H} \) is a (induced) subhypergraph.

1. The colorings \( P \) and \( \tilde{P} \) of \( \mathcal{H} \) are extendable to the colorings of \( \mathcal{H}' \), and
2. in every proper coloring of \( \mathcal{H}' \),
   (a) any two vertices belonging to distinct classes of \( P \) are assigned distinct colors, and
   (b) each class of \( \tilde{P} \) is monochromatic.

To extend \( \mathcal{H} \) into some \( \mathcal{H}' \) in the required way, we first choose a dummy spanning tree \( T_i \) inside each partition class \( X_i \) of \( \tilde{P} \). If the subhypergraph \( \mathcal{H} \) is not required to be induced in \( \mathcal{H}' \), then we simply replace each edge \( xy \) of \( T_i \) by a \( C \)-edge \((x, y)\). Otherwise, we put an even \((x, y)\)-invertor on each such \((x, y)\), with internal vertices not in \( X \). Different even invertors should be internally disjoint. These invertors ensure (b).

So far the internal vertices of the invertors may get new colors, hence the number of possible colors for \( \mathcal{H}' \) may increase. In order to avoid this, we insert one \( C \)-edge of size 3, \((x = x_0, x_1, z)\), for each even \((x, y)\)-invertor, where \( x_1 \) is the internal vertex, and \( z \) is any vertex such that \( x \) and \( z \) belong to distinct color classes \( X_i \) under \( P \). Thus, we force the colors on the \((x, y)\)-invertor to alternate between the colors of \( x \) and \( z \).

Next, for each pair \((i, j)\) such that \( \bar{X}_i \) and \( \bar{X}_j \) are contained in two distinct classes of \( P \), we choose two vertices \( x_i \in \bar{X}_i \) and \( x_j \in \bar{X}_j \), and build an odd \((x_i, x_j)\)-invertor on them. Then (a) is satisfied, too. Again the newly added intermediate vertices in odd invertors should be distinct for the distinct vertex pairs of \( \mathcal{H} \), and should not be in \( X \). Since colors on invertors alternate, the colors on an odd invertor are all the same as the two colors at the endpoints. Hence, no new colors outside \( X \) can occur on internal vertices of newly created invertors. Note that if the subhypergraph need not be induced, then we may simply take \((x_i, x_j)\) as a \( D \)-edge, instead of connecting the two vertices by an invertor.

It is clear that \( P \) is the unique coloring of \( \mathcal{H}' \) on \( X \) with \( t \) colors, and \( \tilde{P} \) is the unique coloring of \( \mathcal{H}' \) on \( X \) with \( \tilde{t} \) colors, because each class of \( \tilde{P} \) is monochromatic (due to the presence of even invertors) and no pair of vertices belonging to distinct classes of \( P \) can get the same color (by the odd invertors). If we only use even and odd invertors with distinct intermediate vertices, then \( \mathcal{H} \) is an induced subhypergraph.
of $H'$. And if we only add $C$-edges and $D$-edges to construct $H'$, then $H$ is just a subhypergraph of $H'$ which is usually not reduced. □

**Corollary 2.3.** Let $H = (X, C, D)$ be a mixed hypergraph, and $t$ an integer such that $r_t > 0$. Then there exists a uc hypergraph $H' = (X', C', D')$ with $H$ as its (induced) subhypergraph and $\chi(H') = t = \bar{\chi}(H')$.

**Proof.** Apply the same proof as in the theorem above, with a strict $t$-coloring $\mathcal{P} = \mathcal{P}$. □

Since the chromatic spectrum may contain gaps [15], the condition $r_t > 0$ cannot be replaced by the weaker assumption $\chi(H) \leq t \leq \bar{\chi}(H)$ in this corollary.

### 3. Separation on UC subhypergraphs

In this section we investigate the situation where uc mixed hypergraphs are separators. Let $C(x)$ ($D(x)$) denote the set of $C$-edges ($D$-edges) containing vertex $x \in X$.

**Definition 3.1.** In a connected mixed hypergraph $H = (X, C, D)$, the set $X_0 \subset X$ (the induced subhypergraph $H[X_0]$) is called a **separator** if there exist nonempty pairwise disjoint subsets $X_1, X_2, \ldots, X_k$, $k \geq 2$, such that

$$X_1 \cup X_2 \cup \cdots \cup X_k \cup X_0 = X$$

and for every $x \in X_i$ and every $y \in X_j$ with $1 \leq i < j \leq k$ we have

$$C(x) \cap C(y) = D(x) \cap D(y) = \emptyset.$$

The induced subhypergraphs $H[X_i \cup X_0]$, $i = 1, \ldots, k$, are called the **derived subhypergraphs** of $H$ (with respect to the separator $X_0$). Thus, the mixed subhypergraph induced by $X_1 \cup X_2 \cup \cdots \cup X_k$ is disconnected and has at least $k$ connected components.

