Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential✩

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Abstract

In this paper, an existence theorem is obtained for periodic solutions of a second-order discrete Hamiltonian system with a change of sign in the potential by the minimax methods in the critical point theory.
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1. Introduction

We denote by N, Z, R the sets of all natural numbers, integers and real numbers, respectively. For a, b ∈ Z, define Z(a) = {a, a + 1, ...}, Z(a, b) = {a, a + 1, ..., b} when a ≤ b. In this paper, we deal with the existence of periodic solutions for a discrete second-order Hamiltonian system

\[ \Delta^2 x_{n-1} + b(n) \nabla V(x_n) = 0, \quad n \in \mathbb{Z}, \]

where b ∈ C(R, R) and there exists a positive integer m such that for any t ∈ R, b(t + m) = b(t), Δx_n = x_{n+1} - x_n, Δ^2 x_n = Δ(Δx_n), V ∈ C^1(R, R), ∇V(z) denotes the gradient of V(z) in z for z ∈ R.

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Let \( p \) be a given positive integer. We will study the existence of \( pm \)-periodic solutions for Eq. (1.1). As usual, such periodic solutions will be called subharmonic solutions.

We may regard Eq. (1.1) as being a discrete analogue of the following second-order differential equation with \( A(t) \equiv 0 \):

\[
\ddot{x}(t) + A(t)x(t) + b(t)\nabla V(x(t)) = 0, \quad \forall t \in \mathbb{R},
\]

where \( b \in C(\mathbb{R}, \mathbb{R}) \), \( b(t + m) = b(t) \) for any \( t \in \mathbb{R} \), \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \), \( A(\cdot) \) is a continuous, \( T \)-periodic matrix-valued function.

For the case \( A(t) \equiv 0 \), in 1978, Rabinowitz [13] proved the existence of \( T \)-periodic solution of (1.2) with \( b(t) > 0 \) and \( V \) superquadratic. Since then, many authors have discussed the problem (1.2).

When \( b \) does not change its sign, there has been much progress on the existence and multiplicity of periodic solutions of (1.2) (see [12,14]).

When \( b \) changes its sign, many excellent results have been worked out on the existence of periodic and subharmonic solutions for Eq. (1.2). Lassoued in [10,11] proved the existence of \( T \)-periodic solutions under the assumptions that \( V \) is homogeneous and strictly convex. Afterwards, Ben Naoum, Troestler and Willem in [5] obtained some existence results when \( V \) is only homogeneous. Girardi and Matzeu in [7] considered the more general case and obtained some existence and multiplicity results for \( T \)-periodic and subharmonic solutions. Chen and Long in [6] and Tang and Wu in [17] considered the case that \( \int_0^T b(t) \, dt = 0 \), \( b \not\equiv 0 \), which is different from the case \( \int_0^T b(t) \, dt \neq 0 \), when \( V \) is neither convex nor homogeneous.

For the case \( A(t) \neq 0 \), there are also some existence results for periodic solutions and homoclinic solutions of (1.2) (see [1,2,18]). For the recent progress in this direction, one can refer to [15,16].

The main idea of these papers is to reduce the problem of finding the periodic solutions of differential equations to the problem of seeking the critical points of suitable variational functional. It is natural for us to think that critical point theory may be applied to prove the existence of periodic solutions of difference equations.

In fact, for the general form of (1.1) as

\[
\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z},
\]

there are many new results of existence of nontrivial periodic solutions. By using the critical point theory, Guo and Yu have proved the existence of periodic and subharmonic solutions of (1.3) when \( f(t, z) \) is superlinear in the second variable \( z \) in [8] and when \( f(t, z) \) is sublinear in the second variable \( z \) in [9]. Yu and Long and Guo in [19] have proved the existence of subharmonic solutions with prescribed minimal period for the discrete forced pendulum equation. Zhou, Yu and Guo in [21] have proved the existence of periodic solutions of higher-dimensional discrete systems when \( f(t, z) \) is neither superlinear nor sublinear and generalized the results in [8]. As for boundary value problem, Agarwal, Perera and O’Regan [3] studied the existence of positive solutions of Eq. (1.2) with homogeneous boundary value problem by using Mountain Pass Lemma. In this direction, one can find more results in [4,20].

