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On the divisibility of meet and join matrices

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Abstract

Let $(P, \leqslant) = (P, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a meet-closed subset of P and let $f : P \to \mathbb{Z}_+$ be a function. We characterize the matrix divisibility of the join matrix $[S]_f = [f(x_i \vee x_j)]$ by the meet matrix $(S)_f = [f(x_i \wedge x_j)]$ in the ring $\mathbb{Z}^{n\times n}$ in terms of the usual divisibility in $\mathbb{Z}$, and we present two algorithms for constructing certain classes of meet-closed sets S such that $(S)_f$ divides $[S]_f$. As an example we present the lattice-theoretic structure of all meet-closed sets with at most five elements possessing the matrix divisibility property. Finally, we show that our methods solve some open problems in the divisor lattice, concerning the divisibility of GCD and LCM matrices.

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1. Introduction

Let $(P, \leqslant) = (P, \wedge, \vee)$ be a lattice, let $S = \{x_1, x_2, \ldots, x_n\}$ be a subset of P and let $f : P \to \mathbb{C}$ be a function. The meet matrix $(S)_f$ and the join matrix $[S]_f$ on S with respect to f are defined by $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$.

Bhat [22] and Haukkanen [6] introduced meet matrices and Korkee and Haukkanen [19] introduced join matrices. Explicit formulae for the determinant and the inverse of meet and join matrices are presented in [6,18,19,22] (see also [2,16,24]). Most of these formulae are presented

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on meet-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$) and join-closed sets S (i.e., $x_i, x_j \in S \Rightarrow x_i \vee x_j \in S$). Recently Korkee and Haukkanen [20] presented a method for calculating $\det(S)_f$, $(S)_f^{-1}$, $\det[S]_f$ and $[S]_f^{-1}$ on all sets S and functions f.

It is well known that $(\mathbb{Z}_+, |) = (\mathbb{Z}_+, \gcd, \mathrm{lcm})$ is a lattice, where $|$ is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet and join matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$ and LCM matrices $([S]_f)_{ij} = f(\mathrm{lcm}(x_i, x_j))$. The study of GCD and LCM matrices is considered to have begun in 1876, when Smith [26] presented his famous determinant formulae. For general accounts of GCD and LCM matrices, see [10,19]. The GCUD and LCUM matrices, which are unitary analogies of GCD and LCM matrices, are also special cases of meet and join matrices, see [9,17].

Bourque and Ligh [4,5] were the first to study the divisibility of GCD and LCM matrices in the ring $\mathbb{Z}^{n\times n}$ (i.e., when $[S]_f = M(S)_f$ for some $M \in \mathbb{Z}^{n\times n}$). Hong [11–13] has studied this subject extensively. See also [8].

In this paper we study the divisibility of meet and join matrices, the subject of Bourque, Ligh and Hong in a more general level. We present a characterization for the matrix divisibility of the join matrix by the meet matrix in the ring $\mathbb{Z}^{n\times n}$ in terms of the usual divisibility in $\mathbb{Z}$, where S is a meet-closed set and f is an integer-valued function on P (see Theorem 3.1). We also present two inductive algorithms for constructing certain classes of lattice-theoretic structures of meet-closed sets S such that $(S)_f$ divides $[S]_f$ under certain conditions on f (see Theorem 3.2). For example, all chains and x_1-sets (i.e., $x_i \wedge x_j = x_1$ for all $i \neq j$) can be constructed using our algorithms, and thus they possess this divisibility property. All meet-closed sets satisfying the divisibility property can be divided into two classes: those that can be constructed using our algorithms in Theorem 3.2 and those that should be treated otherwise, for example using our Theorem 3.1. As an example we find for all meet-closed sets S with at most five elements a necessary and sufficient condition on f for the divisibility property; we classify the conditions on f on the basis of the lattice-theoretic structure of S. Finally, the new contributions of this study to the divisor lattice are described in Section 5. For example, we show that Conjecture 3.1 in Hong [15] holds.

2. Preliminaries

Let $(P, \leqslant)$ be a locally finite poset and let g be a complex-valued function on $P \times P$ such that $g(x, y) = 0$ whenever $x \nleqslant y$. We say that g is an incidence function of P. If g and h are incidence functions of P, their sum $g + h$ is defined by $(g + h)(x, y) = g(x, y) + h(x, y)$ and their convolution $g * h$ is defined by $(g * h)(x, y) = \sum_{x\leqslant z\leqslant y} g(x, z)h(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the unity δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The incidence function ζ is defined by $\zeta(x, y) = 1$ if $x \leqslant y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ (with respect to convolution). On the basis of the recursive property [27, p. 116] the values of μ are always integers.

Definition 2.1. We say that f is an order-preserving function from the poset $(P, \leqslant)$ into the poset $(Q, \sqsubseteq)$ if

$$x \leqslant y \Rightarrow f(x) \sqsubseteq f(y) \tag{2.1}$$

for all $x, y \in P$.

Throughout the remainder of this paper we set $(P, \leqslant) = (P, \wedge, \vee)$ to be a lattice such that all principal order ideals of P are finite, f to be a complex-valued function on P, and S to be a finite subset of P, where $S = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_j \Rightarrow i < j$. The assumptions imply that P has the least element, which we denote by 0.

We say that S is an a-set if $x_i \wedge x_j = a$ for all $i \neq j$. We say that S is lower-closed if $(x_i \in S, y \in P, y \leqslant x_i) \Rightarrow y \in S$. We say that S is meet-closed if $x_i, x_j \in S \Rightarrow x_i \wedge x_j \in S$. It is clear that a lower-closed set is always meet-closed but the converse need not hold.

Definition 2.2. We say that f is a semi-multiplicative function on P if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{2.2}$$

for all $x, y \in P$.

The order-preserving property is a poset-theoretic concept [3], but it also appears in number theory. For example, integer-valued totients possess the order-preserving property $x|y \Rightarrow f(x)|f(y)$, see [7]. For semi-multiplicative arithmetical functions, see [23, p. 49] or [25, p. 237]. Note that totients are also semi-multiplicative, and all completely multiplicative arithmetical functions [21,25] are totients in the sense of [7].

Definition 2.3. The $n \times n$ matrices $(S)_f$ and $[S]_f$, where $((S)_f)_{ij} = f(x_i \wedge x_j)$ and $([S]_f)_{ij} = f(x_i \vee x_j)$, are called the meet and the join matrix on S with respect to f.

Let $\mathbb{Z}^{n\times m}$ denote the set of $n \times m$ matrices with integer elements. If $A \in \mathbb{Z}^{n\times m}$, then $(A)_{(i)}$ and $(A)_j$ denote the ith row and the jth column of A, respectively. Note that by $\mathbf{1}_n$ and $\mathbf{0}_n$ we denote the $1 \times n$ row vectors $\mathbf{1}_n = (1, 1, \ldots, 1)$ and $\mathbf{0}_n = (0, 0, \ldots, 0)$. For $a \in \mathbb{Z}$ we denote $a|A$ if $a|(A)_{ij}$ for all i and j.

Definition 2.4. Let $A, B \in \mathbb{Z}^{n\times n}$. We say that A divides B (in the ring $\mathbb{Z}^{n\times n}$ under addition and multiplication of matrices), written as $A|B$, if there exists $M \in \mathbb{Z}^{n\times n}$ such that $B = MA$.

Note that since $\mathbb{Z}^{n\times n}$ is not a commutative ring, it matters on which side of A the matrix M occurs in Definition 2.4. If A and B are symmetric, then clearly $B = MA \Leftrightarrow B = AM^{\mathrm{T}}$. In this paper we consider meet and join matrices, and these are symmetric matrices.