The next theorem shows that if a uc hypergraph is a separator then both the upper and lower chromatic numbers, and also the chromatic polynomial, can be computed recursively.

**Theorem 3.2.** Let $H = (X, C, D)$ be a connected mixed hypergraph, and $H_0 = H[X_0]$ be a uniquely colorable separator with derived subhypergraphs $H_1 = H[X_1 \cup X_0]$ and $H_2 = H[X_2 \cup X_0]$. Then the following equalities hold:

1. $\chi(H) = \max\{\chi(H_1), \chi(H_2)\}$;
2. $\bar{\chi}(H) = \bar{\chi}(H_1) + \bar{\chi}(H_2) - \bar{\chi}(H_0)$;
3. $P(H, \lambda) = P(H_1, \lambda)P(H_2, \lambda)/P(H_0, \lambda)$.

**Proof.** (1) Denote $\chi(H) = \bar{\chi}_1$, $\chi(H_1) = \bar{\chi}_1$, $\chi(H_2) = \bar{\chi}_2$, $\chi(H_0) = \bar{\chi}_0$, $\bar{\chi}(H) = \bar{\chi}$, $\bar{\chi}(H_1) = \bar{\chi}_1$, $\bar{\chi}(H_2) = \bar{\chi}_2$, and $\bar{\chi}(H_0) = \bar{\chi}_0$. Obviously, $\chi \geq \max\{\bar{\chi}_1, \bar{\chi}_2\}$ since every
coloring of $\mathcal{H}$ induces a proper coloring on its subhypergraphs. Conversely, considering an arbitrary $\chi_1$-coloring of $\mathcal{H}_1$ and an arbitrary $\chi_2$-coloring of $\mathcal{H}_2$ one can permute the colors in, say, $\mathcal{H}_2$ in such a way that $\mathcal{H}_0$ becomes colored identically in both $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence, $\chi \leq \max\{\chi_1, \chi_2\}$.

(2) It is clear that $\tilde{\chi} \leq \tilde{\chi}_1 + \tilde{\chi}_2 - \tilde{\chi}_0$, since every strict coloring of $\mathcal{H}$ with $\tilde{\chi}$ colors induces a strict coloring on each of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_0$, and the colors of $\mathcal{H}_0$ appear in both derived subhypergraphs. To prove the converse inequality, color $\mathcal{H}_1$ with the colors $1, 2, \ldots, \tilde{\chi}_1$, and color $\mathcal{H}_2$ with the colors $\tilde{\chi}_1 + 1 - \tilde{\chi}_0$, $\tilde{\chi}_1 + 2 - \tilde{\chi}_0$, $\ldots$, $\tilde{\chi}_1 + \tilde{\chi}_2 - \tilde{\chi}_0$. If the colorings of $\mathcal{H}_0$ in $\mathcal{H}_1$ and in $\mathcal{H}_2$ do not coincide, then permute the colors in such a way that $\mathcal{H}_0$ becomes colored identically. Hence, we obtain $\chi \geq \tilde{\chi}_1 + \tilde{\chi}_2 - \tilde{\chi}_0$.

(3) Let $\ell = \rho(\mathcal{H}_0, \lambda)$. Since $\mathcal{H}_0$ is a uc subhypergraph of $\mathcal{H}_1$, the set of all $\rho(\mathcal{H}_1, \lambda)$ colorings of $\mathcal{H}_1$ can be partitioned into $\ell$ equal classes, such that each class contains exactly $\rho(\mathcal{H}_1, \lambda)/\ell$ colorings. Similarly, the $\rho(\mathcal{H}_2, \lambda)$ colorings of $\mathcal{H}_2$ can be partitioned into $\ell$ equal classes, and each such class contains exactly $\rho(\mathcal{H}_2, \lambda)/\ell$ colorings.

