MATHEMATICS

NOTES ON BANACH FUNCTION SPACES, II

BY

W. A. J. LUXEMBURG [1]) AND A. C. ZAANEN

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This note is a sequal to the first note with the same title in these Proceedings, the contents of which are assumed to be known.

5. *The Fatou property*

We recall that nonnegative functions in M are denoted by $u, v, \ldots$. The notation $u_n \uparrow u$ will mean that $u_n(x) \uparrow u(x)$ as $n \to \infty$ almost everywhere on X. The main topic in this section is a certain property which a function seminorm ϱ may or may not possess, and which turns out to be stronger than the Riesz–Fischer property.

Definition 5.1 (*Fatou property*). *The function seminorm ϱ is said to have the Fatou property whenever $u_n \uparrow u$ implies $\varrho(u_n) \uparrow \varrho(u)$.*

(*Weak Fatou property*). *The function seminorm ϱ is said to have the weak Fatou property whenever $u_n \uparrow u$ and $\lim \varrho(u_n) < \infty$ implies that $\varrho(u) < \infty$.*

(*Fatou null property*). *The function seminorm ϱ is said to have the Fatou null property whenever it follows from $u_n \uparrow u$ and $\varrho(u_n) = 0$ for all $n \in N$ that $\varrho(u) = 0$.*

The reason why these various properties are called Fatou properties will become clear from Theorem 5.7. Note that any function norm has the Fatou null property.

Lemma 5.2. *The Fatou property implies the weak Fatou property, and the weak Fatou property implies the Fatou null property.*

Proof. The first statement is evident. For the proof of the second statement, assume that ϱ has the weak Fatou property, and $u_n \uparrow u$ with $\varrho(u_n) = 0$ for all $n \in N$. If $\varrho(u) > 0$ we set $v_n = nu_n$, so $v_n \uparrow v$ and $\varrho(v_n) = 0$ for all $n \in N$. Since $v \geqslant nu_n \geqslant ku_n$ for all $n \geqslant k$, we have $v \geqslant ku$ for every $k \in N$, so $\varrho(v) \geqslant k\varrho(u)$, which implies $\varrho(v) = \infty$. Hence, $v_n \uparrow v$, $\varrho(v_n) = 0$ for all $n \in N$, and $\varrho(v) = \infty$; this contradicts the hypothesis that ϱ has the weak Fatou property.

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Theorem 5.3. *If ϱ has the weak Fatou property, and hence surely if ϱ has the Fatou property, then ϱ has the Riesz–Fischer property, and so L_ϱ is then complete.*

Proof. Let ϱ have the weak Fatou property, and $\sum\varrho(u_n) < \infty$. Setting $s_n = u_1 + \ldots + u_n$, we have $\varrho(s_n) \leqslant \sum\varrho(u_n)$ for all $n \in N$, so $\lim \varrho(s_n) < \infty$. Since $s_n \uparrow \sum u_n$, it follows from the weak Fatou property that $\varrho(\sum u_n) < \infty$.

The proof shows that the weak Fatou property implies the Riesz–Fischer property in the strong sense that $\sum\varrho(u_n) < \infty$ implies $\varrho(\sum u_n') < \infty$ for every choice of $u_n' \equiv u_n (n \in N)$.

It will be proved now that the Fatou property is preserved under the formation of suprema. The same does not hold for the weak Fatou property (cf. Example 5.6 (iii)).

Theorem 5.4. *If ϱ_τ is a function seminorm with the Fatou property for every τ in the index set $\{\tau\}$, then $\varrho = \sup \varrho_\tau$ has the Fatou property.*

Proof. It was already observed in section 3 that $\varrho = \sup \varrho_\tau$ is a function seminorm. Let $u_n \uparrow u$. Since it is evident that $\alpha = \lim \varrho(u_n) \leqslant \varrho(u)$, it will be sufficient to show that $\alpha > \beta$ for any $\beta < \varrho(u)$. Hence, let $\beta < \varrho(u)$. Then $\varrho_\tau(u) > \beta$ for some $\tau \in \{\tau\}$, so $\varrho_\tau(u_n) > \beta$ for all $n \geqslant n_0$ since ϱ_τ has the Fatou property. It follows that $\varrho(u_n) > \beta$ for all $n \geqslant n_0$, and so surely $\alpha = \lim \varrho(u_n) > \beta$.