We associate each $f(z)$ with the incidence function $f(0, z)$. Thus by the notation $(f * \mu)(z)$ we mean the convolution $(f * \mu)(0, z) = \sum_{0\leqslant w\leqslant z} f(0, w)\mu(w, z)$. Let S be meet-closed and define

$$\Delta_{S,k} = \sum_{z\leqslant x_k;\, z\nleqslant x_1,\ldots,x_{k-1}} (f * \mu)(z) \tag{2.3}$$

for all $x_k \in S$. Haukkanen [6] shows that $(S)_f$ is invertible if and only if $\Delta_{S,k} \neq 0$ for all $x_k \in S$. Moreover, if S is a lower-closed set, then $\Delta_{S,k} = (f * \mu)(x_k)$.

Let g be an incidence function of P. By g_S we denote the restriction of g on $S \times S$. By [1, p. 139] we can associate each incidence function g_S uniquely with the $n \times n$ upper triangular matrix g_S, where $(g_S)_{ij} = g_S(x_i, x_j)$. Note that by μ_S we do not mean $(\zeta^{-1})_S$ but the Möbius function $\mu_S = (\zeta_S)^{-1}$ of S. Further, Korkee [17, Theorem 2] obtains a representation of μ_S for meet-closed sets S in terms of μ of P, where

$$\mu_S(x_i, x_j) = \sum_{x_i \leqslant z \leqslant x_j;\, z \nleqslant x_i, \ldots, x_{j-1}} \mu(x_i, z). \tag{2.4}$$

The values of μ_S are also integers.

3. General results

In this section we examine the divisibility of $[S]_f$ by $(S)_f$ in $\mathbb{Z}^{n\times n}$, and therefore we assume that f is an integer-valued function on P.

In Theorem 3.1 we characterize $(S)_f | [S]_f$ on meet-closed sets in terms of the usual divisibility of integers.

Theorem 3.1. *Let S be a meet-closed set such that $(S)_f$ is invertible (i.e., $\Delta_{S,j} \neq 0$ for all $x_j \in S$, where the $\Delta_{S,j}$'s are as given in (2.3)). Then $(S)_f | [S]_f$ if and only if $\Delta_{S,j} | ([S]_f \mu_S)_j$ for all $j = 1, 2, \ldots, n$.*

Proof. Let S be meet-closed and let $(S)_f^{-1}$ exist. By [18, Theorem 7.1] we have $(S)_f^{-1} = \mu_S \Delta^{-1} \mu_S^{\mathrm{T}}$, where $\Delta = \mathrm{diag}(\Delta_{S,1}, \Delta_{S,2}, \ldots, \Delta_{S,n})$. Note that μ_S^{T}, $[S]_f \mu_S \in \mathbb{Z}^{n\times n}$ and $\zeta_S^{\mathrm{T}} = (\mu_S^{\mathrm{T}})^{-1} \in \mathbb{Z}^{n\times n}$. Now the following statements are equivalent.

(a) $(S)_f | [S]_f$,
(b) $[S]_f (S)_f^{-1} = [S]_f \mu_S \Delta^{-1} \mu_S^{\mathrm{T}} \in \mathbb{Z}^{n\times n}$,
(c) $[S]_f \mu_S \Delta^{-1} = [S]_f \mu_S \Delta^{-1} \mu_S^{\mathrm{T}} \zeta_S^{\mathrm{T}} \in \mathbb{Z}^{n\times n}$,
(d) $\Delta_{S,j} | ([S]_f \mu_S)_j$ for all $j = 1, 2, \ldots, n$. □

In the following results we have to make further assumptions on f. Clearly it suffices that the assumptions hold at least on those sets S that we are dealing with. However, we construct sets S inductively and we do not want to mention every time that the assumptions for f should also hold for the extended sets. Thus we state the following (excessively strong) assumption.

Remark 3.1. Unless otherwise stated, we let f be an order-preserving and semi-multiplicative function from $(P, \leqslant)$ into $(\mathbb{Z}_+, |)$, and all sets (specified by x_i's) are written so that $x_i < x_j \Rightarrow i < j$.

We next examine which elements of P, denoted as x_{n+1}, could be adjoined to S so that the divisibility also holds for the extended set $S \cup \{x_{n+1}\}$. The following Theorem 3.2 contains two construction methods based on the following concept of an admissible partition.

Definition 3.1. We say that $S \cup \{x_{n+1}\}$ is an admissible partition with binding element x_p if $S \cup \{x_{n+1}\}$ is meet-closed and there exists an element $x_p \in S$ such that $x_1, x_2, \ldots, x_{p-1} < x_p < x_{n+1}$ and $x_{p+1}, \ldots, x_n \nleqslant x_{n+1}$. (Note that the binding element is unique for fixed x_{n+1}.) See Fig. 3.1 (M_1).

Theorem 3.2. *Let S be a meet-closed set such that $(S)_f$ is invertible (i.e., $\Delta_{S,j} \neq 0$ for all $x_j \in S$). Consider the following construction methods.*

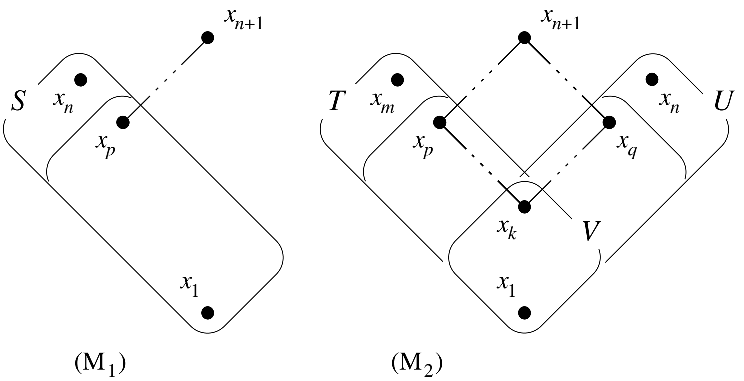

Fig. 3.1. Construction methods.

(M$_1$) *Let* $S' = S \cup \{x_{n+1}\}$ *be an admissible partition with the binding element* x_p *and let* $(S')_f$ *be invertible (i.e.,* $\Delta_{S,j} \neq 0$ *for all* $x_j \in S$ *and* $f(x_{n+1}) \neq f(x_p)$*). Then* $(S)_f|[S]_f \Leftrightarrow (S')_f|[S']_f$.

(M$_2$) *Let* $S'' = S \cup \{x_{n+1}\} = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_m, x_{m+1}, \ldots, x_n, x_{n+1}\}$ *be a meet-closed set, where* $V = \{x_1, \ldots, x_k\}$, $T = V \cup \{x_{k+1}, \ldots, x_m\}$ *and* $U = V \cup \{x_{m+1}, \ldots, x_n\}$, *and let* $x_k = \max V$ *and* $x_i \wedge x_j \in V$ *whenever* $x_i \in T, x_j \in U$. *Let* $T \cup \{x_{n+1}\}$ *be an admissible partition with the binding element* $x_p \in T\backslash V$ *and let* $U \cup \{x_{n+1}\}$ *be an admissible partition with the binding element* $x_q \in U\backslash V$, *where* $x_k = x_p \wedge x_q$ *and* $x_{n+1} = x_p \vee x_q$. *Further, let* $x_p = x_k \vee x_i$ *for* $k < i \leqslant p$, *let* $x_q = x_k \vee x_j$ *for* $m < j \leqslant q$, *and let* $x_k \leqslant x_i \leqslant x_{n+1} \Rightarrow x_i \in \{x_k, x_p, x_q, x_{n+1}\}$ *hold. Let* $(S'')_f$ *be invertible (i.e.,* $\Delta_{S,j} \neq 0$ *for all* $x_j \in S$ *and* $f(x_{n+1}) - f(x_p) - f(x_q) + f(x_k) \neq 0$*). Then* $(S)_f|[S]_f \Leftrightarrow (S'')_f|[S'']_f$.