Combining every coloring from each class of $\mathcal{H}_1$ with every coloring from the corresponding class of $\mathcal{H}_2$ gives a coloring of $\mathcal{H}$. Hence, the total number of all the colorings of $\mathcal{H}$ is

$$P(\mathcal{H}, \lambda) = \frac{\rho(\mathcal{H}_1, \lambda)}{\ell} \cdot \frac{\rho(\mathcal{H}_2, \lambda)}{\ell} = \frac{\rho(\mathcal{H}_1, \lambda) \cdot \rho(\mathcal{H}_2, \lambda)}{(\rho(\mathcal{H}_0, \lambda))^2}.$$

It may happen that $\mathcal{H}_0$ is a separator also in $\mathcal{H}_1$ or in $\mathcal{H}_2$. In such a situation, by the same reasoning the above theorem can be generalized as follows:

**Corollary 3.3.** In a mixed hypergraph $\mathcal{H} = (X, \mathcal{E}, \mathcal{D})$, if $\mathcal{H}_0 = H[X_0]$ is a uniquely colorable separator, and $\mathcal{H}_1 = \mathcal{H}[X_1 \cup X_0]$, $\mathcal{H}_2 = \mathcal{H}[X_2 \cup X_0], \ldots, \mathcal{H}_k = \mathcal{H}[X_k \cup X_0]$ are the derived subhypergraphs (with $k \geq 2$), then the following equalities hold:

1. $\chi(\mathcal{H}) = \max\{\chi(\mathcal{H}_1), \chi(\mathcal{H}_2), \ldots, \chi(\mathcal{H}_k)\}$;
2. $\tilde{\chi}(\mathcal{H}) = \tilde{\chi}(\mathcal{H}_1) + \tilde{\chi}(\mathcal{H}_2) + \ldots + \tilde{\chi}(\mathcal{H}_k) - (k-1)\tilde{\chi}(\mathcal{H}_0)$;
3. $P(\mathcal{H}, \lambda) = \rho(\mathcal{H}_1, \lambda) \cdot \rho(\mathcal{H}_2, \lambda) \cdot \ldots \cdot \rho(\mathcal{H}_k, \lambda) \cdot (\rho(\mathcal{H}_0, \lambda))^{1-k}$.

**Corollary 3.4** (Berge [4]). In a graph $G = (X, \mathcal{D})$, if a subgraph $G_0 = G[X_0]$ is a clique and a separator at the same time, and $G_1 = G[X_1 \cup X_0]$, $G_2 = G[X_2 \cup X_0], \ldots$, $G_k = G[X_k \cup X_0]$ are the derived subgraphs, then the following equalities hold:

1. $\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots, \chi(G_k)\}$;
2. $P(G, \lambda) = \rho(G_1, \lambda) \cdot \rho(G_2, \lambda) \cdot \ldots \cdot \rho(G_k, \lambda) \cdot (\rho(G_0, \lambda))^{1-k}$.

**Proof.** The assertion follows immediately because every clique is a uniquely colorable graph. □

A graph is called chordal if every cycle of length at least four has two consecutive adjacent vertices. (These graphs are also termed triangulated or rigid circuit
Theorem 3.5 (Voloshin [30]). If \( G = (X, \mathcal{D}) \) is a chordal graph with an arbitrary separator \( G_0 = G[X_0] \), and \( G_1 = G[X_1 \cup X_0], G_2 = G[X_2 \cup X_0], \ldots, G_k = G[X_k \cup X_0] \) are the derived subgraphs, then the following equality holds:

\[
P(G, \lambda) = P(G_1, \lambda)P(G_2, \lambda) \cdots P(G_k, \lambda) P(G_0, \lambda)^{1-k}.
\]

Since a separator is not necessarily a uniquely colorable subgraph in a chordal graph (although every minimal separator is a clique), it follows that the above equality is not sufficient for the separator to be uniquely colorable.

4. Recursive operations

In this section, we consider the uc hypergraphs from two different points of view; first, how they can be applied to simplify the original structure, and second, which of them can be built up recursively.

