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Alternating-offers bargaining with one-sided uncertain deadlines: an efficient algorithm

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Abstract

In the arena of automated negotiations we focus on the principal negotiation protocol in bilateral settings, i.e. the alternating-offers protocol. In the scientific community it is common the idea that bargaining in the alternating-offers protocol will play a crucial role in the automation of electronic transactions. Notwithstanding its prominence, literature does not present a satisfactory solution to the alternating-offers protocol in real-world settings, e.g. in presence of uncertainty. In this paper we game theoretically analyze this negotiation problem with one-sided uncertain deadlines and we provide an efficient solving algorithm. Specifically, we analyze the situation where the values of the parameters of the buyer are uncertain to the seller, whereas the parameters of the seller are common knowledge (the analysis of the reverse situation is analogous). In this particular situation the results present in literature are not satisfactory, since they do not assure the existence of an equilibrium for every value of the parameters. From our game theoretical analysis we find two choice rules that apply an action and a probability distribution over the actions, respectively, to every time point and we find the conditions on the parameters such that each choice rule can be singularly employed to produce an equilibrium. These conditions are mutually exclusive. We show that it is always possible to produce an equilibrium where the actions, at any single time point, are those prescribed either by the first choice rule or by the second one. We exploit this result for developing a solving algorithm. The proposed algorithm works backward by computing the equilibrium from the last possible deadline of the bargaining to the initial time point and by applying at each time point the actions prescribed by the choice rule whose conditions are satisfied. The computational complexity of the proposed algorithm is asymptotically independent of the number of types of the player whose deadline is uncertain. With linear utility functions, it is $O(m \cdot \overline{T})$ where m is the number of the issues and $\overline{T}$ is the length of the bargaining.

Keywords: Automated negotiations; Game theory; Multiagent systems

1. Introduction

Automated negotiation is a promising scenario of computer science where artificial intelligence can play a crucial role: it can automate software agents allowing them to negotiate each other on behalf of users for buying and selling

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items [21]. This automation, as stated in literature, can lead to more effective negotiations since software agents work faster than humans and are more prone in finding efficient agreements [35].

Several negotiation settings can be found in the electronic commerce arena. The most common ones are usually bilateral: a buyer and a seller negotiate a contract over one or more issues. In this paper we consider the principal setting in bilateral negotiations: the *bargaining* [28]. In a bargaining two agents must reach an agreement regarding how to distribute objects or a monetary amount and each player prefers to reach an agreement, rather than abstain from doing so; however, each agent prefers that agreement which most favors her interests. A real-world example that depicts this situation is a negotiation between a service provider and a customer over the price and the quality level of a service.

Classically, the study of bargaining is carried out employing game-theoretical tools [28,30] wherein one distinguishes the negotiation *protocol* and the negotiation *strategies*: the protocol sets the negotiation rules, specifying which actions are allowed and when [32]; the strategies define the behavior of an agent in any possible agent's decision node. Strategies can be *pure* or *mixed*.[1] For any decision node of the game a pure strategy prescribes one action; a mixed strategy prescribes probability distributions over the actions. Given a protocol, the game-theoretical approach postulates that rational agents should employ strategies that maximize their payoffs [29]. In this paper we *a priori* assume agents to be rational; such an assumption will be supported *a posteriori* by the results provided in the paper. Indeed, we will show that the problem of computing a solution with rational agents is tractable.

The principal protocol for bilateral bargaining, the *alternating-offers* protocol, pioneered by Ståhl in [37], has reached an outstanding place in literature thanks to Rubinstein in [33]. It is considered to be the most satisfactory model of bargaining present in literature. Basically, an agent starts by offering a value for the issue under dispute (e.g., a price) to her opponent. The opponent can accept the offer or make a counteroffer. If a counteroffer is made, the process is repeated until one of the agents accepts. Rubinstein's alternating-offers model is not accurate enough to capture all the aspects involved in the electronic commercial transactions, where, typically, agents have reservation values and deadlines, negotiate over multiple issues, and have uncertain information. Therefore, refinements and extensions of [33] are commonly employed in computer science community to provide a more satisfactory model [10]. Examples of real-world applications that employ bargaining techniques can be found in [1,2,24,26,31].

The solution of the classic Rubinstein's protocol is well known in the literature [28]. On the contrary, the study of the alternating-offers protocol in presence of extensions and refinements is hard and still open. Specifically, the two crucial problems concern the development of algorithmic techniques to find equilibria in presence of *issue multiplicity* and *information incompleteness*. The problem of bargaining efficiently over multiple issues when information is complete has been recently addressed in [6,7] and refined in [12]. The equilibrium strategies can be easily computed by extending the classic backward induction method [14]. The computational complexity is $O(m \cdot \overline{T})$, where m is the number of issues and $\overline{T}$ is the length of the bargaining. In presence of incomplete information, it is customary in game theory to introduce probability distributions over the parameters that are not known by the agents. Notwithstanding, the analysis of bargaining with uncertain information is currently more a series of examples than a coherent set of results. Game theory provides an appropriate solution concept for extensive-form games with uncertain information, i.e., the *sequential equilibrium* [22], but no solving technique to find it. We recall that the backward induction method can be employed with success exclusively in presence of complete information [14]. Moreover, economic studies only provide equilibria in very narrow settings of information uncertainty, focusing mainly on discount factors and reservation values. For instance, in [34] Rubinstein analyzes a scenario with uncertainty over two possible discount factors of one of the two agents, while in [3] Chatterjee and Samuelson analyze a scenario with uncertainty over the reservation values of both the buyer and the seller, where each player can be of two types. An interested reader can find an exhaustive survey on bargaining with uncertain information in [4].

The employment of the alternating-offers protocol in electronic commerce has put attention on the role of the deadlines in the negotiation. The infinite horizon assumption, which is usually made in game theory literature, is not realistic in real-world applications [36]. Furthermore, agents' deadlines are usually uncertain, not being known *a priori* by the agents themselves. Notwithstanding the importance of uncertain deadlines in negotiations, only few works have deeply analyzed their effects in the alternating-offers protocol with discount factors, discrete time, and rational agents, and no satisfactory solution is currently known. This prevents the employment of autonomous rational

[1] For the sake of simplicity, we use in the paper, as Kreps and Wilson in [22], the term "strategies" in the place of appropriate game theoretical term "behavioral strategies".

agents in real-world applications and pushes scientific community to develop solutions for this bargaining problem. Classic results concerning the presence of deadlines in bargaining are the followings. In [25] Ma and Manove consider a complete information finite horizon alternating-offers model without temporal discounting with continuous time and players' option of strategic delay. In [13] Fershtman and Seidmann study a complete information bargaining model with random proposer and a deadline. In [16] Gneezy et al. study a variation of the *ultimatum* game. In [36] Sandholm and Vulkan analyze a slight variation of the *war-of-attrition* game: the surplus can be divided, time is continuous, the deadlines are uncertain, and there are not discount factors. In [10] Fatima et al. study the alternating-offers protocol with uncertainty over deadlines and reservation values in presence of bounded rational agents: more precisely, agents must employ predefined bidding tactics based on the *negotiation decision functions* paradigm [8]. Only recently some attempts have been made in computer science community to achieve the solution of the alternating-offers protocol with uncertain deadlines and rational agents. The first attempt is by Fatima et al. in [11], where they present an algorithm that produces equilibrium strategies in presence of two-sided uncertainty. Their algorithm searches in the space of the strategies finding equilibria in pure strategies in time linear in the length of the bargaining and polynomial in the number of agents' types. However, bargaining with only pure strategies in the alternating-offers protocol with uncertain deadlines is not satisfactory, since, as showed in [5], for some values of the parameters there is not any equilibrium in pure strategies. As it is customary in game theory, such a problem can be overtaken by resorting to mixed strategies; we recall indeed that any game admits at least one sequential equilibrium in mixed strategies by Kreps and Wilson's theorem [22].

