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Perturbative non-equilibrium thermal field theory to all orders in gradient expansion

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ABSTRACT

We present a new perturbative formulation of non-equilibrium thermal field theory, based upon non-homogeneous free propagators and time-dependent vertices. The resulting time-dependent diagrammatic perturbation series are free of pinch singularities without the need for quasi-particle approximation or effective resummation of finite widths. After arriving at a physically meaningful definition of particle number densities, we derive master time evolution equations for statistical distribution functions, which are valid to all orders in perturbation theory and to all orders in a gradient expansion. For a scalar model, we perform a perturbative loopwise truncation of these evolution equations, whilst still capturing fast transient behaviour, which is found to be dominated by energy-violating processes, leading to the non-Markovian evolution of memory effects.

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1. Introduction

The description of out-of-equilibrium many-body field-theoretic systems is of increasing relevance in theoretical and experimental physics at the *density frontier*. Examples range from the early Universe to the deconfined phase of QCD, the quark–gluon plasma, relevant at heavy-ion colliders, such as RHIC and the LHC as well as the internal dynamics of compact astro-physical phenomena and condensed matter systems.

In this Letter, we present the key concepts of a new perturbative approach to non-equilibrium thermal quantum field theory, where master time evolution equations for macroscopic observables are derived from first principles. A comprehensive exposition of this new formulation is provided in [1]. In contrast to semi-classical approaches based on the Boltzmann equation [2–9], this new approach allows the systematic incorporation of finite-width and off-shell effects without the need for effective resummations. Furthermore, having a well-defined underlying perturbation theory that is free of pinch singularities, these time evolution equations may be truncated in a perturbative loopwise sense, whilst retaining all orders of the time behaviour. Several studies appeared in the literature [10–40] proposing quantum-corrected transport equations, based upon systems of Kadanoff–Baym equations [41], functional renormalization group approaches [42] or expansion of the Liouville–von Neumann equation [43,44]. Whilst retaining all orders in perturbation theory, the existing approaches often rely on the truncation of gradient expansions [45,46] in time derivatives, quasi-particle approximations or *ad hoc* ansaetze in order to obtain calculable expressions or extract meaningful observables. In this new perturbative formalism, the loopwise-truncated evolution equations are built from non-homogeneous free propagators and time-dependent vertices. This diagrammatic approach encodes both spatial and temporal inhomogeneity already from tree-level, without resorting to any such approximations.

2. Canonical quantization

We begin by highlighting the details of the canonical quantization of a real scalar field pertinent to a perturbative treatment of non-equilibrium thermal field theory.

The *time-independent* Schrödinger-picture field operator, denoted by a subscript S, may be written in the familiar plane-wave decomposition

$$\Phi_{\rm S}(\mathbf{x};\tilde{t}_i) = \int \frac{{\rm d}^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E(\mathbf{p})}\big(a_{\rm S}(\mathbf{p};\tilde{t}_i)e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\rm S}^{\dagger}(\mathbf{p};\tilde{t}_i)e^{-i\mathbf{p}\cdot\mathbf{x}}\big), \qquad (1)$$

where $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$ and $a_{\rm S}^{\dagger}(\mathbf{p};\tilde{t}_i)$ and $a_{\rm S}(\mathbf{p};\tilde{t}_i)$ are the usual single-particle creation and annihilation operators. It is essential to emphasize that we define the Schrödinger, Heisenberg and interaction (Dirac) pictures to be coincident at the finite *micro*scopic boundary time $\tilde{t}_i$, i.e.

$$\Phi_{\rm S}(\mathbf{x};\tilde{t}_i) = \Phi_{\rm H}(\tilde{t}_i,\mathbf{x};\tilde{t}_i) = \Phi_{\rm I}(\tilde{t}_i,\mathbf{x};\tilde{t}_i). \qquad (2)$$

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It is at this picture-independent boundary time $\tilde{t}_i$ that initial conditions must be specified. The dependence upon the boundary time $\tilde{t}_i$ is separated from other arguments by a semi-colon.

The *time-dependent* interaction-picture operator $\Phi_{\rm I}(x;\tilde{t}_i)$ is obtained via the unitary transformation

$$\Phi_{\rm I}(x;\tilde{t}_i) = e^{iH_{\rm S}^0(x_0-\tilde{t}_i)}\Phi_{\rm S}(\mathbf{x};\tilde{t}_i)e^{-iH_{\rm S}^0(x_0-\tilde{t}_i)}, \tag{3}$$

where $H_{\rm S}^0$ is the free part of the Hamiltonian in the Schrödinger picture. This yields

$$\Phi_{\rm I}(x;\tilde{t}_i) = \int \frac{{\rm d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\mathbf{p})}\big(a_{\rm I}(\mathbf{p},0;\tilde{t}_i)e^{-iE(\mathbf{p})x_0}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\rm I}^\dagger(\mathbf{p},0;\tilde{t}_i)e^{iE(\mathbf{p})x_0}e^{-i\mathbf{p}\cdot\mathbf{x}}\big), \tag{4}$$

where $a_{\rm I}(\mathbf{p},x_0;\tilde{t}_i) = a_{\rm I}(\mathbf{p},0;\tilde{t}_i)e^{-iE(\mathbf{p})x_0}$ and its Hermitian conjugate are the *time-dependent* interaction-picture annihilation and creation operators. These operators satisfy the canonical commutation relation

$$\big[a_{\rm I}(\mathbf{p},x_0;\tilde{t}_i), a_{\rm I}^\dagger(\mathbf{p}',x_0';\tilde{t}_i)\big] = (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}')e^{-iE(\mathbf{p})(x_0-x_0')}, \tag{5}$$

with all other commutators vanishing. Note the presence of an overall phase $e^{-iE(\mathbf{p})(x_0-x_0')}$ in (5) for $x_0 \neq x_0'$.