However no similar results were obtained in the literature for Eq. (1.1) in the case that \( b \) changes its sign. Motivated by [6,17], the main purpose of this paper is to give some sufficient conditions for the existence of periodic solutions to (1.1) when \( b \) changes its sign. The main approach is the critical point theory. Our main results are as follows.
Theorem 1.1. Suppose that \( b(n) \) and \( V(x) \) satisfy the following assumptions:

(B) \( b \in C(\mathbb{R}, \mathbb{R}) \), there exists a positive integer \( m \) such that for any \( n \in \mathbb{Z} \), \( b(n+m) = b(n) \), \( b(n) \neq 0 \), and \( \sum_{n=1}^{m} b(n) = 0 \);

(V) \( V(x) = a|x|^\beta + F(x) \), \( \forall x \in \mathbb{R} \), where \( a > 0 \), \( \beta > 2 \) and \( F(x) \in C^1(\mathbb{R}, \mathbb{R}) \);

\((F_1)\) there exist \( \alpha_0 \in (0, 2B^{-1}\sin^2(\frac{\pi}{pm})) \) and \( r_0 > 0 \) such that

\[
|F(x)| \leq \alpha_0 |x|^2, \quad \forall |x| \leq r_0,
\]

where \( B = \max\{b(n): n \in \mathbb{Z}(1,m)\} \);

\((F_2)\) there exists \( r_1 > 0 \) such that

\[
|\nabla F(x)| \leq \frac{\alpha_1}{B(1 + |x|)}, \quad \forall |x| \geq r_1,
\]

where \( \alpha_1 = \max_{\rho \in [0, \sqrt{pmr_0}]} h(\rho) \), \( h(\rho) = [2\sin^2(\frac{\pi}{pm}) - B\alpha_0]^2 - \frac{aB}{C_1}\rho^\beta \), \( C_1 > 0 \) is a constant which can be referred to (2.3).

Then Eq. (1.1) has at least one nontrivial \( pm \)-periodic solution.

Remark 1.1. Condition (B) implies that \( m > 1 \) and condition \((F_1)\) implies that \( F(0) = 0 \).

Corollary 1.1. Suppose that \( b(t) \) and \( V(x) \) satisfy assumptions (B) and (V), and assume that \( F(x) \equiv 0 \) for all \( x \in \mathbb{R} \), then Eq. (1.1) has at least one nontrivial \( pm \)-periodic solution.

2. Variational structure and a basic lemma

In this section, we recall some basic facts which will be used in the proof of main results. Let \( x = \{x_n\}_{n \in \mathbb{Z}} = (\ldots, x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, x_2, \ldots, x_n, \ldots)^T \), where \( .^T \) is the transpose of a vector or a matrix.

For any given positive integers \( p \) and \( m \), \( E_{pm} \) is defined by

\[ E_{pm} = \{ x = \{x_n\}: x_{n+p} = x_n, \; x_n \in \mathbb{R}, \; n \in \mathbb{Z} \} \]

\( E_{pm} \) can be equipped with the inner product \( \langle \cdot, \cdot \rangle_{E_{pm}} \) and norm \( \| \cdot \|_{E_{pm}} \) as follows:

\[
\langle x, y \rangle_{E_{pm}} = \sum_{j=1}^{pm} x_j y_j, \quad \forall x, y \in E_{pm},
\]

\[
\|x\|_{E_{pm}} = \sqrt{\sum_{j=1}^{pm} x_j^2}, \quad \forall x \in E_{pm}.
\]

It is easy to see that \((E_{pm}, \langle \cdot, \cdot \rangle)\) is a finite dimensional Hilbert space and linearly homeomorphic to \( \mathbb{R}^{pm} \). For convenience, we identify \( x \in E_{pm} \) with \( x = (x_1, x_2, \ldots, x_{pm})^T \).

Define functional \( I \) on \( E_{pm} \) as follows:

\[
I(x) = \sum_{n=1}^{pm} \left[ \frac{1}{2} (\Delta x_n)^2 - b(n) V(x_n) \right], \quad \forall x \in E_{pm}.
\]
A similar argument to that of [8] shows that $x \in E_{pm}$ is a critical point of $I$ if and only if $x = \{x_n\}$ is a $pm$-periodic solution of (1.1).

For $pm > 2$, we rewrite functional $I(x)$ in the form

$$I(x) = \frac{1}{2}x^T Ax - \sum_{n=1}^{pm} b(n)V(x_n),$$

(2.2)

where $x = (x_1, x_2, \ldots, x_{pm})^T$.