In Fig. 3.1 we illustrate the idea of Theorem 3.2. The method (M$_1$) allows us to insert (or remove) an element above a binding element so that divisibility remains unchanged. The method (M$_2$) allows us to join together (or separate) two incomparable binding elements.

Proof of Theorem 3.2. We first prove (M$_1$). Consider $[S']_f\mu_{S'}$. By using the partitioned matrices [28, p. 36] we have

$$\zeta_{S'} = \begin{bmatrix} \zeta_S & \mathbf{e}^{\mathrm{T}} \\ \mathbf{0}_n & 1 \end{bmatrix}, \quad \mu_{S'} = \begin{bmatrix} \mu_S & \mathbf{g}^{\mathrm{T}} \\ \mathbf{0}_n & 1 \end{bmatrix}, \quad [S']_f = \begin{bmatrix} [S]_f & \mathbf{h}^{\mathrm{T}} \\ \mathbf{h} & f(x_{n+1}) \end{bmatrix} \tag{3.1}$$

and thus

$$[S']_f\mu_{S'} = \begin{bmatrix} [S]_f\mu_S & [S]_f\mathbf{g}^{\mathrm{T}} + \mathbf{h}^{\mathrm{T}} \\ \mathbf{h}\mu_S & \mathbf{h}\mathbf{g}^{\mathrm{T}} + f(x_{n+1}) \end{bmatrix}, \tag{3.2}$$

where $\mathbf{e} = (\mathbf{1}_p, \mathbf{0}_{n-p})$, $\mathbf{g} = (\mathbf{0}_{p-1}, -1, \mathbf{0}_{n-p})$ and $(\mathbf{h})_j = f(x_{n+1} \vee x_j)$ for $j = 1, 2, \ldots, n$. Clearly $\mathbf{h}\mathbf{g}^{\mathrm{T}} + f(x_{n+1}) = 0$. By semi-multiplicativity and the definition of S' we have

$$(\mathbf{h})_j = \frac{f(x_{n+1})f(x_j)}{f(x_{n+1} \wedge x_j)} = \frac{f(x_{n+1})f(x_j)}{f(x_p \wedge x_j)} = \frac{f(x_{n+1})f(x_p \vee x_j)}{f(x_p)} \tag{3.3}$$

for $j = 1, 2, \ldots, n$. Thus $\mathbf{h} = \frac{f(x_{n+1})}{f(x_p)}([S]_f)_{(p)}$, and by (3.2) we have

$$[S']_f \mu_{S'} = \begin{bmatrix} [S]_f \mu_S & \frac{f(x_{n+1})-f(x_p)}{f(x_p)}([S]_f)_p \\ \frac{f(x_{n+1})}{f(x_p)}([S]_f \mu_S)_{(p)} & 0 \end{bmatrix}. \tag{3.4}$$

On the other hand, since $(S)_f^{-1}$ and $(S')_f^{-1}$ exist, we have $\Delta_{S',j} = \Delta_{S,j} \neq 0$ for $j = 1, 2, \ldots, n$ and by the Möbius inversion formula [1, p. 152] we have

$$\begin{aligned} \Delta_{S',n+1} &= \sum_{z \leqslant x_{n+1}; z \nleqslant x_1,\ldots,x_n} (f*\mu)(z) \\ &= \sum_{z \leqslant x_{n+1}} (f*\mu)(z) - \sum_{z \leqslant x_p} (f*\mu)(z) = f(x_{n+1}) - f(x_p) \neq 0. \end{aligned} \tag{3.5}$$

Now let $(S)_f | [S]_f$. By Theorem 3.1 $\Delta_{S',j} = \Delta_{S,j} | ([S]_f \mu_S)_j$, and since $f(x_p) | f(x_{n+1})$, by (3.4) we have $\Delta_{S',j} | ([S']_f \mu_{S'})_j$ for $j = 1, 2, \ldots, n$. Further $\Delta_{S',n+1} | ([S']_f \mu_{S'})_{n+1}$, since $\Delta_{S',n+1} = f(x_{n+1}) - f(x_p)$ and $f(x_p) | ([S]_f)_p$. Thus by Theorem 3.1 we have $(S')_f | [S']_f$. If $(S')_f | [S']_f$, then by (3.4) we have $(S)_f | [S]_f$. Thus (M_1) holds.

Second we prove (M_2). Consider $[S'']_f \mu_{S''}$. Clearly $(\mu_{S''})_{i,n+1} = 0$ whenever $p < i \leqslant m$ or $q < i \leqslant n$. If $1 \leqslant i \leqslant p$ or $m < i \leqslant q$, then

$$\begin{aligned} (\mu_{S''})_{i,n+1} &= -\sum_{x_i \leqslant x_r \leqslant x_p} (\mu_S)_{ir} - \sum_{x_i \leqslant x_r \leqslant x_q} (\mu_S)_{ir} + \sum_{x_i \leqslant x_r \leqslant x_k} (\mu_S)_{ir} \\ &= -\delta_S(x_i, x_p) - \delta_S(x_i, x_q) + \delta_S(x_i, x_k) = \begin{cases} 1 & \text{if } i = k, \\ -1 & \text{if } i = p, q, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.6}$$

Adapting the notation (3.1) to $\mathbf{e} = (\mathbf{1}_k, \mathbf{1}_{p-k}, \mathbf{0}_{m-p}, \mathbf{1}_{q-m}, \mathbf{0}_{n-q})$, we have

$$[S'']_f \mu_{S''} = \begin{bmatrix} [S]_f \mu_S & [S]_f \mathbf{g}^{\mathrm{T}} + \mathbf{h}^{\mathrm{T}} \\ \mathbf{h} \mu_S & \mathbf{h}\mathbf{g}^{\mathrm{T}} + f(x_{n+1}) \end{bmatrix}, \tag{3.7}$$

where $\mathbf{g} = (\mathbf{0}_{k-1}, 1, \mathbf{0}_{p-k-1}, -1, \mathbf{0}_{m-p}, \mathbf{0}_{q-m-1}, -1, \mathbf{0}_{n-q})$ and $(\mathbf{h})_j = f(x_{n+1} \vee x_j)$ for $j = 1, 2, \ldots, n$. Then

$$\mathbf{h}\mathbf{g}^{\mathrm{T}} + f(x_{n+1}) = f(x_{n+1}) - f(x_{n+1}) - f(x_{n+1}) + f(x_{n+1}) = 0. \tag{3.8}$$

By semi-multiplicativity and arguments similar to those for (3.3), we have

$$(\mathbf{h})_j = \begin{cases} \frac{f(x_{n+1})}{f(x_p)}([S]_f)_{pj} = \frac{f(x_{n+1})}{f(x_q)}([S]_f)_{qj} & \text{if } 1 \leqslant j \leqslant k, \\ \frac{f(x_{n+1})}{f(x_p)}([S]_f)_{pj} = ([S]_f)_{qj} & \text{if } k < j \leqslant m, \\ \frac{f(x_{n+1})}{f(x_q)}([S]_f)_{qj} = ([S]_f)_{pj} & \text{if } m < j \leqslant n. \end{cases} \tag{3.9}$$

Since $(\mu_S)_{ij} = 0$ whenever $k < i \leqslant m$ and $m < j \leqslant n$, we have

$$(\mathbf{h}\mu_S)_j = \begin{cases} \frac{f(x_{n+1})}{f(x_p)}([S]_f \mu_S)_{pj} & \text{if } 1 \leqslant j \leqslant m, \\ \frac{f(x_{n+1})}{f(x_q)}([S]_f \mu_S)_{qj} & \text{if } m < j \leqslant n. \end{cases} \tag{3.10}$$