In this paper we game theoretically study the problem of bargaining one issue in the alternating-offers protocol with one-sided uncertainty over deadlines and we provide an efficient algorithm to compute it. Exactly, we analyze the situation where the values of the parameters of the buyer are uncertain to the seller, whereas the parameters of the seller are common knowledge (the analysis of the reverse situation is analogous). We show also how our result can be easily extended to the multiple issue situation exploiting the result presented in [6]. The extension of our result to the two-sided situation is not easy instead, and it will be explored in future works. From our game theoretical analysis we find two choice rules that apply an action and a probability distribution over the actions, respectively, to each time point and we find the conditions on the parameters such that each choice rule can be singularly employed to produce an equilibrium.[2] These conditions are mutually exclusive. We show that it is always possible to produce an equilibrium where the actions at any single time point are those prescribed either by the first choice rule or by the second one. We exploit this result for developing a solving algorithm. Differently from [11,12], our algorithm does not search for the optimal actions of the agents at each time point among all the available ones, but it applies the actions prescribed by the choice rule whose conditions are satisfied. The computational complexity of our algorithm is asymptotically independent of the number of types of the agent whose deadline is uncertain, being $O(m \cdot \overline{T})$ where m is the number of issues and $\overline{T}$ is the length of the bargaining. The two choice rules we present are not the only ones that can be employed to produce an equilibrium. Indeed, since the alternating-offers protocol with uncertain deadlines admits more equilibria, other choice rules could be employed. The choice rules we propose have the peculiarity to produce only one equilibrium for all the values of the parameters and guarantee that there is not any other set of choice rules that allows one for computing faster the solution through dynamic programming techniques.

The paper is organized as follows. Section 2 reviews the alternating-offers model and the solution with complete information. Section 3 states the problem introducing the appropriate solution concept and the solutions currently available in the state of the art. Sections 4 and 5 game theoretically analyze the alternating-offers protocol with one-sided uncertainty over deadlines in pure strategies and mixed strategies, respectively. Section 6 presents our solving algorithm and shows how it can be extended to the multiple issue situation. Section 7 concludes the paper. In Appendix A we report the proofs of the main theoretical results and in Appendix B we report the formulas to compute the equilibrium mixed strategy in presence of more than two types.

2. Complete information alternating-offers

In this section we review the basis of the alternating-offers protocol with deadlines, in order to introduce notations, models, and techniques. We present (Section 2.1) the model of the alternating-offers bargaining with deadlines and (Section 2.2) its known solution with complete information.

[2] The first choice rule has been preliminarily presented in [5].

2.1. Bargaining model

We study a discrete time finite horizon alternating-offers bargaining protocol on one continuous issue (e.g., a price). Formally, the buyer agent $\mathbf{b}$ and the seller agent $\mathbf{s}$ can act at times $t \in \mathbb{N}$. The *player function* $\iota : \mathbb{N} \to \{\mathbf{b}, \mathbf{s}\}$ returns the agent that acts at time point t and is such that $\iota(t) \neq \iota(t+1)$. Possible actions $\sigma^t_{\iota(t)}$ of agent $\iota(t)$ at any time point $t > 0$ are:

(i) *offer*(x), where $x \in \mathbb{R}$ is the value of the issue to negotiate,
(ii) *exit*,
(iii) *accept*,

whereas at time point $t = 0$ the only allowed actions are (i) and (ii). If $\sigma^t_{\iota(t)} = accept$ the bargaining stops and the *outcome* is (x, t), where x is the value such that $\sigma^{t-1}_{\iota(t-1)} = offer(x)$. This is to say that the agents agree on the value x at time point t. If $\sigma^t_{\iota(t)} = exit$ the bargaining stops and the outcome is *NoAgreement*. Otherwise the bargaining continues to the next time point.

Each agent i has an utility function $U_i : (\mathbb{R} \times \mathbb{N}) \cup \{NoAgreement\} \to \mathbb{R}$, that represents her gain over the possible bargaining outcomes. Each utility function U_i depends on three parameters of agent i:

- the *reservation price* $RP_i \in \mathbb{R}^+$,
- the *temporal discount factor* $\delta_i \in (0, 1]$,
- the *deadline* $T_i \in \mathbb{N}$, $T_i > 0$.

More precisely, if the outcome of the bargaining is an agreement (x, t), then the utility functions $U_\mathbf{b}$ and $U_\mathbf{s}$ are respectively:

$$U_\mathbf{b}(x,t) = \begin{cases} (RP_\mathbf{b} - x) \cdot (\delta_\mathbf{b})^t & \text{if } t \leqslant T_\mathbf{b} \\ -1 & \text{otherwise} \end{cases}$$

$$U_\mathbf{s}(x,t) = \begin{cases} (x - RP_\mathbf{s}) \cdot (\delta_\mathbf{s})^t & \text{if } t \leqslant T_\mathbf{s} \\ -1 & \text{otherwise} \end{cases}$$

If the outcome is *NoAgreement*, then $U_\mathbf{b}(NoAgreement) = U_\mathbf{s}(NoAgreement) = 0$. Notice that the assignment of a strictly negative value (we have chosen by convention the value -1) to U_i after agent i's deadline allows one to capture the essence of the deadline concept: an agent, after her deadline, strictly prefers to exit the negotiation rather than to reach any agreement.

According to classic works in literature, we assume the *feasibility* of the agreement, i.e., $RP_\mathbf{b} > RP_\mathbf{s}$, and the *rationality* of the agents, i.e., it is common knowledge that each agent will act to maximize her utility.

2.2. Complete information solution

When the information is complete the appropriate solution concept for a game like the one we are dealing with is the *subgame perfect equilibrium* [18]. Rigorously speaking, the protocol described above is not a finite horizon game: the deadlines are not in the protocol, but in the agent's utility functions, and the agents are allowed to offer and counteroffer also after their deadlines have expired. Nevertheless, it is essentially a finite horizon game: a rational agent will give up bargaining after her deadline. Therefore, subgame perfect equilibrium strategies can be found employing the *backward induction* method [14]. In what follows we informally summarize the backward induction construction (see [28] for more details).