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
-1 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}_{pm \times pm}.$$

We denote the eigenvalues of $A$ by

$$\lambda_k = 4 \sin^2 \left( \frac{k\pi}{pm} \right), \quad k = 0, 1, \ldots, pm - 1,$$

the minimal positive eigenvalue of $A$ by $\lambda_{\min} \triangleq 4 \sin^2 \left( \frac{\pi}{pm} \right)$, the maximal eigenvalue of $A$ by $\lambda_{\max}$ (see [19]).

**Remark 2.1.** For the case $pm = 2$, $A$ has a different form

$$\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}.$$

However, in this special case, the eigenvalues of $A$ are $\lambda_0 = 0$ and $\lambda_1 = 4$ by direct computation, thus the argument need not be changed and we omit it.

The eigenspace of $A$ associated with $\lambda_0 = 0$ is

$$Z = \{x \in E_{pm}: x = \{v\}, \ v \in \mathbb{R}\}.$$ 

Clearly $Z$ is an invariant subspace of $E_{pm}$ with respect to $A$. Let $Y$ be the direct orthogonal complement of $E_{pm}$ to $Z$, that is, $E_{pm} = Y \oplus Z$.

For any $x \in E_{pm}$, set

$$\|x\|_\gamma = \left( \sum_{n=1}^{pm} |x_n|^\gamma \right)^{\frac{1}{\gamma}}$$

for $\gamma > 1$.

By the discrete analogy of Hölder inequality, $\| \cdot \|_\gamma$ is a norm on $E_{pm}$ and $\|x\|_2 = \|x\|_{E_{pm}}$. Furthermore, $E_{pm}$ is equivalent to the finite dimensional Hilbert space $\mathbb{R}^{pm}$, hence $\| \cdot \|_2$ and $\| \cdot \|_\gamma$ are equivalent, that is, there exist constants $C_1$ and $C_2$ such that $C_1 \geq C_2 > 0$, and

$$C_1 \|x\|_\gamma \leq \|x\|_2 \leq C_2 \|x\|_\gamma, \quad \forall x \in E_{pm}. \quad (2.3)$$

For convenient, we denote $\|x\|_2$ by $\|x\|$ in this paper.

We will make use of Linking Theorem introduced in [14] by Rabinowitz to obtain the critical points of $I(x)$. Let us first recall this theorem.
Linking Theorem. Let $H = H_1 \oplus H_2$ be a real Banach space, where $H_1$ is a finite dimensional closed subspace of $H$ and $H_2 = H_1^\bot$. Suppose that $I(\cdot) \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale condition (PS condition) and the following conditions:

(1) there exist constants $\rho > 0$ and $\alpha > 0$ such that $I(x) \geq \alpha, \forall x \in H_2 \cap \partial B_\rho$, where $B_\rho = \{x \in H : \|x\|_H \leq \rho\}$, $\partial B_\rho$ denotes the boundary of $B_\rho$;

(2) there exist $e \in H_2, \|e\|_H = 1, R_1 > 0, R_2 > \rho$ such that $I(x)|_{\partial Q} \leq \frac{\alpha}{2}$, where $Q = \{x : x \in H, x = z + \lambda e, z \in H_1, \|z\|_H \leq R_1, \lambda \in (0, R_2)\}$.

Then $I$ possesses a critical value $c \geq \alpha$, where

$$c = \inf_{h \in \Gamma} \max_{x \in Q} I(h(x))$$

and $\Gamma = \{h \in C(\bar{Q}, H) : h_{|\partial Q} = \text{id}\}$, where $\text{id}$ denotes the identity operator.

3. Proofs of the main result

In order to prove Theorem 1.1 by using Linking Theorem, we only need to prove the following lemmas. Before proving these lemmas, we give two propositions.

Proposition 3.1. Under condition $(F_2)$, there exists $M_0 > 0$ such that

$$|\nabla F(x)| \leq \frac{\alpha_1}{B(1 + |x|)} + M_0, \forall x \in \mathbb{R}.$$  

Proof. Let $M_0 = \max\{|\nabla F(x)| : |x| \leq r_1\}$.

By $(F_2)$, we have

$$|\nabla F(x)| \leq \frac{\alpha_1}{B(1 + |x|)} + M_0, \forall x \in \mathbb{R}.$$  

The proof of Proposition 3.1 is complete.