In quantum statistical mechanics, we are interested in the Ensemble Expectation Values (EEVs) of operators at a fixed *micro*scopic time of observation $\tilde{t}_f$. Such EEVs are obtained by taking the trace with the density operator $\rho(\tilde{t}_f;\tilde{t}_i)$, i.e.

$$\langle\bullet\rangle_t = \mathcal{Z}^{-1}(t)\,{\rm Tr}\,\rho(\tilde{t}_f;\tilde{t}_i)\bullet, \tag{6}$$

where $\mathcal{Z}(t) = {\rm Tr}\,\rho(\tilde{t}_f;\tilde{t}_i)$ is the partition function, which is time-dependent in the presence of external sources. We have introduced the *macro*scopic time $t = \tilde{t}_f - \tilde{t}_i$, which is the interval between the *micro*scopic boundary and observation times.

Consider the following observable, which is the EEV of a two-point product of field operators:

$$\mathcal{O}(\mathbf{x},\mathbf{y},\tilde{t}_f;\tilde{t}_i) = \mathcal{Z}^{-1}(t)\,{\rm Tr}\,\rho(\tilde{t}_f;\tilde{t}_i)\Phi(\tilde{t}_f,\mathbf{x};\tilde{t}_i)\Phi(\tilde{t}_f,\mathbf{y};\tilde{t}_i). \tag{7}$$

As shown in [1], it is not necessary to specify the picture in which the operators of the RHS of (7) are to be interpreted, since all operators are evaluated at *equal times*. In addition, the observable $\mathcal{O}$ is invariant under simultaneous time translations of the boundary and observation times and depends only on the macroscopic time t: $\mathcal{O}(\mathbf{x},\mathbf{y},\tilde{t}_f;\tilde{t}_i) \equiv \mathcal{O}(\mathbf{x},\mathbf{y},\tilde{t}_f-\tilde{t}_i;0) \equiv \mathcal{O}(\mathbf{x},\mathbf{y},t)$. Notice that $\mathcal{O}$ depends upon 7 independent coordinates: the spatial coordinates $\mathbf{x}$ and $\mathbf{y}$ and the macroscopic time t.

The density operator $\rho(\tilde{t}_f;\tilde{t}_i)$ of a time-dependent and spatially inhomogeneous background is non-diagonal in the Fock space and contains an intractable incoherent sum of all possible n to m multiparticle correlations, see [1]. We may account for our ignorance of the exact form of this density operator by defining the bilinear EEVs

$$\big\langle a_{\rm I}(\mathbf{p},\tilde{t}_f;\tilde{t}_i)a_{\rm I}^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)\big\rangle_t = (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}') + 2E^{\frac{1}{2}}(\mathbf{p})E^{\frac{1}{2}}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t), \tag{8a}$$

$$\big\langle a_{\rm I}^\dagger(\mathbf{p}',\tilde{t}_f;\tilde{t}_i)a_{\rm I}(\mathbf{p},\tilde{t}_f;\tilde{t}_i)\big\rangle_t = 2E^{\frac{1}{2}}(\mathbf{p})E^{\frac{1}{2}}(\mathbf{p}')f(\mathbf{p},\mathbf{p}',t), \tag{8b}$$

consistent with the canonical commutation relation (5), where $f(\mathbf{p},\mathbf{p}',t) = f^*(\mathbf{p}',\mathbf{p},t)$. The *statistical distribution function* $f(\mathbf{p},\mathbf{p}',t)$ is related to the particle number density $n(\mathbf{q},\mathbf{X},t)$ via the Wigner transform

$$n(\mathbf{q},\mathbf{X},t) = \int \frac{{\rm d}^3\mathbf{Q}}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{X}}f(\mathbf{q}+\mathbf{Q}/2,\mathbf{q}-\mathbf{Q}/2,t), \tag{9}$$

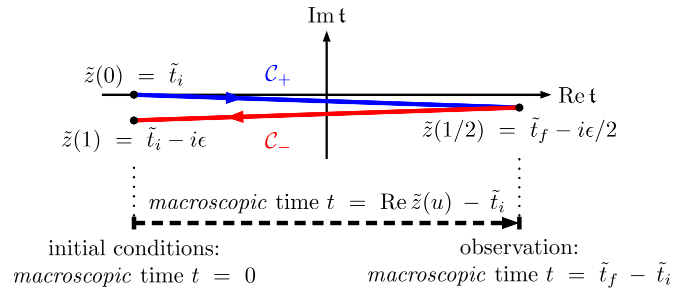


Fig. 1. The closed-time path, $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$. The relationship between *micro*scopic and *macro*scopic times is indicated by a dashed black arrow.

where we have introduced the relative and central momenta $\mathbf{Q} = \mathbf{p} - \mathbf{p}'$ and $\mathbf{q} = (\mathbf{p}+\mathbf{p}')/2$, conjugate to the central and relative coordinates $\mathbf{X} = (\mathbf{x}+\mathbf{y})/2$ and $\mathbf{R} = \mathbf{x} - \mathbf{y}$, respectively. Observe that spatial homogeneity is broken by the explicit dependence of $f(\mathbf{p},\mathbf{p}',t)$ on the two three-momenta $\mathbf{p}$ and $\mathbf{p}'$. In the thermodynamic equilibrium limit, we have the correspondence $f(\mathbf{p},\mathbf{p}',t) \to f_{\rm eq}(\mathbf{p},\mathbf{p}') = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')f_{\rm B}(E(\mathbf{p}))$, where $f_{\rm B}(x) = (e^{\beta x}-1)^{-1}$ is the Bose–Einstein distribution function and β is the inverse thermodynamic temperature.