4.1. Subset contraction

The next observation points out the relevance of uniquely colorable mixed hypergraphs with respect to structural reduction. Let \( \mathcal{H}' = (X', \mathcal{C}', \mathcal{D}') \) be a mixed hypergraph, and suppose that it contains a subhypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) which is uniquely colorable. Assume that \( X_1 \cup \cdots \cup X_t \) is the unique coloring of \( \mathcal{H} \). Denote by \( \mathcal{D}'' \) the set of \( \mathcal{D} \)-edges \( D \in \mathcal{D}' \setminus \mathcal{D} \) meeting more than one color class \( X_i \) (\( 1 \leq i \leq t \)), and by \( \mathcal{C}'' \) the set of \( \mathcal{C} \)-edges \( C \in \mathcal{C}' \) sharing more than one vertex with some \( X_i \). We now make the following structural modifications. First, remove \( \mathcal{C}'' \cup \mathcal{D}'' \) from \( \mathcal{H}' \), and then contract each \( X_i \) to a single vertex; i.e., take \( t \) new vertices \( x_1, \ldots, x_t \not\in X' \), and replace each \( \mathcal{D} \)-edge \( D \in \mathcal{D}' \setminus \mathcal{D}'' \) by \( (D \setminus X) \cup \{x_i; D \cap X_i \neq \emptyset\} \), and similarly, replace each \( \mathcal{C} \)-edge \( C \in \mathcal{C}' \setminus \mathcal{C}'' \) by \( (C \setminus X) \cup \{x_i; C \cap X_i \neq \emptyset\} \). \( \mathcal{H}'/\mathcal{H} \) denotes the mixed hypergraph obtained.

One of the interesting particular cases occurs when \( \mathcal{H}' \) itself is uniquely colorable. Then we can apply the above operation with \( X = X' \), choosing any uniquely colorable subhypergraph \( \mathcal{H} \subseteq \mathcal{H}' \). For instance, taking \( \mathcal{H} = \mathcal{H}' \) leads to the following assertion.

Lemma 4.1. If a mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) is uniquely colorable, then the hypergraph \( \mathcal{H}/\mathcal{H} \) is a complete graph with \( t \) vertices.

Proof. Indeed, in a unique feasible partition of \( X \) the union of any two color classes contains some \( \mathcal{D} \)-edge which, under the transformation, becomes a \( \mathcal{D} \)-edge of size 2. In this way we obtain all edges of a complete graph with \( t \) vertices. Moreover, by
the definition of proper coloring, each $\mathcal{C}$-edge meets some color class in at least two vertices. Thus, in the reduction all the $\mathcal{C}$-edges of $\mathcal{H}$ are removed. \hfill $\square$

In general, let $\mathcal{H}'=(X',\mathcal{C}',\mathcal{D}')$ be a mixed hypergraph containing a uniquely colorable subhypergraph $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$ with coloring $X_1 \cup \cdots \cup X_t$. Then $\mathcal{H}'/\mathcal{H}$ contains a complete graph on $t$ vertices, as a $\mathcal{D}$-subhypergraph. The other $\mathcal{D}$-edges of $\mathcal{H}'/\mathcal{H}$ are those inherited from the $\mathcal{D}$-edges of $\mathcal{H}'$ meeting at most one class $X_i$ ($1 \leq i \leq t$), and the other $\mathcal{C}$-edges of $\mathcal{H}'/\mathcal{H}$ are those inherited from the $\mathcal{C}$-edges of $\mathcal{H}'$ sharing at most one vertex with each $X_i$ ($1 \leq i \leq t$). In particular, all $C \in \mathcal{C}$ get removed.

**Lemma 4.2** (Contraction lemma). With the above notation, there is a one-to-one correspondence between the colorings of $\mathcal{H}'$ and those of $\mathcal{H}'/\mathcal{H}$. In particular,

1. $\mathcal{H}'$ is colorable if and only if $\mathcal{H}'/\mathcal{H}$ also is,
2. $\mathcal{H}'$ is uniquely colorable if and only if $\mathcal{H}'/\mathcal{H}$ also is,
3. $\mathcal{H}'$ is weakly uniquely colorable if and only if $\mathcal{H}'/\mathcal{H}$ also is,
4. $P(\mathcal{H}',\lambda)=P(\mathcal{H}'/\mathcal{H},\lambda)$.

It is worth mentioning one particular case explicitly, in which the contraction can be applied efficiently, namely, in any invertor, where all vertices of the same parity may be identified.

### 4.2. Orderings and unique colorings

It is an interesting problem to characterize the structure of uc hypergraphs. In contrast to complete graphs, to do this in full generality is definitely very hard. The first investigations in this direction—motivated by a previous version of the present paper—have been made in [26].

Below we give a general method to construct uc hypergraphs sequentially, similar to the way it can be done with the complete graph, by adding new vertices one by one to an initial uc hypergraph.