Proposition 3.2. Under condition $(F_2)$, there exists $M_1 > 0$ such that for any $|x| \geq r_1$, there exists a constant $\tau_x \in (0, 1)$ such that

$$|F(x)| \leq M_1 + \frac{\alpha_1 |x| + \alpha_1 r_1}{B(1 + |r_1 + \tau_x (x - r_1)|)}, \forall |x| \geq r_1.$$  

Proof. Since $F \in C^1(\mathbb{R}, \mathbb{R})$, it follows from the mean value theorem that for any $x \in \mathbb{R}$, there exists $\tau_x \in (0, 1)$ such that

$$F(x) - F(r_1) = \nabla F(r_1 + \tau_x (x - r_1)) \cdot (x - r_1).$$

Thus, by using condition $(F_2)$, we have

$$|F(x)| \leq |F(r_1)| + |F(x) - F(r_1)|$$

$$= |F(r_1)| + |\nabla F(r_1 + \tau_x (x - r_1))| \cdot |x - r_1|$$

$$\leq F(r_1) + \frac{\alpha_1 |x| + \alpha_1 r_1}{B(1 + |r_1 + \tau_x (x - r_1)|)}.$$
for all \( |x| \geq r_1 \). Since the set \( \{ x : |x| = r_1 \} \) is compact, there exists \( M_1 > 0 \) such that

\[
|F(x)| \leq M_1 + \frac{\alpha_1 |x| + \alpha_1 r_1}{B(1 + |r_1 + \tau x - r_1|)}
\]

for all \( |x| \geq r_1 \). The proof of Proposition 3.2 is complete. \( \square \)

**Lemma 3.1.** Under the assumptions of Theorem 1.1, the functional \( I \) satisfies the PS condition, i.e., for any sequence \( \{ x^{(k)} \} \) such that \( I(x^{(k)}) \) is bounded and \( I'(x^{(k)}) \to 0 \) as \( k \to \infty \), there exists a subsequence of \( \{ x^{(k)} \} \) which is convergent in \( E_{pm} \).

**Proof.** Recall that \( E_{pm} \) is identified with \( \mathbb{R}^{pm} \). Consequently, in order to prove that \( I \) satisfies PS condition, we only need to prove that \( \{ x^{(k)} \} \) is bounded.

Suppose that \( \{ x^{(k)} \} \) is unbounded, then we can assume, going to a subsequence if necessary, that \( \{ x^{(k)} \} \to \infty \) as \( k \to \infty \).

Since \( I(x^{(k)}) \) is bounded and \( I'(x^{(k)}) \to 0 \) as \( k \to \infty \), there exists \( M_2 > 0 \) such that

\[
I(x^{(k)}) = \frac{1}{2} (x^{(k)})^T A x^{(k)} - \sum_{n=1}^{pm} b(n) [a \beta |x_n^{(k)}|^{\beta-2} x_n^{(k)} x_n + \nabla F(x_n^{(k)}) x_n] \leq M_2
\]

for all \( k \in \mathbb{N} \), and \( k \) large enough such that

\[
\langle I'(x^{(k)}), x^{(k)} \rangle = \langle A x^{(k)}, x \rangle - \sum_{n=1}^{pm} b(n) [a \beta |x_n^{(k)}|^{\beta-2} x_n^{(k)} x_n + \nabla F(x_n^{(k)}) x_n] \geq -||x||.
\]

Set \( y_n^{(k)} = x_n^{(k)}/||x^{(k)}|| \), then \( ||y^{(k)}|| = 1 \) and \( \{ y^{(k)} \} \) is bounded so that it has a subsequence, say \( y^{(k)} \), which converges to \( y^{(0)} \).

Assume \( y^{(k)} = \tilde{y}^{(k)} + \hat{y}^{(k)} \), where \( \hat{y}_n^{(k)} = \frac{1}{pm} \sum_{l=1}^{pm} y_l^{(k)} \), then \( \sum_{n=1}^{pm} \hat{y}_n^{(k)} = 0 \). It is easy to see that

\[
\tilde{y}^{(k)} \to \tilde{y}^{(0)}.
\]

By (3.1) and (3.2), we have

\[
\beta I(x^{(k)}) - \langle I'(x^{(k)}), x^{(k)} \rangle + \sum_{n=1}^{pm} b(n) \left[ \beta F(x_n^{(k)}) - \nabla F(x_n^{(k)}) \cdot x_n^{(k)} \right] = \left( \frac{\beta}{2} - 1 \right) \sum_{n=1}^{pm} (\Delta x_n^{(k)})^2
\]

for all \( k \in \mathbb{N} \).

