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Least-Squares Solutions of Multi-Valued Linear Operator Equations in Hilbert Spaces

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Let M be a linear manifold in $H_1 \oplus H_2$, where H_1 and H_2 are Hilbert spaces. Two notions of least-squares solutions for the *multi-valued* linear operator equation (inclusion) $y \in M(x)$ are introduced and investigated. The main results include (i) equivalent conditions for least-squares solvability, (ii) properties of a least-squares solution, (iii) characterizations of the set of all least-squares solutions in terms of algebraic operator parts and generalized inverses of *linear manifolds*, and (iv) best approximation properties of generalized inverses and operator parts of *multi-valued* linear operators. The principal tools in this investigation are an abstract adjoint theory, orthogonal operator parts, and orthogonal generalized inverses of linear manifolds in Hilbert spaces.

1. INTRODUCTION

Let M be a linear manifold in $H_1 \oplus H_2$, where H_1 and H_2 are Hilbert spaces. We view M as a *multi-valued linear operator* (or as a linear relation) by taking $M(x) := \{y \mid \{x, y\} \in M\}$. The domain, range, and null space of M are defined, respectively, by

$$\operatorname{Dom} M := \{x \in H_1 \mid \{x, y\} \in M \text{ for some } y \in H_2\},$$

$$\operatorname{Range} M := \{y \in H_2 \mid \{x, y\} \in M \text{ for some } x \in H_1\},$$

$$\operatorname{Null} M := \{x \in H_1 \mid \{x, 0\} \in M\}.$$

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In this paper, we introduce and investigate two notions of *least-squares solutions* (LSS) for the multi-valued linear operator equation (or inclusion)

$$y \in M(x),$$

where $y \in H_2$ is given. If $M(0) = \{0\}$, then M is (the graph of) a single-valued linear operator from H_1 to H_2. We are primarily interested in the situation when this is *not* the case. We shall refer to a "single-valued linear operator" simply as an "operator."

The main results which are developed in Section 3 include (i) equivalent conditions for least-squares solvability, (ii) characterizations of the set of all least-squares solutions in terms of algebraic operator parts and generalized inverses of multi-valued linear operators, (iii) properties of a least-squares solution, and (iv) best approximation properties of generalized inverses and operator parts for multi-valued mappings. The crucial tools in this development are an abstract adjoint theory (or adjoint subspaces), orthogonal operator parts and orthogonal generalized inverses of linear manifolds in Hilbert spaces. The essential aspects of these tools that are needed in the proofs are delineated in Section 2.

Throughout this paper, H_1, H_2, and H_3 denote Hilbert spaces. The inner product in any of these spaces is denoted by $\langle\ ,\ \rangle$ and the induced norm by $\|\cdot\|$. The following are standard notations (see [1]), but for convenience we define them. For any sets $A, B \subset H_1 \oplus H_2$ and $Y \subset H_3 \oplus H_1$,

$$\begin{aligned} AY &:= \{\{x, y\} \in H_3 \oplus H_2 \mid \{x, z\} \in Y, \{z, y\} \in A\}, \\ \alpha A &:= \{\{x, \alpha y\} \mid \{x, y\} \in A\}, \qquad \alpha \in \mathbb{C}, \\ A \dotplus B &:= \{a + b \mid a \in A, b \in B\}, \\ A + B &:= \{\{x, y + z\} \mid \{x, y\} \in A, \{x, z\} \in B\}. \end{aligned}$$

The *adjoint* (subspace) of $A \subset H_1 \oplus H_2$ is defined by

$$A^* := \{\{y, -x\} \in H_2 \oplus H_1 \mid \{x, y\} \in A^\perp\},$$

where $A^\perp$ denotes the orthogonal complement of A. Useful properties of adjoints of linear manifolds are:

$$\begin{aligned} &A^{**} = A^c, \qquad \text{where } A^c \text{ denotes the closure of } A, \\ &(\lambda A)^* = \bar{\lambda} A^* \qquad \text{for } \lambda \in \mathbb{C}, \\ &(AB)^* \supset B^* A^*, \qquad (A + B)^* \supset A^* + B^*. \end{aligned}$$