By the definition of S'' we have $([S]_f)_{ik} = ([S]_f)_{ip}$ if $k < i \leqslant m$, and similarly $([S]_f)_{ik} = ([S]_f)_{iq}$ if $m < i \leqslant n$. Thus

$$([S]_f\mathbf{g}^{\mathrm{T}} + \mathbf{h}^{\mathrm{T}})_{(i)} = ([S]_f)_{ik} - ([S]_f)_{ip} - ([S]_f)_{iq} + (\mathbf{h}^{\mathrm{T}})_{(i)} = \begin{cases} f(x_k) - f(x_p) - f(x_q) + f(x_{n+1}) & \text{if } 1 \leqslant i \leqslant k, \\ 0 & \text{if } k < i \leqslant n. \end{cases} \tag{3.11}$$

On the other hand, since $(S)_f^{-1}$ and $(S'')_f^{-1}$ exist, we have $\Delta_{S'',j} = \Delta_{S,j}$ for $j = 1, 2, \ldots, n$ and

$$\begin{aligned}\Delta_{S'',n+1} &= \sum_{z \leqslant x_{n+1};\, z \nleqslant x_1,\ldots,x_n} (f * \mu)(z) \\ &= \sum_{z \leqslant x_{n+1}} (f * \mu)(z) - \sum_{z \leqslant x_p} (f * \mu)(z) - \sum_{z \leqslant x_q} (f * \mu)(z) \\ &\quad + \sum_{z \leqslant x_k} (f * \mu)(z) = f(x_{n+1}) - f(x_p) - f(x_q) + f(x_k) \neq 0.\end{aligned} \tag{3.12}$$

Now let $(S)_f | [S]_f$. Then by Theorem 3.1 $\Delta_{S'',j} = \Delta_{S,j} | ([S]_f \mu_S)_j$ for $j = 1, 2, \ldots, n$, and since $f(x_p), f(x_q) | f(x_{n+1})$, by (3.7) and (3.10) we have $\Delta_{S'',j} | ([S'']_f \mu_{S''})_j$ for $j = 1, 2, \ldots, n$. Further, by (3.7), (3.8) and (3.11) we have $\Delta_{S'',n+1} | ([S'']_f \mu_{S''})_{n+1}$. Therefore $(S'')_f | [S'']_f$. If $(S'')_f | [S'']_f$, then clearly also $(S)_f | [S]_f$. Thus (M_2) holds. □

Application of Theorem 3.2 goes inductively as follows. Let S be a meet-closed set with n elements. Suppose that S can be constructed from $S \setminus \{x_n\}$ using the method (M_1) or (M_2) of Theorem 3.2, and suppose that $S \setminus \{x_n\}$ satisfies the divisibility property if and only if a certain condition on f holds. Then S satisfies the divisibility property if and only if the same condition on f holds. (It is possible that the divisibility property holds in $S \setminus \{x_n\}$ for all f and therefore also in S for all f, and it is possible that the divisibility property does not hold in $S \setminus \{x_n\}$ for any f and therefore does not hold either in S for any f.) If S cannot be constructed from $S \setminus \{x_n\}$ using (M_1) or (M_2), then the divisibility condition should be found using other methods, for instance, using Theorem 3.1. Examples are provided in Corollaries 3.1 and 3.2 and in Sections 4 and 5.

Chains and x_1-sets are meet-closed and are easy to construct inductively using the method (M_1). Thus we obtain the following two corollaries. The requirement of semi-multiplicativity in Corollary 3.1 is irrelevant (see Remark 3.1), since every f is semi-multiplicative on chains.

Corollary 3.1. *Let S be a chain such that $(S)_f$ is invertible (i.e., $f(x_1) \neq 0$ and $f(x_k) \neq f(x_{k-1})$ for $k = 2, 3, \ldots, n$). Then $(S)_f | [S]_f$.*

Corollary 3.2. *Let S be an x_1-set such that $(S)_f$ is invertible (i.e., $f(x_1) \neq 0$ and $f(x_k) \neq f(x_1)$ for $k = 2, 3, \ldots, n$). Then $(S)_f | [S]_f$.*

4. Divisibility on meet-closed sets with at most five elements

In this section we provide concrete examples on the application of Theorems 3.1 and 3.2. We find for all meet-closed sets S with at most five elements a necessary and sufficient condition on

f in order that the divisibility property holds. For each set S we apply the method (M_1) or (M_2) of Theorem 3.2 if possible, and otherwise we apply part (c) of the proof of Theorem 3.1. We classify the sets S on the basis of their lattice-theoretic structure. In Section 5 we use these results to solve certain open problems on the divisibility of GCD and LCM matrices.

4.1. Cases $n = 1, 2, 3$

Case $n = 1$. Let $S = \{x_1\}$ (which is always meet-closed) and let $(S)_f$ be invertible. Then $(S)_f = [S]_f = f(x_1) \neq 0$ and thus $(S)_f | [S]_f$.

Case $n = 2$. Let $S = \{x_1, x_2\}$ be meet-closed and let $(S)_f$ be invertible. Then S is a chain and by Corollary 3.1 we have $(S)_f | [S]_f$.

Case $n = 3$. Let $S = \{x_1, x_2, x_3\}$ be meet-closed and let $(S)_f$ be invertible. Then S is either a chain with $x_1 < x_2 < x_3$ or an x_1-set with $x_1 = x_2 \wedge x_3$. By Corollaries 3.1 and 3.2 we have $(S)_f | [S]_f$.

Before examining sets with 4 elements we present the following corollary.

Corollary 4.1. *Let S be a meet-closed set with at most three elements such that $(S)_f$ is invertible. Then $(S)_f | [S]_f$.*

4.2. Case $n = 4$

When we construct all possible meet-closed sets with four elements (from those having three elements), we obtain exactly five different classes $4_A, 4_B, 4_C, 4_D, 4_E$ presented in Fig. 4.1. In each class the white point stands for the last added element.

Let $S \in 4_A, 4_B, 4_C, 4_D$ (meaning that S is a set whose Hasse diagram is isomorphic to 4_A, 4_B, 4_C or 4_D). Since S can be constructed by (M_1), we have $(S)_f | [S]_f$. Let $S \in 4_E$. Now S can be constructed by (M_2), and thus $(S)_f | [S]_f$, if the white point represents the join (and not merely an upper bound) of the two incomparable elements. Otherwise 4_E needs further investigation.

To be more precise, let $S \in 4_E$, where $x_1 = x_2 \wedge x_3$ and $x_2 \vee x_3 \leqslant x_4$, and let $(S)_f$ be invertible. Then $\Delta = \mathrm{diag}(\Delta_{S,1}, \ldots, \Delta_{S,4})$ is invertible, that is, $\Delta_{S,1} = f(x_1)$, $\Delta_{S,2} = f(x_2) - f(x_1)$,

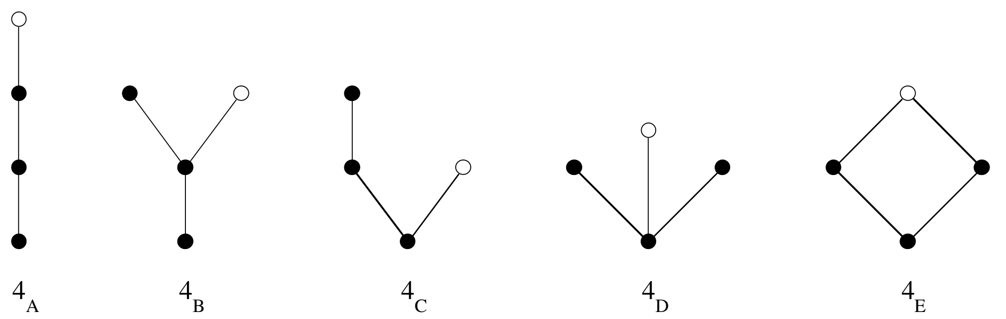

Fig. 4.1. Meet-closed sets with four elements.