On the other hand, by using Proposition 3.2, we have

\[
\sum_{n=1}^{pm} b(n) \left[ \beta W(x_n^{(k)}) - \nabla F(x_n^{(k)}) x_n^{(k)} \right] \leq \sum_{n=1}^{pm} B \left[ \beta M_1 + \frac{\alpha_1 \beta |x_n^{(k)}| + \alpha_1 r_1}{B(1 + |r_1 + \tau (x_n^{(k)} - r_1)|)} + \frac{\alpha_1 |x_n^{(k)}|}{B(1 + |x_n^{(k)}|)} \right]
\]

\[
\leq (\beta + 1) \alpha_1 \sum_{n=1}^{pm} |x_n^{(k)}| + pm \beta (\alpha_1 r_1 + M_1 B)
\]
for \( k \) large enough.

Thus we have

\[
\left( \frac{\beta}{2} - 1 \right) \sum_{n=1}^{pm} (\Delta x_n^{(k)})^2 \\
\leq \beta M_2 + \| x^{(k)} \| + (\beta + 1) \alpha_1 \sum_{n=1}^{pm} |x_n^{(k)}| + pm \beta (\alpha_1 r_1 + BM_1) \\
\leq \beta [M_2 + pm (\alpha_1 r_1 + BM_1)] + [1 + (\beta + 1) \alpha_1 \sqrt{pm}] \| x^{(k)} \|
\]

for \( k \) large enough, which implies that \( \| \Delta \tilde{y}^{(k)} \| \to 0 \), as \( k \to \infty \). Therefore

\[
4 \sin^2 \left( \frac{\pi}{pm} \right) \| \tilde{y}^{(k)} \| \leq \sum_{n=1}^{pm} (\Delta \tilde{y}_n^{(k)})^2
\]

implies that

\[
\| \tilde{y}^{(k)} \| \to 0 \tag{3.4}
\]

as \( k \to \infty \).

By (3.3) and (3.4), we have \( y^{(k)} \to \bar{y}^{(0)} = y^{(0)} \) as \( k \to \infty \) and \( \| y^{(0)} \| = 1 \). Choose \( y \) such that \( y \in E_{pm} \) and \( \| y \| < \infty \), it follows from (3.2) that

\[
a \beta \left| \sum_{n=1}^{pm} b(n) |{y}_n^{(k)}|^{-2} \{ y_n^{(k)} \} y_n \right| \\
\leq \| x^{(k)} \|^{1-\beta} \| I'(x^{(k)}) \| + \| x^{(k)} \|^{2-\beta} \{ A(x^{(k)}) \} y + \| x^{(k)} \|^{1-\beta} \\
\cdot \sum_{n=1}^{pm} b(n) \nabla F(x_n^{(k)}) y_n \\
\leq \| x^{(k)} \|^{1-\beta} \| y \| + \lambda_{\max} \| x^{(k)} \|^{2-\beta} \| y \| + \alpha_1 \sqrt{pm} \| x^{(k)} \|^{1-\beta} \| y \|
\]

\to 0

as \( k \to \infty \).

By the arbitrariness of \( y \), one has

\[
b(n) |{y}_n^{(0)}|^{-2} \{ y_n^{(0)} \} = 0
\]

for all \( n \in \mathbb{Z}(1, pm) \). Since \( y^{(0)} = \bar{y}^{(0)} \neq 0 \), we have \( b(n) \equiv 0 \) for all \( n \in \mathbb{Z}(1, pm) \), which contradicts the condition that \( b(n) \neq 0 \). Hence \( I(\cdot) \) satisfies the PS condition. The proof of Lemma 3.1 is complete. \( \square \)