2. Operator Parts of Subspaces

Let M be a vector space in $H_1 \oplus H_2$, the (external) direct sum of two Hilbert spaces H_1, H_2. A vector space $R \subset H_1 \oplus H_2$ is called an *algebraic* operator part of M if R is the graph of a *linear operator* such that M is the (internal) algebraic direct sum of R and $\{0\} \oplus M(0)$. If an algebraic operator part is also (topologically) closed in $H_1 \oplus H_2$, then it is called an *operator part*. These concepts were introduced by E. A. Coddington, and have been extensively studied in [1, 3]. (Recall that a vector space V is said to be the internal direct sum of subspaces S_1 and S_2 of V if every element $v \in V$ can be uniquely written as $v = v_1 + v_2$, where $v_1 \in S_1$ and $v_2 \in S_2$. In contrast, if V_1 and V_2 are given vector spaces, then the vector space V of all ordered pairs (v_1, v_2), where $v_i \in V_i$, with the standard algebraic operations, is called the external direct sum of V_1 and V_2. It is well known that if V is the internal direct sum of S_1 and S_2, then V is isomorphic to the external direct sum of S_1 and S_2. From now on we shall drop the adjectives "external" and "internal" for direct sums.)

We next introduce a notation S_M. Suppose that $M(0)$ is *closed* in H_2 and let $\mathscr{P}$ denote the *orthogonal* projector from H_2 onto $M(0)$. Then we define

$$S_M := [\operatorname{graph}(I - \mathscr{P})] M$$
$$= \{\{g, (I - \mathscr{P})(y)\} \mid \{g, y\} \in M\}.$$

It is easy to check that S_M is an algebraic operator part of M such that S_M is orthogonal to $\{0\} \oplus M(0)$, and $\operatorname{Dom} S_M = \operatorname{Dom} M$, $\operatorname{Range} S_M = (\operatorname{Range} M) \cap (M(0))^{\perp}$. Moreover, S_M is closed if and only if M is closed. We emphasize here that throughout this paper, the notation S_M is reserved for the above algebraic operator part of M *only* when $M(0)$ is closed. It is clear from the definition that $S_{\lambda M} = \lambda S_M$ for any $\lambda \in \mathbb{C}$.

Proposition 2.1. (1) *Let A, B be vector spaces in $H_1 \oplus H_2$, such that $A(0), B(0)$ are closed. Then*: (i) $(A \dot{+} B)(0) = A(0) \dot{+} B(0) \dot{+} \operatorname{Range}(S_A - S_B)$. (ii) *If $(A \dot{+} B)(0)$ is closed, then*

$$S_{A \dot{+} B} = \big\{\{g, (I - \mathscr{P})(S_A(a) + S_B(g - a))\} \mid g \in \operatorname{Dom} A \dot{+} \operatorname{Dom} B$$
$$\text{and } a \in \operatorname{Dom} A \text{ such that } g - a \in \operatorname{Dom} B\big\},$$

where $\mathscr{P}$ is the orthogonal projector from H_2 onto $(A \dot{+} B)(0)$. (iii) *If $A(0) \dot{+} B(0)$ is closed, then*

$$S_{A+B} = (\operatorname{graph}(I - \mathscr{Q}))(S_A + S_B),$$

where $\mathscr{Q}$ is the orthogonal projector from H_2 onto $(A + B)(0) = A(0) \dot{+} B(0)$.