Let $\mathcal{H}=(X,\mathcal{C},\mathcal{D})$ be a mixed hypergraph. Assume that $c$ is a strict $t$-coloring of $\mathcal{H}$. Now consider a mixed hypergraph $\mathcal{H}'$ constructed by adding a vertex $y$ to the vertex set $X$, and adding a family $\mathcal{C}_y$ of $\mathcal{C}$-edges to $\mathcal{C}$, where each $\mathcal{C}$-edge $C \in \mathcal{C}_y$ contains $y$, and a family $\mathcal{D}_y$ of $\mathcal{D}$-edges to $\mathcal{D}$, where each $\mathcal{D}$-edge $D \in \mathcal{D}_y$ contains $y$.

A $\mathcal{C}$-edge $C \in \mathcal{C}_y$ is called influencing with respect to the coloring $c$ of $\mathcal{H}$ if all its vertices different from $y$ are colored with distinct colors in $c$. Analogously, a $\mathcal{D}$-edge $D \in \mathcal{D}_y$ is called influencing with respect to $c$ if all its vertices different from $y$ are colored with the same color in $c$. Influencing $\mathcal{C}$-edges and $\mathcal{D}$-edges define all the restrictions for extending the coloring $c$ of $\mathcal{H}$ to the vertex $y$.

Let $c(C)$ ($c(D)$) be the set of colors used by the vertices in the $\mathcal{C}$-edge $C$ ($\mathcal{D}$-edge $D$) in the coloring $c$ of $\mathcal{H}$. Let $FS(y)=\bigcap\{c(C): C \in \mathcal{C}_y, C \text{ is an influencing } \mathcal{C}\text{-edge}\}$. 

Thus, $\text{FS}(y)$ is precisely the set of colors one of which must be used by $y$ when extending the coloring $c$ of $\mathcal{H}$ to the vertex $y$. We call $\text{FS}(y)$ the *Forcing Set* of $y$.

Let $\text{VS}(y) = \bigcup \{ c(D) : D \in \mathcal{D}_y, D$ is an influencing $\mathcal{D}$-edge $\}$. Thus, $\text{VS}(y)$ is the set of colors which must not be used by $y$ when extending the coloring $c$ of $\mathcal{H}$ to the vertex $y$. We call $\text{VS}(y)$ the *Veto Set* of $y$.

If there exist influencing $\mathcal{D}$-edges yielding $\text{FS}(y) = \emptyset$, then the coloring $c$ of the mixed hypergraph $\mathcal{H}$ is not extendable to the vertex $y$. Also, if some influencing $\mathcal{D}$-edges imply $\text{VS}(y) = \{ 1, 2, \ldots, t \}$, then any influencing $\mathcal{D}$-edge makes it impossible to extend the coloring $c$ of $\mathcal{H}$ to $y$. The same is valid under the weaker condition that $\emptyset \neq \text{FS}(y) \subseteq \text{VS}(y)$.

On the other hand, the following definition can be used in constructing larger uc hypergraphs from smaller ones.

**Definition 4.3.** The vertex $y$ is called *uniquely colorable* in extending the strict coloring $c$ of $t$ colors from $\mathcal{H}$ to $\mathcal{H}'$ if one of the following two conditions holds:

1. there is no influencing $\mathcal{D}$-edge and $|\text{VS}(y)| = t$, or
2. $|\text{FS}(y) \setminus \text{VS}(y)| = 1$.

In extending the coloring $c$ of the mixed hypergraph $\mathcal{H}$ to the coloring of $\mathcal{H}'$, the first condition of this definition means the unique possibility to color the vertex $y$ with a new color, while the second condition determines the unique old color for $y$. Obviously, if $\mathcal{H}$ is uniquely colorable, and the vertex $y$ is uniquely colorable in extending the coloring of $\mathcal{H}$, then $\mathcal{H}'$ is uniquely colorable as well. Furthermore, we have the following result.

**Theorem 4.4.** Let $\mathcal{H}$, $\mathcal{H}'$ and $y$ be as above. Assume that $\mathcal{H}$ is uniquely colorable. Then $\mathcal{H}'$ is uniquely colorable if and only if the vertex $y$ is uniquely colorable in extending the coloring of $\mathcal{H}$.

**Proof.** Let $c : X \to \{ 1, 2, \ldots, t \}$ be the unique coloring of $\mathcal{H}$. The sufficiency of the above condition is obvious: since $\mathcal{H}$ is uniquely colorable, and there is only one way to extend this unique coloring to $y$, it follows that $\mathcal{H}'$ is also uniquely colorable.

