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Possibilistic risk aversion with many parameters

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Abstract

The study of risk aversion of an agent confronted by a risk situations with several parameters is an important topic of risk theory. It is tackled traditionally with probabilistic methods. When these do not offer an appropriate shaping we can use Zadeh's possibility theory . In this paper a possibilistic model of risk aversion with several parameters is proposed. The notion of possibilistic risk premium vector is introduced as a measure of an agent's risk aversion to a situation with several risk parameters. The main result of the paper is an approximate calculation formula of this indicator. The way we can apply this model in risk aversion evaluation in grid computing is sketched out.

Keywords: Fuzzy number; possibilistic risk aversion; possibilistic risk premium vector

1. Introduction

In economic and financial activity risk situations with several parameters often appear. An agent's risk aversion to such situations is a topic which concerned several authors [9], [16], [17], [19], etc. They tackled this topic by probabilistic methods. The risk situation with several parameters is represented by a random vector, and the agent's attitude to risk is represented by a multidimensional utility function.

In [9] the notion of risk premium vector was introduced as a measure of multidimensional risk aversion. It generalizes the notion of risk premium studied by Arrow [1] and Pratt [21] for unidimensional risk.

Probability theory can describe appropriately only those situations of uncertainty for which the events occur with a big frequency and for which we have large databases. For other phenomena of uncertainty possibility theory initiated by Zadeh in [25] and developed mainly by Dubois and Prade [7], [8] is preferred.

Possibilistic models were successfully applied in decision making problems in conditions of uncertainty, fuzzy neural networks, fuzzy cooperative games, portfolio problem, etc. (see [3], [8], [12], [20]).

In [13], [15] two possibilistic models of unidimensional risk aversion have been proposed. The study of multidimensional risk aversion by possibilistic methods has been started in [14] by defining a notion of possibilistic risk premium and by proving an approximate calculation formula.

The transition from probabilistic models to possibilistic models has two main components: random variables are replaced with fuzzy variables and probabilistic indicators (expected value, variance, covariance, etc.) are replaced

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with possibilistic indicators. To define possibilistic risk premium the notion of possibilistic expected utility had to be first introduced (see [13]-[15]).

This paper treats multidimensional possibilistic risk aversion from a more general perspective, similar to that of [9]. Fuzzy numbers are the most important class of possibilistic distributions. They are often chosen to represent situations of uncertainty. In our case risk parameters will be represented by fuzzy numbers. The main concept of the paper is possibilistic risk premium vector associated with a possibilistic vector, a weighting function and a multidimensional utility function. This indicator measure the aversion of an agent in front of a risk situation with several parameters. For the particular case when all components of this vector are equal, the notion of possibilistic risk premium of [14] is found.

The paper is organized as follows. Section 2 recalls the definition of fuzzy numbers and the main indicators associated with them: expected value, variance and covariance. In Section 3 the notion of multidimensional possibilistic expected utility by means of γ-level sets is introduced. By particularization, possibilistic expected value [2], [11] and some notions of possibilistic variance and covariance [2], [11], [26] are obtained from it. These indicators will be used in the next section to evaluate multidimensional possibilistic risk premium.

Section 4 develops a model of possibilistic risk aversion with several parameters. The mathematical framework of this possibilistic model is specified by a possibilistic vector (representing the risk situation), a multidimensional utility function (representing the attitude of an agent to a risk situation) and a weighting function. Using multidimensional possibilistic expected utility the notion of possibilistic risk premium vector is defined as a measure of an agent's risk aversion to a risk situation with several parameters. The main result of the paper is an approximate calculation formula of this risk indicator. The formula expresses this indicator in terms of utility function, possibilistic expected value and possibilistic covariance.

Section 5 contains an application of this model to evaluate risk aversion in grid computing.

2. Possibilistic indicators of fuzzy numbers

In this section we recall the definition of fuzzy number and some of its properties. By [2], [3], [10], [11], [26] we will present some of the possibilistic indicators associated with fuzzy numbers (expected value, variance, covariance) needed to formulate definitions and results of multidimensional possibilistic risk aversion theory (see Section 4).

Let X be a set of states. A fuzzy subset of X (=fuzzy set) is a function $A : X \to [0,1]$. For any $x \in X$ the number $A(x)$ is the degree of membership of x to A.

Let A be a fuzzy set in X. A is normal if there exists $x \in X$ such that $A(x) = 1$. The support of A is defined by $\sup(A) = \{x \in X \mid A(x) > 0\}$.