(2) *Suppose that* $A \subset H_2 \oplus H_3$ *and* $B \subset H_1 \oplus H_2$ *are vector spaces such that* $A(0)$, $B(0)$ *are closed. Then*: (i) $(AB)(0) = A(0) \dot{+} S_A(B(0) \cap \operatorname{Dom} A)$, *orthogonal sum.* (ii) *If* $(AB)(0)$ *is closed, then*

$$S_{AB} = \{\{g, (I - \mathscr{Q})\, S_A(S_B(g) + k\} \mid k \in B(0) \textit{ and } g \in \operatorname{Dom} B \textit{ such that } S_B(g) + k \in \operatorname{Dom} A\},$$ *where* $\mathscr{Q}$ *is the orthogonal projector from* H_3 *onto* $(AB)(0)$.

Proof. Take $x \in (A \dot{+} B)(0)$. Then $x = a_2 + b_2$ for some a_1 such that $\{a_1, a_2\} \in A$, $\{-a_1, b_2\} \in B$. Since S_A and S_B are algebraic operator parts *of* A and B, respectively, it follows that $a_2 = S_A(a_1) + k_1$, $b_2 = -S_B(a_1) + k_2$ for some $k_1 \in A(0)$, $k_2 \in B(0)$. Thus

$$x = S_A(a_1) - S_B(a_1) + k_1 + k_2 \in A(0) \dot{+} B(0) \dot{+} \operatorname{Range}(S_A - S_B).$$

Hence

$$(A \dot{+} B)(0) \subset A(0) \dot{+} B(0) \dot{+} \operatorname{Range}(S_A - S_B).$$

It is easy to check that

$$(A \dot{+} B)(0) \supset A(0) \dot{+} B(0) \dot{+} \operatorname{Range}(S_A - S_B).$$

This proves (1-i). To prove (ii) of (1), let $\mathscr{P}$ be as in the theorem. Then

$$S_{A \dot{+} B} = \{\{g, (I - \mathscr{P})(h)\} \mid \{g, h\} \in A \dot{+} B\}. \qquad (*)$$

Now $\{g, h\} \in A \dot{+} B$ if and only if

$$g \in \operatorname{Dom} A \dot{+} \operatorname{Dom} B, \qquad h = S_A(a_1) + S_B(b_1) + k_1 + k_2$$

for some $k_1 \in A(0)$, $k_2 \in B(0)$, $a_1 \in \operatorname{Dom} A$, $b_1 \in \operatorname{Dom} B$ such that $a_1 + b_1 = g$. Since $A(0) \dot{+} B(0) \subset (A \dot{+} B)(0)$, $(I - \mathscr{P})(k_1 + k_2) = 0$. Thus $(*)$ combined with the above argument proves (ii) of (1). We now prove (iii) of (1). Let $\mathscr{Q}$ be as in the theorem. Then

$$S_{A+B} = \{\{g, (I - \mathscr{Q})(h)\} \mid \{g, h\} \in A + B\}. \qquad (**)$$

Take $\{g, h\} \in A + B$. Then $g \in \operatorname{Dom} A \cap \operatorname{Dom} B$ and $h = p + q$ for some p, q such that $\{g, p\} \in A$, $\{g, q\} \in B$. Let

$$p = S_A(g) + k_1, \qquad q = S_B(g) + k_2$$

for some $k_1 \in A(0)$, $k_2 \in B(0)$. Then

$$(I - \mathscr{Q})(h) = (I - \mathscr{Q})(S_A + S_B)(g)$$

as $k_1 + k_2 \in (A + B)(0)$. This together with $(**)$ yields (iii) of (1). Part (2) can be proved in a similar way. ∎

DEFINITION. Let $M \subset H_1 \oplus H_2$ be a vector space such that Null M is *closed*. Let $\mathscr{P}$ be the *orthogonal* projector from H_1 onto Null M, and $\mathscr{P}^+$ the *orthogonal* projector from H_2 onto Null M^*. Let M^{-1} be the inverse relation of M. Define a vector space $M^\#$ by

$$M^\# := [\text{graph}(I - \mathscr{P})]\, M^{-1} [\text{graph}(I - \mathscr{P}^+)].$$

Then $M^\#$ is called the *orthogonal generalized inverse* of M. (If M is the graph of a closed densely defined linear operator, then $M^\#$ is precisely the *graph* of the Moore–Penrose inverse of that operator.)

