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Optimal management of the machine repair problem with working vacation: Newton's method

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ABSTRACT

This paper studies the M/M/1 machine repair problem with working vacation in which the server works with different repair rates rather than completely terminating the repair during a vacation period. We assume that the server begins the working vacation when the system is empty. The failure times, repair times, and vacation times are all assumed to be exponentially distributed. We use the MAPLE software to compute steady-state probabilities and several system performance measures. A cost model is derived to determine the optimal values of the number of operating machines and two different repair rates simultaneously, and maintain the system availability at a certain level. We use the direct search method and Newton's method for unconstrained optimization to repeatedly find the global minimum value until the system availability constraint is satisfied. Some numerical examples are provided to illustrate Newton's method.

1. Introduction

There are many studies on the machine repair problems with vacation policy in different frameworks in recent years. But, to the best of our knowledge, there has been no research that explores machine repair problems combined with working vacation. In most queueing systems with single/multiple vacations considered in the literature, it is assumed that a server stops working completely during vacation periods. In this paper, we consider a more general case so that the server can work at a different repair rate during the vacation period.

We consider the M/M/1 machine repair problem with working vacation in which the server works with different repair rates rather than completely terminating the repair during a vacation period. Such a vacation has been referred to as the working vacation (WV) (Servi and Finn [1]). The server begins a working vacation of random length when the system becomes empty. When a working vacation is over and the system is empty, the server starts another working vacation. Whenever the server returns from a working vacation and finds that the system is not empty, the server switches to another repair rate.

Wang [2] used a recursive method to develop steady-state analytic solutions to the M/M/1 machine repair problem with two types of server breakdowns. Wang and Kuo [3] considered the $M/E_k/1$ machine repair problems with a non-reliable server. They constructed a profit model to determine the optimum number of operating machines at a maximum profit. Recently, Wang et al. [4] used the direct search with steepest decent method to find the global maximum value of the profit function until the availability, balking, and reneging constraints are satisfied. Gupta [5] examined the M/M/1 machine interference problem with warm spares and server vacations with exhaustive service. Gupta [5] proposed an algorithm

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to calculate the steady-state probability distribution of the number of failed machines in the system, and then obtained the results of various system performance measures. Jain et al. [6] used the recursive method to investigate the bilevel control policy for the machine repair model with warm standbys and two repairmen. Machine repair problems and vacation queueing models have been studied by several researchers. It is well known that the queueing system with server vacations is useful to model a system in which the server has additional task during a vacation. Ke [7] considered the machine inference problem under two vacation policies with an unreliable server and state-dependent service rate. A comprehensive survey on the machine interference problem, including vacation model, was examined by Haque and Armstrong [8]. Recently, Ke and Wang [9] applied the matrix-geometric method to derive steady-state solutions for the M/M/R machine repair problem under two vacation policies with two types of spares.

Queueing models with working vacation have been studied by many researchers. For models involving server vacations, Doshi [10] first conducted a survey on vacation queueing models. The GI/M/1 queues with server vacations have been analyzed by several authors, such as Chatterjee and Mukherjee [11], Karaesmen and Gupta [12], and Tian [13–15]. Chatterjee and Mukherjee [11] investigated GI/M/1 queue with server vacations and exhaustive service discipline. Chatterjee and Mukherjee [11] utilized the embedded Markov chain technique to obtain the steady-state probability distributions of queue length at pre-arrival and at random epochs, respectively. Karaesmen and Gupta [12] developed the queue length distribution at arrival and random epochs for a finite capacity GI/M/1 queue with server vacations. Karaesmen and Gupta [12] also presented heuristic algorithms to calculate the blocking probability. Tian [13–15] analyzed the GI/M/1 queueing system with a single exponential vacation, phase-type vacations, and exponential vacations, respectively. Fuhrmann and Cooper [16] investigated the M/G/1 queue with generalized vacations, and they demonstrated that the M/G/1 decomposition property holds. Lee [17] used a combination of the supplementary variable and sample biasing techniques to analyze the M/G/1/N queue with vacation and exhaustive service discipline. Servi and Finn [1] first introduced the concept of the working vacation and investigated the M/M/1 queueing model with working vacation. Baba [18] studied GI/M/1 queue with multiple working vacations which extended the Servi and Finn model. For the finite capacity GI/M/1/WV queue with multiple working vacations, Banik et al. [19] derived the system size distributions at pre-arrival and at arbitrary epochs, the blocking probability and the mean waiting time in the system. Further, Wu and Takagi [20] extended M/M/1/WV queue to an M/G/1/WV queue. Li et al. [21] examined the GI/M/1 queue with two policies: working vacations and vacation interruption. The main results in Li et al. [21] are to develop the mean queue length and the mean waiting time by using the matrix analysis method.