The presence of deadlines in the agents' utility functions induces a time point $\overline{T}$ where the game, if it is rationally played, stops. This time point is the earliest of the two deadlines, formally, $\overline{T} = \min\{T_\mathbf{b}, T_\mathbf{s}\}$. Indeed, after $\overline{T}$ no agent can gain positive utility by bargaining, being *NoAgreement* the equilibrium outcome of the subgame starting from $t = \overline{T}$. The peculiarity of the time point $\overline{T}$ with respect to any other time point $t < \overline{T}$ is that the optimal action agent $\iota(\overline{T})$ can make, if she does not accept her opponent's offer, is to make *exit*. Instead, at any time point $t < \overline{T}$ the agents strictly prefer to make an offer rather than to make *exit*. From time point $t = \overline{T} - 1$ back, the optimal actions of agent

$\iota(t)$ at time point t can be found in two steps. In the first step we find the best offer agent $\iota(t)$ can make at t: it is the offer that gives agent $\iota(t+1)$ the same utility of making at $t+1$ her best offer, if $t < \overline{T}-1$, and *exit*, if $t = \overline{T}-1$. We denote such an offer by $x^*(t)$. In the second step, we find the offers made by agent $\iota(t-1)$ at $t-1$ that agent $\iota(t)$ would accept at t: they are all the offers that give agent $\iota(t)$ an utility equal to or greater than offering $x^*(t)$. The rule whereby agent $\iota(t)$ chooses her optimal action at t is therefore: if $t = \overline{T}$, she accepts any offer that gives her an utility equal to or greater than zero, otherwise she makes *exit*, and, if $t < \overline{T}$, she accepts any offer that gives her an utility equal to or greater than offering $x^*(t)$, otherwise she offers $x^*(t)$.

For the sake of simplicity, let $\iota(\overline{T}) = \mathbf{s}$; the backward induction construction with $\iota(\overline{T}) = \mathbf{b}$ is analogous. The unique equilibrium outcome of the subgame starting from time point $t = \overline{T}$ is *NoAgreement*, since $\mathbf{s}$ makes *exit*. Being zero the utility of *NoAgreement*, $\mathbf{s}$ would accept any offer made by $\mathbf{b}$ at $t = \overline{T}-1$ that gives her an utility equal to or greater than zero. Formally, she accepts any offer y such that $U_\mathbf{s}(y,\overline{T}) \geqslant 0$, namely, $y \geqslant RP_\mathbf{s}$.

Consider the subgame starting from time point $t = \overline{T}-1$. The optimal offer $x^*(\overline{T}-1)$ which $\mathbf{b}$ can make is $RP_\mathbf{s}$. Such an offer leads to the agreement $(RP_\mathbf{s}, \overline{T})$, which gives $\mathbf{b}$ an utility greater than the utility of the outcome she would reach in the subgame starting from $t = \overline{T}$, i.e., *NoAgreement*, while $\mathbf{s}$ is indifferent between *NoAgreement* and $(RP_\mathbf{s}, \overline{T})$. It can be easily observed that all the other available actions lead to outcomes that give $\mathbf{b}$ an utility strictly lower than offering $RP_\mathbf{s}$. More precisely, any other offer y that $\mathbf{s}$ would accept gives $\mathbf{b}$ a utility lower than offering $RP_\mathbf{s}$, being $y > RP_\mathbf{s}$; any offer that $\mathbf{s}$ would not accept gives $\mathbf{b}$ an utility of zero, since $\mathbf{s}$ will exit; and *exit* gives $\mathbf{b}$ zero. Therefore, $\mathbf{b}$ would accept at $t = \overline{T}-1$ any offer that gives her an utility equal to or greater than offering $RP_\mathbf{s}$, otherwise she offers $RP_\mathbf{s}$. Formally, she accepts any offer y such that $U_\mathbf{b}(y, \overline{T}-1) \geqslant U_\mathbf{b}(RP_\mathbf{s}, \overline{T})$, namely, $y \leqslant RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}$.

Consider the subgame starting from time point $t = \overline{T}-2$. The optimal offer $x^*(\overline{T}-2)$ which $\mathbf{s}$ can make is $RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}$. Such an offer leads to the agreement $(RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}, \overline{T}-1)$ which gives $\mathbf{s}$ an utility greater than the utility of the outcome she would reach in the subgame starting from $t = \overline{T}-1$, i.e. $(RP_\mathbf{s}, \overline{T})$, while $\mathbf{b}$ is indifferent between $(RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}, \overline{T}-1)$ and $(RP_\mathbf{s}, \overline{T})$. The utility $U_\mathbf{s}(RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}, \overline{T}-1)$ is positive, since, being $\delta_\mathbf{b} \in (0,1)$, it holds $RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b} > RP_\mathbf{s}$. Also in this case, as it happens in the subgame starting from $t = \overline{T}-1$, all the other available actions give $\mathbf{s}$ an utility strictly lower than offering $RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}$. Therefore, $\mathbf{s}$ would accept at $t = \overline{T}-2$ any offer that gives her an utility equal to or greater than offering $RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}$, otherwise she offers $RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}$. Formally, she accepts any offer y such that $U_\mathbf{s}(y, \overline{T}-2) \geqslant U_\mathbf{s}(RP_\mathbf{b} - (RP_\mathbf{b} - RP_\mathbf{s})\delta_\mathbf{b}, \overline{T}-1)$, namely, $y \geqslant RP_\mathbf{s} + (RP_\mathbf{b} - RP_\mathbf{s})(1-\delta_\mathbf{b})\delta_\mathbf{s}$.

This reasoning can be inductively carried back to the beginning of the game producing a sequence of agreements $(x^*(t), t+1)$s where each agreement $(x^*(t), t+1)$ is the equilibrium outcome of the subgame starting from time point t. Due to the presence of the discount factors, this sequence has the property that at each time point t it holds $U_{\iota(t)}(x^*(t), t+1) > U_{\iota(t)}(x^*(t+1), t+2)$ and $U_{\iota(t+1)}(x^*(t), t+1) = U_{\iota(t+1)}(x^*(t+1), t+2)$. On the equilibrium path the agents agree at time point $t = 1$, since agent $\iota(0)$ offers $x^*(0)$ at $t = 0$ and agent $\iota(1)$ accepts it at $t = 1$.

In order to provide a recursive formula for $x^*(t)$, we introduce the notion of *backward propagation*, whose definition is independent of the protocol we are studying.

Definition 1. Given $x \in \mathbb{R}$ and agent $i \in \{\mathbf{b}, \mathbf{s}\}$, we call *one-step backward propagation* of x along the isoutility curves of i the value $x_{\leftarrow i} \in \mathbb{R}$ such that $U_i(x_{\leftarrow i}, t-1) = U_i(x, t)$ for any time $t \in \{1, \dots, T_i\}$. (The notion is well defined because $x_{\leftarrow i}$ does not depend on the choice of $t \in \{1, \dots, T_i\}$.) Given $x \in \mathbb{R}$ and a sequence $s = \langle i_1, i_2, \dots, i_n \rangle \in \{\mathbf{b}, \mathbf{s}\}^n$, we call *backward propagation* of x along the curves of s the value $x_{\leftarrow s} \in \mathbb{R}$ such that

$$x_{\leftarrow s} = \begin{cases} (x_{\leftarrow s'})_{\leftarrow i_n} \text{ where } s' = \langle i_1, i_2, \dots, i_{n-1} \rangle & \text{if } n > 1 \\ x_{\leftarrow i_1} & \text{if } n = 1 \end{cases}$$

Two examples of multi-step propagations are $x_{\leftarrow \langle \mathbf{b}, \mathbf{s} \rangle} = (x_{\leftarrow \mathbf{b}})_{\leftarrow \mathbf{s}}$ and $x_{\leftarrow \langle \mathbf{b}, \mathbf{s}, \mathbf{b} \rangle} = ((x_{\leftarrow \mathbf{b}})_{\leftarrow \mathbf{s}})_{\leftarrow \mathbf{b}}$. For multi-step propagations we usually employ a shorter notation for repeated subsequences of agents; for example, $x_{\leftarrow \mathbf{b}3[\mathbf{bs}]}$ stands for $x_{\leftarrow \langle \mathbf{b}, \mathbf{b}, \mathbf{s}, \mathbf{b}, \mathbf{s}, \mathbf{b}, \mathbf{s} \rangle}$.