The study of generalized inverses of *multi-valued* linear operators in Banach space was initiated by the authors in [3], where a comprehensive theory is developed with applications to differential subspaces and general boundary-value problems. In this paper, we will only need elementary properties of the orthogonal generalized inverse. It is proved in [3] that $M^\#$ is a linear *operator* such that

$$M^\# = S_{M^{-1}} \dot{+} (\text{Null } M^* \oplus \{0\}), \qquad \text{direct sum,}$$
$$\text{Dom } M^\# = \text{Range } M \dot{+} \text{Null } M^*,$$
$$\text{Range } M^\# = \text{Range } S_{M^{-1}} = (\text{Dom } M) \cap (\text{Null } M)^\perp.$$

Moreover, if M is closed, then $M^\#$ is closed and $(M^\#)^*$ is the orthogonal generalized inverse of M^*. Furthermore, when M is closed, $M^\#$ is continuous if and only if Range M is closed.

PROPOSITION 2.2. *Let $M \subset H_1 \oplus H_2$ be a vector space such that* Null M *is closed. Let $\mathscr{P}$ and $\mathscr{P}^+$ be the orthogonal projectors from H_1 and H_2 onto* Null M *and* Null M^*, *respectively. Then*

$$MM^\# = \big\{\{x, (I - \mathscr{P}^+)(x) + s\} \mid s \in M(0), x \in \text{Dom } M^\#\big\},$$
$$M^\# M = \big\{\{x, (I - \mathscr{P})(x)\} \mid x \in \text{Dom } M\big\}.$$

Proof. This can be found in [3]. ∎

The preceding properties of generalized inverses of multi-valued linear operators should be contrasted with those in the case of an operator; see [5]. In particular, it should be noted from Proposition 2.2 that in the case when $M(0) \neq \{0\}$, $MM^\#$ is not a single-valued orthogonal projector.

3. Least-Squares Solutions of Multi-Valued Linear Operator Equations

Definition. Let $M \subset H_1 \oplus H_2$ be an arbitrary given vector space. Let $y \in H_2$. Then $u \in H_1$ is called a *least-squares solution* (LSS) of the inclusion $y \in M(x)$ if $u \in \operatorname{Dom} M$ and

$$d(y, \operatorname{Range} M) = \|y - z\|$$

for some $z \in M(u)$, where $d(y, \operatorname{Range} M)$ is the distance between y and Range M.

Note that if such a z exists, then it is unique. Of course, u need not be unique. Also, if M is an operator, then the above definition coincides with the usual definition of a least-squares solution of an operator equation.

Proposition 3.1. [I] *Let $y \in H_2$. Then the following statements are equivalent*:

(i) $y \in M(x)$ *has a* LSS.

(ii) $(I - \mathscr{P}^+)(y) \in \operatorname{Range} M$, *where* $\mathscr{P}^+$ *is the orthogonal projector from* H_2 *onto* Null M^*.

(iii) $y \in \operatorname{Null} M^* \dot{+} \operatorname{Range} M$.

[II] *Let $y \in H_1$. Then the following statements are equivalent*:

(i) $y \in M^*(x)$ *has a* LSS.

(ii) $(I - \mathscr{P})(y) \in \operatorname{Range} M^*$, *where* $\mathscr{P}$ *is the orthogonal projector from* H_1 *onto* Null M^c.

(iii) $y \in \operatorname{Null} M^c \dot{+} \operatorname{Range} M^*$.