3. Schwinger–Keldysh CTP formalism

We require a path-integral approach to generating EEVs for products of field operators. Such an approach is provided by the Schwinger–Keldysh CTP formalism [47,48].

In order to obtain a generating functional of EEVs, we insert unitary evolution operators to the left and right of the density operator in the partition function $\mathcal{Z}(t) = {\rm Tr}\,\rho(\tilde{t}_f;\tilde{t}_i)$, yielding

$$\mathcal{Z}[\rho,J_\pm,t] = {\rm Tr}\big[\bar{\rm T}e^{-i\int_{\Omega_t}{\rm d}^4x\,J_-(x)\Phi_{\rm H}(x)}\big]\rho_{\rm H}(\tilde{t}_f;\tilde{t}_i) \times \big[{\rm T}e^{i\int_{\Omega_t}{\rm d}^4x\,J_+(x)\Phi_{\rm H}(x)}\big], \tag{10}$$

in the Heisenberg picture, where Ω_t is the temporally-bounded spacetime hypervolume $[-t/2,t/2]\times\mathbb{R}^3$. We stress that (10) differs fundamentally from existing interpretations of the CTP formalism [49,50]. Specifically, the Heisenberg-picture density operator $\rho_{\rm H}(\tilde{t}_f;\tilde{t}_i)$, which is explicitly time-dependent in the presence of the external sources $J_\pm$, is evaluated at the *time of observation* $\tilde{t}_f$ and *not* the *initial time* $\tilde{t}_i$. In our approach, the role of the unitary evolution operators is to enable us to generate EEVs for products of field operators as given in (7) by functional differentiation with respect to the external sources. The resulting EEVs are evaluated at the *time of observation*.

We may interpret the evolution operators in (10) as defining a closed contour $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ in the complex-time plane (t-plane, ${\rm t} \in \mathbb{C}$), as shown in Fig. 1, which is the union of two anti-parallel branches: $\mathcal{C}_+$, running from $\tilde{t}_i$ to $\tilde{t}_f - i\epsilon/2$; and $\mathcal{C}_-$, running from $\tilde{t}_f - i\epsilon/2$ back to $\tilde{t}_i - i\epsilon$. A small imaginary part $\epsilon = 0^+$ is added to separate the two, essentially coincident, branches. We may introduce an explicit parametrization of this contour $\tilde{z}(u)$ [1], where u increases monotonically along $\mathcal{C}$, which allows the definition of a path-ordering operator ${\rm T}_{\mathcal{C}}$. We emphasize that, in our formalism, this contour evolves in time, with each branch having length t.

Following the notation of [49,50], we denote fields confined to the positive and negative branches of the CTP contour by $\Phi_\pm(x) \equiv \Phi(x^0 \in \mathcal{C}_\pm,\mathbf{x})$. We then define the doublets

$$\Phi^a(x) = \big(\Phi_+(x),\Phi_-(x)\big), \tag{11a}$$

$$\Phi_a(x) = \eta_{ab}\Phi^b(x) = \big(\Phi_+(x),-\Phi_-(x)\big), \tag{11b}$$

where the CTP indices $a, b = 1, 2$ and $\eta_{ab} = \text{diag}(1, -1)$ is an $\mathbb{SO}(1,1)$ 'metric.'

Inserting into (10) complete sets of eigenstates of the Heisenberg field operator, we derive a path-integral representation of the CTP generating functional [1], which depends on the path-ordered propagator

$$i\Delta^{ab}(x, y, \tilde{t}_f; \tilde{t}_i) \equiv \langle \mathrm{T}_{\mathcal{C}}[\Phi^a(x; \tilde{t}_i)\Phi^b(y; \tilde{t}_i)]\rangle_t = i\begin{bmatrix} \Delta_{\mathrm{F}}(x, y, \tilde{t}_f; \tilde{t}_i) & \Delta_<(x, y, \tilde{t}_f; \tilde{t}_i) \\ \Delta_>(x, y, \tilde{t}_f; \tilde{t}_i) & \Delta_{\mathrm{D}}(x, y, \tilde{t}_f; \tilde{t}_i) \end{bmatrix}. \tag{12}$$

For $x^0, y^0 \in \mathcal{C}_+$, the path-ordering $\mathrm{T}_{\mathcal{C}}$ is equivalent to the standard time-ordering T and we obtain the time-ordered Feynman propagator $i\Delta_{\mathrm{F}}(x, y, \tilde{t}_f; \tilde{t}_i)$. On the other hand, for $x^0, y^0 \in \mathcal{C}_-$, $\mathrm{T}_{\mathcal{C}}$ is equivalent to anti-time-ordering $\bar{\mathrm{T}}$ and we obtain the anti-time-ordered Dyson propagator $i\Delta_{\mathrm{D}}(x, y, \tilde{t}_f; \tilde{t}_i)$. For $x^0 \in \mathcal{C}_+$ and $y^0 \in \mathcal{C}_-$, x^0 is always 'earlier' than y^0, yielding the absolutely-ordered negative-frequency Wightman propagator $i\Delta_<(x, y, \tilde{t}_f; \tilde{t}_i)$. Conversely, for $y^0 \in \mathcal{C}_+$ and $x^0 \in \mathcal{C}_-$, we obtain the positive-frequency Wightman propagator $i\Delta_>(x, y, \tilde{t}_f; \tilde{t}_i)$.