On the basis of the notion of backward propagation the values of $x^*(t)$ can be easily defined as follows:

$$x^*(t-1) = \begin{cases} RP_{\iota(t)} & \text{if } t = \overline{T} \\ (x^*(t))_{\leftarrow \iota(t)} & \text{if } t < \overline{T} \end{cases}$$

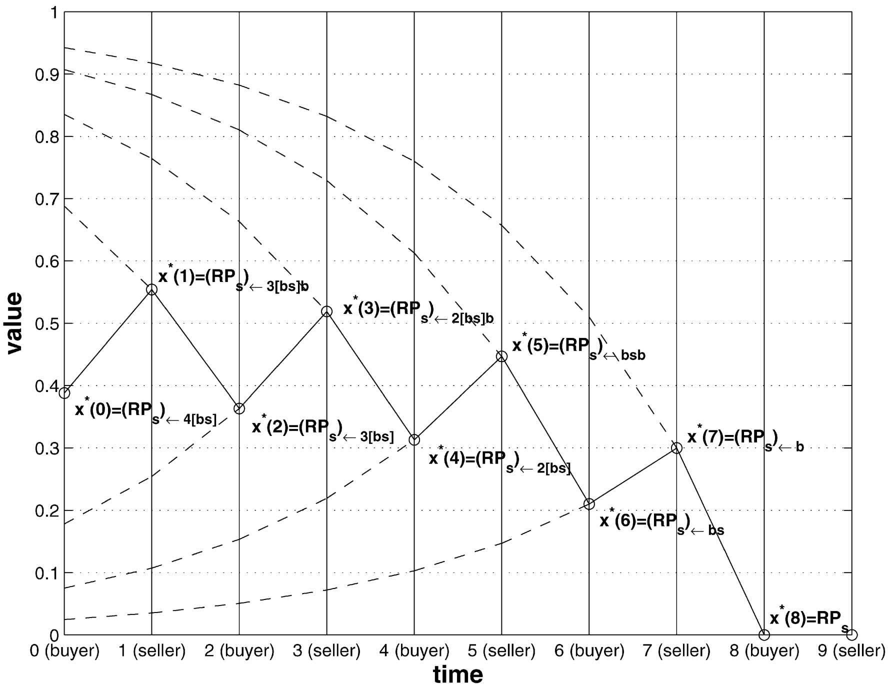

Fig. 1. Backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.7$, $\delta_{\mathbf{s}} = 0.7$, $T_{\mathbf{b}} = 9$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$; at each time point t the optimal offer $x^*(t)$ that $\iota(t)$ can make is marked; the dashed lines are isoutility curves.

where the formulas to compute $(x^*(t))_{\leftarrow\mathbf{b}}$ and $(x^*(t))_{\leftarrow\mathbf{s}}$ are: $(x^*(t))_{\leftarrow\mathbf{b}} = RP_{\mathbf{b}} - (RP_{\mathbf{b}} - x^*(t))\delta_{\mathbf{b}}$ and $(x^*(t))_{\leftarrow\mathbf{s}} = RP_{\mathbf{s}} + (x^*(t) - RP_{\mathbf{s}})\delta_{\mathbf{s}}$.

Fig. 1 shows an example of backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.7$, $\delta_{\mathbf{s}} = 0.7$, $T_{\mathbf{b}} = 9$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$. We report in the figure for any time point t the optimal offer $x^*(t)$ agent $\iota(t)$ can make; the dashed lines are agents' isoutility curves. The time point from which we can apply the backward induction method, $\overline{T} = \min\{T_{\mathbf{b}}, T_{\mathbf{s}}\}$, is $\overline{T} = 9$. At $t = 9$ agent $\iota(9) = \mathbf{s}$ will accept any offer equal to or greater than 0, being $RP_{\mathbf{s}} = 0$. The optimal offer $x^*(8)$ of $\mathbf{b}$ at $t = 8$ is thus $RP_{\mathbf{s}} = 0$. The optimal offer $x^*(7)$ of $\mathbf{s}$ at $t = 7$ is $(x^*(8))_{\leftarrow\mathbf{b}} = RP_{\mathbf{b}} - (RP_{\mathbf{b}} - x^*(8))\delta_{\mathbf{b}} = 0.3$. Analogously, the optimal offer $x^*(6)$ of $\mathbf{b}$ at $t = 6$ is $(x^*(7))_{\leftarrow\mathbf{s}} = RP_{\mathbf{s}} + (x^*(7) - RP_{\mathbf{s}})\delta_{\mathbf{s}} = 0.21$. The process continues to the initial time point $t = 0$.

The equilibrium strategies can be easily defined by specifying, according to $x^*(t)$, the rules the agents employ to choose their optimal action at each time point t:

$$\sigma^*_{\mathbf{b}}(t) = \begin{cases} t = 0 & \textit{offer}(x^*(0)) \\ 0 < t < \overline{T} & \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant x^*(t-1) \\ \textit{offer}(x^*(t)) & \textit{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{b}} & \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant RP_{\mathbf{b}} \\ \textit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{b}} < t & \textit{exit} \end{cases} \tag{1}$$

$$\sigma^*_{\mathbf{s}}(t) = \begin{cases} t = 0 & \textit{offer}(x^*(0)) \\ 0 < t < \overline{T} & \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant x^*(t-1) \\ \textit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{s}} & \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant RP_{\mathbf{s}} \\ \textit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{s}} < t & \textit{exit} \end{cases} \tag{2}$$

The above strategies constitute the unique subgame perfect equilibrium of bargaining in presence of deadlines with complete information (see [28]). The equilibrium outcome is $(x^*(0), 1)$. They can be computed in time linear in the length of the bargaining.

3. Problem statement and known solutions

In this section we state (Section 3.1) the problem of bargaining with the alternating-offers protocol in presence of one-sided uncertain deadlines, by enriching the complete information bargaining model presented in the previous section and introducing appropriate solution concepts. We review (Section 3.2) the solutions currently available in literature and we present (Section 3.3) the solution suggested in this paper.

3.1. Model enrichment and appropriate solution concept

We consider single-issue alternating-offers bargaining when one of the two agents does not exactly know her opponent's deadline. We will assume that the uncertain deadline is the buyer's; the case wherein the uncertain deadline is the seller's can be treated analogously.

As is customary in game theory to avoid situations that cannot be faced, an incomplete information game is casted into an imperfect information game with the introduction of probability distributions over the unknown parameters. In our case we assume that **b**'s possible deadlines are distributed according to a probability distribution on $\mathbb{R}^+$ which is common knowledge between the agents. We further assume that the support of the probability distribution is bounded; and, since the agents can act only at time points $t \in \mathbb{N}$, we can assume, without loss of generality, that the probability distribution is finite on $\mathbb{N}$. We denote by $\mathbf{T_b} = \{T_{\mathbf{b}_1}, \dots, T_{\mathbf{b}_n}\}$ the set of possible deadlines of **b**, by $P^0_{\mathbf{b}} = \{\omega^0_{\mathbf{b}_1}, \dots, \omega^0_{\mathbf{b}_n}\}$ the pertinent probability distribution, and by $BT^0_{\mathbf{b}}$ the couple $BT^0_{\mathbf{b}} = \langle \mathbf{T_b}, P^0_{\mathbf{b}} \rangle$. Without loss of generality, we assume that for any $i \in [1, n-1]$ it holds $T_{\mathbf{b}_i} < T_{\mathbf{b}_{i+1}}$. We denote by $\mathbf{b}_i$ the type of **b** whose deadline is $t = T_{\mathbf{b}_i}$. Agent **b**'s real deadline is known only to **b** herself: it is her *private information*.