Theorem 5.5 (*Amemiya's theorem*, [1]). *The function seminorm ϱ has the weak Fatou property if and only if there exists a finite constant $k \geqslant 1$ such that $u_n \uparrow u$ implies $\varrho(u) \leqslant k \lim \varrho(u_n)$.*

Proof. If there exists such a constant k then ϱ has evidently the weak Fatou property. Assume now that ϱ has the weak Fatou property, but there exists no such k. Then there exists for every $k \in N$ a sequence $u_{nk} \uparrow u_k$ (as $n \to \infty$) such that

$$(1) \qquad \varrho(u_k) > k^3 \lim_n \varrho(u_{nk}).$$

It is impossible that $\lim_n \varrho(u_{nk}) = 0$ for some k, since in view of the Fatou null property (which follows from the weak Fatou property) this would imply $\varrho(u_k) = 0$, in contradiction to (1). Hence, multiplying by suitable constants, we may assume that $\lim_n \varrho(u_{nk}) = k^{-2}$, and so $\varrho(u_k) > k$ for every $k \in N$. Now, let $v_n = u_{n1} + u_{n2} + \ldots + u_{nn}$ for all $n \in N$. Then $v_n \uparrow v$ and $\lim \varrho(v_n) \leqslant \sum_k \lim_n \varrho(u_{nk}) \leqslant \sum k^{-2} < \infty$. But $v = \sup v_n \geqslant \sup_n u_{nk} = u_k$ for all $k \in N$, so $\varrho(v) = \infty$. This contradicts the hypothesis that ϱ has the weak Fatou property.

The weak Fatou property was introduced, for (partially) ordered vectorspaces, by I. Amemiya [1]; in the first papers on general Banach function spaces it was always assumed that ϱ has the Fatou property. The Fatou null property, which will play an important part in the development of the theory, is introduced here for the first time in print, as far as we are aware.

Example 5.6. (i) The weak Fatou property does not imply the Fatou property. Indeed, let $X=N$, μ discrete measure, and

$$\varrho(u)=\sup u(n)+a \limsup u(n),$$

where a is a positive constant. Obviously, ϱ is a function norm, and $\varrho(u)\leqslant (a+1)||u||_\infty$ where $||u||_\infty=\sup u(n)$ is the l_∞ norm of u.

Let the sequence $\{u_k;\, k\in N\}$ be defined by $u_k(n)=1$ for $n\leqslant k$ and $u_k(n)=0$ for $n>k$. Then $\varrho(u_k)=1$ for every k, and $u_k\uparrow u$ with $u(n)=1$ for every n. Hence $\varrho(u_k)\uparrow 1$, but $\varrho(u)=1+a>1$. This shows that ϱ does not have the Fatou property.

We shall prove now that $u_k\uparrow u$ implies $\varrho(u)\leqslant (a+1)\lim \varrho(u_k)$ for every sequence $\{u_k;\, k\in N\}$, and it will follow then immediately that ϱ has the weak Fatou property. For the proof we may assume that $\lim \varrho(u_k)<\infty$. Since $u_k\uparrow u$, we have $||u_k||_\infty\uparrow ||u||_\infty$, and $||u_k||_\infty$ is bounded since $\lim\varrho(u_k)<\infty$. Hence, given $\varepsilon>0$, we have $||u||_\infty<||u_k||_\infty+\varepsilon$ for $k\geqslant k_0$. It follows that

$$\varrho(u)\leqslant (a+1)||u||_\infty<(a+1)||u_k||_\infty+(a+1)\varepsilon\leqslant (a+1)\varrho(u_k)+(a+1)\varepsilon$$

for $k\geqslant k_0$, and so $\varrho(u)\leqslant(a+1)\lim\varrho(u_k)$. The particular sequence $\{u_k\}$ considered in the preceding paragraph shows that $k=a+1$ is the best constant in Amemiya's theorem for this example.