The main objectives of this paper are the following:

(1) apply an efficient MAPLE program to compute the steady-state probabilities and various system performance measures;
(2) develop the expected cost function per machine per unit time to determine the joint optimal values of M, μ_v and μ_B at minimum cost until the system availability constraint is satisfied;
(3) use the direct search method and Newton's method for unconstrained optimization to find the global minimum value until the system availability constraint is satisfied.

2. The machine repair model

We consider a machine repair model with M identical operating machines which are maintained by a single repairman. It is assumed that each of the operating machines fails according to a Poisson process with parameter λ. Each of the operating machines fails independently of the state of the others. When the system is empty, the server begins a working vacation, and the vacation duration follows an exponential distribution with mean duration $1/\eta$. When a working vacation terminates and the system is empty, the server starts another working vacation. Repair times during a vacation period are according to exponential distribution with mean $1/\mu_v$. Repair times during a normal busy period are according to exponential distribution with mean $1/\mu_B$. When an operating machine fails, it is immediately sent to a repair facility where it is repaired in the order of their breakdowns; that is the first-come, first-served discipline. Failed machines arrive at the server from a single waiting line. The repairman can repair only one machine at a time and the failed machines have to wait in the queue until the repairman is free.

3. Steady-state results

We consider an M/M/1 machine repair problem with working vacation. The server works with different repair rates rather than completely terminates repair service during a normal busy period, and the server begins a working vacation with mean duration when the system is empty. We first set up the steady-state equations and then use an efficient MAPLE program to calculate the steady-state probability.

3.1. Steady-state equations

The states of the system are presented by pairs $\{(i, n)|i = 0, 1; n = 0, 1, 2, \ldots, M\}$, where $i = 0$ denotes that the server is on working vacation, $i = 1$ denotes that the server is on the normal busy period, and n is the number of failed machines in the system. We define the following steady-state probabilities as follows:

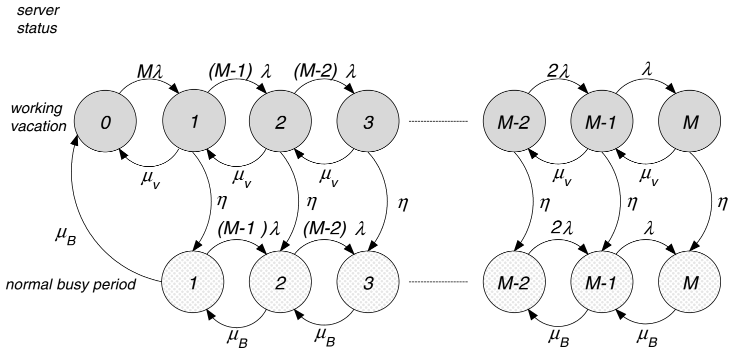

Fig. 1. State transition rate diagram for the M/M/1 machine repair problem with working vacation.

$P_0(n) \equiv$ probability that there are n failed machines in the system when the server is on working vacation, $n = 0, 1, 2, \ldots, M$.

$P_1(n) \equiv$ probability that there are n failed machines in the system when the server is on normal busy period, $n = 1, 2, \ldots, M$.

Relating to Fig. 1, steady-state equations of the machine repair model are given by

$$M\lambda P_0(0) = \mu_B P_1(1) + \mu_v P_0(1), \tag{1}$$

$$[(M-n)\lambda + \mu_v + \eta]P_0(n) = (M-n+1)\lambda P_0(n-1) + \mu_v P_0(n+1), \quad 1 \le n \le M-1 \tag{2}$$

$$(\mu_v + \eta)P_0(M) = \lambda P_0(M-1), \tag{3}$$

$$[(M-1)\lambda + \mu_B]P_1(1) = \mu_B P_1(2) + \eta P_0(1), \tag{4}$$

$$[(M-n)\lambda + \mu_B]P_1(n) = (M-n+1)\lambda P_1(n-1) + \mu_B P_1(n+1) + \eta P_0(n), \quad 2 \le n \le M-1 \tag{5}$$

$$\mu_B P_1(M) = \lambda P_1(M-1) + \eta P_0(M). \tag{6}$$

3.2. Matrix-geometric solutions

A matrix-geometric method is used to analyze the problem further as there is no way of solving (1)–(6) in a recursive manner to develop the closed-form expressions for the steady-state probabilities $P_0(n)$ and $P_1(n)$, where $n = 0, 1, 2, \ldots, M$. We will implement the matrix-geometric method to simplify the computation of the stationary probabilities in the following.