**Lemma 3.2.** Under the conditions \((V)\) and \((F_1)\), the functional \( I(\cdot) \) satisfies condition \((1)\).

**Proof.** Under the assumptions of Theorem 1.1, it follows from (2.1)–(2.3) that for any \( x \in Y \),

\[
\| x \| \leq \sqrt{pmr_0}.
\]
\[ I(x) = \frac{1}{2} \sum_{n=1}^{pm} (\Delta x_n)^2 - \sum_{n=1}^{pm} \left[ ab(n)|x_n|^\beta + b(n)F(x_n) \right] \]
\[ = \frac{1}{2} x^T Ax - \sum_{n=1}^{pm} \left[ ab(n)|x_n|^\beta + b(n)F(x_n) \right] \]
\[ \geq \frac{1}{2} \lambda_{\min} \|x\|^2 - \sum_{n=1}^{pm} \left[ ab(n)|x_n|^\beta + b(n)F(x_n) \right] \]
\[ = \frac{1}{2} \sin^2 \left( \frac{\pi}{pm} \right) \|x\|^2 - aB \|x\|_\beta^\beta - B\alpha_0 \|x\|^2 \]
\[ = \left( \frac{1}{2} \sin^2 \left( \frac{\pi}{pm} \right) - B\alpha_0 \right) \|x\|^2 - aB \|x\|_\beta^\beta \]
\[ \geq \left( \frac{1}{2} \sin^2 \left( \frac{\pi}{pm} \right) - B\alpha_0 \right) \|x\|^2 - \frac{aB}{C^\beta} \|x\|_\beta^\beta. \]

Choose \( \rho \in (0, \sqrt{pmr_0}] \) such that
\[ \alpha_1 = \left( \frac{1}{2} \sin^2 \left( \frac{\pi}{pm} \right) - B\alpha_0 \right) \rho^2 - \frac{aB}{C^\beta} \rho^\beta > 0, \]
where \( \alpha_1 \) is defined in condition \((F_2)\) of Theorem 1.1. Then we have
\[ I(x) \geq \alpha_1 > 0 \] (3.5)
for all \( x \in Y \) and \( \|x\| = \rho \). Therefore the functional \( I(\cdot) \) satisfies condition (1). The proof of Lemma 3.2 is complete. \( \square \)

**Lemma 3.3.** Under the conditions \((B), (V) \) and \((F_2)\), the functional \( I(\cdot) \) satisfies condition (2).

**Proof.** Since \( \sum_{n=1}^{m} b(n) = 0 \) and \( b(n) \neq 0 \), there exist \( n_1, n_2 \in \mathbb{Z}(1, m) \) such that \( b(n_1) > 0 \) and \( b(n_2) < 0 \). Let \( N_1 = \{n: n \in \mathbb{Z}(1, m), b(n) > 0\}, N_2 = \{n: n \in \mathbb{Z}(1, m), b(n) < 0\} \).

Choose \( e \in Y \) such that \( \|e\| = 1, e_{n+m} = e_n \) for all \( n \in \mathbb{Z}, e(n) \neq 0 \) for all \( n \in N_1 \) and \( e(n) = 0 \) for all \( n \in N_2 \).

Since \( E_{pm} = Y \oplus Z \) and \( Az = 0 \) for all \( z \in Z \), it is easy to see that
\[ I(z + re) = \frac{1}{2} \left[ \langle A(z + re), z + re \rangle - \sum_{n=1}^{pm} \left[ ab(n)|z_n + re_n|^\beta + b(n)F(z_n + re_n) \right] \right] \]
\[ = \frac{r^2}{2} \langle Ae, e \rangle - \sum_{n=1}^{pm} \left[ ab(n)|v + re_n|^\beta + b(n)F(v + re_n) \right]. \] (3.6)

Let \( C_3 = \frac{1}{\sqrt{pm}+1}, \)
\[ R_1 = \left[ \frac{\lambda_{\max}}{aB \left( \sum_{n \in N_1} b(n) |e_n|^2 \right)} \right]^{\frac{1}{p-2}}. \]
Suppose that \( R_2 > 0 \) is a constant such that
\[
\frac{C_3^2 R_2^2}{2} \lambda_{\text{max}} - apC_3^\beta R_2^\beta \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \cdot \left[ \sum_{n \in N_1} b(n)e_n^2 \right]^{\beta/2}
+ \sqrt{pmC_3 R_2(\alpha_1 + BM_0)} = 0,
\]

and then for any \( R > R_2, \)
\[
\frac{C_3^2 R_2^2}{2} \lambda_{\text{max}} - apC_3^\beta R_2^\beta \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \cdot \left[ \sum_{n \in N_1} b(n)e_n^2 \right]^{\beta/2}
+ \sqrt{pmC_3 R(\alpha_1 + BM_0)} < 0.
\]

Define \( Q(R) = \{ z + re : z = \{ v \} \in Z, \ |v| \leq R, \ r \in [0, C_3 R] \}, \) where
\[
R \geq \max \left\{ R_1, R_2, \frac{r_1}{1 - C_3}, \sqrt{pmC_3^{-1} r_0} \right\},
\]
then \( \partial Q = G_1 \cup G_2 \cup G_3, \) where
\[
G_1 = \{ z \in Z : |v| \leq R \},
G_2 = \{ z + re : |v| = R, \ r \in [0, C_3 R] \},
G_3 = \{ z + re : |v| \leq R, \ r = C_3 R \}.
\]