By means of a Legendre transform of the CTP generating functional [1], we derive the respective Cornwall–Jackiw–Tomboulis effective action [51], from which the CTP Schwinger–Dyson equation

$$\Delta^{-1}_{ab}(x, y, \tilde{t}_f; \tilde{t}_i) = \Delta^{0,-1}_{ab}(x, y) + \Pi_{ab}(x, y, \tilde{t}_f; \tilde{t}_i) \tag{13}$$

is obtained, where $\Delta^{-1}_{ab}(x, y, \tilde{t}_f; \tilde{t}_i)$ and $\Delta^{0,-1}_{ab}(x, y)$ are the resummed and free inverse CTP propagators, respectively, and $\Pi_{ab}(x, y, \tilde{t}_f; \tilde{t}_i)$ is the CTP self-energy, analogous in form to (12).

4. Master time evolution equations for particle number densities

In order to count both on-shell and off-shell contributions systematically, we 'measure' the number of charges, rather than quanta of energy. This avoids any need to identify 'single-particle' energies by means of a quasi-particle approximation. We begin by relating the Noether charge

$$\mathcal{Q}(x_0; \tilde{t}_i) = -i\int \mathrm{d}^3\mathbf{x}\big(\pi_{\mathrm{H}}(x; \tilde{t}_i)\Phi_{\mathrm{H}}(x; \tilde{t}_i) - \text{H.c.}\big) \tag{14}$$

to a charge density operator $\mathcal{Q}(\mathbf{q}, \mathbf{X}, X_0; \tilde{t}_i)$ via

$$\mathcal{Q}(X_0; \tilde{t}_i) = \int \mathrm{d}^3\mathbf{X}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\mathcal{Q}(\mathbf{q}, \mathbf{X}, X_0; \tilde{t}_i), \tag{15}$$

where $\pi_{\mathrm{H}}(x; \tilde{t}_i)$ is the conjugate momentum operator to $\Phi_{\mathrm{H}}(x; \tilde{t}_i)$. By taking the equal-time EEV of $\mathcal{Q}(\mathbf{q}, \mathbf{X}, X_0; \tilde{t}_i)$ and extracting the positive- and negative-frequency particle components, we arrive at the following definition of the particle number density in terms of off-shell propagators [1]:

$$n(\mathbf{q}, \mathbf{X}, t) = \lim_{X_0 \to t} 2\int \frac{\mathrm{d}q_0}{2\pi}\int \frac{\mathrm{d}^4 Q}{(2\pi)^4}\, e^{-iQ\cdot X} \times \theta(q_0) q_0 i\Delta_<\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right), \tag{16}$$

using the translational invariance of the CTP contour.

By partially inverting the CTP Schwinger–Dyson equation in (13), we derive the following master time evolution equation for the statistical distribution function $f(\mathbf{q} + \frac{\mathbf{Q}}{2}, \mathbf{q} - \frac{\mathbf{Q}}{2}, t)$ [1]:

$$\begin{aligned} &\partial_t f\left(\mathbf{q} + \frac{\mathbf{Q}}{2}, \mathbf{q} - \frac{\mathbf{Q}}{2}, t\right) \\ &\quad - 2\iint \frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}Q_0}{2\pi}\, e^{-iQ_0 t}\mathbf{q}\cdot\mathbf{Q}\theta(q_0)\Delta_<\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right) \\ &\quad + \iint \frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}Q_0}{2\pi}\, e^{-iQ_0 t}\theta(q_0)\bigg(\mathscr{F}\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right) \\ &\quad + \mathscr{F}^*\left(q - \frac{Q}{2}, q + \frac{Q}{2}, t; 0\right)\bigg) \\ &= \iint \frac{\mathrm{d}q_0}{2\pi}\frac{\mathrm{d}Q_0}{2\pi}\, e^{-iQ_0 t}\theta(q_0)\bigg(\mathscr{C}\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right) \\ &\quad + \mathscr{C}^*\left(q - \frac{Q}{2}, q + \frac{Q}{2}, t; 0\right)\bigg), \end{aligned} \tag{17}$$

where we have introduced

$$\begin{aligned} &\mathscr{F}\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right) \\ &\quad \equiv -\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, i\Pi_{\mathcal{P}}\left(q + \frac{Q}{2}, k, t; 0\right) \\ &\quad \times i\Delta_<\left(k, q - \frac{Q}{2}, t; 0\right), \end{aligned} \tag{18a}$$

$$\begin{aligned} &\mathscr{C}\left(q + \frac{Q}{2}, q - \frac{Q}{2}, t; 0\right) \\ &\quad \equiv \frac{1}{2}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\bigg[i\Pi_>\left(q + \frac{Q}{2}, k, t; 0\right)i\Delta_<\left(k, q - \frac{Q}{2}, t; 0\right) \\ &\quad - i\Pi_<\left(q + \frac{Q}{2}, k, t; 0\right)\bigg(i\Delta_>\left(k, q - \frac{Q}{2}, t; 0\right) \\ &\quad - 2i\Delta_{\mathcal{P}}\left(k, q - \frac{Q}{2}, t; 0\right)\bigg)\bigg]. \end{aligned} \tag{18b}$$

It is important to emphasize that (17) provides a self-consistent time evolution equation for f valid *to all orders* in perturbation theory and to *all orders* in gradient expansion. The terms on the LHS of (17) may be associated with the total derivative in the phase space $(\mathbf{X}, \mathbf{p})$, which appears in the classical Boltzmann transport equation [52]. The expression $\mathscr{F}$ in (18a) is the *force* term, generated by the potential due to the dispersive part of the self-energy, and the $\mathscr{C}$ in (18b) are the *collision* terms.