In the following we consider that X is the set **R** of real numbers. For any $\gamma \in [0,1]$, the γ –level set of a fuzzy set A in **R** is defined by

$$[A]^\gamma = \begin{cases} \{x \in R \mid A(x) \geq \gamma\} & if \quad \gamma > 0 \\ cl(\sup(A)) & if \quad \gamma = 0 \end{cases}$$

(cl(sup(A)) is the topological closure of the set $\sup(A) \subseteq R$).

A fuzzy number is a fuzzy set of **R** normal, fuzzy convex, continuous and with bounded support.

Let A be a fuzzy number and $\gamma \in [0,1]$. Then $[A]^\gamma$ is a closed and convex subset of **R**. We denote $a_1(\gamma) = \min[A]^\gamma$ and $a_2(\gamma) = \max[A]^\gamma$. Hence, $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$.

A non–negative and monotone increasing function $f : [0,1] \to R$ is a weighting function if it satisfies the normality condition

$$\int_0^1 f(\gamma) d\gamma = 1.$$

We fix a fuzzy number A and a weighting function f. Assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0,1]$. The possibilistic expected value of A w.r.t. f was defined in [11], [20] by

$$E(f,A) = \frac{1}{2}\int_0^1 (a_1(\gamma) + a_2(\gamma)) f(\gamma) d\gamma .$$

The possibilistic variance of A w.r.t. f is defined by

$$Var(f,A) = \frac{1}{2}\int_0^1 [(a_1(\gamma) - E(f,A))^2 + (a_2(\gamma) - E(f,A))^2] f(\gamma) d\gamma .$$

If $f(\gamma) = 2\gamma$ for any $\gamma \in [0,1]$, then $E(f,A)$ is the crisp possibilistic mean value introduced in [2], p. 318, and $Var(f,A)$ is the second possibilistic variance defined in [2], p. 324.

Let A, B be two fuzzy numbers and f a weighting function. Assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$ for all $\gamma \in [0,1]$. The possibilistic covariance $Cov(f, A, B)$ of A and B w.r.t. f is defined in [26], p. 261 by

$$Cov(f,A,B) = \frac{1}{2}\int_0^1 [(a_1(\gamma) - E(f,A))(b_1(\gamma) - E(f,B)) + (a_2(\gamma) - E(f,A))(b_2(\gamma) - E(f,B))] f(\gamma) d\gamma$$

If $f(\gamma) = 2\gamma$ for any $\gamma \in [0,1]$, then $Cov(f, A, B)$ is the second possibilistic covariance defined in [2] p. 324.

A triangular fuzzy number $A = (a, \alpha, \beta)$ is defined by the function $A : R \to [0,1]$:

$$A(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & if \quad a - \alpha \le t \le a \\ 1 - \dfrac{t-a}{\beta} & if \quad a \le t \le a + \beta \\ 0 & otherwise \end{cases}$$

If $\alpha=\beta$ then the triangular fuzzy number A is called symmetric and it is denoted $A=(a,\alpha)$.

Recall from [7] that if $A = (a, \alpha, \beta)$ then $[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta]$ for all $\gamma \in [0,1]$, hence $a_1(\gamma) = a - (1-\gamma)\alpha, a_2(\gamma) = a + (1-\gamma)\beta$.

Assume that $f(\gamma) = 2\gamma$ for all $\gamma \in [0,1]$. Then $E(f,A) = a + \dfrac{\beta - \alpha}{6}$.

If $A = (a, \alpha)$ then $E(f, A) = a$.

A possibilistic vector has the form $(A_1, \ldots, A_n)$ where $A_1, \ldots, A_n$ are fuzzy numbers.

3. Possibilistic expected utility: the multidimensional case

Probability theory of multidimensional risk aversion is built on the notion of expected utility (by [9]). We consider a risk situation with n parameters represented by a random vector $\vec{X} = (X_1, \ldots, X_n)$. If $u : R^n \to R$ is a continuous utility function then $u(X_1, \ldots, X_n)$ is a random variable. The expected value $E(u(X_1, \ldots, X_n))$ of $u(X_1, \ldots, X_n)$ is called $\vec{X}$'s expected utility w.r.t. u.