The solution concept employed in extensive-form games with complete information, namely, subgame perfect equilibrium, is not satisfactory when information is imperfect. Specifically, it does not have the power to cut the so called *incredible threats* [14], i.e., Nash equilibria that are non-reasonable given the sequential structure of the game. The most common refinement of the subgame perfect equilibrium concept in presence of information imperfectness is the *sequential equilibrium* of Kreps and Wilson [22]. We now review this concept.

Rational agents try to maximize their expected utilities relying on their beliefs about the opponent's private information [38] and such beliefs are updated during the game, depending on which actions have been actually made [22]. The set of beliefs held by each agent over the other's private information after every possible sequence of actions in the game is called a *system of beliefs* and is usually denoted by μ. These beliefs are probabilistic and their values at time point $t = 0$ are given data of the problem. How beliefs evolve during the game is instead part of the solution which should be found for the game. A solution of an incomplete information bargaining is therefore a suitable couple $a = \langle \mu, \sigma \rangle$ called *assessment*.

An assessment $a = \langle \mu, \sigma \rangle$ must be such that the strategies in σ are mutual best responses given the probabilistic beliefs in μ (*rationality*); and the beliefs in μ must reasonably depend on the actions prescribed by σ (*consistency*). Different solution concepts differ on how they specify these two requirements.

For a sequential equilibrium $a^* = \langle \mu^*, \sigma^* \rangle$, with $\sigma^* = \langle \sigma^*_{\mathbf{b}}, \sigma^*_{\mathbf{s}} \rangle$, the rationality requirement is specified as *sequential rationality*. Informally, after every possible sequence of actions S, on or off the equilibrium path, the strategy $\sigma^*_{\mathbf{s}}$ must maximize **s**'s expected utility given **s**'s beliefs prescribed by μ for S, and given that **b** will act according to $\sigma^*_{\mathbf{b}}$ from there on and *vice versa*. The notion of consistency is defined as follows: assessment a is *consistent in the sense of Kreps and Wilson* (or simply *consistent*) if there exists a sequence a_n of assessments, each with fully mixed strategies and such that the beliefs are updated according to Bayes' rule, that converges to a. By Kreps and Wilson's theorem any extensive-form game in incomplete information admits at least one sequential equilibrium in mixed strategies [22].

Moreover, as is customary in economic studies, e.g. Rubinstein's [34], we consider only *stationary* systems of beliefs, namely, if **s** believes a **b**'s type with zero probability at time point $\bar{t}$, then she will continue to believe such a type with zero probability at any time point $t > \bar{t}$.

3.2. Solutions known in literature

The computation of a solution of an extensive-form game with imperfect information is known to be a hard task to tackle. Contrary to what happens with complete information games, classic game theory does not provide any solving technique to find sequential equilibria. From [19] there is long standing literature in computer science, e.g. [15,20,23], that studies algorithms to find Nash equilibria and refinements, e.g. sequential equilibria [27], searching in the space of the strategies. These algorithms have two main drawbacks that make them inapplicable in solving bargaining situations: they work only with games with finite strategies and they produce equilibrium strategies but not systems of beliefs. The lack of a solving algorithm for games as bargaining has pushed researchers to game theoretically study each possible specific setting and develop relative algorithms. Well known examples of alternating-offers settings studied in literature are: in [34] Rubinstein analyzes one-sided uncertainty over the discount factors with only two types, in [5] Di Giunta and Gatti analyze in pure strategies one-sided uncertainty over deadlines, in [3] Chatterjee and Samuelson analyze uncertainty over reservation prices where the agents can be of two possible types, and in [12] Fatima et al. consider a slight variation of the alternating-offers protocol where there are multiple issues to negotiate and analyze uncertainty over the weights of the issues.

The best known results for bargaining in the alternating-offers protocol with uncertain deadlines are due to Fatima et al. in [9,10] and in [11]. In [9,10] they study the alternating-offers protocol with two-sided uncertainty over deadlines and bounds on the agents' rationality, more precisely, agents are self-constrained to play predefined tactics based on the *negotiation decision functions* paradigm [8]. In [11] they consider a slight variation of the alternating-offers protocol where there are multiple issues. The bargaining model they consider is exactly the one presented in Section 2.1, except that the agents' utility functions, offers, and acceptance are defined on a tuple of issues, instead of a single issue. The solution of this model with complete information is analogous to the solution of the single issue model presented in Section 2.2 and can be easily obtained by extending, as showed in [7], the offers' backward propagation we defined in Section 2.2. In presence of a single issue the solution of the multiple issue model and the one of the single issue model collapse. Fatima et al. analyze the situation where the deadlines are uncertain and propose an algorithm that finds equilibrium assessments in pure strategies for all the values of the parameters. Basically, their algorithm searches in the space of the strategies exploiting the backward induction from the last possible deadline to $t = 0$ with agents' initial beliefs, and, once the optimal strategies at time point $t = 0$ have been found, the system of beliefs is designed to be consistent with them. The computational complexity of their algorithm is linear in the length of the bargaining and polynomial in the number of agents' types. This work fails in finding equilibria for some values of parameters. Indeed, as [5] shows, for a non-null measure subset of the space of the parameters there is not any equilibrium assessment in pure strategies.[3] The reason behind the failure of [11] in producing equilibrium strategies for some settings of parameters is that in each step of backward induction they limit the search to the space of the strategies, but they do not verify the existence of a consistent system of beliefs such that the found strategy is sequentially rational. As a result, once their algorithm has produced the agents' strategies at $t = 0$ and has designed the system of beliefs consistent with them, the strategies could be not sequentially rational given the designed system of beliefs. This happens for all the values of the parameters such that there is not any equilibrium in pure strategies [5]. We show an example where their algorithm fails in Section 4.1.3.

Analogously to the result presented in [5], it can be easily showed that for a non-null measure subset of the space of the parameters there is not any equilibrium assessment in fully mixed strategies. Therefore, in the provision of a satisfactory solution to the bargaining with uncertain deadlines, it is necessary to employ both pure and mixed strategies.

3.3. Overview of our solution

In this paper we go beyond the state of art along two directions: our algorithm (i) finds a sequential equilibrium for all the values of the parameters and (ii) requires less computational time than the algorithms presented in literature so far. In particular, the computational time required by the proposed algorithm is asymptotically the same time required to compute the solution with complete information, being linear in the length of the bargaining and asymptotically independent of the number of types of the agent whose deadline is uncertain. Furthermore there is not any equilibrium assessment that can be computed backward faster than ours. We present our result by degrees in Sections 4, 5, and 6 as follows.