(ii) The Fatou null property does not imply the weak Fatou property. Indeed, any function norm ϱ has the Fatou null property, but if L_ϱ is not complete (e.g. as in Example 4.9 (ii)), then ϱ does not have the weak Fatou property (since the weak Fatou property implies completeness of L_ϱ by Theorem 5.3).

(iii) The weak Fatou property is not always preserved under formation of suprema. Indeed, let $X=N$, μ discrete measure, and let the sequence $\{\varrho_m;\, m\in N\}$ of function norms be defined by

$$\varrho_m(u)=\sup u(n)+m\limsup u(n).$$

We set $\varrho=\sup\varrho_m$. Obviously, $\varrho(u)=\sup u(n)$ if $\limsup u(n)=0$ and $\varrho(u)=\infty$ if $\limsup u(n)>0$. In view of (i) every ϱ_m has the weak Fatou property, but ϱ does not possess this property as shown by the sequence $\{u_k;\, k\in N\}$ satisfying $u_k(n)=1$ for $n\leqslant k$ and $u_k(n)=0$ for $n>k$.

(iv) The Riesz–Fischer property does not imply the weak Fatou property, not even for norms. Indeed, let ϱ be the same function norm as in the preceding part (iii). Evidently, L_ϱ is isomorphic (algebraically and isometrically) to the subspace (c_0) of l_∞, and hence L_ϱ is complete. It follows that ϱ has the Riesz–Fischer property. But, as shown in (iii), ϱ does not have the weak Fatou property.

(v) For seminorms the Riesz–Fischer property and the Fatou null property are independent. Indeed, every norm has the Fatou null property but there exist norms which do not possess the Riesz–Fischer property. Conversely, if $X=N$, μ discrete measure, and $\varrho(u)=\limsup u(n)$, then ϱ has the Riesz–Fischer property but not the Fatou null property. Another example is the Marcinkiewicz space with exponent $p\,(1\leqslant p<\infty)$ in Example 4.9 (iv). In this example the seminorm ϱ has the Riesz–Fischer property, but not the Fatou null property as shown by the sequence $u_n(x)=\chi_{[-n,n]}(x)$.

The next theorem, due to its parallelism to Fatou's well-known lemma for integrals, will show why the Fatou property is called by that name.

Theorem 5.7. *The function seminorm ϱ has the weak Fatou property if and only if there exists a finite constant $k\geqslant 1$ such that*

$$\varrho(\liminf u_n)\leqslant k \liminf \varrho(u_n)$$

for every sequence $\{u_n;\, n\in N\}$. The seminorm ϱ has the Fatou property if and only if this inequality is always satisfied with $k=1$.

Proof. If ϱ has the weak Fatou property there exists, by Amemiya's theorem, a finite constant $k\geqslant 1$ such that $v_n\uparrow v$ implies $\varrho(v)\leqslant k\lim \varrho(v_n)$. Now, let $\{u_n;\, n\in N\}$ be an arbitrary sequence. We set $v_n=\inf(u_n, u_{n+1}, \ldots)$ for every $n\in N$. Then $v_n\uparrow \liminf u_n$ and so $\varrho(\liminf u_n)\leqslant k\lim\varrho(v_n)$. But $\varrho(v_n)\leqslant\varrho(u_k)$ for all $k\geqslant n$, so $\varrho(v_n)\leqslant\liminf\varrho(u_k)$ for every $n\in N$. It follows that

$$\varrho(\liminf u_n)\leqslant k\lim\varrho(v_n)\leqslant k\liminf\varrho(u_n).$$

Conversely, if $\varrho(\liminf u_n)\leqslant k\liminf\varrho(u_n)$ for some finite $k\geqslant 1$ and every sequence $\{u_n\}$, then the particular case that $u_n\uparrow u$ shows that ϱ has the weak Fatou property.