We consider a risk situation with n parameters for which a possibilistic modeling is needed (e.g., when we have few data on risk situation). Then the risk situation will be described by a possibilistic vector. By definition, an *n*–dimensional possibilistic vector has the form $\vec{A}=(A_1,...,A_n)$ where $A_1,...,A_n$ are fuzzy numbers. Then $A_1,...,A_n$ describe *n* risk parameters.

To develop a theory of risk aversion corresponding to this case an appropriate notion of possibilistic expected utility is needed.

Let $f:[0,1]\to R$ be a weighting function and $g:R^n\to R$ a continuous function.We consider a possibilistic vector $(A_1,...,A_n)$ where $[A]^\gamma=[a_i(\gamma),b_i(\gamma)]$ for any $i = 1, . . . ,n$ and $\gamma\in[0,1]$. We define the possibilistic expected utility of $(A_1,...,A_n)$ w.r.t. *f* and *g* by

$$E(f,g(A_1,...,A_n))=\frac{1}{2}\int_0^1[g(a_1(\gamma),...,a_n(\gamma))+g(b_1(\gamma),...,b_n(\gamma))]f(\gamma)d\gamma\ .$$

If $n = 1$ we obtain the notion of possibilistic expected utility of [13]. For $n = 1$ and $g(x) = x$ for each $x\in R$ we have $E(f, g(A)) = E(f,A)$.

Remark 3.1 Let $n = 2$ and $g(x, y) = (x - E(f,A_1))(y - E(f,A_2))$ for any $x,y\in R$. Then $E(f, g(A_1,A_2)) = cov(f,A_1,A_2)$.

In this section in the following we fix a possibilistic vector $(A_1, . . . ,A_n)$ and a weighting function $f:[0,1]\to R$. Assume that $[A_i]^\gamma=[a_i(\gamma),b_i(\gamma)]$ for any $\gamma\in[0,1]$.

The following propositions will be used in the next section to prove an approximate calculation formula for the possibilistic risk premium vector.

Proposition 3.2 Let $g:R^n\to R, h:R^n\to R$ be two continuous functions and $a,b\in R$. We consider the function $u:R^n\to R$ defined by $u(x_1,...,x_n)=ag(x_1,...,x_n)+bh(x_1,...,x_n)$ for any $(x_1,...,x_n)\in R^n$. Then $E(f,u(A_1,...,A_n))=aE(f,g(A_1,...,A_n))+bE(f,h(A_1,...,A_n))$.

Proposition 3.3 Let $g:R^n\to R, h:R^n\to R$ be two continuous functions such that $g(x_1,...,x_n)\le h(x_1,...,x_n)$ for any $(x_1,...,x_n)\in R^n$. Then $E(f,g(A_1,...,A_n))\le E(f,h(A_1,...,A_n))$.

Proposition 3.4 Let n continuous functions $g_i:R\to R, i=1,...,n$ and $a_1,...,a_n\in R$. We consider the function $g:R^n\to R$ defined by $g(x_1,...,x_n)=\sum_{i=1}^n a_i g_i(x_i)$ for any $(x_1,...,x_n)\in R^n$. Then $E(f,g(A_1,...,A_n))=\sum_{i=1}^n a_i E(f,g_i(A_i))$.

Proposition 3.5 Let n^2 continuous functions $g_{ij}:R^2\to R, i,j=1,...,n$ and $a_{ij}\in R,\ i,j=1,...,n$. We consider the function $g:R^n\to R$ defined $g(x_1,...,x_n)=\sum_{i,j=1}^n a_{ij}g_{ij}(x_i,x_j)$ for any $(x_1,...,x_n)\in R^n$. Then

$$E(f,g(A_1,...,A_n))=\sum_{i,j=1}^n a_{ij}E(f,g_{ij}(A_i,A_j))\ .$$

4. Multidimensional possibilistic risk aversion

In this section we will introduce the possibilistic risk premium vector as a measure of risk aversion of an agent to a possibilistic risk situation with several parameters. Then we prove an approximate calculation formula of this indicator.

We recall first some elements of multidimensional probabilistic risk aversion [9], [16], [17], [19], etc. They will be the inspiring source for the notions and results on multidimensional possibilistic risk aversion from this section.

The setting in which probability theory of multidimensional risk aversion is developed has two components:

• a random vector $\vec{X} = (X_1,...,X_n)$ representing a risk situation with *n* parameters;

• a utility function $u : R^n \to R$, which describes the agent's attitude to the risk situation.

Assume that the utility function has the class C^2 and is strictly increasing in each argument.