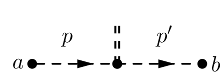
5. Non-homogeneous diagrammatics

Let us consider a simple scalar theory, with one heavy real scalar field Φ and one light pair of complex scalar fields $(\chi^\dagger, \chi)$, described by the Lagrangian

$$\begin{aligned} \mathcal{L} = {}& \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \partial_\mu\chi^\dagger\partial^\mu\chi - m^2\chi^\dagger\chi \\ & - g\Phi\chi^\dagger\chi - \cdots, \end{aligned} \tag{19}$$

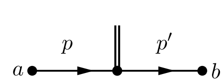
where the ellipsis contains omitted self-interactions. This model yields the following set of modified Feynman rules:

- Sum over all topologically distinct diagrams at a given order in perturbation theory.
- Assign to each Φ-propagator line a factor of



$$= i\Delta^{0,ab}_{\Phi}(p, p', \tilde{t}_f; \tilde{t}_i).$$

  The set of non-homogeneous free propagators is listed in Table 1.
- Assign to each χ-propagator line a factor of



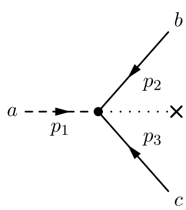
$$= i\Delta^{0,ab}_{\chi}(p, p', \tilde{t}_f; \tilde{t}_i).$$

Table 1
The non-homogeneous free scalar propagators, where $\tilde{f}(p,p',t) = \theta(p_0)\theta(p'_0)f(\mathbf{p},\mathbf{p}',t) + \theta(-p_0)\theta(-p'_0)f^*(-\mathbf{p},-\mathbf{p}',t)$, $\theta(p_0)$ is the unit step function and $\varepsilon(p_0)$ is the signum function.

Propagator	Double-momentum representation
Feynman (Dyson)	$i\Delta^0_{\mathrm{F(D)}}(p,p',\tilde{t}_f;\tilde{t}_i) = \frac{(-)i}{p^2-M^2+(-)i\epsilon}(2\pi)^4\delta^{(4)}(p-p') + 2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)e^{i(p_0-p'_0)\tilde{t}_f}2\pi\lvert 2p'_0\rvert^{1/2}\delta(p'^2-M^2)$
+ (−)ve-freq. Wightman	$i\Delta^0_{>(<)}(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\theta(+(-)p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p') + 2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)\tilde{f}(p,p',t)e^{i(p_0-p'_0)\tilde{t}_f}2\pi\lvert 2p'_0\rvert^{1/2}\delta(p'^2-M^2)$
Retarded (advanced)	$i\Delta^0_{\mathrm{R(A)}}(p,p') = \frac{i}{(p_0+(-)i\epsilon)^2-\mathbf{p}^2-M^2}(2\pi)^4\delta^{(4)}(p-p')$
Pauli–Jordan	$i\Delta^0(p,p') = 2\pi\varepsilon(p_0)\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p')$
Hadamard	$i\Delta^0_1(p,p',\tilde{t}_f;\tilde{t}_i) = 2\pi\delta(p^2-M^2)(2\pi)^4\delta^{(4)}(p-p') + 2\pi\lvert 2p_0\rvert^{1/2}\delta(p^2-M^2)2\tilde{f}(p,p',t)e^{i(p_0-p'_0)\tilde{t}_f}2\pi\lvert 2p'_0\rvert^{1/2}\delta(p'^2-M^2)$
Principal-part	$i\Delta^0_{\mathcal{P}}(p,p') = \mathcal{P}\frac{i}{p^2-M^2}(2\pi)^4\delta^{(4)}(p-p')$

The double lines occurring in the CTP propagators reflect the violation of three-momentum due to the non-homogeneous statistical distribution function $f(\mathbf{p},\mathbf{p}',t)$.

- Assign to each three-point vertex a factor of



$$= -ig\eta_{abc}(2\pi)^4\delta_t^{(4)}\left(\sum_{i=1}^{3}p_i\right),$$

where $\eta_{abc\cdots} = 1$, $a = b = \cdots = 1$; $\eta_{abc\cdots} = -1$, $a = b = \cdots = 2$ and $\eta_{abc\cdots} = 0$ otherwise. Due to the finite upper and lower bounds on the interaction-dependent time integrals, the energy–momentum delta function is replaced by

$$\delta_t^{(4)}\left(\sum_{i=1}^{3}p_i\right) \equiv \delta_t\left(\sum_{i=1}^{3}p_{0,i}\right)\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{20}$$

in which energy conservation is systematically violated by the analytic weight function

$$\delta_t\left(\sum_{i=1}^{3}p_{0,i}\right) \equiv \frac{t}{2\pi}\operatorname{sinc}\left[\left(\sum_{i=1}^{3}p_{0,i}\right)t/2\right]. \tag{21}$$

This violation of energy conservation, shown diagrammatically by the dotted line terminated in a cross, results from the uncertainty principle, since the observation of the system is made over a finite time interval. We ignore additional statistical contributions to vertices that result from a possible non-Gaussian density operator (for a discussion, see [1]).

- Associate with each external vertex a phase

$$e^{ip_0\tilde{t}_f},$$

where p_0 is the energy flowing *into* the vertex. This phase results from the proper consideration of the Wick contraction and field-particle duality relations.

- Contract all internal CTP indices.
- Integrate with the measure

$$\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$

over the four-momentum associated with each contracted pair of CTP indices.