Necessity can be proved by contradiction. Assume that $y$ is not uniquely colorable in extending $c$. First, we assume that there exists no influencing $\mathcal{D}$-edge of $y$ under the unique coloring $c$ of $\mathcal{H}$, and $|\text{VS}(y)| < t$. Then we have at least two ways to color the vertex $y$, one way is to use an old color, the other one is to use a new color. Second, we assume that there exists some influencing $\mathcal{D}$-edge(s) of $y$ under the coloring $c$. Then either $|\text{FS}(y) \setminus \text{VS}(y)| = 0$, which implies that $\mathcal{H}'$ is uncolorable, a contradiction; or else, $|\text{FS}(y) \setminus \text{VS}(y)| \geq 2$, which implies that there are at least two ways to color the vertex $y$, again a contradiction.

This theorem suggests a method to construct new uc hypergraphs from any given uc hypergraph by consecutively adding one uc vertex each time.
The unique colorability of $H'$ does not imply the unique colorability of $H$, as can be easily shown by examples. Therefore, the ordering of the vertex set is very important in decompositions of uc hypergraphs, if such decompositions exist.

**Definition 4.5.** A mixed hypergraph $H = (X, \mathcal{C}, \mathcal{D})$ is called uc-orderable if there exists an ordering of the vertex set $X = \{x_1, x_2, \ldots, x_n\}$ with the following property: each subhypergraph $H_i = H[X_i]$ induced by the vertex set $X_i = \{x_1, x_2, \ldots, x_i\}$ is uniquely colorable, and in extending its unique coloring $c_i$ each vertex $x_{i+1}$ is a uc vertex, for $i = 1, 2, \ldots, n - 1$.

The corresponding sequence $x_1, \ldots, x_n$ will be called a uc-ordering of $H$.

Some particular uc-orderable mixed hypergraphs, namely those for which all edges (both $\mathcal{C}$ and $\mathcal{D}$) can be represented as vertex sets of subtrees of a tree, have been characterized in [25].

**Definition 4.6.** Suppose that the mixed hypergraph $H = (X, \mathcal{C}, \mathcal{D})$ is uc-orderable, and $x_1, \ldots, x_n$ is an uc-ordering. Call $H$ strongly critical if it does not remain uc-orderable whenever one of the following two operations is applied:

1. the removal of a $\mathcal{D}$-edge or $\mathcal{C}$-edge,
2. the removal of a vertex from a $\mathcal{D}$-edge, i.e., the replacement of $\mathcal{D}$ by $\mathcal{D} \setminus \{x\}$ for some $x \in \mathcal{D}$.

A possible third condition, dealing with $\mathcal{C}$-edges, will be discussed briefly after Proposition 4.8.

Observe that the above two operations are acting in opposite direction. That is, the removal of a $\mathcal{D}$-edge or $\mathcal{C}$-edge weakens the constraints, creating the possibility for more than one coloring (and the hypergraph trivially remains colorable), while shrinking a $\mathcal{D}$-edge strengthens the conditions, possibly making the hypergraph uncolorable.

In the next two results, $X_i$ and $H_i$ are meant as in Definition 4.5.

**Theorem 4.7.** If $H = (X, \mathcal{C}, \mathcal{D})$ is uc-orderable and strongly critical, then $(X, \emptyset, \mathcal{D}) = (X, \mathcal{D})$ is a graph.

**Proof.** Suppose $D \in \mathcal{D}$. Let $x = x_{i+1}$ be the vertex with largest subscript in $D$; i.e., $x_{i+1} \in D \subseteq X_{i+1}$. Assuming that $|D| \geq 3$, we derive a contradiction as follows. If $D$ is not influencing with respect to $x$, then we can simply remove $D$, and $H$ remains uc-orderable. If $D$ is influencing, we shrink $D$ by removing a vertex distinct from $x$, and $H$ remains uc-orderable. This can always be done, because $D \setminus \{x\}$ is monochromatic in the unique coloring of $H_i$. □

One can also obtain an upper bound on the degree of vertices in the above graph $(X, \mathcal{D})$. Note that the degree of a vertex $x$ is the number of $\mathcal{D}$-edges containing $x$ (i.e., $\mathcal{C}$-edges are not counted here).
Proposition 4.8. If a uc-orderable hypergraph $\mathcal{H} = (X, \mathcal{C}, D)$ is strongly critical, then in the graph $(X, D)$, each vertex $x_{i+1}$ is adjacent to at most $\chi(\mathcal{H}_i)$ vertices of $\mathcal{H}_i$.

