Letter to the Editor

A note on preconditioned AOR method for $L$-matrices

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Abstract

In this note, some errors in a recent article by Li et al. (Improvements of preconditioned AOR iterative methods for $L$-matrices, J. Comput. Appl. Math. 206 (2007) 656–665) are pointed out and some correct results are presented.

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1. Introduction

In this note, we consider the following linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n,$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. The basic iterative method for solving the linear system (1) can be expressed as

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \quad k = 0, 1, \ldots,$$

where $x_0$ is an initial vector and $A = M - N$ is a splitting of $A$. The $M^{-1}N$ is called an iteration matrix of the basic iterative method.

Throughout the paper, let $A = I - L - U$, where $I$ is the identity matrix, and $L$ and $U$ are strictly lower triangular and strictly upper triangular matrices, respectively. Then the iteration matrix of the AOR iterative method [2] for solving the linear system (1) is

$$T_{r,\omega} = (I - rL)^{-1}((1 - \omega)I + (\omega - r)L + \omega U),$$

where $\omega$ and $r$ are real parameters with $\omega \neq 0$.

We now transform the original linear system (1) into the preconditioned linear system

$$PAx = Pb,$$

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where \( P \) is called a preconditioner. Then the basic iterative method for solving the linear system (4) is

\[
x_{k+1} = M_p^{-1}N_p x_k + M_p^{-1} P b, \quad k = 0, 1, \ldots,
\]

where \( x_0 \) is an initial vector and \( PA = M_p - N_p \) is a splitting of \( PA \).

The preconditioner \( P \) introduced in [3] is of the form \( P = I + S_x \), where

\[
S_x = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & 0 & \ldots & 0
\end{pmatrix}
\]

and \( x \) is a real parameter. In this paper, we consider the preconditioned linear system of the form

\[
\tilde{A}x = \tilde{b},
\]

where \( \tilde{A} = (I + S_x)A \) and \( \tilde{b} = (I + S_x)b \). Let \( S_x U = D' + L' \), where \( D' \) is a diagonal matrix and \( L' \) is a strictly lower triangular matrix. Then, from \( S_x L = 0 \) we have

\[
\tilde{A} = (I + S_x)(I - L - U) = I - L - U + S_x - S_x U = \tilde{D} - \tilde{L} - \tilde{U},
\]

where \( \tilde{D} = I - D' \), \( \tilde{L} = L - S_x + L' \) and \( \tilde{U} = U \). If we apply the AOR iterative method to the preconditioned linear system (6), then we get the preconditioned AOR iterative method whose iteration matrix is

\[
\tilde{T}_{r_0} = (\tilde{D} - r \tilde{L})^{-1}((1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega \tilde{U}).
\]

This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results. In Section 3, we provide correct results for some errors given in [3].

2. Preliminaries

A matrix \( A = (a_{ij}) \in \mathbb{R}^{n\times n} \) is called a Z-matrix if \( a_{ij} \leq 0 \) for \( i \neq j \), and it is called an L-matrix if \( A \) is a Z-matrix and \( a_{ii} > 0 \) for \( i = 1, 2, \ldots, n \). For a vector \( x \in \mathbb{R}^n \), \( x \geq 0 \) denotes that all components of \( x \) are nonnegative (positive). For two vectors \( x, y \in \mathbb{R}^n \), \( x \geq y \) means that \( x - y \geq 0 \) (\( x - y > 0 \)). These definitions carry immediately over to matrices. For a square matrix \( A \), \( \rho(A) \) denotes the spectral radius of \( A \), and \( A \) is called irreducible if the directed graph of \( A \) is strongly connected [4].

Some useful results which we refer to later are provided below.

**Theorem 2.1** (Varga [4]). Let \( A \geq 0 \) be an irreducible matrix. Then

(a) \( A \) has a positive eigenvalue equal to \( \rho(A) \).
(b) \( A \) has an eigenvector \( x > 0 \) corresponding to \( \rho(A) \).
(c) \( \rho(A) \) is a simple eigenvalue of \( A \).

**Theorem 2.2** (Berman and Plemmons [1]). Let \( A \geq 0 \) be a matrix. Then the following hold:

(a) If \( Ax \geq \beta x \) for a vector \( x \geq 0 \) and \( x \neq 0 \), then \( \rho(A) \geq \beta \).
(b) If \( Ax \leq \gamma x \) for a vector \( x > 0 \), then \( \rho(A) \leq \gamma \). Moreover, if \( A \) is irreducible and if \( \beta x \leq Ax \leq \gamma x \), equality excluded, for a vector \( x \geq 0 \) and \( x \neq 0 \), then \( \beta < \rho(A) < \gamma \) and \( x > 0 \).