- Consider the combinatorial symmetry factors, where appropriate.

These non-homogeneous Feynman rules encode the absolute spacetime dependence of the system starting from *tree level*.

6. Absence of pinch singularities

The perturbation series built from the non-homogeneous Feynman rules in Section 5 are free of the pinch singularities previously thought to spoil such perturbative treatments of non-equilibrium field theory, see e.g. [25,53–55]. In our formulation, this absence of pinch singularities is ensured by two factors: (i) the violation of energy conservation at early times and (ii) the statistical distribution functions in free CTP propagators are evaluated at the *time of observation*. The latter (ii) is in contrast to existing approaches in which *free* propagators do not evolve and depend only on the *initial* distributions.

Consider the following one-loop insertion to the propagator:

$$\begin{aligned} i\Delta^{(1),ab}(p,p',\tilde{t}_f;\tilde{t}_i) = {} & i\Delta^{0,ab}(p,p',\tilde{t}_f;\tilde{t}_i) \\ & + i\Delta^{0,ac}(p,q,\tilde{t}_f,\tilde{t}_i)i\Pi^{(1)}_{cd}(q,q',\tilde{t}_f;\tilde{t}_i) \\ & \times i\Delta^{0,db}(q',p',\tilde{t}_f,\tilde{t}_i). \end{aligned} \tag{22}$$

Potential pinch singularities arise from terms like

$$\delta(p^2-M^2)\delta(p_0-p'_0)\delta(p'^2-M^2). \tag{23}$$

However, at early times, energy is not conserved through the loop insertion. As a result, these terms are analytic, becoming

$$\delta(p^2-M^2)\delta_t(p_0-p'_0)\delta(p'^2-M^2), \tag{24}$$

where $\delta_t(p_0-p'_0)$ is given in (21). At late times, $t\to\infty$,

$$\lim_{t\to\infty}\delta_t(p_0-p'_0) = \delta(p_0-p'_0) \tag{25}$$

and energy conservation is restored. However, in the same limit the system must have thermalized. In this case, the statistical distribution functions appearing in free propagators will be the equilibrium distributions for which pinch singularities are known to cancel by virtue of the Kubo–Martin–Schwinger (KMS) relation [56]. At intermediate times, pinch singularities grow like a power law in t, which will always occur more slowly than the exponential approach to equilibrium. Thus, the perturbation series is free of pinch singularities for all times [1].

Given the systematic diagrammatics of this approach, we may therefore truncate the master time evolution equations in (17) in a perturbative loopwise sense. If the statistical distribution functions are tempered for all times, any ultra-violet divergences may be renormalized by the usual zero-temperature counter-terms, whilst infra-red divergences may be regularized by the partial resummation of thermal masses, see [1].

7. Time-dependent one-loop width

To illustrate the distinctive features of our perturbative formalism, let us consider two isolated but coincident subsystems $\mathscr{S}_\Phi$

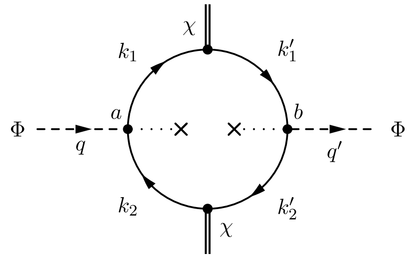


Fig. 2. The one-loop Φ self-energy $i\Pi^{(1)}_{\Phi,ab}(q,q',\tilde{t}_f;\tilde{t}_i)$.

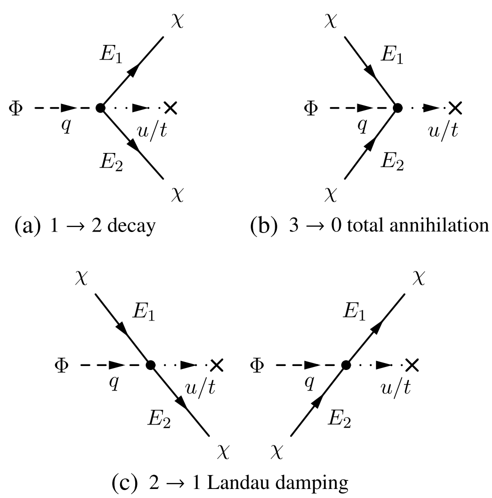


Fig. 3. The four evanescent processes contributing to the one-loop time-dependent Φ width.

and $\mathscr{S}_\chi$, both separately in thermodynamic equilibrium and at the same temperature $T = 10$ GeV with the interactions switched off. The subsystem $\mathscr{S}_\Phi$ contains only the field Φ with mass $M = 1$ GeV and $\mathscr{S}_\chi$, only the χ fields of mass $m = 0.01$ GeV. At $t = 0$, we turn on the interactions and allow the system $\mathscr{S} = \mathscr{S}_\Phi \cup \mathscr{S}_\chi$ to re-thermalize.