We first analyze the alternating-offers protocol with uncertain deadlines in pure strategies (Section 4). We find an assessment in pure strategies a_p that is an equilibrium under some conditions on the parameters. We show that, except

[3] In measure theory, a null set is a set that is negligible for the purposes of the measure in question [17]. As commonly done in literature to study sets in Euclidean n-space $\mathbb{R}^n$, we use Lebesgue measure.

for a null measure subset of the space of the parameters, a_p is an equilibrium whenever there is an equilibrium in pure strategies. Moreover, we show that there is not any other equilibrium assessment in pure strategies that can be computed backward faster than a_p. We derive from a_p a choice rule that applies to each time point the optimal actions of the agents and we find the conditions on the parameters such that the choice rule can be employed to produce an equilibrium. Then, we resort to mixed strategies in analyzing the alternating-offers protocol with uncertain deadlines (Section 5). We find an assessment in mixed strategies a_m that is an equilibrium for all the values of the parameters such that a_p is not an equilibrium. More precisely, the conditions on the parameters such that a_m is an equilibrium are complementary to the conditions that make a_p an equilibrium. Therefore, there is always exactly one assessment, either a_p or a_m, that is an equilibrium. Moreover, no assessment in mixed strategies can be computed backward faster than a_m. We derive from a_m a choice rule of the agents and the relative conditions on the parameters. Finally, we provide (Section 6) our solving algorithm and we show how it can be extended in presence of multiple issues.

4. Equilibrium analysis in pure strategies

In this section we present (Section 4.1) a specific assessment in pure strategies that is an equilibrium for a non-null measure subset of the space of the parameters;[4] then we show (Section 4.2) that no equilibrium assessment in pure strategies can be computed backward faster than ours and that, except for a null measure subset of the space of the parameters, there is not any equilibrium in pure strategies when our assessment is not an equilibrium.

4.1. A pure strategy equilibrium assessment

Differently from what happens in presence of complete information, game theory does not provide any solving technique to find equilibrium assessments when information is imperfect. Indeed, the backward induction method cannot be employed because it does not consider the possible evolution of the beliefs. Here we exploit the idea behind the backward induction method combining it with an *a priori* fixed system of beliefs. More precisely, the method we will employ is the following:

(i) *a priori* fix a reasonable system of beliefs $\overline{\mu}$,
(ii) use backward induction to find the optimal strategies σ^* given $\overline{\mu}$,
(iii) identify possible anomalies in the use of backward induction, i.e. situations where the produced strategies σ^* are not sequentially rational,
(iv) *a posteriori* prove the consistency of the assessment.

We now describe the application of our method.

4.1.1. Fixing the system of beliefs

We base our system of beliefs on the idea that $\mathbf{b}$ can signal her type on the equilibrium path only at her real deadline, namely, a buyer $\mathbf{b}_i$ can signal her type only at $t = T_{\mathbf{b}_i}$. Implicitly, this means that at any time point t all the types $\mathbf{b}_i$s whose deadline has not expired (i.e., $T_{\mathbf{b}_i} \geqslant t$) have the same equilibrium strategies but the type $\mathbf{b}_i$ with $T_{\mathbf{b}_i} = t$ (if she exists). Although such an assumption seems very restrictive, we show in Section 4.2 that, if there is an equilibrium assessment in pure strategies, then it respects such an assumption. In other words, bargaining in the alternating-offers protocol with uncertain deadlines in pure strategies does not admit any equilibrium assessment where $\mathbf{b}$ can signal her type at a time point t different from her real deadline.

We fix our system of beliefs such that, after any sequence of actions, $\mathbf{s}$ just excludes those deadlines $T_{\mathbf{b}_i}$s among the initially possible ones that have already expired and normalizes the probabilities of the future ones. Notice that among all the possible systems of beliefs that satisfy the property discussed above, ours is the simplest one: it employs the same upgrading rule both on and off the equilibrium path.

We assume, for the sake of generality, that any time point $t \in [0, T_{\mathbf{b}_n}]$ is a possible buyer's deadline. We denote by $\omega_{\mathbf{b}}^{t_1}(t_2)$ the probability, calculated at $t = t_1$ with $\overline{\mu}$, that $\mathbf{b}$'s deadline is at $t = t_2$, and by $\Omega_{\mathbf{b}}^{t_1}(t_2) = \sum_{t=t_2}^{+\infty} \omega_{\mathbf{b}}^{t_1}(t)$ the

[4] This result has been preliminarily presented in [5].

cumulative probability, calculated at $t = t_1$ with $\overline{\mu}$, that $\mathbf{b}$'s deadline is at $t \geqslant t_2$, respectively. The value of $\omega^0_{\mathbf{b}}(t)$ is set according to the initial beliefs $BT^0_{\mathbf{b}}$. Our system of beliefs $\overline{\mu}$ can be written as:

$$\overline{\mu}(t) = \begin{cases} \text{for any } \tau \text{ such that } \tau < t & \omega^t_{\mathbf{b}}(\tau) = 0 \\ \text{for any } \tau \text{ such that } t \leqslant \tau & \omega^t_{\mathbf{b}}(\tau) = \frac{\omega^0_{\mathbf{b}}(\tau)}{\Omega^0_{\mathbf{b}}(t)} \end{cases}$$

4.1.2. Backward induction with the fixed system of beliefs

Given the system of beliefs $\overline{\mu}$, we can find the optimal strategies using backward induction. However, in this context the use of backward induction is more involved than in the complete information bargaining and requires some explanations. More precisely, two issues are crucial: the determination of the time point $\overline{T}$ from which the backward induction construction starts and the determination of the optimal offers along the construction.

We first focus on the determination of $\overline{T}$. With complete information the backward induction construction can start at the earliest of the two deadlines, while with our incomplete information framework, the earliest deadline is not *a priori* known. Nevertheless, the backward induction construction can start from $\overline{T} = \min\{\max_i\{T_{\mathbf{b}_i}\}, T_{\mathbf{s}}\}$, because it is *a priori* known that after such a time point the agents will exit the negotiation; if the bargaining process reaches the time point $\overline{T}$, then agent $\iota(\overline{T})$ would accept any non-negative utility offer; therefore, at time point $t = \overline{T} - 1$ agent $\iota(\overline{T} - 1)$, if she make an offer, would offer $\iota(\overline{T})$'s reservation price.

We now focus on the determination of the sequence of the optimal offers $x^*(t)$s from time point $t = \overline{T} - 1$ to time point $t = 0$. Although $\mathbf{b}$'s types can be many, the sequence of the optimal offers $x^*(t)$s is the same for all the $\mathbf{b}$'s types. This is because $\overline{\mu}$ prescribes that all $\mathbf{b}$'s types whose deadline is beyond t have the same optimal action at t and therefore, if they make an offer, they make the same offer; instead, $\mathbf{b}$'s types whose deadline is at t or before do not make any offer, but *exit*. Like in the complete information setting described in Section 2, it is possible to backwardly infer the sequence of the offers $x^*(t)$s that the agents would do if they choose to make an offer. But there are some complications in the construction of the sequence of offers $x^*(t)$s. In the complete information case the offers composing this sequence have two properties:

(i) $x^*(t)$ is the value propagated backward from time point $t + 1$,
(ii) $x^*(t)$ is the value to be propagated backward at time point $t - 1$.

Essentially, this makes an offer $x^*(t)$ to be the optimal offer $\mathbf{s}$ can make at time point t and, at the same time, the optimal offer $\mathbf{b}$ can accept at time point $t + 1$. With imperfect information in bargaining, instead, not holding in general properties (i) and (ii), the optimal offer of $\mathbf{s}$ at time point t could be different from the optimal offer that $\mathbf{b}$ would accept at time point $t + 1$. In what follows we preliminarily introduce these topics informally and then we report the exact formulas.