The corresponding transition rate matrix $\mathbf{Q}$ of this Markov chain has the block-tridiagonal form:

$$\mathbf{Q} = \begin{bmatrix} \widehat{B}_0 & \widehat{C}_0 & & & & & \\ \widehat{A}_1 & B_1 & C_1 & & & & \\ & A_2 & B_2 & C_2 & & & \\ & & A_2 & B_3 & C_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_2 & B_{M-1} & C_{M-1} \\ & & & & & A_2 & B_M \end{bmatrix}.$$

The rate matrix $\mathbf{Q}$ of this state process is similar to the quasi-birth and death type, and this class of Markov process has been extensively studied by Neuts [22]. Each element of the matrix $\mathbf{Q}$ is listed in the following:

$$\widehat{A}_1 = \begin{bmatrix} \mu_v \\ \mu_B \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \mu_v & 0 \\ 0 & \mu_B \end{bmatrix}$$

$$\widehat{B}_0 = -M\lambda, \qquad B_n = \begin{bmatrix} -(M-n)\lambda - \mu_v - \eta & \eta \\ 0 & -(M-n)\lambda - \mu_B \end{bmatrix}, \quad \text{for } 1 \le n \le M,$$

$$\widehat{C}_0 = [M\lambda,\ 0], \qquad C_n = \begin{bmatrix} (M-n)\lambda & 0 \\ 0 & (M-n)\lambda \end{bmatrix} = (M-n)\lambda\mathbf{I}, \quad \text{for } 1 \le n \le M-1,$$

where A_1 is a matrix of order 1×2, $\widehat{C}_0$ is a matrix of order 2×1, $\mathbf{I}$ is the identity matrix of order 2, $A_2, B_n (1 \le n \le M)$ and $C_n (1 \le n \le M-1)$ are square matrices of order 2.

Let $\mathbf{P}$ be the corresponding steady-state probability vector of $\mathbf{Q}$. By partitioning the vector $\mathbf{P}$ as $\mathbf{P} = \{P_0, P_1, P_2, \ldots, P_{M-1}, P_M\}$, where $P_0 = P_0(0)$ is a nonnegative real number and $P_n = \{P_0(n), P_1(n)\}$, $(1 \le n \le M)$ is a row vector of dimension 2. By solving the steady-state equations $\mathbf{PQ} = \mathbf{0}$, it follows that

$$\begin{aligned}
&P_0\widehat{B}_0 + P_1\widehat{A}_1 = 0,\\
&P_0\widehat{C}_0 + P_1B_1 + P_2A_2 = 0\\
&P_{n-1}C_{n-1} + P_nB_n + P_{n+1}A_2 = 0, \quad \text{for } 2 \le n \le M-1\\
&P_{M-1}C_{M-1} + P_MB_M = 0.
\end{aligned}$$

Thus, we obtain after routine substitutions:

$$P_M = -P_{M-1}C_{M-1}B_M^{-1} = P_{M-1}X_M, \tag{7}$$

$$P_n = P_{n-1}X_n, \quad \text{for } 2 \le n \le M-1, \tag{8}$$

$$P_1 = -P_0\widehat{C}_0(B_1 + X_2A_2), \tag{9}$$

$$P_0(\widehat{B}_0 - \widehat{C}_0(B_1 + X_2A_2)\widehat{A}_1) = 0, \tag{10}$$

where $X_n = -(M-n+1)\lambda(B_n + X_{n+1}A_2)^{-1}$, $2 \le n \le M-1$ are square matrices of order 2. Furthermore, we have $X_M = -\lambda B_M^{-1}$.

The limitation part for the proposed approach is that the matrix B_M must be nonsingular, that is, the determinant of B_M is not equal to zero. Therefore, the following inequalities must satisfy the proposed approach:

$$\begin{aligned}
&\mu_V + \eta \ne 0,\\
&\mu_B \ne 0.
\end{aligned}$$

Eq. (10) determines P_0 up to a multiplicative constant. The other Eqs. (7)–(9) determine $P_M, P_{M-1}, \ldots, P_2, P_1$, up to the same constant, which is uniquely determined by the following normalizing equation

$$P_0(0) + \sum_{n=1}^{K} P_n\mathbf{e} = 1,$$

where $\mathbf{e}$ is a column vector with each component equal to one. We can solve $P_0(0)$, P_n and $P_j(n)$ for $j = 0, 1$ and $1 \le n \le M$ by using the computer software MAPLE.

4. System performance measures

We define the following system performance measures of the machine repair problem with working vacation.

$E[N_0] \equiv$ the expected number of failed machines in the system when the server is on working vacation.

$E[N_1] \equiv$ the expected number of failed machines in the system when the server is on normal busy period.

$E[N] \equiv$ the expected number of failed machines in the system.

$E[O] \equiv$ the expected number of operating machines in the system.

$MA \equiv$ machine availability (the fraction of the time that the machines are working).

$OU \equiv$ operative utilization (the fraction of the busy repairman).