For any \( z \in G_1, \) one has
\[
I(z) = \frac{1}{2} z^T A z - \sum_{n=1}^{pm} [ab(n)|v|^\beta + b(n)F(v)] = - \sum_{n=1}^{pm} b(n)V(v) = 0.
\] (3.7)

For all \( z + re \in G_2, \) it follows from (3.7) and the choose of \( e \) that
\[
I(z + re) = \frac{1}{2} r^2 e^T A e - p \sum_{n \in N_1} b(n) [V(v + re_n) - V(v)] = I_1 + I_2,
\]
where
\[
I_1 = \frac{1}{2} r^2 e^T A e - ap \sum_{n \in N_1} b(n)[|v + re_n|^\beta - |v|^\beta],
I_2 = -p \sum_{n \in N_1} b(n) [F(v + re_n) - F(v)].
\]

It follows from Hölder inequality that
\[
\sum_{n \in N_1} b(n)|v + re_n|^\beta \geq \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \cdot \left[ \sum_{n \in N_1} b(n)|v + re_n|^2 \right]^{\beta/2}
\]
for all \( v \in \mathbb{R} \) and \( r \geq 0. \)

Hence we have
\[
I_1 \leq \frac{r^2}{2} \lambda_{\text{max}} + apR^\beta \sum_{n \in N_1} b(n) - ap \left[ \sum_{n \in N_1} b(n) (R^2 + r^2 e_n^2) \right]^{\beta/2} \cdot \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}}
\triangleq g(r)
\]
for all $|v| = R$ and $r \geq 0$.

By a direct computation, we have

$$
g'(r) = r\lambda_{\text{max}} - apr\beta \left[ \sum_{n \in N_1} b(n) \left( R^2 + r^2 |e_n|^2 \right)^{\frac{\beta-2}{2}} \cdot \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \cdot \left( \sum_{n \in N_1} b(n) |e_n|^2 \right) \right].
$$

In view of $\beta > 2$, it follows from the choice of $R_1$ that $g'(r) \leq 0$ for all $0 \leq r \leq C_3R$ and $R \geq R_1$.

It is obvious that $g(0) = 0$, hence we have $g(r) \leq 0$ for all $0 \leq r \leq C_3R$ and $R \geq R_1$, so that

$$I_1 \leq g(r) \leq 0 \quad (3.8)$$

for all $0 \leq r \leq C_3R$ and $R \geq R_1$.

On the other hand, it follows from the mean value theorem that there exists $\xi_n \in [0, r]$ such that

$$I_2 \leq p \sum_{n \in N_1} b(n) \cdot |\nabla F(v + \xi_n e_n)| \cdot |re_n| \leq rpB \sum_{n \in N_1} |\nabla F(v + \xi_n e_n)| \cdot |e_n|.$$

Since $|v + \xi_n e_n| \geq |v| - |\xi_n e_n| \geq R - r \geq R - C_3R \geq r_1$, we have

$$I_2 \leq pBr \sum_{n \in N_1} \frac{\alpha_1 |e_n|}{B(1 + |v + \xi_n e_n|)} \leq \frac{pr\alpha_1}{1 + R - r} \sum_{n \in N_1} |e_n| \leq \frac{pr\alpha_1 \sqrt{pm}}{p(1 + R - r)} \leq \frac{\alpha_1}{2}.$$

Hence we have, by (3.8) and (3.9),

$$I(z + re) \leq \frac{\alpha_1}{2} \quad (3.10)$$

for all $z + re \in G_2$.