Proof. If $x_{i+1}$ is preceded by more than $\chi(\mathcal{H}_i)$ of its neighbors in $(X, D)$, then some color in the unique coloring of $\mathcal{H}_i$ is associated with more than one influencing $D$-edge for $x_{i+1}$. In this situation, however, we can delete one of those $D$-edges, and the color still remains excluded at $x_{i+1}$, keeping the vertex uniquely colorable. This contradicts the assumption that $\mathcal{H}$ is strongly critical.

As regards $\mathcal{C}$-edges, it is also clear that if the Forcing Set is created as the intersection of the color sets of more than one influencing $\mathcal{C}$-edge, then the same Forcing Set can also be obtained by just one suitably chosen (newly added) $\mathcal{C}$-edge. Therefore, if we are interested in the sparsest possible uc-orderable mixed hypergraphs, we can assume without the loss of generality that every vertex $x_{i+1}$ is incident to at most one $\mathcal{C}$-edge $C \subseteq X_{i+1}$. Perhaps these sparse structures are simple enough to be recognized efficiently.

We conclude this section with three examples of uc hypergraphs; the first two are uc-orderable but the third one has no uc-ordering.

Example 1. Let $\mathcal{H} = (X, \mathcal{C}, D)$, where $X = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{C} = \{C_1, C_2, C_3, C_4\} = \\{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 4, 5\}, \{1, 5, 6\}\}$ and $D = \\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}\}$. Then $\tilde{\chi}(H) = \chi(H) = 2$, $R(\mathcal{H}) = (0, 1, 0, 0, 0, 0)$ and the uc-ordering is $1, 2, 3, 4, 5, 6$.

Example 2. Let $\mathcal{H} = (X, \mathcal{C}, D)$, where $X = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{C} = \{C_1, C_2, C_3\} = \\{\{1, 2, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}\}$, and

$D = \\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 5\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{1, 4\}, \{3, 4\}\}$.

Then $\tilde{\chi}(H) = \chi(H) = 3$, $R(\mathcal{H}) = (0, 0, 1, 0, 0, 0)$ and the uc-ordering is $1, 2, 3, 4, 5, 6$.

Example 3. $\mathcal{H}_{3}^{2}$ denotes the family of all 3-element subsets of the vertex set $X = \{1, 2, 3, 4, 5\}$. Let $\mathcal{H} = (X, \mathcal{C}, D)$ where $\mathcal{C} = \mathcal{H}_{3}^{2}$, and $D = \mathcal{H}_{3}^{2} - \{1, 2, 3\}$. Then $\tilde{\chi}(H) = \chi(H) = 2$ and $R(\mathcal{H}) = (0, 1, 0, 0, 0)$. There exists no uc ordering. The unique 2-coloring assigns 1, 2, 3 with one color, and 4, 5 with the other color.

5. Algorithmic complexity

Finally, we deal with some complexity issues. Here, we show that many problems concerning mixed hypergraph colorings are computationally intractable.

Theorem 5.1. The following decision problems, taking a general mixed hypergraph $\mathcal{H}$ as input, are NP-complete.

(1) For a given integer $r \geq 2$, is the lower chromatic number of $\mathcal{H}$ at most $r$?
(2) For a given integer \( r \geq 1 \), is the upper chromatic number of \( \mathcal{H} \) at least \( r \)?

(3) Does \( \overline{\chi}(\mathcal{H}) > \chi(\mathcal{H}) \) hold?

(4) Is \( \mathcal{H} \) colorable?

Moreover,

(5) For each \( r \geq 3 \), it is co-NP-complete to decide whether \( \mathcal{H} \) is uniquely \( r \)-colorable, if a strict \( r \)-coloring is also given in the input.

In particular, the recognition problem of uniquely \( r \)-colorable mixed hypergraphs is co-NP-hard for every fixed \( r \geq 3 \).

Proof. Membership in NP for (1)–(4) can be witnessed by

(1) a proper \( r \)-coloring,
(2) a strict \( r' \)-coloring for some \( r' \geq r \),
(3) a proper \( r \)-coloring and a strict \( r' \)-coloring for some \( r' > r \geq 1 \),
(4) a proper coloring,

and that in co-NP for (5) by

(5) a proper coloring different from the one given in the input.

Clearly, both the descriptions of those colorings and the testing of their feasibility takes polynomial (in fact, linear) time. (As usual, the term linearity is meant linear in the input size, that is \( O(\sum_{E \in \mathcal{C} \cup \mathcal{D}} |E|) \) for \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \).) Hence, we need to prove that the problems listed above are NP-hard (resp. co-NP-hard for (5)).