The expressions for $E[N_0]$, $E[N_1]$, $E[N]$ and $E[O]$ are obtained as follows:

$$E[N_0] = \sum_{n=1}^{M} nP_0(n), \tag{11}$$

$$E[N_1] = \sum_{n=1}^{M} nP_1(n), \tag{12}$$

$$E[N] = E[N_0] + E[N_1] = \sum_{n=1}^{M} n[P_0(n) + P_1(n)], \tag{13}$$

$$E[O] = M - E[N_0] - E[N_1] = M - \sum_{n=1}^{M} n[P_0(n) + P_1(n)]. \tag{14}$$

The machine availability and the operative utilization are defined as:

$$MA = 1 - \frac{E[N]}{M} = \frac{E[O]}{M} = 1 - \frac{1}{M}\sum_{n=1}^{M} n[P_0(n) + P_1(n)], \tag{15}$$

$$OU = 1 - P_0(0). \tag{16}$$

Table 1
The machine availability *MA* and the operative utilization *OU* ($\mu_v = 1, \mu_B = 2, \eta = 0.3$).

	MA			*OU*		
M/λ	0.1	0.2	0.3	0.1	0.2	0.3
1	0.919	0.850	0.790	0.081	0.150	0.210
2	0.914	0.835	0.765	0.160	0.289	0.393
3	0.908	0.819	0.737	0.235	0.414	0.547
4	0.903	0.802	0.710	0.307	0.524	0.670
5	0.900	0.785	0.683	0.376	0.619	0.764
6	0.892	0.767	0.657	0.440	0.699	0.833
7	0.885	0.750	0.634	0.500	0.765	0.884
8	0.879	0.733	0.612	0.556	0.818	0.921
9	0.872	0.716	0.590	0.608	0.860	0.947
10	0.865	0.700	0.568	0.655	0.894	0.966
11	0.858	0.683	0.545	0.697	0.921	0.979
12	0.850	0.667	0.520	0.736	0.942	0.988
13	0.843	0.649	0.493	0.771	0.959	0.993
14	0.835	0.630	0.466	0.802	0.971	0.997
15	0.828	0.610	0.440	0.829	0.981	0.998

Table 2
The machine availability *MA* and the operative utilization *OU* ($\mu_v = 1, \mu_B = 2, \lambda = 0.2$).

	MA			*OU*		
M/η	0.1	0.2	0.3	0.1	0.2	0.3
1	0.839	0.845	0.850	0.160	0.155	0.150
2	0.821	0.829	0.835	0.310	0.298	0.289
3	0.799	0.810	0.819	0.447	0.428	0.414
4	0.775	0.791	0.802	0.567	0.543	0.524
5	0.749	0.770	0.785	0.671	0.641	0.619
6	0.722	0.750	0.767	0.755	0.722	0.699
7	0.695	0.729	0.750	0.821	0.787	0.765
8	0.670	0.710	0.733	0.870	0.838	0.818
9	0.647	0.692	0.716	0.906	0.878	0.860
10	0.627	0.675	0.700	0.931	0.908	0.894
11	0.611	0.659	0.683	0.949	0.932	0.921
12	0.597	0.644	0.667	0.962	0.950	0.942
13	0.586	0.630	0.649	0.972	0.964	0.959
14	0.576	0.614	0.630	0.980	0.974	0.971
15	0.566	0.597	0.610	0.986	0.983	0.981

We choose $\mu_v = 1$, $\mu_B = 2$, $\eta = 0.3$, vary the number of operating machines M from 1 to 15, and vary the failure rate λ from 0.1 to 0.3. We observe from Table 1 that (i) the machine availability *MA* decreases as M increases; (ii) the machine availability *MA* decreases as λ increases ; (iii) the operative utilization *OU* increases as M increases ; and (iv) the operative utilization *OU* increases as λ increases.

We select $\mu_v = 1$, $\mu_B = 2$, $\lambda = 0.2$, vary the number of operating machines M from 1 to 15, and vary the working vacation rate η from 0.1 to 0.3. From Table 2, we find that (i) the machine availability *MA* decreases as M increases; (ii) the machine availability *MA* increases as η increases; (iii) the operative utilization *OU* increases as M increases; and (iv) the operative utilization *OU* decreases as η increases.

5. Cost analysis

We first develop a steady-state expected cost function per machine per unit time for the M/M/1 machine repair problem with working vacation, in which three decision variables M, μ_v, and μ_B are considered. The discrete variable M is required to be a natural number, and the continuous variables μ_v and μ_B are positive numbers. Our main objective is to determine the optimum number of operating machines M, say M^* and the optimum value of repair rate (μ_v, μ_B), say (μ_v^*, μ_B^*), so as to minimize this function.

5.1. Cost function

Let A_v denote the probability that at least one machine is operating, and A_0 represent the minimum fraction of one machine is operating. We select the following cost elements:

$C_0 \equiv$ cost per unit time per failed machine in the system when the server is on working vacation,
$C_1 \equiv$ cost per unit time per failed machine in the system when the server is on normal busy period,

Table 3
The expected cost $F(M, \mu_v, \mu_B)$ for $\lambda = 0.4, 0.5, 0.6$.