Proof. [I] Assume (i). Let u be a LSS of $y \in M(x)$. Then $u \in \operatorname{Dom} M$ and $d(y, \operatorname{Range} M) = \|y - z\|$, $z \in M(u)$. Now $d(y, \operatorname{Range} M) = d(y, (\operatorname{Range} M)^c) = \|y - (I - \mathscr{P}^+)(y)\|$. It follows from the best approximation property of an orthogonal projection in Hilbert space that $z := (I - \mathscr{P}^+)y \in M(u)$. Thus, (i) implies (ii). Now assume (ii). Then $(I - \mathscr{P}^+)y = z$ for some $z \in \operatorname{Range} M$. Thus, $y = \mathscr{P}^+(y) + z \in \operatorname{Null} M^* \dot{+} \operatorname{Range} M$, and so (ii) implies (iii). To prove that (iii) implies (i), let $y = k + z$, $z \in M(u)$ for some $u \in \operatorname{Dom} M$, $k \in \operatorname{Null} M^*$. Then $d(y, \operatorname{Range} M) = \|\mathscr{P}^+(y)\| = \|\mathscr{P}^+(k)\| = \|k\| = \|y - z\|$. Thus u is a LSS. This completes the proof of [I]. Part [II] is the dual of [I]; it follows from it by replacing M by M^* and by noting that $M^{**} = M^c$. ∎

Remark. Note that $\operatorname{Null} M^* \dot{+} \operatorname{Range} M$ is always dense in H_2. It is closed if and only if Range M is closed.

Proposition 3.2. *Let $\mathscr{P}$ and $\mathscr{P}^+$ be as in Proposition* 3.1.

[I] *Let $y \in H_2$ be given. Then the following statements are equivalent*:

(i) $u \in H_1$ *is a* LSS *of* $y \in M(x)$.

(ii) $u \in \text{Dom } M$ *and* $(I - \mathscr{P}^+)(y) \in M(u)$.

(iii) $u \in \text{Dom } M$ *and* $y \in M(u) \dotplus \text{Null } M^*$.

(iv) $u \in \text{Dom } M$ *and* $M(u) \subset y \dotplus (\text{Range } M)^\perp$.

[II] *Let $y \in H_1$ be given. Then the following statements are equivalent*:

(i) $u \in H_2$ *and u is a* LSS *of* $y \in M^*(x)$.

(ii) $u \in \text{Dom } M^*$ *and* $(I - \mathscr{P})(y) \in M^*(u)$.

(iii) $u \in \text{Dom } M^*$ *and* $y \in M^*(u) \dotplus \text{Null } M^c$.

(iv) $u \in \text{Dom } M^*$ *and* $M^*(u) \subset y \dotplus (\text{Range } M^*)^\perp$.

Proof. Assume (i). Then $d(y, \text{Range } M) = \|y - z\|$ for some $z \in M(u)$ and hence $y - z = \mathscr{P}^+(y)$. Thus, $(I - \mathscr{P}^+)(y) \in M(u)$ and so (i) implies (ii). Assume (ii) holds. Since $d(y, \text{Range } M) = \|y - (I - \mathscr{P}^+)(y)\|$ and $(I - \mathscr{P}^+)(y) \in M(u)$, it follows that u is a LSS. Thus (ii) implies (i). It is clear that (ii) implies (iii). Also, since $(\text{Range } M)^\perp = \text{Null } M^*$, (iii) implies (iv). Finally to show that (iv) implies (i), let $k = z - y$ for some $z \in M(u)$, $k \in (\text{Range } M)^\perp$. Then $d(y, \text{Range } M) = \|\mathscr{P}^+(y)\| = \|\mathscr{P}^+(z - k)\| = \|\mathscr{P}^+(k)\| = \|y - z\|$. This shows that u is a LSS and completes the proof of [I]. Again, part [II] is the dual of part [I]. ∎

Remark. Suppose that M is an operator. If $\text{Dom } M^* = H_2$, or equivalently, M^c is an operator and $\text{Dom } M^*$ is closed, then (I-iii) of Proposition 3.2 can be rewritten as follows: u is a LSS of $Mx = y$ if and only if $M^*Mu = M^*y$, which is the usual "normal equation" characterization for a least-squares solution for, say, a bounded linear operator equation in Hilbert space. Of course, this characterization is false if $\text{Dom } M^* \neq H_2$.