Some results in [3] which have some errors are given below.

**Lemma 2.3** ([3, Lemma 3]). Let \( T_{r_0} \) and \( \tilde{T}_{r_0} \) be defined by (3) and (8). If \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \) and \( A \) is an irreducible L-matrix with \( 0 < a_{1n}a_{n1} < \alpha(\alpha \geq 1) \), then \( T_{r_0} \) and \( \tilde{T}_{r_0} \) are nonnegative and irreducible.
Theorem 2.4 ([3, Theorem 1]). Let \( T_{r,\omega} \) and \( \tilde{T}_{r,\omega} \) be defined by (3) and (8). If \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \) and \( A \) is an irreducible \( L \)-matrix with \( 0 < a_{11}a_{13} < \alpha(\alpha \geq 1) \), then

(a) \( \rho(\tilde{T}_{r,\omega}) < \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) < 1 \).
(b) \( \rho(\tilde{T}_{r,\omega}) = \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) = 1 \).
(c) \( \rho(\tilde{T}_{r,\omega}) > \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) > 1 \).

3. Correct results

Lemma 2.3 is correct for \( \alpha > 1 \), but it is not correct for \( \alpha = 1 \). More specifically, it is not true that \( \tilde{T}_{r,\omega} \) is irreducible for \( \alpha = 1 \). The following example shows that Lemma 2.3 is not true for \( \alpha = 1 \).

Example 3.1. Consider a \( 3 \times 3 \) matrix \( A \) of the form

\[
A = \begin{pmatrix}
1 & -0.1 & -0.1 \\
0 & 1 & -0.1 \\
-0.1 & 0 & 1
\end{pmatrix}.
\]

Clearly, \( A \) is an irreducible \( L \)-matrix and \( 0 < a_{13}a_{31} < \alpha = 1 \). Then

\[
\tilde{A} = (I + S_1)A = \begin{pmatrix}
1 & -0.1 & -0.1 \\
0 & 1 & -0.1 \\
0 & -0.01 & 0.99
\end{pmatrix}.
\]

Assume that \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \). Since \( \tilde{A} = \tilde{D} - \tilde{L} - \tilde{U} \), from (8) one obtains

\[
\tilde{T}_{r,\omega} = \begin{pmatrix}
1 - \omega & 0.1\omega & 0.1\omega \\
0 & 1 - \omega & 0.1\omega \\
0 & \omega(1 - r)/99 & 1 - \omega + r\omega/990
\end{pmatrix},
\]

which is clearly nonnegative, but reducible.

Notice that Theorem 2.4 was proved in [3] under the assumption that \( \tilde{T}_{r,\omega} \) is irreducible. Since it is not true that \( \tilde{T}_{r,\omega} \) is irreducible for \( \alpha = 1 \), Theorem 2.4 is not generally true for \( \alpha = 1 \). The correct result corresponding to Theorem 2.4 for \( \alpha = 1 \) is provided in the following theorem.

Theorem 3.2. Let \( T_{r,\omega} \) and \( \tilde{T}_{r,\omega} \) be defined by (3) and (8). Assume that \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \) and \( \alpha = 1 \). If \( A \) is an irreducible \( L \)-matrix with \( 0 < a_{11}a_{13} < 1 \), then

(a) \( \rho(\tilde{T}_{r,\omega}) \leq \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) < 1 \).
(b) \( \rho(\tilde{T}_{r,\omega}) = \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) = 1 \).
(c) \( \rho(\tilde{T}_{r,\omega}) \geq \rho(T_{r,\omega}) \) if \( \rho(T_{r,\omega}) > 1 \).

Proof. From Lemma 2.3, \( T_{r,\omega} \) is nonnegative and irreducible. Thus, from Theorem 2.1 there exists a vector \( x > 0 \) such that \( T_{r,\omega}x = \lambda x \), where \( \lambda = \rho(T_{r,\omega}) \). From \( T_{r,\omega}x = \lambda x \) and \( S_2L = 0 \), one easily obtains

\[
((1 - \omega)I + (\omega - r)L + \omega U)x = \lambda(I - rL)x,
\]

\[
\omega S_2Ux = (\lambda + \omega - 1)S_2x.
\]