λ/M	3	4	5	6	7	8	9	10	11
0.4	89.2	71.9	62.4	56.8	53.6	52.0	**51.4**	51.5	52.2
0.5	92.8	76.2	67.5	62.8	60.5	**59.8**	60.1	61.1	62.5
0.6	96.3	80.5	72.6	68.7	**67.3**	67.4	68.3	69.9	72.0

Table 4
The expected cost $F(M, \mu_v, \mu_B)$ for $\eta = 0.4, 0.6, 0.8$.

η/M	3	4	5	6	7	8	9	10	11
0.4	92.7	76.1	67.3	62.5	60.1	**59.3**	59.5	60.4	61.9
0.6	92.6	75.9	67.0	62.1	59.6	**58.6**	58.7	59.5	61.0
0.8	92.5	75.8	66.8	61.8	59.1	**58.0**	58.1	58.9	60.3

$C_2 \equiv$ fixed cost for every repair rate of the working vacation,
$C_3 \equiv$ fixed cost for every repair rate of the normal busy period.

Using the definitions of these cost elements listed above, the total expected cost function per machine per unit time is given by

$$F(M, \mu_v, \mu_B) = \frac{C_0E[N_0] + C_1E[N_1] + C_2\mu_v + C_3\mu_B}{M}. \tag{17}$$

The cost minimization problem can be presented mathematically as

$$\underset{M,\mu_v,\mu_B}{\text{Minimize}}\ F(M, \mu_v, \mu_B)$$

Subject to: $A_v \geq A_0$.

The cost parameters in (17) are assumed to be linear in the expected number of the indicated quantity, and it would have been a hard task to develop analytic results for the optimum value (M^*, μ_v^*, μ_B^*) because the expected cost function is highly non-linear and complex. We first use the direct search method to find the optimal value of the number of operating machines M, say M^* when μ_v and μ_B are fixed. Next, we fix M^* and use Newton's method to find the optimal value of (μ_v, μ_B), say (μ_v^*, μ_B^*).

5.2. *Direct search method*

Since M is a discrete variable, we use direct substitution of successive values of M into the cost function until the minimum value of $F(M, \mu_v, \mu_B)$, say $F(M^*, \mu_v, \mu_B)$ is achieved and the constraint $A_v \geq A_0$ is satisfied. The following numerical results are provided by considering cost parameters as follows:

$$C_0 = \$100/\text{day}, \qquad C_1 = \$150/\text{day}, \qquad C_2 = \$50/\text{day}, \qquad C_3 = \$15/\text{day}.$$

The cost minimization problem can be illustrated mathematically as

$$F(M^*, \mu_v, \mu_B) = \underset{M}{\text{Minimize}}\, F(M, \mu_v, \mu_B)$$

Subject to: $A_v \geq A_0$.

We first fix $A_0 = 0.9$, $(\mu_v, \mu_B) = (3, 5)$, $\eta = 0.3$, vary the number of operating machines M from 3 to 11, and choose different values of $\lambda = 0.4, 0.5, 0.6$. We observe from Table 3 that a minimum expected cost per day (a) of \$51.4 is achieved at $M^* = 9$ for $\lambda = 0.4$, (b) of \$59.8 is achieved at $M^* = 8$ for $\lambda = 0.5$, (c) of \$67.3 is achieved at $M^* = 7$ for $\lambda = 0.6$. Fig. 2 depicts the various values of λ on (i) the expected cost $F(M, \mu_v, \mu_B)$, and (ii) the optimal number of operating machines $M$to be assigned to the server.

Next, we fix $A_0 = 0.9$, $(\mu_v, \mu_B) = (3, 5)$, $\lambda = 0.5$, vary the number of operating machines M from 3 to 11, and choose different values of $\eta = 0.4, 0.6, 0.8$. We can see from Table 4 that a minimum expected cost per day (a) of \$59.3 is achieved at $M^* = 8$ for $\eta = 0.4$, (b) of \$58.6 is achieved at $M^* = 8$ for $\eta = 0.6$, (c) of \$58.0 is achieved at $M^* = 8$ for $\eta = 0.8$. Fig. 3 plots the different values of η on (i) the expected cost $F(M, \mu_v, \mu_B)$, and (ii) the optimal number of operating machines $M$to be assigned to the server.

Moreover, the minimum expected cost $F(M, \mu_v, \mu_B)$ and the values of the system performance measures A_v, $E[N_0]$, $E[N_1]$, $E[O]$, MA and OU, at the optimum values $M^*$are shown in Table 5 for different values of (λ, η). From Table 5, we find that (i) $F(M^*, \mu_v, \mu_B)$ increases as λ increases or η decreases; (ii) M^* decreases as λ increases; and (iii) η rarely affects M^* when λ is fixed.

The minimum expected cost $F(M, \mu_v, \mu_B)$ and the values of the system performance measures A_v, $E[N_0]$, $E[N_1]$, $E[O]$, MA and OU, at the optimum values M^* are shown in Table 6 for different values of (μ_v, μ_B). It appears from Table 6 that (i) $F(M^*, \mu_v, \mu_B)$ decreases as μ_v or μ_B increases; and (ii) M^* increases as μ_v or μ_B increases.