First, when $\iota(t) = \mathbf{s}$ the optimal offer $x^*(t)$ is not generally the value propagated backward from time point $t + 1$, being $x^*(t)$ generally different from $(x^*(t + 1))_{\leftarrow \mathbf{b}}$. This is because different $\mathbf{b}$'s types can have different maximum acceptable offers at time point $t + 1$ (e.g., $\mathbf{b}_1$ accepts any offer $\leqslant x_1$ and $\mathbf{b}_2$ accepts any offer $\leqslant x_2$ with $x_1 < x_2$), and the determination of what offer is to propagate backward depends on the beliefs of $\mathbf{s}$. This happens at any time point t where $\iota(t) = \mathbf{s}$ and the time point $t + 1$ is a possible deadline of $\mathbf{b}$. In this situation, $\mathbf{b}$'s type whose deadline is at time point $t + 1$, obtaining a non-positive utility from the continuation of the game, will accept any offer equal to or lower than $RP_{\mathbf{b}}$, otherwise she will make *exit*. Instead, since all $\mathbf{b}$'s types whose deadlines are beyond the time point $t + 1$ gain a positive utility from the continuation of the game, they will accept any offer equal to or lower than $(x^*(t + 1))_{\leftarrow \mathbf{b}}$, otherwise they will offer $x^*(t + 1)$. The value of $(x^*(t + 1))_{\leftarrow \mathbf{b}}$, determined by backward induction from $\overline{T}$ to $t + 1$, is obviously lower than $RP_{\mathbf{b}}$. Let $EU_{\mathbf{s}}(\sigma, t)$ be the expected utility of $\mathbf{s}$ in making the strategy σ at time point t. The expected utility $EU_{\mathbf{s}}(\sigma, t)$ when $\sigma = \textit{offer}(x)$ is:

$$EU_{\mathbf{s}}(\textit{offer}(x), t) = \begin{cases} x > RP_{\mathbf{b}} & \Omega^t_{\mathbf{b}}(t+2) \cdot U_{\mathbf{s}}(x^*(t+1), t+2) \\ RP_{\mathbf{b}} \geqslant x > (x^*(t+1))_{\leftarrow \mathbf{b}} & \omega^t_{\mathbf{b}}(t+1) \cdot U_{\mathbf{s}}(x, t+1) + \Omega^t_{\mathbf{b}}(t+2) \cdot U_{\mathbf{s}}(x^*(t+1), t+2) \\ (x^*(t+1))_{\leftarrow \mathbf{b}} \geqslant x & U_{\mathbf{s}}(x, t+1) \end{cases}$$

That is, any offer $x > RP_{\mathbf{b}}$ will be rejected by all $\mathbf{b}$'s types and the type whose deadline is at $t + 1$ will make *exit*, whereas all the other types whose deadline has not expired will make the offer $x^*(t + 1)$ that $\mathbf{s}$ will accept at $t + 2$;

any offer $RP_{\mathbf{b}} \geqslant x > (x^*(t+1))_{\leftarrow \mathbf{b}}$ will be accepted by the type whose deadline is at $t+1$, but it will be rejected by all the other types to make the offer $x^*(t+1)$ that **s** will accept at $t+2$; any offer $(x^*(t+1))_{\leftarrow \mathbf{b}} \geqslant x$ will be accepted by all the types. We recall that $\omega^t_{\mathbf{b}}(t+1)$ is the probability calculated at t with $\overline{\mu}$ that **b**'s deadline is at t and $\Omega^t_{\mathbf{b}}(t+2)$ is the probability calculated at t with $\overline{\mu}$ that **b**'s deadline is beyond $t+1$. In order to maximize $EU_{\mathbf{s}}(\sigma, t)$, **s** chooses her optimal action between offering $(x^*(t+1))_{\leftarrow \mathbf{b}}$ and offering $RP_{\mathbf{b}}$. Notice that all the other possible offers that **s** can make are dominated by making her best offer between $(x^*(t+1))_{\leftarrow \mathbf{b}}$ and $RP_{\mathbf{b}}$.

Second, when $\iota(t) = \mathbf{s}$ the optimal offer $x^*(t)$ is not generally the value to be propagated backward, being $x^*(t-1)$ generally different from $(x^*(t))_{\leftarrow \mathbf{s}}$. To determine the value to be propagated backward we need to introduce the notion of *equivalent value* of an offer x at time point t: given an offer x made by **s** at time point t, the equivalent value of x, denoted by $e(x,t)$, is the value such that $U_{\mathbf{s}}(e(x,t),t) = EU_{\mathbf{s}}(\mathit{offer}(x),t)$. Clearly, the equivalent value of an offer that will be accepted with a probability equal to 1 is the offer itself. The right value to be propagated backward is the equivalent value of $x^*(t)$; we denote it by $e^*(t)$. Easily, as optimal offers of agent $\iota(t) = \mathbf{s}$ could be rejected, **s** could accept at time point t an offer $\bar{x}$ such that $EU_{\mathbf{s}}(\mathit{accept}, t) = U_{\mathbf{s}}(\bar{x}, t) = EU_{\mathbf{s}}(\mathit{offer}(RP_{\mathbf{b}}), t)$. Summarily, if **b** makes an offer at a time point t greater than or equal to $(e^*(t+1))_{\leftarrow \mathbf{s}}$, then **s** accepts it; if **b** makes an offer lower than $(e^*(t+1))_{\leftarrow \mathbf{s}}$, then **s** rejects it and counteroffers $x^*(t+1)$.

Formulas to find the equivalent value $e(x,t)$ and the sequence of the optimal offers $x^*(t)$s can be easily derived from the formula of $EU_{\mathbf{s}}(\mathit{offer}(x),t)$:

- when $\iota(t) = \mathbf{b}$, the optimal offer $x^*(t)$ is $x^*(t) = (e^*(t+1))_{\leftarrow \mathbf{s}}$ and, being such an offer surely accepted by **s** at $t+1$, the relative equivalent is the offer itself, i.e. $e^*(t) = x^*(t)$;
- when $\iota(t) = \mathbf{s}$, the equivalent value of an offer x depends on **s**'s beliefs and it is:

$$e(x,t) = \begin{cases} x > RP_{\mathbf{b}} & \Omega^t_{\mathbf{b}}(t+2) \cdot ((x^*(t+1))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}} \\ RP_{\mathbf{b}} \geqslant x > (x^*(t+1))_{\leftarrow \mathbf{b}} & \omega^t_{\mathbf{b}}(t+1) \cdot (x - RP_{\mathbf{s}}) + \Omega^t_{\mathbf{b}}(t+2) \cdot ((x^*(t+1))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}} \\ (x^*(t+1))_{\leftarrow \mathbf{b}} \geqslant x & \Omega^t_{\mathbf{b}}(t+1) \cdot (x - RP_{\mathbf{s}}) + RP_{\mathbf{s}} \end{cases}$$

  the optimal offer $x^*(t)$ is the offer between $RP_{\mathbf{b}}$ and $(e^*(t+1))_{\leftarrow \mathbf{b}}$ that maximizes $e(x,t)$, and consequently $e^*(t) = e(x^*(t),t)$. Notice that for **s** maximizing $e(x,t)$ corresponds to maximizing $EU_{\mathbf{s}}(\mathit{offer}(x),t)$.