An element $\vec{x} = (x_1,...,x_n) \in R^n$ is called a commodity vector. For the utility function $u : R^n \to R$ we denote for any *i, j* = 1, . . . ,*n*:

$$u_i(x_1,...,x_n) = \frac{\partial u(x_1,...,x_n)}{\partial x_i}, u_{ij}(x_1,...,x_n) = \frac{\partial^2 u(x_1,...,x_n)}{\partial x_i \partial x_j}$$

In vectorial notation we have $u_i(\vec{x}) = \frac{\partial u(\vec{x})}{\partial x_i}, u_{ij}(\vec{x}) = \frac{\partial^2 u(\vec{x})}{\partial x_i \partial x_j}$.

From [9] we recall the notion of risk premium vector.

Definition 4.1 A risk premium vector $\vec{\rho}(\vec{X},u) = (\rho_1,...,\rho_n)$ (associated with the random vector $\vec{X}$ and the utility function *u*) is defined as a solution of the equation:

(1) $E(u(X_1,...,X_n)) = u(E(X_1) - \rho_1,...,E(X_n) - \rho_n)$.

By [9] equation (1) can have several solutions $(\rho_1,...,\rho_n)$.

We denote $e_i = E(X_i)$ for *i*= 1, . . . ,*n*.

Proposition 4.2 [9] An approximate solution of equation (1) is given by

(2) $\rho_i \approx -\frac{1}{2}\sum_{j=1}^{n} \frac{1}{u_j(\vec{e})} Cov(X_i, X_j) u_{ij}(\vec{e})$ *for* $i = 1,...,n$.

To build a possibility theory of multidimensional risk aversion we consider a setting with the following components:

• a weighting function $f : [0,1] \to R$;

• a possibilistic vector $\vec{A} = (A_1,...,A_n)$;

• an n–dimensional utility function $u : R^n \to R$

The possibilistic vector $\vec{A}$ models the risk situation and the utility function u describes the agent's attitude to $\vec{A}$.

We introduce now the notion of possibilistic risk premium vector.

Definition 4.3 A possibilistic risk premium vector $\vec{\rho}(\vec{A}, f, u) = (\rho_1,...,\rho_n)$ (associated with the possibilistic vector $\vec{A}$, the weighting function *f* and the utility function *u*) is defined as a solution of the equation

(3) $E(f, u(A_1,...,A_n)) = u(E(f, A_1) - \rho_1,...,E(f, A_n) - p_n)$.

$E(f, u(A_1,...,A_n))$ is the possibilistic expected utility introduced in the previous section.

Definition 4.3 can be confronted with Definition 4.1. The utility function *u* appears in both definitions, but the probabilistic expected value (resp. probabilistic expected utility) of Definition 4.1 was replaced by possibilistic expected value (resp. possibilistic expected utility).

Equation (3) does not have a unique solution $\vec{\rho} = (\rho_1,...,\rho_n)$.

A possibilistic risk premium vector $\vec{\rho} = (\rho_1,...,\rho_n)$ is an indicator of risk aversion of the agent represented by u. Equality (3) expresses that agent u is willing to pay the amount $\rho_1,...,\rho_n$ to realize the possibilistic expected utility $E(f, u(A_1,..., A_n))$.

In [14] a notion of possibilistic risk premium was defined as follows.

Definition 4.4 A possibilistic risk premium $\rho = \rho(\vec{A}, f, u)$ (associated with $\vec{A}$,*f* and *u*) is a solution of the equation

(4) $E(f, u(A_1,...,A_n)) = u(E(f, A_1) - \rho ,..., E(f, A_n) - p)$.

If ρ is a possibilistic risk premium associated with $\vec{A}$, *f* and *u*, then $\vec{\rho} = (\rho,....,\rho)$ is a possibilistic risk premium vector associated with $\vec{A}$, f and u.

Remark 4.5 In equations (3) and (4) assume that $E(f,u(A_1,...,A_n))$ is finite.

Remark 4.6 (4) is an equation in ρ and it can have several solutions. To see it let us consider the bidimensional possibilistic vector (A_1, A_2) and a utility function $u(x_1, x_2) = x_1 x_2$. Then equation (4) becomes

$$(E(f,A_1)-\rho)(E(f,A_2)-\rho)=E(f,u(A_1,A_2))$$

Taking $f(\gamma)=2\gamma$ for $\gamma\in[0,1]$ and $A_1=(r_1,\alpha_1)$, $A_2=(r_2,\alpha_2)$ we have $E(f,A_1)=r_1$, $E(f,A_2)=r_2$ and

$$E(f,u(A_1,A_2))=\int_0^1[(r_1-(1-\gamma)\alpha_1)(r_2-(1-\gamma)\alpha_2)+(r_1+(1-\gamma)\alpha_1)(r_2+(1-\gamma)\alpha_2)]\gamma d\gamma$$

Then equation (4) takes the form $(r_1-\rho)(r_2-\rho)=E(f,u(A_1,A_2))$ and we can determine distinct solutions.