The one-loop non-local Φ self-energy is shown in Fig. 2. Neglecting back-reaction on the subsystem $\mathscr{S}_\chi$, the one-loop time-dependent Φ width is then given by the following integral:

$$\Gamma^{(1)}_\Phi(q,t) = \frac{g^2 t}{64\pi^3 M} \sum_{\alpha_1,\alpha_2=\pm1} \int \mathrm{d}^3\mathbf{k}\, \frac{\alpha_1\alpha_2}{E_1E_2} \times \operatorname{sinc}\big[(q_0 - \alpha_1E_1 - \alpha_2E_2)t\big] \times \big(1 + f_{\mathrm{B}}(\alpha_1E_1) + f_{\mathrm{B}}(\alpha_2E_2)\big), \tag{26}$$

where $E_1 \equiv E_\chi(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$ and $E_2 \equiv E_\chi(\mathbf{q} - \mathbf{k})$. The violation of energy conservation, due to the sinc function in (26), leads to otherwise-forbidden contributions from $\alpha_1, \alpha_2 = -1$ (total annihilation) and $\alpha_1 = -\alpha_2$ (Landau damping). In addition, the kinematically-allowed phase space for $1 \to 2$ decays is expanded. These evanescent processes are shown in Fig. 3, where we have defined the evanescent action

$$u \equiv (q_0 - \alpha_1E_1 - \alpha_2E_2)t, \tag{27}$$

quantifying the degree of energy non-conservation. For $t \to \infty$, we recover the known equilibrium result, since

$$\lim_{t\to\infty} \frac{t}{\pi} \operatorname{sinc}\big[(q_0 - \alpha_1E_1 - \alpha_2E_2)t\big] = \delta(q_0 - \alpha_1E_1 - \alpha_2E_2). \tag{28}$$

In Fig. 4, we plot the ratio

$$\bar{\Gamma}^{(1)}_\Phi\big(|\mathbf{q}|,t\big) = \frac{\Gamma^{(1)}_\Phi(|\mathbf{q}|,t)}{\Gamma^{(1)}_\Phi(|\mathbf{q}|,t\to\infty)} \tag{29}$$

of the time-dependent one-loop Φ width to its late-time equilibrium value as a function of Mt for $q^2 = M^2$. In addition, we plot the separate contributions of the processes shown in Fig. 3.

8. Non-Markovian oscillations

In Fig. 4, we observe that the oscillations in the Φ width have time-dependent frequencies. This non-Markovian behaviour is inherent to truly out-of-equilibrium quantum systems, exhibiting so-called *memory effects*. Moreover, due to the Lorentz boost of ultra-violet modes relative to the rest frame of the heat bath, these memory effects persist for timescales much longer than the $1/M$ that would be expected for effects resulting from the uncertainty principle.

In terms of the evanescent action u in (27) and in the high-temperature limit $T \gg M$, we may show quantitatively that the frequencies of these non-Markovian oscillations are given by

$$\omega^{(b)}_1(q,u,t) = q_0 - \frac{(q_u^2(t) - |\mathbf{q}|^2 + m_1^2 - m_2^2)q_u(t) + b\alpha_\theta|\mathbf{q}|\lambda^{1/2}(q_u^2(t) - |\mathbf{q}|^2, m_1^2, m_2^2)}{2(q_u^2(t) - |\mathbf{q}|^2)}, \tag{30a}$$

$$\omega^{(b)}_2(q,u,t) = \frac{(q_u^2(t) - |\mathbf{q}|^2 - m_1^2 + m_2^2)q_u(t) - b\alpha_\theta|\mathbf{q}|\lambda^{1/2}(q_u^2(t) - |\mathbf{q}|^2, m_1^2, m_2^2)}{2(q_u^2(t) - |\mathbf{q}|^2)}, \tag{30b}$$

with $b, \alpha_\theta = \pm1$, $\lambda(x,y,z) = (x^2 - y^2 - z^2)^2 - 4y^2z^2$. In addition, we have introduced the evanescent energy

$$q_u(t) \equiv q_0 - \frac{u}{t} \tag{31}$$

and we have quoted the result with different masses m_1 and m_2, for generality. Notice that in the limit $t \to \infty$, $q_u(t) \to q_0$ and we obtain the usual time-independent kinematics. To the best of our knowledge, such a quantitative analysis of the non-Markovian evolution of memory effects has not been reported previously in the literature.

9. Loopwise-truncated time evolution equations

Truncating the master time evolution equation (17) to leading order in a perturbative loopwise expansion, we obtain the following one-loop transport equation for the Φ statistical distribution function:

$$\begin{aligned}
\partial_t f_\Phi\big(|\mathbf{q}|,t\big) &= -\frac{g^2}{2} \sum_{\alpha,\alpha_1,\alpha_2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{2E_\Phi(\mathbf{q})} \frac{1}{2E_\chi(\mathbf{k})} \frac{1}{2E_\chi(\mathbf{q}-\mathbf{k})} \\
&\quad \times \frac{t}{2\pi} \operatorname{sinc}\big[\big(\alpha E_\Phi(\mathbf{q}) - \alpha_1E_\chi(\mathbf{k}) - \alpha_2E_\chi(\mathbf{q}-\mathbf{k})\big)t/2\big] \\
&\quad \times \big\{\pi + 2\,\mathrm{Si}\big[\big(\alpha E_\Phi(\mathbf{q}) + \alpha_1E_\chi(\mathbf{k}) + \alpha_2E_\chi(\mathbf{q}-\mathbf{k})\big)t/2\big]\big\} \\
&\quad \times \big\{\big[\theta(-\alpha) + f_\Phi\big(|\mathbf{q}|,t\big)\big] \\
&\quad \times \big[\theta(\alpha_1)\big(1 + f_\chi\big(|\mathbf{k}|,t\big)\big) + \theta(-\alpha_1)f^C_\chi\big(|\mathbf{k}|,t\big)\big] \\
&\quad \times \big[\theta(\alpha_2)\big(1 + f^C_\chi\big(|\mathbf{q}-\mathbf{k}|,t\big)\big) + \theta(-\alpha_2)f_\chi\big(|\mathbf{q}-\mathbf{k}|,t\big)\big] \\
&\quad - \big[\theta(\alpha) + f_\Phi\big(|\mathbf{q}|,t\big)\big]
\end{aligned}$$