If ϱ has the Fatou property, then the inequality for increasing sequences holds with $k=1$, and the above proof shows then that the inequality for arbitrary sequences holds as well with $k=1$. Conversely, if $\varrho(\liminf u_n)\leqslant$ $\leqslant\liminf\varrho(u_n)$ holds for arbitrary sequences, then in particular for increasing sequences, so ϱ has the Fatou property.

If ϱ has the Fatou property, then

$$\varrho(u\chi_{E_n})\uparrow\varrho(u\chi_E) \tag{2}$$

for any $u\in M^+$ and any sequence $E_n\uparrow E$. It may be asked whether (2) is already sufficient in order that ϱ have the Fatou property. The answer is no as shown by the following example. Let X consist of one point with $\mu(X)=1$, and let $\varrho(u)=0$ for u finite and $\varrho(u)=\infty$ for $u=\infty$. Then (2) holds but ϱ does not have the Fatou property. We shall prove now that if ϱ satisfies (2) and possesses, besides that, the Fatou null property, then ϱ has the Fatou property. A similar theorem for the weak Fatou property will be proved later (in section 7).

Lemma 5.8. *If $u \in M^+$ and $E = \{x : u(x) = +\infty\}$, then either $\varrho(u\chi_E) = 0$ or $\varrho(u\chi_E) = \infty$. If ϱ has the Fatou null property, then $\varrho(u\chi_E) = 0$ if and only if $\varrho(\chi_E) = 0$ (and, hence, $\varrho(u\chi_E) = \infty$ if and only if $\varrho(\chi_E) > 0$).*

Proof. Since $\infty \cdot u\chi_E = u\chi_E$ it follows easily that $\infty \cdot \varrho(u\chi_E) = \varrho(u\chi_E)$, and so $\varrho(u\chi_E)$ is either zero or $+\infty$. It is evident that $\varrho(u\chi_E) = 0$ implies $\varrho(\chi_E) = 0$. Conversely, if ϱ has the Fatou null property, then $\varrho(\chi_E) = 0$ implies $\varrho(u\chi_E) = 0$.

Theorem 5.9. *The following conditions on the seminorm ϱ are equivalent:*

(i) *ϱ has the Fatou property.*

(ii) *ϱ has the Fatou null property, and $\varrho(u\chi_{E_n}) \uparrow \varrho(u\chi_E)$ for any $u \in M^+$ and any sequence $E_n \uparrow E$.*

Proof. It is evident that (i) implies (ii). Conversely, let (ii) hold, and assume that $u_n \uparrow u$. We have to show that $\varrho(u_n) \uparrow \varrho(u)$. Let

$$E = \{x : u(x) = +\infty\}.$$

There are two cases, either $\varrho(u\chi_E) < \infty$ or $\varrho(u\chi_E) = \infty$. If $\varrho(u\chi_E) < \infty$, then $\varrho(u\chi_E) = 0$ by the preceding lemma (i.e., E is now a strong ϱ-null set), so that making u and all u_n zero on E affects neither $\varrho(u)$ nor any of the $\varrho(u_n)$. Hence, we may assume that u is finitevalued. Let $D = \{x : u(x) > 0\}$, $0 < \varepsilon < 1$, and $E_n = \{x : u_n(x) > (1-\varepsilon)u(x)\}$. Then $E_n \uparrow D$, so

$$\varrho(u\chi_{E_n}) \uparrow \varrho(u),$$

and it follows that

$$\varrho(u_n) \geqslant (1-\varepsilon)\varrho(u\chi_{E_n}) \uparrow (1-\varepsilon)\varrho(u).$$

This shows that $\lim \varrho(u_n) \geqslant (1-\varepsilon)\varrho(u)$ for all such ε, so $\varrho(u_n) \uparrow \varrho(u)$.