In the unidimensional case if the utility function $u:R^n\to R$ is injective then the solution of (4) is unique.

We denote $m_i=E(f,A_i), i=1,...,n$ and $\vec{m}=(m_1,...,m_n)$.

Proposition 4.7 An approximate solution of equation (3) is given by

(5) $$\rho_i^0\approx-\frac{1}{2}\sum_{j=1}^{n}\frac{1}{u_i(\vec{m})}Cov(f,A_i,A_j)u_{ij}(\vec{m}) \text{ for } i=1,...,n.$$

Proof. By applying the Taylor formula for $u:R^n\to R$ and by neglecting the Taylor remainder of second order, one obtains:

$$u(\vec{x})\approx u(\vec{m})-\sum_{i=1}^{n}(x_i-m_i)\frac{\partial u(\vec{m})}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n}(x_i-m_i)(x_j-m_j)\frac{\partial^2 u(\vec{m})}{\partial x_i\partial x_j}.$$

With the notations from the beginning of the section this relation can be written

$$u(\vec{x})\approx u(\vec{m})-\sum_{i=1}^{n}(x_i-m_i)u_i(\vec{m})+\frac{1}{2}\sum_{i,j=1}^{n}(x_i-m_i)(x_j-m_j)u_{ij}(\vec{m}).$$

Consider the functions $g:R^n\to R$ and $h:R^n\to R$ defined by

$$g(\vec{x})=\sum_{i=1}^{n}(x_i-m_i)u_i(\vec{m});$$

$$h(\vec{x})=\sum_{i,j=1}^{n}(x_i-m_i)(x_j-m_j)u_{ij}(\vec{m}).$$

According to Proposition 3.2 we have

(6) $$E(f,u(A_1,...,A_n))\approx u(\vec{m})-E(f,g(A_1,...,A_n))+\frac{1}{2}E(f,h(A_1,...,A_n)).$$

We consider the functions $g_i:R\to R, i=1,...,n$ defined by

$g_i(x_i)=x_i-m_i$ for any $x_i\in R$.

Then $g(\vec{x})=\sum_{i=1}^{n}u_i(\vec{m})g_i(x_i)$ for any $\vec{x}\in R^n$. By Proposition 3.4 we have

$$E(f,g(A_1,...,A_n))=\sum_{i=1}^{n}u_i(\vec{m})E(f,g_i(A_i)).$$

By Proposition 3.2 $E(f,g_i(A))=E(f,A_i)-m_i=0$ for any $i=1,...,n$.

Therefore $E(f,g(A_1,...,A_n))=0$. Replacing in (6) it follows

(7) $$E(f,u(A_1,...,A_n))\approx u(\vec{m})+\frac{1}{2}E(f,h(A_1,...,A_n))$$

A straightforward application of Proposition 3.5 and Remark 3.1 shows that

$$E(f,u(A_1,...,A_n))\approx u(\vec{m})+\frac{1}{2}\sum_{i,j=1}^{n}u_{ij}(\vec{m})Cov(f,A_i,A_j)$$

Thus (7) becomes

(8) $E(f,u(A_1,...,A_n)) \approx u(\vec{m}) + \frac{1}{2}\sum_{i,j=1}^{n} u_{ij}(\vec{m})Cov(f,A_i,A_j)$

By applying again the Taylor formula and omitting the Taylor remainder of first order, it follows that for all $\rho_1,...,\rho_n \in R$ we have

(9) $u(m_1-\rho_1,...,m_n-\rho_n) \approx u(\vec{m}) - \sum_{i=1}^{n} \rho_i u_i(\vec{m})$.

If $\rho_1^0,...,\rho_n^0$ are real numbers defined in (5), then a simple calculation shows that

(10) $\sum_{i=1}^{n} \rho_i^0 u_i(\vec{m}) = -\frac{1}{2}\sum_{i,j=1}^{n} u_{ij}(\vec{m})Cov(f,A_i,A_j)$.