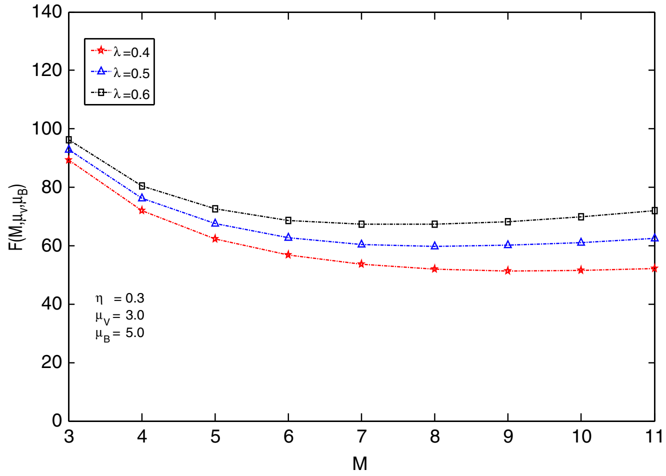

Fig. 2. The expected cost $F(M, \mu_v, \mu_B)$ for $\lambda = 0.4, 0.5, 0.6$.

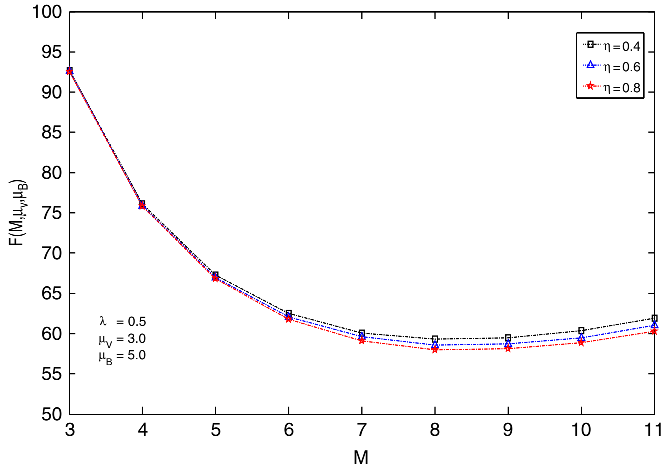

Fig. 3. The expected cost $F(M, \mu_v, \mu_B)$ for $\eta = 0.4, 0.6, 0.8$.

Table 5
System performance measures of the machine repair problem with working vacation ($\mu_v = 3$, $\mu_B = 5$).

(λ, η)	(0.4, 0.3)	(0.5, 0.3)	**(0.6, 0.3)**	(0.5, 0.4)	(0.5, 0.6)	(0.5, 0.8)
M^*	9	8	**7**	8	8	8
$F(M^*, \mu_v, \mu_B)$	51.36	59.78	**67.29**	59.30	58.58	58.05
A_v	0.9997	0.9987	**0.9961**	0.9995	0.9993	0.9995
$E[N_0]$	1.575	1.654	**1.633**	1.463	1.190	1.004
$E[N_1]$	0.531	0.586	**0.551**	0.687	0.831	0.927
$E[O]$	6.893	5.761	**4.815**	5.850	5.979	6.069
MA	0.766	0.720	**0.688**	0.731	0.747	0.759
OU	0.795	0.825	**0.830**	0.814	0.798	0.785

Table 6
System performance measures of the machine repair problem with working vacation ($\lambda = 0.5$, $\eta = 0.3$).

(μ_v, μ_B)	(1.5, 4.0)	**(2.0, 4.0)**	(2.5, 4.0)	(3.0, 4.5)	(3.0, 5.0)	(3.0, 5.5)
M^*	6	**6**	7	8	8	9
$F(M^*, \mu_v, \mu_B)$	65.70	**63.46**	61.91	60.27	59.78	59.60
A_v	0.9810	**0.9907**	0.9964	0.9987	0.9987	0.99904
$E[N_0]$	1.718	**1.472**	1.494	1.581	1.654	1.958
$E[N_1]$	0.583	**0.490**	0.660	0.710	0.586	0.720
$E[O]$	3.700	**4.038**	4.846	5.709	5.761	6.321
MA	0.617	**0.673**	0.692	0.714	0.720	0.702
OU	0.860	**0.813**	0.827	0.832	0.825	0.864

Table 7
Newton-Quasi method in searching the optimal solution ($\lambda = 0.6,\ \eta = 0.3$).

No. of Iterations	0	1	2	3	4
$F(M^*, \mu_v, \mu_B)$	67.2914	66.7762	66.7758	66.7758	66.7758
A_v	0.99608	0.99804	0.99807	0.99807	0.99807
M^*	7	7	7	7	7
μ_v	3.0	3.613186	3.628003	3.628037	3.628037
μ_B	5.0	5.192347	5.180117	5.180171	5.180171

Table 8
Newton-Quasi method in searching the optimal solution ($\lambda = 0.5, \eta = 0.3$).