In the second case we have $\varrho(u\chi_E) = \infty$. Then $\varrho(u) = \infty$, and $\varrho(\chi_E) > 0$ by the preceding lemma (since ϱ has the Fatou null property). Note that $u_n \uparrow \infty$ on E. We choose any $\alpha > 0$, and let $F_n = \{x : u_n(x) > \alpha\} \cap E$. Then $F_n \uparrow E$, so $\varrho(\chi_{F_n}) \uparrow \varrho(\chi_E)$. It follows that $\varrho(u_n) \geqslant \alpha\varrho(\chi_{F_n}) \uparrow \alpha\varrho(\chi_E)$, so $\lim \varrho(u_n) \geqslant \alpha\varrho(\chi_E)$ for every $\alpha > 0$. But then $\lim \varrho(u_n) = \infty$, and so $\varrho(u_n) \uparrow \varrho(u)$.

In the remainder of this section we investigate the properties of a seminorm ϱ having the Fatou null property. In this case any ϱ-null set E, i.e. any set E satisfying $\varrho(\chi_E) = 0$, is a strong ϱ-null set, and hence $\varrho(f\chi_E) = 0$ for any $f \in M$ (indeed, $\varrho(f\chi_E) = \varrho(\infty \cdot \chi_E) = 0$). Obviously, countable unions of ϱ-null sets are now ϱ-null sets.

Theorem 5.10. *Given the function seminorm ϱ with the Fatou null property, there exists a maximal ϱ-null set X_0, i.e. a ϱ-null set X_0 such that $Y = X - X_0$ does not have any ϱ-null subset of positive measure. The set X_0 is μ-uniquely determined.*

Proof. Assume first that $\mu(X)<\infty$, and let $\alpha=\sup\{\mu(E): \varrho(\chi_E)=0\}$. Then, since finite unions of ϱ-null sets are ϱ-null sets, there exists an ascending sequence $\{E_n; n \in N\}$ with $\varrho(\chi_{E_n})=0$ for all n and $\mu(E_n)\uparrow\alpha$. Let $E_n\uparrow E_0$. Then $\varrho(\chi_{E_0})=0$ in view of the Fatou null property, and $\mu(E_0)=\alpha$, so by the definition of α the set $X-E_0$ does not have any ϱ-null subset of positive measure. It follows easily that E_0 is μ-uniquely determined.

If $\mu(X)=\infty$, we set $X=\bigcup_1^\infty X_k$ with all X_k disjoint and of finite measure. Each X_k has a μ-uniquely determined maximal ϱ-null subset E_k. It follows easily that $X_0=\bigcup_1^\infty E_k$ is the desired maximal ϱ-null set in X.

It is an immediate consequence that $f\equiv g$ holds if and only if $f=g$ holds almost everywhere on $Y=X-X_0$. Furthermore, $\varrho(f)=\varrho(f\chi_Y)$ for every $f\in M$. Hence, there is no loss of generality if we delete X_0 from X and consider only the restriction of each function on the set $Y=X-X_0$. In other words, instead of the equivalence class $[f]$ we consider the common restriction on Y of all $g\in[f]$. The function seminorm ϱ becomes then a function norm on the collection of all μ-measurable functions on Y. In the particular case that ϱ has the Riesz–Fischer property (i.e., the case that L_ϱ is complete) we have therefore $\varrho(\sum v_n)\leqslant\sum\varrho(v_n)$ for any sequence $\{v_n; n\in N\}$ defined on Y (cf. Theorem 4.2). But then, if $\{u_n; n\in N\}$ is any given sequence of nonnegative functions on X, we have just as well that $\varrho(\sum u_n)\leqslant\sum\varrho(u_n)$, since $\varrho(u)=\varrho(u\chi_Y)$ for any u. Hence, the functions u_n' appearing in the Riesz–Fischer inequality $\varrho(\sum u_n')\leqslant\sum\varrho(u_n)$ for seminorms may now be chosen arbitrarily subject only to the condition that $u_n'\equiv u_n$ for every $n\in N$. Conversely, it is easy to show that if the seminorm ϱ has the Riesz–Fischer property in this strong sense, then ϱ has the Fatou null property.

California Institute of Technology-
Utrecht State University and
Leiden State University

REFERENCES

1. AMEMIYA, I., A generalization of Riesz-Fischer's theorem, Journal Math. Soc. of Japan **5**, 353–354 (1953).