Relations (8), (9) and (10) show that

$E(f,u(A_1,...,A_n)) \approx u(m_1-\rho_1^0,...,m_n-\rho_n^0)$

therefore $(\rho_1^0,...,\rho_n^0)$ is an approximate solution of equation (4).

Remark 4.8 The previous proposition is the possibilistic version of Proposition 4.2. We remark that the probabilistic covariances $Cov(X_i,X_j)$ appear in (2), and the possibilistic covariances $Cov(f,A_i,A_j)$ appear in (5).

Proposition 4.9 An approximate solution of equation (4) is given by

(11) $$\rho^0 \approx -\frac{1}{2}\frac{\sum_{i,j=1}^{n} Cov(f,A_i,A_j)u_{ij}(\vec{m})}{\sum_{i=1}^{n} u_i(\vec{m})}$$

Proof. We will use relation (8) of the proof of Proposition 4.7. By applying the Taylor formula and omitting the Taylor remainder of the first order, it follows for any $\rho \in R$

$u(m_1-\rho,...,m_n-\rho) \approx u(\vec{m}) - \rho\sum_{i=1}^{n} u_i(\vec{m})$.

In particular, for $\rho = \rho^0$

(12) $u(m_1-\rho^0,...,m_n-\rho^0) \approx u(\vec{m}) - \rho^0\sum_{i=1}^{n} u_i(\vec{m})$.

Using (8), (11), (12) it follows easily that

$E(f,u(A_1,...,A_n)) \approx u(m_1-\rho^0,...,m_n-\rho^0)$,

therefore ρ^0 is an approximate solution of equation (4).

In [9] the notation

(13) $r^{ij}(\vec{x}) = -\frac{u_{ij}(\vec{x})}{u_i(\vec{x})}$ *for any* $\vec{x} \in R^n$

is introduced. With this notation, the approximate solution $(\rho_1^0,...,\rho_n^0)$ of equation (3)given by (5) is written as

(14) $\rho_i^0 = \frac{1}{2}\sum_{j=1}^{n} Cov(f,A_i,A_j)r^{ij}(\vec{m})$.

Example 4.10 We consider the family of utility functions from Example 3.1 of [9]

$u(x_1,x_2) = -\theta_1[e^{-x_1}+e^{-x_2}] - \theta_2 e^{-x_1-x_2}$, where θ_1,θ_2 are two real parameters.

Let $A_1 = (r_1,\alpha_1,\beta_1)$ *and* $A_2 = (r_2,\alpha_2,\beta_2)$ be triangular fuzzy numbers and $f(\gamma) = 2\gamma, \gamma \in [0,1]$ be a weighting function. By [26], in this case we have

$$m_1 = E(f, A_1) = r_1 + \frac{\beta_1 - \alpha_1}{6}; m_2 = E(f, A_2) = r_2 + \frac{\beta_2 - \alpha_2}{6};$$

$$Var(f, A_1) = Cov(f, A_1, A_1) = \frac{\alpha_1^2 + \beta_1^2 + \alpha_1\beta_1}{18};$$

$$Var(f, A_2) = Cov(f, A_2, A_2) = \frac{\alpha_2^2 + \beta_2^2 + \alpha_2\beta_2}{18};$$

$$Cov(f, A_1, A_2) = Cov(f, A_2, A_1) = \frac{\alpha_1\alpha_2 + \beta_1\beta_2 + (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{36}.$$

By formula (14), an approximate solution of the equation $E(f, u(A_1, A_2)) = u(m_1 - \rho_1, m_2 - \rho_2)$ will have the form

(15) $\rho_i^0 = \frac{1}{2}\sum_{j=1}^{2} Cov(f, A_i, A_j) r^{ij}(\vec{m})$, i=1,2.