(1) The assertion follows immediately from the fact that the Hypergraph \( r \)-Colorability problem is NP-hard for every \( r \geq 2 \). For \( r = 2 \) this was first proved by Lovász [19]; and to increase \( r \) by 1, one can take a new vertex and join it to the initial hypergraph \( \mathcal{H} \) (viewed as a \( \mathcal{D} \)-hypergraph) completely with \( \mathcal{D} \)-edges of size 2, to create a slightly larger instance \( \mathcal{H}' \). (Hence, for general \( r \), a complete graph \( K_{r-2} \) is joined with \( \mathcal{H} \).)

(2) Let \( r' = \max \{r, 2\} \). We take the example \( \mathcal{H}' \) of (1)—with parameter \( r' = 2 \) instead of \( r \) if \( r = 1 \)—together with all the \((r' + 1)\)-element subsets of the vertex set as \( \mathcal{E} \)-edges, to create a larger instance \( \mathcal{H}'' \). By definition, \( \overline{\chi}(\mathcal{H}'') > 0 \) if and only if \( \mathcal{H}'' \) is colorable. And in the present construction this happens precisely when \( \chi(\mathcal{H}') \leq r' \) holds, because the \( \mathcal{E} \)-edges exclude all strict colorings with more than \( r' \) colors. Since \( \mathcal{H}' \) and \( \mathcal{H}'' \) differ just by a polynomial number \( O(n^{r'+1}) \) of edges, NP-hardness follows from (1).

(3) For the hardness of deciding whether the inequality \( \overline{\chi}(\mathcal{H}) > \chi(\mathcal{H}) \) holds, we consider the restricted class of \textit{planar} graphs. Hence, let \( G \) be planar, and view the edges of \( G \) as \( \mathcal{D} \)-edges. Moreover, insert all 5-tuples as \( \mathcal{E} \)-edges, to obtain the mixed hypergraph \( \mathcal{H} \). A vertex partition properly colors the \( \mathcal{E} \)-subhypergraph of \( \mathcal{H} \) if and only if it has at most four color classes. Since \( \chi(G) \leq 4 \) by the Four Color Theorem [2,3], it follows that \( \mathcal{H} \) is \textit{colorable} and \( \overline{\chi}(\mathcal{H}) = 4 \). Observe further that \( \overline{\chi}(\mathcal{H}) > \chi(\mathcal{H}) \) holds if and only if \( \chi(G) \leq 3 \). Since it is NP-complete to decide whether a planar graph
is 3-colorable (this is so even if the maximum degree is assumed to be at most 4, see [10]), the hardness of (3) follows.

(4) The argument given for (2) proves this case, too. Alternatively, as observed in [29], each instance of the List Coloring problem can be transformed in an equivalent way to a special instance of Mixed Hypergraph Colorability. Since the former is NP-complete already for some rather restricted instances (proved in [16]; cf. also the surveys [28,17]), it follows that deciding colorability is hard also for some fairly small classes of mixed hypergraphs.

(5) Greenwell and Lovász proved [11] that if $G$ is a connected graph with $\chi(G) > r$, then the direct product $G \times K_r$—where two product vertices $(u_1, u_2)$ and $(v_1, v_2)$ of $G_1 \times G_2$ are adjacent if and only if $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$—has a unique $r$-coloring. It is easy to see (and is also contained implicitly in [11]) that even the converse of this assertion is valid. Indeed, the trivial coloring is with the color classes $V(G) \times \{x\}$, where $x$ runs over the vertices of $K_r$; and if $V_1 \cup \cdots \cup V_r$ is a proper $r$-coloring of $G$, then even the sets $V_i \times V(K_r)$ ($i = 1, \ldots, r$) properly color $G \times K_r$. Since graph $r$-colorability is NP-complete for every $r > 2$, it follows that deciding whether an input graph (given together with a proper $r$-coloring) is uniquely $r$-colorable is co-NP-complete.

Now, let $r \geq 3$ be a fixed integer, and $G$ any graph. Similar to the previous cases, we view $G$ as a $\mathcal{D}$-hypergraph, and insert all $(r + 1)$-tuples of vertices as $\mathcal{E}$-edges. This mixed hypergraph has upper chromatic number at most $r$, and therefore is uniquely $r$-colorable if and only if $G$ also is. Since $\mathcal{H}$ is constructed from $G$ in $O(n^{r+1})$ steps (i.e., in polynomial time for every fixed $r$), co-NP-hardness follows. $\square$

References