$\Delta_{S,3} = f(x_3) - f(x_1)$ and $\Delta_{S,4} = f(x_4) - f(x_3) - f(x_2) + f(x_1)$ are all nonzero. We have

$$[S]_f = \begin{bmatrix} f(x_1) & f(x_2) & f(x_3) & f(x_4) \\ f(x_2) & f(x_2) & \frac{f(x_2)f(x_3)}{f(x_1)} & f(x_4) \\ f(x_3) & \frac{f(x_2)f(x_3)}{f(x_1)} & f(x_3) & f(x_4) \\ f(x_4) & f(x_4) & f(x_4) & f(x_4) \end{bmatrix}, \quad \mu_S = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

and thus $[S]_f \mu_S \Delta^{-1} = A_{4_{\mathrm{E}}}$, where

$$A_{4_{\mathrm{E}}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{f(x_2)}{f(x_1)} & 0 & \frac{f(x_2)}{f(x_1)} & \frac{f(x_1)f(x_4)-f(x_2)f(x_3)}{f(x_1)[f(x_4)-f(x_3)-f(x_2)+f(x_1)]} \\ \frac{f(x_3)}{f(x_1)} & \frac{f(x_3)}{f(x_1)} & 0 & \frac{f(x_1)f(x_4)-f(x_2)f(x_3)}{f(x_1)[f(x_4)-f(x_3)-f(x_2)+f(x_1)]} \\ \frac{f(x_4)}{f(x_1)} & 0 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

By the order-preserving property and Theorem 3.1 we find that $(S)_f | [S]_f$ if and only if $(A_{4_{\mathrm{E}}})_{2,4} \in \mathbb{Z}$. In the proof of the next theorem it appears that the only possibility is $f(x_1)f(x_4) = f(x_2)f(x_3)$ (which means by semi-multiplicativity that $f(x_4) = f(x_2 \vee x_3)$).

Theorem 4.1. *Let S be a meet-closed set with four elements such that $(S)_f$ is invertible.*

(a) *If $S \in 4_{\mathrm{A}}, 4_{\mathrm{B}}, 4_{\mathrm{C}}, 4_{\mathrm{D}}$, then $(S)_f | [S]_f$.*
(b) *Let $S \in 4_{\mathrm{E}}$, where $x_1 = x_2 \wedge x_3$ and $x_2 \vee x_3 \leqslant x_4$. Then $(S)_f | [S]_f$ if and only if $f(x_4) = f(x_2 \vee x_3)$.*

Proof. On the basis of the discussion at the beginning of this subsection it suffices to prove (b) in the only if direction. Let $S \in 4_{\mathrm{E}}$, where $x_1 = x_2 \wedge x_3$ and $x_2 \vee x_3 \leqslant x_4$, and let $(S)_f | [S]_f$. Denote $f(x_1) = a \geqslant 1$. By the order-preserving property, the existence of Δ^{-1} and semi-multiplicativity we have $f(x_2) = ab$, $f(x_3) = ac$, $f(x_2 \vee x_3) = f(x_2)f(x_3)/f(x_1) = abc$ and $f(x_4) = abcd$, where $b, c > 1$ and $d \geqslant 1$. We prove that $d = 1$. Suppose to the contrary that $d > 1$. By $(A_{4_{\mathrm{E}}})_{2,4}$ in (4.2), we have $(bcd - c - b + 1) | (bcd - bc)$. Since the both sides of | are positive, we have $bcd - c - b + 1 \leqslant bcd - bc$ and further $bc + 1 \leqslant b + c$. This is a contradiction and so $d = 1$ and $f(x_4) = f(x_2 \vee x_3)$. □

4.3. Case $n = 5$

When we construct all possible meet-closed sets with 5 elements from $4_{\mathrm{A}}, 4_{\mathrm{B}}, \ldots, 4_{\mathrm{E}}$, we obtain exactly 15 different classes $5_{\mathrm{A}}, 5_{\mathrm{B}}, \ldots, 5_{\mathrm{O}}$ presented in Fig. 4.2.

Let $S \in 5_{\mathrm{A}}, 5_{\mathrm{B}}, 5_{\mathrm{C}}, 5_{\mathrm{D}}$, $S \in 5_{\mathrm{F}}, 5_{\mathrm{G}}, 5_{\mathrm{H}}$ or $S \in 5_{\mathrm{K}}, 5_{\mathrm{L}}$. Since S can be constructed by (M_1) from $4_{\mathrm{A}}, 4_{\mathrm{B}}, 4_{\mathrm{C}}$ or 4_{D}, we have $(S)_f | [S]_f$. Let $S \in 5_{\mathrm{M}}, 5_{\mathrm{N}}, 5_{\mathrm{O}}$. Since S can be constructed by (M_1) from 4_{E}, we obtain that $(S)_f | [S]_f$ if $f(x_4) = f(x_2 \vee x_3)$.

Let $S \in 5_{\mathrm{E}}$. Now S can be constructed by (M_2) from 4_{B}, and thus $(S)_f | [S]_f$, if the white point represents the join of the two incomparable elements. Otherwise 5_{E} needs further investigation. Also 5_{I} and 5_{J} must be treated separately, since they cannot be constructed at all by Theorem 3.2.

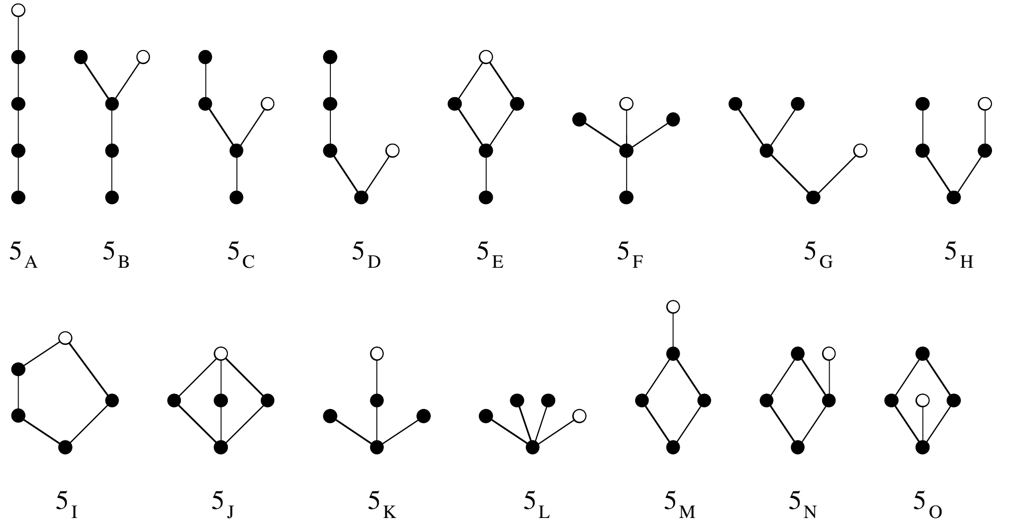

Fig. 4.2. Meet-closed sets with five elements.