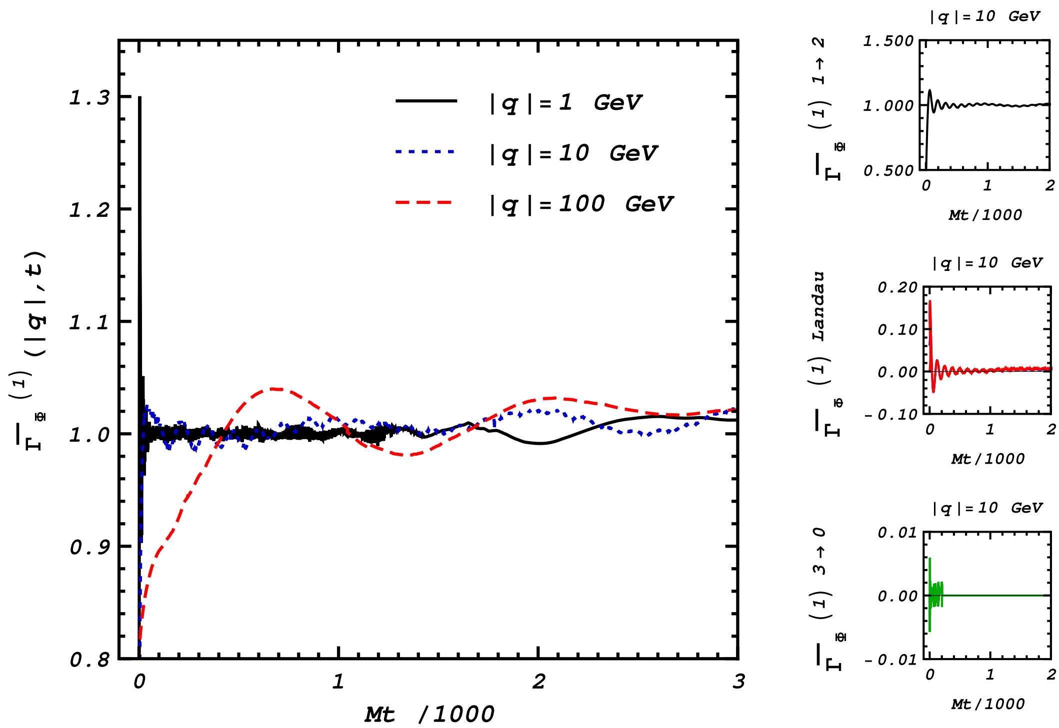


Fig. 4. *Left*: the ratio $\bar{\Gamma}_{\Phi}^{(1)}$ versus Mt for on-shell decays with $|\mathbf{q}| = 1$ GeV (solid black), 10 GeV (blue, in the web version, dotted) and 100 GeV (red, in the web version, dashed). *Right*: separate contributions to $\bar{\Gamma}_{\Phi}^{(1)}$ for $|\mathbf{q}| = 10$ GeV. Landau-damping contributions are equal up to numerical errors.

$$\times \left[\theta(\alpha_1) f_\chi\left(|\mathbf{k}|, t\right) + \theta(-\alpha_1)\left(1 + f_\chi^C\left(|\mathbf{k}|, t\right)\right)\right]$$
$$\times \left[\theta(\alpha_2) f_\chi^C\left(|\mathbf{q}-\mathbf{k}|, t\right) + \theta(-\alpha_2)\left(1 + f_\chi\left(|\mathbf{q}-\mathbf{k}|, t\right)\right)\right]\Big\}, \tag{32}$$

where $\alpha, \alpha_1, \alpha_2 = \pm 1$. The second and third lines of (32) encode the early-time violation of energy conservation. Replacing these lines by the Markovian approximation

$$2\pi\theta(\alpha)\delta\left(E_\Phi(\mathbf{q}) - \alpha_1 E_1(\mathbf{k}) - \alpha_2 E_2(\mathbf{q}-\mathbf{k})\right), \tag{33}$$

we recover the semi-classical Boltzmann equation. However, given the equilibrium initial conditions of our model, this artificial imposition of energy conservation along with the properties of the Bose–Einstein distribution ensure that the RHS of (32) is zero for all times. Thus, the semi-classical Boltzmann equation cannot describe the re-thermalization of our simple model. This is true also for gradient expansions of Kadanoff–Baym equations when truncated to zeroth order in time derivatives. Hence, it is only when energy-violating effects are systematically considered, as in this new perturbative approach with all gradients included, that the dynamics of this re-thermalization is properly captured.

It is clear that (32) describes only decay and inverse decay processes in the topologies shown in Fig. 3. However, higher-multiplicity decays and scatterings can be systematically incorporated by consistently truncating the master time evolution equation in (17) to a higher number of loops.

10. Conclusions

We have obtained master time evolution equations for particle number densities that are valid to all orders in perturbation theory and to all orders in gradient expansion. The underlying perturbation series are built from non-homogeneous free propagators and explicitly time-dependent vertices. Due to the systematic treatment of finite boundary and observation times, these diagrammatic series remain free of pinch singularities for all times. We are therefore able to truncate the time evolution equations in a perturbative loopwise sense, whilst keeping all orders in gradient expansion and capturing the dynamics on all timescales. This includes the prompt transient behaviour, which we have shown to be dominated by energy-violating processes that lead to non-Markovian evolution of memory effects. By virtue of our approach, we have been able to provide the first quantitative analysis of these memory effects.

The foreseeable applications of this new formalism span high-energy physics, astro-particle physics, cosmology and condensed matter physics. Dedicated studies of such applications will be the subject of future works.

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