We now characterize the set of all least-squares solutions in terms of algebraic operator parts and generalized inverses of multi-valued linear operators.

Theorem 3.3. *Assume that* $y \in \text{Range } M \dotplus \text{Null } M^*$ *and let* $\mathscr{P}^+$ *be the orthogonal projector from* H_2 *onto* $\text{Null } M^*$. *Then we have the following*:

(1) (i) *For any algebraic operator part R of M^{-1}, the coset*

$$R(I - \mathscr{P}^+)(y) \dotplus \text{Null } M$$

is the set of all least-squares solutions of $y \in M(x)$.

(ii) *If* Null M *is closed, then* $M^{\#}(y) \dotplus$ Null M *is the set of all least-squares solutions of* $y \in M(x)$.

(iii) *If* Null M *is closed and* $y \in$ Range M, *then* $M^{\#}(y) \dotplus$ Null M *is the set of all solutions of* $y \in M(x)$.

(2) *Assume that* Null M *is closed. Then*

(i) $\|M^{\#}(y)\| \leqslant \|u\|$ *for all least-squares solutions* u *of* $y \in M(x)$; *equality holds only if* $u = M^{\#}(y)$.

(ii) *Assume further that* $M(0)$ *is closed. Then*

$$d(y, \text{Range } M) = \|y - S_M M^{\#}(y) - \mathscr{Q}(I - \mathscr{P}^+)(y)\|,$$

where $\mathscr{Q}$ *is the orthogonal projector from* H_2 *onto* $M(0)$. *Moreover, the map*

$$y \mapsto S_M M^{\#}(y) + \mathscr{Q}(I - \mathscr{P}^+)(y)$$

on Dom $M^{\#}$ *into* H_2 *is continuous.*

Proof. (i) *of* (1). It follows from Proposition 3.2 that u is a least-squares solution of $y \in M(x)$ if and only if $\{u, (I - \mathscr{P}^+)(y)\} \in M$, or equivalently, $(I - \mathscr{P}^+)(y) \in$ Range M and $u = R(I - \mathscr{P}^+)(y) + k$ for some $k \in$ Null M. Since $y \in$ Range $M \dotplus$ Null M^*, $(I - \mathscr{P}^+)(y) \in$ Range $M =$ Dom R. Thus

$$R(I - \mathscr{P}^+)(y) \dotplus \text{Null } M$$

is the set of all least-squares solutions of $y \in M(x)$.

(ii) *of* (1). Since Null M is closed, $S_{M^{-1}}$ is an algebraic operator part of M^{-1}. Thus by taking R as $S_{M^{-1}}$ in (i), we see that

$$S_{M^{-1}}(I - \mathscr{P}^+)(y) \dotplus \text{Null } M = M^{\#}(y) \dotplus \text{Null } M$$

is the set of all least-squares solutions.

(iii) *of* (1). Since $S_{M^{-1}}$ is an algebraic operator part of M^{-1}, $S_{M^{-1}}(y) \dotplus$ Null M is the set of all solutions of $y \in M(x)$. Since $y \in$ Range M, $y = (I - \mathscr{P}^+)(y)$. Thus

$$S_{M^{-1}}(y) = S_{M^{-1}}(I - \mathscr{P}^+)(y) = M^{\#}(y),$$

and so $M^{\#}(y) \dotplus$ Null M is the set of all solutions of $y \in M(x)$.

(i) *of* (2). Let u be a least-squares solution of $y \in M(x)$. Then $u = M^{\#}(y) + k$ for some $k \in$ Null M. Since $M^{\#}(y) \in (\text{Null } M)^{\perp}$, it follows that

$$\|u\|^2 = \|M^{\#}(y)\|^2 + \|k\|^2 \geqslant \|M^{\#}(y)\|^2.$$

Suppose u is a least-squares solution of $y \in M(x)$ such that $\|u\| \leqslant \|M^{\#}(y)\|$. We can write u as $M^{\#}(y) + k$ for some $k \in \operatorname{Null} M$. It follows that $\|M^{\#}(y)\|^2 + \|k\|^2 \leqslant \|M^{\#}(y)\|^2$, and so $k = 0$. Thus $u = M^{\#}(y)$. This proves (i) of (2).