To be more precise, in the following let $(S)_f$ be invertible and denote $\Delta = \mathrm{diag}(\Delta_{S,1}, \ldots, \Delta_{S,5})$. First, let $S \in 5_\mathrm{E}$, where $x_1 < x_2$, $x_2 = x_3 \wedge x_4$ and $x_3 \vee x_4 \leqslant x_5$. Now $\Delta_{S,1} = f(x_1)$, $\Delta_{S,2} = f(x_2) - f(x_1)$, $\Delta_{S,3} = f(x_3) - f(x_2)$, $\Delta_{S,4} = f(x_4) - f(x_2)$ and $\Delta_{S,5} = f(x_5) - f(x_4) - f(x_3) + f(x_2)$ are all nonzero. Thus we have $[S]_f \mu_S \Delta^{-1} = A_{5_\mathrm{E}}$, where

$$A_{5_\mathrm{E}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \frac{f(x_2)}{f(x_1)} & 0 & 1 & 1 & 1 \\ \frac{f(x_3)}{f(x_1)} & 0 & 0 & \frac{f(x_3)}{f(x_2)} & \frac{f(x_2)f(x_5)-f(x_3)f(x_4)}{f(x_2)[f(x_5)-f(x_4)-f(x_3)+f(x_2)]} \\ \frac{f(x_4)}{f(x_1)} & 0 & \frac{f(x_4)}{f(x_2)} & 0 & \frac{f(x_2)f(x_5)-f(x_3)f(x_4)}{f(x_2)[f(x_5)-f(x_4)-f(x_3)+f(x_2)]} \\ \frac{f(x_5)}{f(x_1)} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.3}$$

and clearly $(S)_f | [S]_f$ if and only if $(A_{5_\mathrm{E}})_{3,5} \in \mathbb{Z}$. Second, let $S \in 5_\mathrm{I}$, where $x_1 < x_2 < x_3$, $x_1 < x_4$ and $x_2 \vee x_4 \leqslant x_3 \vee x_4 \leqslant x_5$. Now $\Delta_{S,1} = f(x_1)$, $\Delta_{S,2} = f(x_2) - f(x_1)$, $\Delta_{S,3} = f(x_3) - f(x_2)$, $\Delta_{S,4} = f(x_4) - f(x_1)$ and $\Delta_{S,5} = f(x_5) - f(x_4) - f(x_3) + f(x_1)$ are all nonzero. Thus we have $[S]_f \mu_S \Delta^{-1} = A_{5_\mathrm{I}}$, where

$$A_{5_\mathrm{I}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \frac{f(x_2)}{f(x_1)} & 0 & 1 & \frac{f(x_2)}{f(x_1)} & \frac{f(x_2)-f(x_3)+\frac{f(x_1)f(x_5)-f(x_2)f(x_4)}{f(x_1)}}{f(x_5)-f(x_4)-f(x_3)+f(x_1)} \\ \frac{f(x_3)}{f(x_1)} & 0 & 0 & \frac{f(x_3)}{f(x_1)} & \frac{f(x_1)f(x_5)-f(x_3)f(x_4)}{f(x_1)[f(x_5)-f(x_4)-f(x_3)+f(x_1)]} \\ \frac{f(x_4)}{f(x_1)} & \frac{f(x_4)}{f(x_1)} & \frac{f(x_4)}{f(x_1)} & 0 & \frac{f(x_1)f(x_5)-f(x_3)f(x_4)}{f(x_1)[f(x_5)-f(x_4)-f(x_3)+f(x_1)]} \\ \frac{f(x_5)}{f(x_1)} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

and clearly $(S)_f | [S]_f$ if and only if $(A_{5_\mathrm{I}})_{2,5}, (A_{5_\mathrm{I}})_{3,5} \in \mathbb{Z}$. Third, let $S \in 5_\mathrm{J}$, where $x_1 = x_2 \wedge x_3 = x_2 \wedge x_4 = x_3 \wedge x_4$ and $x_2 \vee x_3 \vee x_4 \leqslant x_5$. Now $\Delta_{S,1} = f(x_1)$, $\Delta_{S,2} = f(x_2) - f(x_1)$, $\Delta_{S,3} =$

$f(x_3) - f(x_1)$, $\Delta_{S,4} = f(x_4) - f(x_1)$ and $\Delta_{S,5} = f(x_5) - f(x_4) - f(x_3) - f(x_2) + 2f(x_1)$ are all nonzero. Thus we have $[S]_f \mu_S \Delta^{-1} = A_{5_\mathrm{J}}$, where

$$A_{5_\mathrm{J}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \frac{f(x_2)}{f(x_1)} & 0 & \frac{f(x_2)}{f(x_1)} & \frac{f(x_2)}{f(x_1)} & \frac{f(x_1)[f(x_2)+f(x_5)]-f(x_2)[f(x_3)+f(x_4)]}{f(x_1)[2f(x_1)-f(x_2)-f(x_3)-f(x_4)+f(x_5)]} \\ \frac{f(x_3)}{f(x_1)} & \frac{f(x_3)}{f(x_1)} & 0 & \frac{f(x_3)}{f(x_1)} & \frac{f(x_1)[f(x_3)+f(x_5)]-f(x_3)[f(x_2)+f(x_4)]}{f(x_1)[2f(x_1)-f(x_2)-f(x_3)-f(x_4)+f(x_5)]} \\ \frac{f(x_4)}{f(x_1)} & \frac{f(x_4)}{f(x_1)} & \frac{f(x_4)}{f(x_1)} & 0 & \frac{f(x_1)[f(x_4)+f(x_5)]-f(x_4)[f(x_2)+f(x_3)]}{f(x_1)[2f(x_1)-f(x_2)-f(x_3)-f(x_4)+f(x_5)]} \\ \frac{f(x_5)}{f(x_1)} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.5}$$

and clearly $(S)_f | [S]_f$ if and only if $(A_{5_\mathrm{J}})_{2,5}, (A_{5_\mathrm{J}})_{3,5}, (A_{5_\mathrm{J}})_{4,5} \in \mathbb{Z}$. In the proof of the next theorem we repeatedly need the order-preserving property, existence of Δ^{-1} and semi-multiplicativity; so for the sake of brevity we do not mention these properties each time.

Theorem 4.2. *Let S be a meet-closed set with five elements such that $(S)_f$ is invertible.*

(i) *If $S \in 5_\mathrm{A}, 5_\mathrm{B}, 5_\mathrm{C}, 5_\mathrm{D}$, $S \in 5_\mathrm{F}, 5_\mathrm{G}, 5_\mathrm{H}$ or $S \in 5_\mathrm{K}, 5_\mathrm{L}$, then $(S)_f | [S]_f$.*
(ii) *Let $S \in 5_\mathrm{M}, 5_\mathrm{N}, 5_\mathrm{O}$, where $x_1 = x_2 \wedge x_3$ and $x_2 \vee x_3 \leqslant x_4$. Then $(S)_f | [S]_f$ if and only if $f(x_4) = f(x_2 \vee x_3)$.*
(iii) *Let $S \in 5_\mathrm{E}$, where $x_1 < x_2$, $x_2 = x_3 \wedge x_4$ and $x_3 \vee x_4 \leqslant x_5$. Then $(S)_f | [S]_f$ if and only if $f(x_5) = f(x_3 \vee x_4)$.*
(iv) *If $S \in 5_\mathrm{I}$, then $(S)_f \nmid [S]_f$.*
(v) *Let $S \in 5_\mathrm{J}$, where $x_1 = x_2 \wedge x_3 = x_2 \wedge x_4 = x_3 \wedge x_4$ and $x_2 \vee x_3 \vee x_4 \leqslant x_5$. Then $(S)_f | [S]_f$ if and only if $x_2 \vee x_3 = x_2 \vee x_4 = x_3 \vee x_4 = x_5$ and $f(x_2) = f(x_3) = f(x_4) = mf(x_1)$ (i.e., $f(x_5) = m^2 f(x_1)$), where $m = 3$ or $m = 4$.*

Proof. On the basis of the discussion at the beginning of this subsection we see that (i) and (ii) clearly hold. The proof of (iii) is similar to the proof of (b) in Theorem 4.1.