In (9) one found that $r^{ii}(\vec{x}) = 1$ and $r^{ij}(\vec{x}) = \frac{\theta_2}{\theta_1 e^{x_j} + \theta_2}$ for $i \neq j$. Thus (15) is written as

$$\rho_1^0 = \frac{1}{2}Var(f, A_1) + \frac{1}{2}Cov(f, A_1, A_2) r^{12}(\vec{m}) = \frac{\alpha_1^2 + \beta_1^2 + \alpha_1\beta_1}{36} + \frac{\alpha_1\alpha_2 + \beta_1\beta_2 + (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{72} \frac{\theta_2}{\theta_1 e^{m_2} + \theta_2}$$

$$\rho_2^0 = \frac{1}{2}Var(f, A_2) + \frac{1}{2}Cov(f, A_2, A_1) r^{21}(\vec{m}) = \frac{\alpha_2^2 + \beta_2^2 + \alpha_2\beta_2}{36} + \frac{\alpha_1\alpha_2 + \beta_1\beta_2 + (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{72} \frac{\theta_2}{\theta_1 e^{m_1} + \theta_2}$$

5. Possibilistic risk aversion in grid computing

Grid computing techniques present a lot of interest for commercial applications [6]. In the management of a grid, risk phenomena with several parameters related to the functioning of grid's nodes might appear. The study of risk aversion w.r.t.the functioning of the grid leads to the choice of a multidimensional model. Next we propose a model based on the topic developed in the previous section.

We consider a grid composed of n nodes $N_1, \ldots, N_n$. Assume that the risk situation generated by node N_i is described by the fuzzy number A_i. The possibilistic vector ($A_1, \ldots, A_n$) will represent the functioning of the grid overall (regarding the risk situation). The way nodes N_i, N_j are interrelated is given by $Cov(f, A_i, A_j)$ where f is a conveniently chosen weighting function.

An agent's attitude to the risk situation of the grid is expressed by the utility function $u : R^n \to R$.

In this context possibilistic risk premium vector evaluates the agent's risk aversion to risk situation ($A_1, \ldots, A_n$) . The approximate calculation of this indicator is done by the formula of Proposition 4.7.

Example 5.1 Consider a grid composed of n nodes $N_1, \ldots, N_n$. The risk situation of node N_i is described by the triangular fuzzy number $A_i = (a_i, \alpha_i, \beta_i)$:

$$A_i(t) = \begin{cases} 1 - \frac{a_i - t}{\alpha_i}, if\ a_i - \alpha_i \le t \le a_i \\ 1 - \frac{t - a_i}{\beta_i}, if\ a_i \le t \le a_i + \beta_i \\ 0, otherwise \end{cases}$$

Assume that the weighting function $f : [0,1] \to R$ and the utility function $u : R^n \to R$ are defined by $f(\gamma) = 2\gamma$ for $\gamma \in [0,1]$ and $u(x_1, \ldots, x_n) = -e^{-2(x_1 + \ldots + x_n)}$ for any $(x_1, \ldots, x_n) \in R^n$.

Notice that u has the class C^2 and is strictly increasing in each argument. A simple calculation shows that for any $(x_1, \ldots, x_n) \in R^n$ and $i, j = 1, \ldots, n$ we have

$$u_i(x_1,...,x_n) = 2e^{-2(x_1+...+x_n)}$$
$$u_{ij}(x_1,...,x_n) = -4e^{-2(x_1+...+x_n)}$$

and therefore $\dfrac{u_{ij}(x_1,...,x_n)}{u_i(x_1,...,x_n)} = -2$.

Then by Proposition 4.7 an approximate value of possibilistic risk premium vector associated with this possibilistic context of risk is given by

$$\rho_i^0 = -\frac{1}{2}\sum_{j=1}^{n}\frac{1}{u_i(\vec{m})}Cov(f,A_i,A_j)u_{ij}(\vec{m}) = \sum_{j=1}^{n}Cov(f,A_i,A_j),\ i=1,...,n .$$

By [26], $Cov(f,A_i,A_j) = \frac{1}{36}[\alpha_i\alpha_j + \beta_i\beta_j + (\alpha_i+\beta_i)(\alpha_j+\beta_j)]$ for any $i,j=1,...,n$. Thus

$$\rho_i^0 = \frac{1}{36}\sum_{j=1}^{n}[\alpha_i\alpha_j + \beta_i\beta_j + (\alpha_i+\beta_i)(\alpha_j+\beta_j)]$$
$$= \frac{1}{36}[\alpha_i\sum_{j=1}^{n}\alpha_j + \beta_i\sum_{j=1}^{n}\beta_j + (\alpha_i+\beta_i)\sum_{j=1}^{n}(\alpha_j+\beta_j)] \qquad \text{for } i=1,...,n .$$

6. Conclusions

In this paper a possibilistic model of risk aversion of an agent confronted by a risk situation with several parameters has been proposed.