(9)
Using (7) and (9),
\[ \bar{T}_{r_o}x - \lambda x = (\bar{D} - r \bar{L})^{-1}((1 - \omega)\bar{D} + (\omega - r)\bar{L} + \omega\bar{U} - \lambda(\bar{D} - r \bar{L}))x \]
\[ = (\bar{D} - r \bar{L})^{-1}((1 - \omega - \lambda)\bar{D} + (\omega - r + \lambda r)\bar{L} + \omega\bar{U})x \]
\[ = (\bar{D} - r \bar{L})^{-1}((\omega + \lambda - 1)D' + (\omega - r + \lambda r)(L' - S_2))x \]
\[ = (\bar{D} - \omega \bar{L})^{-1}((\lambda - 1)(D' + r L') + (-\lambda r + \lambda + r - 1)S_2)x \]
\[ = (\lambda - 1)(\bar{D} - r \bar{L})^{-1}(D' + r L' + (1 - r)S_2)x. \] (10)

Notice that \( \bar{D} \), \( \bar{L} \), \( D' \), \( L' \) and \( S_2 \) are all nonnegative by assumption. If \( \lambda < 1 \), then from (10) \( \bar{T}_{r_o}x \leq \lambda x \). Since \( x > 0 \), Theorem 2.2 implies that \( \rho(\bar{T}_{r_o}) \leq \lambda = \rho(T_{r_o}) \). For the cases of \( \lambda = 1 \) and \( \lambda > 1 \), \( \bar{T}_{r_o}x = \lambda x \) and \( \bar{T}_{r_o}x \geq \lambda x \) are obtained from (10), respectively. Hence, the theorem follows from Theorem 2.2. \( \square \)

**Remark 3.3.** In Theorem 3.2, \( \bar{T}_{r_o} \) is not necessarily irreducible for \( z = 1 \). Hence, we cannot conclude that the strict inequality holds in Theorem 3.2.

The following lemma shows that Lemma 2.3 holds for \( z = 1 \) if a special condition is added.

**Lemma 3.4.** Let \( T_{r_o} \) and \( \bar{T}_{r_o} \) be defined by (3) and (8). Assume that \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \) and \( z = 1 \). If \( A \) is an \( L \)-matrix with \( \alpha < a_{1n}a_{n1} < 1 \) and \( A \) is irreducible even for \( a_{n1} \) set to 0, then \( T_{r_o} \) and \( \bar{T}_{r_o} \) are nonnegative and irreducible.

**Proof.** Since \( A \) is irreducible even for \( a_{n1} \) set to 0, \( A = \bar{I} - L - U \) is irreducible and \( \bar{A} = (I + S_1)A = \bar{D} - \bar{L} - \bar{U} \) is also irreducible. Since \( A \) is an irreducible \( L \)-matrix, it is easy to show that \( T_{r_o} \) is nonnegative and irreducible. By assumption, \( \bar{D} \), \( \bar{L} \) and \( \bar{U} \) are all nonnegative and thus \( \bar{T}_{r_o} \) is nonnegative. Note that \( \bar{T}_{r_o} \) can be expressed as
\[ \bar{T}_{r_o} = (1 - \omega)I + \omega(1 - r)\bar{D}^{-1}\bar{L} + \omega\bar{D}^{-1}\bar{U} + \bar{H}, \] (11)
where \( \bar{H} \) is a nonnegative matrix. Since \( \omega \neq 0, r \neq 1 \) and \( \bar{A} \) is irreducible, \( \omega(1 - r)\bar{D}^{-1}\bar{L} + \omega\bar{D}^{-1}\bar{U} \) is irreducible. Hence, \( \bar{T}_{r_o} \) is irreducible from (11). \( \square \)

From Lemma 3.4, one obtains the following theorem which shows that Theorem 2.4 holds for \( z = 1 \) if a special condition is added.

**Theorem 3.5.** Let \( T_{r_o} \) and \( \bar{T}_{r_o} \) be defined by (3) and (8). Assume that \( 0 \leq r \leq \omega \leq 1(\omega \neq 0, r \neq 1) \) and \( z = 1 \). If \( A \) is an \( L \)-matrix with \( \alpha < a_{1n}a_{n1} < 1 \) and \( A \) is irreducible even for \( a_{n1} \) set to 0, then

(a) \( \rho(\bar{T}_{r_o}) < \rho(T_{r_o}) \) if \( \rho(T_{r_o}) < 1 \).

(b) \( \rho(\bar{T}_{r_o}) = \rho(T_{r_o}) \) if \( \rho(T_{r_o}) = 1 \).

(c) \( \rho(\bar{T}_{r_o}) > \rho(T_{r_o}) \) if \( \rho(T_{r_o}) > 1 \).

**References**