We prove (iv) as follows. Let $S \in 5_\mathrm{I}$, where $x_1 < x_2 < x_3$, $x_1 < x_4$ and $x_2 \vee x_4 < x_3 \vee x_4 \leqslant x_5$, and suppose to the contrary that $(S)_f | [S]_f$ holds. Denote $f(x_1) = a \geqslant 1$. Then $f(x_2) = ab$, $f(x_3) = abc$, $f(x_4) = ad$ and $f(x_3 \vee x_4) = f(x_3)f(x_4)/f(x_1) = abcd$, where $b, c, d > 1$. Further $f(x_5) = abcde$, where $e \geqslant 1$. Similar to the proof of Theorem 4.1 (b), it can be shown by $(A_{5_\mathrm{I}})_{3,5}$ in (4.4) that $e = 1$. Thus $f(x_5) = abcd$. Now by $(A_{5_\mathrm{I}})_{2,5}$ we find that $bd - 1 | b(d-1)$ and further $bd - 1 | b(d-1) - (bd-1) = 1 - b$. This is a contradiction and so $(S)_f | [S]_f$ cannot hold. Thus (iv) holds.

Next we prove the if direction of (v). Let $S \in 5_\mathrm{J}$, where $x_1 = x_2 \wedge x_3 = x_2 \wedge x_4 = x_3 \wedge x_4$ and $x_2 \vee x_3 \vee x_4 \leqslant x_5$. Let $x_2 \vee x_3 = x_2 \vee x_4 = x_3 \vee x_4 = x_5$ and denote $f(x_2) = f(x_3) = f(x_4) = mf(x_1)$, where $m = 3$ or $m = 4$. Then $f(x_5) = f(x_2 \vee x_3) = f(x_2)f(x_3)/f(x_1) = m^2 f(x_1)$. Now by (4.5) we find that $A_{5_\mathrm{J}} \in \mathbb{Z}^{n \times n}$ and therefore $(S)_f | [S]_f$. Thus the if direction holds.

Finally we prove the only if direction of (v). Let $S \in 5_\mathrm{J}$, where $x_1 = x_2 \wedge x_3 = x_2 \wedge x_4 = x_3 \wedge x_4$ and $x_2 \vee x_3 \vee x_4 \leqslant x_5$. Let $(S)_f | [S]_f$ and denote $f(x_1) = a \geqslant 1$. We have exactly two possibilities. Either $x_2 \vee x_3 = x_2 \vee x_4 = x_3 \vee x_4 = x_2 \vee x_3 \vee x_4$ or at least one of $x_2 \vee x_3$, $x_2 \vee x_4$, $x_3 \vee x_4$ is $< x_2 \vee x_3 \vee x_4$. First let, say, $x_2 \vee x_3 < x_2 \vee x_3 \vee x_4$. Then $f(x_2) = ab$, $f(x_3) = ac$, $f(x_4) = ad$, where $b, c, d > 1$. Thus we have $f(x_2 \vee x_3) = f(x_2)f(x_3)/f(x_1) = abc$, $f(x_2 \vee x_3 \vee x_4) = f(x_2 \vee x_3)f(x_4)/f(x_1) = abcd$, and further $f(x_5) = abcde$, where $e \geqslant 1$. Now by $(A_{5_\mathrm{J}})_{2,5}$ in (4.5) we find that $(bcde - b - c - d + 2) | b(cde - c - d + 1)$. Since the both sides of

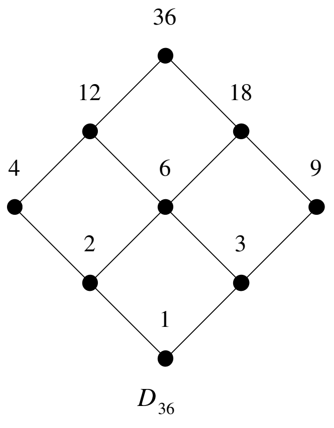

Fig. 4.3. The lattice of the divisors of 36.

| are positive, we have $bcde - b - c - d + 2 \leqslant b(cde - c - d + 1)$ and further $0 \leqslant (b-1)(1 - c - d) - 1$. This is a contradiction and so this case never occurs. Second, let $x_2 \vee x_3 = x_2 \vee x_4 = x_3 \vee x_4 = x_2 \vee x_3 \vee x_4$. Then $f(x_2) = f(x_3) = f(x_4) = ab$ and $f(x_5) = cf(x_2)f(x_3)/f(x_1) = ab^2c$, where $b > 1$ and $c \geqslant 1$. Now by $(A_{5_{\mathrm{J}}})_{2,5}$ in (4.5) we obtain that $(b^2c - 3b + 2)|(b^2(c - 2) + b)$. Let $c > 1$. Since the both sides of | are positive, we have $b^2c - 3b + 2 \leqslant b^2(c-2) + b$ and $2 - 3b \leqslant b - 2b^2$ and further $(b-1)^2 \leqslant 0$. This is a contradiction and so the case $c > 1$ never occurs. Let $c = 1$. Then $(b^2 - 3b + 2)|(b^2 - b)$, i.e., $(b-2)|b$, which means that $b = 3$ or $b = 4$. Thus the only if direction holds and therefore (v) holds. □

4.4. Remarks

Remark 4.1. Suppose that the divisibility property holds in $\{x_1, x_2, \ldots, x_k\}$ if and only if a certain condition on f holds. Suppose that S can be constructed from $\{x_1, x_2, \ldots, x_k\}$ using (M_1) and (M_2) $n - k$ times. Then the divisibility property holds in S if and only if the same condition on f holds. For example, if $n > 5$ and S can be constructed from 5_{I} using (M_1) and (M_2), then the divisibility property does not hold in S, since it does not hold in 5_{I}.

Remark 4.2. All the structures of S mentioned in Sections 4.1–4.3 need not appear on a fixed lattice $(P, \leqslant)$, and thus the structure of $(P, \leqslant)$ also has a bearing on the possibility of the divisibility. For example, if $(P, \leqslant)$ is the sublattice $(D_{36}, |)$ of the divisor lattice $(\mathbb{Z}_+, |)$, where D_{36} is the set of the positive divisors of 36 (see Fig. 4.3), then S cannot be of the form 4_{D} in Fig. 4.1.

5. Application to GCD and LCM matrices

In this section we apply our results to the divisor lattice $(\mathbb{Z}_+, |) = (\mathbb{Z}_+, \gcd, \mathrm{lcm})$. Our results also concern the divisibility of GCUD and LCUM matrices, but we do not include these results here, see [9,17].

We give new explanations for some theorems, answer some conjectures and generalize some results obtained for the divisibility of GCD and LCM matrices in the literature. For the sake of brevity we do not write down our theorems and corollaries in the number-theoretic setting. However, we give some instructions for writing them out.

The symbols $\leqslant$, $\wedge$ and $\vee$ should be replaced with $|$, gcd and lcm. The concepts of meet-closed and lower-closed sets should be replaced with the concepts of gcd-closed and factor-closed sets, respectively. The incidence Möbius function $\mu(x, y)$ should be replaced with the number-theoretic Möbius function $\mu(y/x)$, see [21, p. 300]. If S is gcd-closed, then $\mu_S(x_i, x_j)$ should be replaced with $\sum_{dx_i|x_j; dx_i \nmid x_i,\ldots,x_{j-1}} \mu(d)$, cf., [19, Lemma 6.1] and (2.4). Note that the notation $(f * \mu)(z)$ does not have to be changed.

Let f be an integer-valued function on P. If $f(m) = m$ for all $m \geqslant 1$, we denote $(S)_f = (S)$ and $[S]_f = [S]$.