No. of iterations	0	1	2	3	4
$F(M^*, \mu_v, \mu_B)$	63.4587	62.1104	62.1029	62.1029	62.1029
A_v	0.99907	0.99651	0.99671	0.99671	0.99671
M^*	6	6	6	6	6
μ_v	2.0	2.765030	2.821053	2.821766	2.821766
μ_B	4.0	4.135571	4.086191	4.087125	4.087126

Table 9
Newton-Quasi method in searching the optimal solution from Table 5 ($\mu_v, \mu_B) = (3.0, 5.0)$.

(λ, η)	(0.4, 0.3)	(0.5, 0.3)	(0.5, 0.4)	(0.5, 0.6)	(0.5, 0.8)
M^*	9	8	8	8	8
$F(M^*, \mu_v, \mu_B)$	51.3592	59.7780	59.2983	58.5765	58.0530
A_v	0.99973	0.99873	0.99897	0.99928	0.99945
No. of iteration	4	3	4	5	5
A_v^*	0.99993	0.99960	0.99953	0.99932	0.99883
μ_v^*	3.8565	3.8551	3.5758	2.9520	2.2037
μ_B^*	5.1508	5.2854	5.6086	6.0890	6.4337
$F(M^*, \mu_v^*, \mu_B^*)$	50.3936	58.8005	58.6625	57.8100	56.4284

A_v^* is the value of A_v after iterations.

5.3. *Newton's method*

We fix M^* and use Newton's method for unconstrained optimization to globally search (μ_v, μ_B) until the minimum value of $F(M^*, \mu_v, \mu_B)$, say $F(M^*, \mu_v^*, \mu_B^*)$ and the constraint $A_v \geq A_0$ is satisfied. The cost minimization problem can be illustrated mathematically as

$$F(M^*, \mu_v^*, \mu_B^*) = \underset{\mu_v, \mu_B}{\text{Minimize}}\ F(M^*, \mu_v, \mu_B) \tag{18}$$

$$\text{Subject to: } A_v \geq A_0.$$

The steps of Newton's method for unconstrained optimization can be described as follows:

1. Set $i = 0$, and $\overrightarrow{\mu_i} = [\mu_v, \mu_B]^{\mathrm{T}}$.
2. Set the initial trial solution for $\overrightarrow{\mu_i}$, and compute $F(\overrightarrow{\mu_i})$, where $\mu_v > 0$ and $\mu_B > 0$.
3. Compute the cost gradient $\overrightarrow{\nabla} F(\overrightarrow{\mu_i}) = [\partial F/\partial \mu_v, \partial F/\partial \mu_B]^{\mathrm{T}}|_{\overrightarrow{\mu_i}}$ and the cost Hessian matrix
$$H(\overrightarrow{\mu}) = \begin{bmatrix} \partial^2 F/\partial \mu_v^2 & \partial^2 F/\partial \mu_v \partial \mu_B \\ \partial^2 F/\partial \mu_B \partial \mu_v & \partial^2 F/\partial \mu_B^2 \end{bmatrix}.$$
4. Find the new trial solution $\overrightarrow{\mu}_{i+1} = \overrightarrow{\mu_i} - [H(\overrightarrow{\mu})]^{-1} \overrightarrow{\nabla} F(\overrightarrow{\mu_i})$.
5. Set $i = i + 1$ and repeat steps 3–4 until $A_v \geq A_0$ and $\text{Max}(|\partial F/\partial \mu_v|, |\partial F/\partial \mu_B|) < \varepsilon$, where $\varepsilon = 10^{-7}$ is the tolerance.
6. Find the global minimum value $F(\overrightarrow{\mu_i}) = F(\mu_v^*, \mu_B^*)$.

Two examples are provided to illustrate the above optimization procedure shown in Tables 7 and 8, respectively. We first use the results in Table 5, that is, we choose $(\lambda, \eta) = (0.6, 0.3)$, and select the initial trial solution $(M^*, \mu_v, \mu_B) = (7, 3.0, 5.0)$ with initial value $F(M^*, \mu_v, \mu_B) = 67.29$. Applying Newton's method, we see from Table 7 that after only four iterations, the minimum expected cost converges at this solution $(\mu_v^*, \mu_B^*) = (3.628037, 5.180171)$ with value 66.7758. Next, we utilize the results in Table 6, that is, we fix $(\lambda, \eta) = (0.5, 0.3)$, and choose the initial trial solution $(M^*, \mu_v, \mu_B) = (6, 2.0, 4.0)$ with initial value $F(M^*, \mu_v, \mu_B) = 63.46$. By using Newton's method, we obtain from Table 8 that after only four iterations, the minimum expected cost converges at the solution $(\mu_v^*, \mu_B^*) = (2.821766, 4.087126)$ with value 62.1029. In addition, we also provide other numerical results from Tables 5 and 6 by using Newton's method shown in Tables 9 and 10, respectively. From Tables 9 and 10, it is obvious that the expected cost can be reduced essentially by using Newton's method.