For all $z + re \in G_3$, by the Hölder inequality, we have

$$I_1 \leq \frac{r^2}{2} \lambda_{\text{max}} + ap|v|^\beta \sum_{n \in N_1} b(n) - ap \left[ \sum_{n \in N_1} b(n) \left( |v|^2 + r^2 |e_n|^2 \right)^{\frac{\beta}{2}} \cdot \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \cdot \left( \sum_{n \in N_1} b(n) |e_n|^2 \right)^{\frac{\beta}{2}} \right].$$

By Proposition 3.1, there exists $M_0 > 0$ such that

$$|\nabla F(x)| \leq \frac{\alpha_1}{B(1 + |x|)} + M_0, \quad \forall x \in \mathbb{R}.$$
Then there exists $\xi_n \in [0, r]$ such that
\[
I_2 \leq p \sum_{n \in N_1} b(n) \cdot |\nabla F(v + \xi_n e_n)| \cdot |r e_n| \\
\leq r p B \sum_{n \in N_1} \left[ \frac{\alpha_1}{B(1 + |v + \xi_n e_n|)} + M_0 \right] \cdot |e_n| \leq r \sqrt{pm(\alpha_1 + BM_0)}. \tag{3.12}
\]

By (3.11) and (3.12), we have
\[
I(z + re) \leq \frac{r^2}{2} \lambda_{\text{max}} - a p r^{\beta} \left[ \sum_{n \in N_1} b(n) \right]^{\frac{2-\beta}{2}} \left[ \sum_{n \in N_1} b(n)|e_n|^2 \right]^{\frac{\beta}{2}} + r \sqrt{pm(\alpha_1 + BM_0)} \equiv q(C_3 R)
\]
for all $|v| \leq R$ and $r = C_3 R$.

As $\beta > 2$, it follows from the choose of $R_2$ that $q(C_3 R_2) = 0$ and $q(C_3 R) < 0$ for all $R > R_2$, i.e.,
\[
I(z + re) \leq 0, \quad \forall z + re \in G_3 \tag{3.13}
\]
for all $R \geq R_2$.

It follows from (3.7), (3.10) and (3.13) that
\[
I_{\partial Q} \leq \frac{\alpha_1}{2}.
\]

Hence the functional $I(\cdot)$ satisfies condition (2). The proof of Lemma 3.3 is complete. \(\square\)

Immediately, Theorem 1.1 can be proved by Lemmas 3.1–3.3.

Finally, we give an example to illustrate our result.

**Example 3.1.** For a given integer $m = 4$, consider the difference equation
\[
x_{n+1} - 2x_n + x_{n-1} + b(n) \left[ a|x_n|^\beta + \frac{4dx_n^3}{(1 + x_n^2)^2} \right] = 0, \tag{3.14}
\]
where
\[
b(n) = \begin{cases} 
bn \sin^2\left(\frac{\pi}{4p}\right), & 0 \leq n \leq 1, \\
bn \sin^2\left(\frac{\pi}{4p}\right) + 2bn \sin^2\left(\frac{\pi}{4p}\right), & 1 \leq n \leq 3, \\
b(n) = bn \sin^2\left(\frac{\pi}{4p}\right) - 4bn \sin^2\left(\frac{\pi}{4p}\right), & 3 \leq n \leq 4,
\end{cases}
\]
and $b(4k + n) = b(n)$ for any $k, n \in \mathbb{Z}$, and $b > 0$ is a constant. Let $a > 0$, $\beta > 2$ and
\[
|d| = \min\left\{\alpha_0, \frac{\alpha_1}{2b \sin^2\left(\frac{\pi}{4p}\right)}\right\},
\]
\[
r_0 = \frac{1}{\sqrt{4p}} < 1 \quad \text{and} \quad r_1 > 1. \quad \text{Choose} \quad \alpha_0 \in (0, 2b^{-1}) \quad \text{and} \quad \rho \in (0, 1) \quad \text{such that}
\]
\[
h(\rho) = \left[ (2 - b\alpha_0)\rho^2 - \frac{ab}{C_1^\beta} \rho^\beta \right] \sin^2\left(\frac{\pi}{4p}\right) > 0.
\]
It is easy to see that this equation satisfies conditions (B) and (V). Then it suffices to verify the conditions (F1) and (F2). By a direct computation, we have

\[ |F(x)| = \frac{|d|x^4}{1 + x^4} \leq |d|x^4 \leq |d||x|^2 \leq \alpha_0|x|^2, \quad \forall |x| \leq r_0 < 1, \]

and

\[ |\nabla F(x)| = \frac{4|d| \cdot |x|^3}{(1 + x^4)^2} = \frac{4|d|}{|x|^5 + 2|x| + |x|^{-3}} \leq \frac{|d|}{|x|} \leq \frac{2|d|}{1 + |x|} \leq \frac{\alpha_1}{b \sin^2(\pi/4p)(1 + |x|)} \]

for all \(|x| \geq r_1 > 1\). Hence Eq. (3.14) possesses at least one nontrivial \(4p\)-periodic solution for all positive integers \(p\).

References