(ii) *of* (2). Since $M^{\#}(y)$ is a least-squares solution of $y \in M(x)$,

$$d(y, \operatorname{Range} M) = \|y - s\|$$

for some $s \in M(M^{\#}(y))$. Since $M(0)$ is closed, S_M is an algebraic operator part of M. Therefore, since $\{M^{\#}(y), s\} \in M$, it follows that

$$S_M M^{\#}(s) = (I - \mathscr{Q})(s),$$

and hence

$$s = S_M M^{\#}(s) + \mathscr{Q}(s).$$

On the other hand, by the best approximation property of an orthogonal projection in Hilbert space, $s = (I - \mathscr{P}^+)(y)$. Hence,

$$s = S_M M^{\#}(y) + (I - \mathscr{P}^+)(y).$$

Now, the map defined on $\operatorname{Dom} M^{\#}$ in the theorem is continuous as it coincides with the map $x \mapsto (I - \mathscr{P}^+)(x)$ on $\operatorname{Dom} M^{\#}$. ∎

Another generalization of the notion of a least-squares solution to the case of a multi-valued operator that seems natural is the following: Let g be a given element in H_2. An element $u \in H_1$ is called an *almost least-squares solution* of $g \in M(x)$ if $d(g, \operatorname{Range} M) = d(g, M(u))$. Clearly both the concept of an almost LSS and LSS in the earlier sense reduce to the concept of LSS in the case of a (single-valued) operator.

Suppose that S is a (nonclosed) *dense* vector space in H_2. Define $M := \{0\} \oplus S$ and take any g in H_2 such that $g \notin S$. Then $\operatorname{Range} M \dotplus \operatorname{Null} M^* = S \dotplus S^{\perp} = S$. Thus by part [I] of Proposition 3.1 (or directly from the definition) $g \in M(x)$ has no least-squares solution. However,

$$d(g, \operatorname{Range} M) = d(g, S) = d(g, M(0)),$$

so that the zero vector is an almost least-squares solution of $g \in M(x)$. This example shows that the concepts of a least-squares solution and an "almost" least-squares solution are different, even though they agree in the case of a single-valued operator. In the following we will compare these two concepts more closely.

THEOREM 3.4. *Let $g \in H_2$, $u \in \operatorname{Dom} M$ be given. Let R be an arbitrary, but fixed algebraic operator part of M. Then*

(1) *$d(g, M(u)) = \| g - s\|$ for some $s \in M(u)$ if and only if $g - R(u) \in M(0) \dot{+} (M(0))^{\perp}$. Moreover, if $M(0)$ is closed, then it is always true that*

$$d(g, M(u)) = \| g - s\| \qquad \textit{for some} \quad s \in M(u).$$

(2) *Assume that $g - R(u) \in M(0) + (M(0))^{\perp}$. Then*

$$\mathscr{Q}(g) + (I - \mathscr{Q})\, R(u) \in M(u),$$
$$d(g, M(u)) = \| g - \mathscr{Q}(g) - (I - \mathscr{Q})\, R(u)\|,$$

where $\mathscr{Q}$ is the orthogonal projector from H_2 onto $(M(0))^c$.

(3) *Assume that $M(0)$ is closed. Then*

$$\mathscr{Q}(g) + S_M(u) \in M(u),$$
$$d(g, M(u)) = \| g - \mathscr{Q}(g) - S_M(u)\|,$$

where $\mathscr{Q}$ is the same as the above.