The main contributions of the paper are:

- the study of a notion of possibilistic expected utility associated with a possibilistic vector whose components are fuzzy numbers;
- the definition and the approximate calculation of possibilistic risk premium vector and possibilistic risk premium associated with a possibilistic vector and an n-dimensional utility function;
- the sketch of a possible application of this model in the evaluation of risk aversion which can appear in the functioning of a grid.

The mentioned results are a first step to the study of possibilistic risk aversion with several parameters. We mention the following problems which can be the object of future investigations:

- to obtain a Pratt type theorem to compare the risk aversion of two agents to a risk situations with several parameters;
- to compare multidimensional risk situations (possibilistic versions of Diamond- Stieglitz theory);
- applications in the analysis of portfolio problem (including experimental results);
- the treatment of risk aversion with mixed parameters (some described by random variables, some described by fuzzy numbers) .

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References

1. K.J. Arrow, Essays in the theory of risk bearing, North–Holland, Amsterdam,1970.
2. C. Carlsson and R. Fullér, On the possibilistic mean value and variance of fuzzy numbers, Fuzzy Sets Syst., 122, 2001, 315–326.
3. C. Carlsson and R. Fullér, Fuzzy reasoning in decision making and optimization, Studies in Fuzziness and Soft Computing Series, Vol. 82, Springer,Berlin, Heidelberg, 2002.
4. C. Carlsson, R. Fullér, and P. Majlender, A possibilistic approach to selecting portfolios with highest utility score, Fuzzy Sets Syst., 131, 2002, 13–21.
5. C. Carlsson, R. Fullér, and P. Majlender, Some normative properties of possibility distributions, in: Proceedings of the Third International

Symposium of Hungarian Researchers in Computational Intelligence, Budapest, 2002, 61–71.
6. C. Carlsson and O. Weissmann, Deliverable 4.1 Advanced Risk Assessment, September 2008, AssessGrid– Advanced Risk Assessment and Management for Trustable Grids, http://www.assessgrid.eu.
7. D. Dubois and H. Prade, Fuzzy sets and systems: theory and applications, Academic Press, New York, 1980.
8. D. Dubois and H. Prade, Possibility theory, Plenum Press, New York, 1988.
9. G. T. Duncan, A matrix measure of multivariate local risk aversion, Econometrica, 45, 1977, 895–902.
10. R. Fullér and P. Majlender, On possibilistic dependencies, Turku Centre for Computer Science Technical Report No. 477, 2002.
11. R. Fullér and P. Majlender, On weighted possibilistic mean and variance of fuzzy numbers, Fuzzy Sets Syst., 136, 2003, 365–374.
12. R. Fullér, Introduction to neuro–fuzzy systems, Advances in Soft Computing, Springer, Berlin, Heidelberg, 2000.
13. I. Georgescu, A possibilistic approach to risk aversion, Soft Computing, 2011, in press.
14. I. Georgescu and J. Kinnunen, Multidimensional possibilistic risk aversion, 11th IEEE International Symposium on Computational Intelligence and Informatics CINTI 2010, Budapest, Hungary, 18-20 November 2010, 163– 168.
15. I. Georgescu, Possibilistic risk aversion, Fuzzy Sets Syst., 60, 2009, 2608–2619.
16. E. Karni, On multivariate risk aversion, Econometrica, 47, 1979, 1391–1401.
17. R. E. Kihlstrom and L. J. Mirman, Constants, increasing and decreasing risk aversion with many commodities, Rev. Ec. Studies, XLVIII, 1981, 271–280.
18. J. J. Laffont, The economics of uncertainty and information, MIT Press, Cambridge, 1993.
19. H. Levy and A. Levy, Arrow–Pratt measures of risk aversion: the multivariate case, International Economic Review, 32, 1991, 891–898.
20. P. Majlender, A normative approach to possibility theory and decision support, PhD Thesis, Turku Centre for Computer Science, 2004.
21. J. Pratt, Risk aversion in the small and in the large, Econometrica, 32, 1964, 122–130.
22. J. Quiggin, Generalized expected utility theory, Kluwer-Nijhoff, Amsterdam, 1993.
23. S. Ross, Some stronger measures of risk aversion in the small and in the large with applications, Econometrica, 40, 1981, 621–638.
24. L. A. Zadeh, Fuzzy sets, Information and Control, 8, 1965, 228–253.
25. L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst., 1, 1978, 3–28.
26. W. G. Zhang and Y. L. Wang, A comparative analysis of possibilistic variance and covariance of fuzzy numbers, Fundamenta Informaticae, 79, 2007, 257–261.