Table 10
Newton-Quasi method in searching the optimal solution from Table 6 ($\lambda = 0.5$, $\eta = 0.3$).

(μ_v, μ_B)	(1.5, 4.0)	(2.5, 4.0)	(3.0, 4.5)	(3.0, 5.0)	(3.0, 5.5)
M^*	6	7	8	8	9
$F(M^*, \mu_v, \mu_B)$	65.6958	61.9104	60.2714	59.7780	59.5969
A_v	0.98104	0.99640	0.99867	0.99873	0.99904
No. of iteration	4	4	4	3	4
A_v^*	0.99671	0.99885	0.99960	0.99960	0.99986
μ_v^*	2.8218	3.3387	3.8551	3.8551	4.3709
μ_B^*	4.08713	4.6897	5.2854	5.2854	5.8754
$F(M^*, \mu_v^*, \mu_B^*)$	62.1029	60.2975	58.8005	58.8005	57.5318

A_v^* is the value of A_v after iterations.

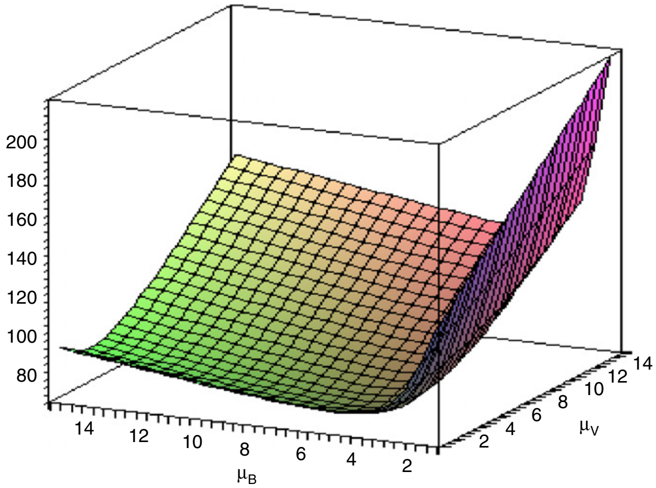

Fig. 4. Plot of $F(M^*, \mu_v, \mu_B)$ for $\lambda = 0.6$, $\eta = 0.3$, $M^* = 7$.

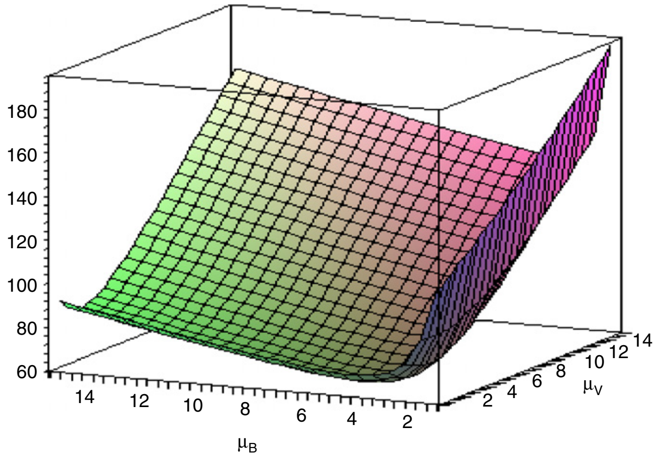

Fig. 5. Plot of $F(M^*, \mu_v, \mu_B)$ for $\lambda = 0.5$, $\eta = 0.3$, $M^* = 6$.

We finally vary the values of μ_v and μ_B, consider two cases: (1) $M^* = 7$, $(\lambda, \eta) = (0.6, 0.3)$; (2) $M^* = 6$, $(\lambda, \eta) = (0.5, 0.3)$, and the values of μ_v and μ_B range from 1.0 to 15.0. The numerical results of $F(M^*, \mu_v, \mu_B)$ for the two cases are depicted in Figs. 4 and 5. The global minimum values $F(M^*, \mu_v, \mu_B)$ for cases 1–2 are shown in Figs. 4 and 5, respectively.

6. Conclusions

In this paper, we considered the M/M/1 machine repair problems with working vacation, in which the server remains working with different repair rates rather than completely terminating the repair during a vacation period. We first established the steady-state equations and applied a matrix-geometric method to derive the steady-state probabilities.

Various system performance measures, such as the expected number of failed machines, the expected number of operating machines, machine availability, and operative utilization, were also calculated. We then developed the expected cost function per machine per unit time to determine the joint optimal values of M, μ_v and μ_B at minimum cost until the system availability constraint is satisfied. In addition, we used the direct search method and Newton's method for unconstrained optimization to determine the optimal values (M^*, μ_v^*, μ_B^*), which satisfy the system availability constraint.

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