Proof. (1) Let R be an arbitrary, but fixed algebraic operator part of M. Then for $u \in \operatorname{Dom} M$,

$$M(u) = R(u) \dot{+} M(0).$$

It follows that

$$d(g, M(u)) = \| g - s\| \qquad \text{for some} \quad s \in M(u)$$

if and only if

$$d(g - R(u), M(0)) = \| g - R(u) - k\| \tag{$*$}$$

for some $k \in M(0)$. Define $M_\infty := \{0\} \oplus M(0)$. Then $M(0) = \operatorname{Range} M_\infty$. By Proposition 3.1, $(*)$ holds for some $k \in M(0)$ if and only if $g - R(u)$ belongs to

$$\operatorname{Range} M_\infty \dot{+} \operatorname{Null}(M_\infty)^* = M(0) \dot{+} (M(0))^{\perp}.$$

This proves the first part of (1). To establish the last part, we choose R to be

$$R := \big\{\{g, (I - \mathscr{P})(h)\} \mid \{g, h\} \in M\big\},$$

where $\mathscr{P}$ is the orthogonal projector from H_2 onto $M(0)$ which is closed by assumption. Then $R(u) \in (M(0))^{\perp}$. It follows that $g - R(u) \in M(0) \dot{+} (M(0))^{\perp}$ if and only if $g \in M(0) \dot{+} (M(0))^{\perp}$.

(2) Let $\mathscr{Q}$ be as in the theorem. Then

$$\begin{aligned} d(g, M(u)) &= d(g - R(u), M(0)) = \|(I - \mathscr{Q})(g - R(u))\| \\ &= \| g - [\mathscr{Q}(g) + (I - \mathscr{Q}) R(u)]\|. \\ &= \| g - R(u) - k\| \end{aligned}$$

for some $k \in M(0)$. Since such a k is unique, it follows that $\mathscr{Q}(g) - \mathscr{Q}R(u) = k \in M(0)$. Thus

$$\mathscr{Q}(g) + (I - \mathscr{Q}) R(u) \in R(u) \dot{+} M(0) = M(u).$$

(3) Since $M(0)$ is closed, S_M is an algebraic operator part of M. Thus the result follows from (2) by replacing R by S_M and noting that $(I - \mathscr{Q}) S_M = S_M$. ∎

COROLLARY 3.5. *Let $M \subset H_1 \oplus H_2$ be a vector space and $g \in H_2$.*

(1) *If u is a least-squares solution of $g \in M(x)$, then u is an "almost" least-squares solution of $g \in M(x)$.*

(2) *Assume that $M(0)$ is closed. Then u is a least-squares solution of $g \in M(x)$ if and only if it is an "almost" least-squares solution of $g \in M(x)$.*

Proof. (1) Suppose that $u \in \operatorname{Dom} M$ and

$$d(g, \operatorname{Range} M) = \| g - z\|$$

for some $z \in M(u)$. Since $M(u) \subset \operatorname{Range} M$, $d(g, \operatorname{Range} M) \leqslant d(g, M(u))$. Thus

$$\| g - z\| = d(g, \operatorname{Range} M) \leqslant d(g, M(u)) \leqslant \| g - z\|,$$

and hence u is an "almost" least-squares solution of $g \in M(x)$.

(2) Assume that

$$d(g, \operatorname{Range} M) = d(g, M(u)).$$

Since $M(0)$ is closed, by Theorem 3.4, $d(g, M(u)) = \| g - z\|$ for some $z \in M(u)$. It follows that

$$d(g, \operatorname{Range} M) = \| g - z\|, \qquad z \in M(u).$$

Thus u is a least-squares solution of $g \in M(x)$. This together with the result of (1) completes the proof of (2). ∎

Some of the preceding results develop *vector* extremal properties (i.e., in terms of $M^{\#}y$) of the orthogonal generalized inverse of $M \subset H_1 \oplus H_2$ under some mild assumptions. The authors have also obtained *operator* extremal properties of $M^{\#}$, extending some of the results of [2] to multi-valued operators. These results will appear elsewhere. The authors have also investigated iterative and regularization methods for equations (or inclusions) involving nondensely defined and/or multi-valued linear operators in Hilbert spaces (see, e.g., [4]).

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