On Theorem 3.1. Bourque and Ligh [4, Theorem 3] show that if S is factor-closed, then $(S)|[S]$. Further, in [5, Theorem 4] they show that if S is factor-closed and f is a multiplicative function such that $f(x_i)$ and $(f * \mu)(x_i)$ are nonzero for all $x_i \in S$, then $(S)_f|[S]_f$. This result cannot be generalized for all lower-closed sets (cf., 5_{I}), but Theorem 3.1 shows what is essential in the proofs of Bourque and Ligh.

In what follows, let f be an order-preserving and semi-multiplicative function from $(\mathbb{Z}_+, |)$ into $(\mathbb{Z}_+, |)$.

On Theorem 3.2. This result is new in the number-theoretic setting. It gives positive answer to Conjecture 3.1 in [15]. In fact, $d \in S \subseteq \mathbb{Z}_+$ is said to be a greatest-type divisor of $x \in S$ if $d|x$ with $d \neq x$ and if $d|y|x$ with $y \in S$ implies $y = d$ or $y = x$. Conjecture 3.1 in [15] states that if S is a gcd-closed set such that for each $x \in S$ the number of greatest-type divisors of x is at most one, then $(S)|[S]$. This result follows from Theorem 3.2, since in this case the set S can be constructed applying (M_1) finite number of times.

On Corollary 3.1. Hong [13, Theorem 5.1] shows that if S is a divisor chain, if $f(x_1)|f(x_i)$ for all $x_i \in S$ and if $(f * \mu)(d) \in \mathbb{Z}$ whenever $d|\mathrm{lcm} S$, then $(S)_f|[S]_f$. By Corollary 3.1 we obtain essentially the same result.

On Corollary 3.2. This result is also new in the number-theoretic setting.

On Corollary 4.1. This result generalizes the result of Hong [11, Theorem 3.1(i)] (and [12, Theorem 3.3(ii)]), which states that $(S)|[S]$ (and thus $\det(S)|\det[S]$) on any gcd-closed set S with at most 3 elements.

On Theorems 4.1 and 4.2. Hong [11, Theorem 3.1(ii)] shows that for each $n \geqslant 4$ there exists a gcd-closed set S with n elements such that $(S)\nmid[S]$. We prove this using our results and the same counterexample as in the proof of [8, Theorem 3.5]. Let $S = \{x_1, x_2, \ldots, x_n\}, n \geqslant 4$, where $x_1 = 1, x_2 = p_1, x_3 = p_2, x_i = p_1 p_2 \cdots p_{i-1}$ for $i = 4, 5, \ldots, n$, and $p_1, p_2, \ldots, p_{n-1}$ are some distinct prime numbers in increasing order. Let $T = \{x_1, x_2, x_3, x_4\}$ and $f(m) = m$ for all $m \geqslant 1$. Since $T \in 4_{\mathrm{E}}$ and $f(x_4) = p_1 p_2 p_3 \neq p_1 p_2 = f(\mathrm{lcm}(x_2, x_3))$, according to Theorem 4.1 (b) we have $(T)_f\nmid[T]_f$. Since S is obtained from T applying (M_1) finite number of times, then also $(S)_f\nmid[S]_f$.

Hong [12, Theorem 3.5] shows that if S is gcd-closed and $x_i < 12$ for $i = 1, 2, \ldots, n$, then $\det(S)|\det[S]$. Next we prove a stronger result, where even $(S)_f|[S]_f$. Consider the sets $T = \{x \in \mathbb{Z}_+ : x < 12\}$ and $U = T \setminus \{7, 8, 11\}$, whose Hasse diagrams are as presented in Fig. 5.1.

Let S be a subset of U with $n = 4$. Since $S = \{1, 2, 3, 6\}$ and $S = \{1, 2, 5, 10\}$ are the only occurrences of 4_{E} in U, and further $6 = \mathrm{lcm}(2, 3)$ and $10 = \mathrm{lcm}(2, 5)$, then $(S)_f|[S]_f$ holds.

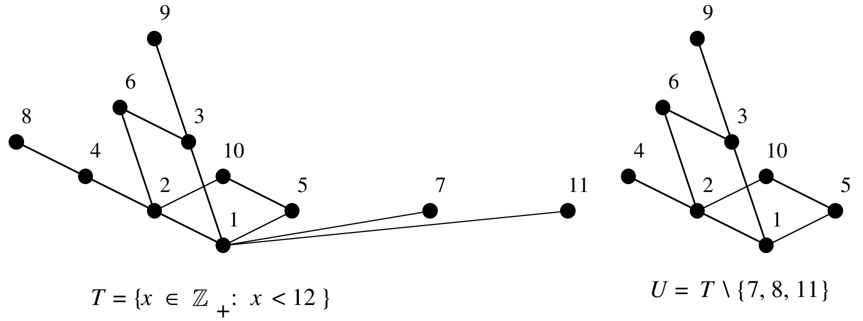

Fig. 5.1. The divisor lattices of I and U.

Let $S \subseteq U$ with $n = 5$. Since there are no occurrences of 5_{E}, 5_{I} and 5_{J} in U, then $(S)_f|[S]_f$ holds. Let $S \subseteq U$ with $n = 6$. Now all gcd-closed subsets of U with 6 elements are given as $U \setminus \{6, 10\}$, $U \setminus \{9, 10\}$, $U \setminus \{6, 9\}$, $U \setminus \{5, 10\}$, $U \setminus \{5, 9\}$, $U \setminus \{5, 6\}$, $U \setminus \{4, 10\}$, $U \setminus \{4, 9\}$, $U \setminus \{4, 6\}$, $U \setminus \{4, 5\}$, $U \setminus \{3, 9\}$ and $U \setminus \{3, 6\}$. Since they can all be constructed by (M_1) and (M_2), then $(S)_f|[S]_f$ holds. Let $S \subseteq U$ with $n = 7$. Now all gcd-closed subsets of U with 7 elements are given as $U \setminus \{10\}$, $U \setminus \{9\}$, $U \setminus \{6\}$, $U \setminus \{5\}$ and $U \setminus \{4\}$. Since they all can be constructed by (M_1) and (M_2), then $(S)_f|[S]_f$ holds. Also $S = U$ can be constructed by (M_1) and (M_2) and thus $(S)_f|[S]_f$ holds when $n = 8$. Note that by (M_1) the elements 7 and 11 can be added to any gcd-closed subset S of U mentioned above. Similarly, 4 can be replaced with 8 or with the pair 4, 8 to any gcd-closed subset S of U mentioned above. Thus $(S)_f|[S]_f$ also holds for any gcd-closed subset S of T.

Hong's [14] Conjecture 5.3 states that if S is a gcd-closed set with odd elements, then the power GCD matrix $[\gcd(x_i, x_j)^m]$, $m \in \mathbb{Z}_+$, divides the power LCM matrix $[\mathrm{lcm}(x_i, x_j)^m]$. By Theorem 4.1(b) we easily find a counterexample, where $S = \{1, 3, 5, 45\}$. We have already announced this counterexample in Mathematical Reviews [MR2039420 (2004j:11028)] and Zentralblatt MATH [Zbl 1047.11022], see also [8].

Example 5.1. Consider the set $T = \{x \in \mathbb{Z}_+ : x < 30\}$. By Theorems 4.1 and 4.2 we have the following results. Let S be a gcd-closed subset of T with $n = 4$. Then $(S)|[S]$ if and only if S is not any of the sets {1,2,3,12}, {1,2,3,18}, {1,2,3,24}, {1,2,5,20}, {1,2,7,28}, {1,3,4,24} or {2,4,6,24}. Let S be a gcd-closed subset of T with $n = 5$. Then $(S)|[S]$ if and only if S is not any of the sets {1,2,3,4,12}, {1,2,3,4,24}, {1,2,3,8,24}, {1,2,4,5,20}, {1,2,4,6,24} or {1,3,4,8,24}.

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