With the system of beliefs $\overline{\mu}$ the agents' optimal action is unique at any time point t, except when the beliefs are such that both offering $RP_{\mathbf{b}}$ and offering $(e^*(t+1))_{\leftarrow \mathbf{b}}$ maximize $EU_{\mathbf{s}}(\sigma,t)$. In all these situations **s** can indifferently offer one between $RP_{\mathbf{b}}$ and $(e^*(t+1))_{\leftarrow \mathbf{b}}$. Being the equivalent value of offering $RP_{\mathbf{b}}$ and $(e^*(t+1))_{\leftarrow \mathbf{b}}$ equal, the optimal offer of **b** at $t-1$ is independent of what offer **s** would actually make at t.

Given $x^*(t)$, the optimal action $\sigma^*_{\iota(t)}(t)$ of agent $\iota(t)$ at time point t is: *exit*, if the $\iota(t)$'s deadline is at t and she has received an offer that gives her negative utility; *accept*, if the $\iota(t)$'s deadline is at $t+1$ and she has received an offer that gives her non-negative utility or if the $\iota(t)$'s deadline is not at $t+1$ and she has received an offer not worse for her than $(e^*(t))_{\leftarrow \iota(t)}$; $\mathit{offer}(x^*(t))$, otherwise. The calculation of $x^*(t)$ can be accomplished recursively on the basis of $e(x,t)$ as follows:

$$x^*(t-1) = \begin{cases} RP_{\iota(t)} & \text{if } t = \overline{T} \\ \begin{cases} \arg\max_{x \in \{RP_{\mathbf{b}}, (e^*(t))_{\leftarrow \mathbf{b}}\}} \{e(x,t-1)\} & \text{if } \iota(t) = \mathbf{b} \\ (e^*(t))_{\leftarrow \mathbf{s}} & \text{if } \iota(t) = \mathbf{s} \end{cases} & \text{if } t < \overline{T} \end{cases}$$

The time required to compute the sequences of offers $x^*(t)$s and equivalents $e^*(t)$s is linear in the length of the bargaining and asymptotically independent of the number of the buyer's types. With respect to the computation of the solution with complete information it is needed at most a maximization between two possible values at any time point where $\iota(t) = \mathbf{s}$. The equilibrium strategies can be defined specifying the agents' choice rules on the basis of $x^*(t)$ and $e^*(t)$ as extension of the ones with complete information reported in (1) and (2). More precisely, they are:

$$\sigma^*_{\mathbf{b}_i}(t) = \begin{cases} t = 0 & \mathit{offer}(x^*(0)) \\ 0 < t < \overline{T} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (x^*(t))_{\leftarrow \mathbf{b}} \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{b}_i} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant RP_{\mathbf{b}} \\ \mathit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{b}_i} < t & \mathit{exit} \end{cases} \tag{3}$$

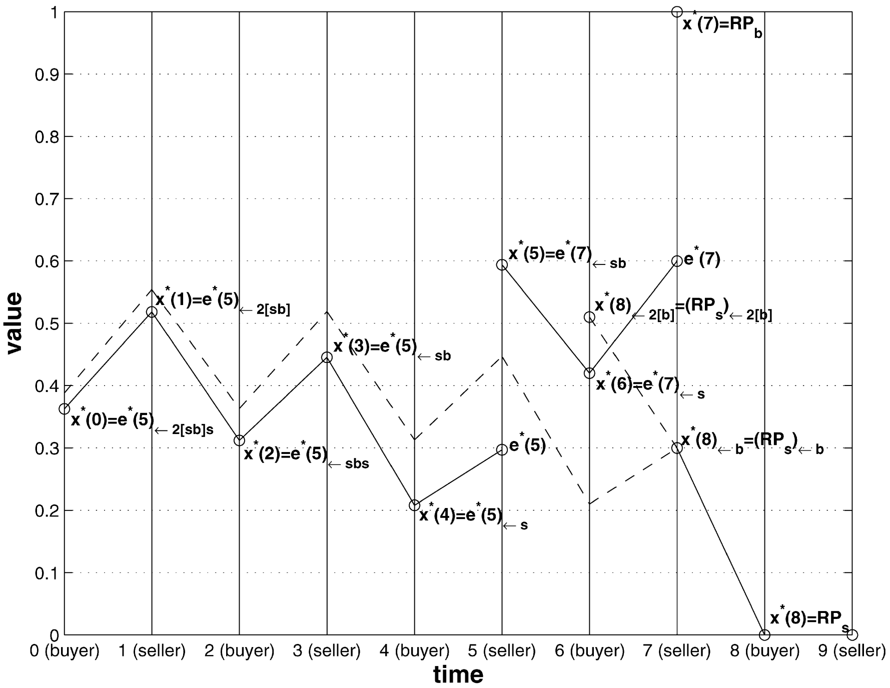

Fig. 2. Backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.7$, $\delta_{\mathbf{s}} = 0.7$, $\langle \mathbf{T_b} = \{5, 8, 9\}, P^0_{\mathbf{b}} = \{0.5, 0.3, 0.2\}\rangle$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$; at each time point t the optimal offer $x^*(t)$ that $\iota(t)$ can make and the relative $e^*(t)$ are reported.

$$\sigma^*_{\mathbf{s}}(t) = \begin{cases} t = 0 & \mathit{offer}(x^*(0)) \\ 0 < t < \overline{T} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant (e^*(t))_{\leftarrow \mathbf{s}} \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{s}} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant RP_{\mathbf{s}} \\ \mathit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{s}} < t & \mathit{exit} \end{cases} \tag{4}$$

Differently from what happens with complete information, the equilibrium agreement can be reached beyond time point $t = 1$. For example, when $\iota(0) = \mathbf{s}$, time point $t = 1$ is a possible $\mathbf{b}$'s deadline but not the real one and $\mathbf{s}$'s beliefs are such that her optimal offer at time point $t = 0$ is $x^*(0) = RP_{\mathbf{b}}$.

Fig. 2 shows an example of backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.7$, $\delta_{\mathbf{s}} = 0.7$, $\langle \mathbf{T_b} = \{5, 8, 9\}, P^0_{\mathbf{b}} = \{0.5, 0.3, 0.2\}\rangle$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$. In the figure we report, at any time point t, the optimal offer $x^*(t)$ agent $\iota(t)$ can make and the relative equivalent value $e^*(t)$; the dashed line from $t = 7$ to $t = 0$ is the construction with complete information; the dashed line that connects $(7, (x^*(8))_{\leftarrow \mathbf{b}})$ to $(6, (x^*(8))_{\leftarrow 2[\mathbf{b}]})$ will be taken into account in the next section. The time point from which we can apply the backward induction method, $\overline{T} = \min\{\max\{\mathbf{T_b}\}, T_{\mathbf{s}}\}$, is $\overline{T} = 9$. Since at time points $t = 9$ and $t = 8$ $\mathbf{s}$ believes $\mathbf{b}$'s deadline to be $t = 9$, the construction in these time points is exactly the one accomplished with complete information. That is, at $t = 9$ agent $\iota(9) = \mathbf{s}$ will accept any offer equal to or greater than 0, being $RP_{\mathbf{s}} = 0$. The optimal offer $x^*(8)$ of $\mathbf{b}$ at $t = 8$ is thus $RP_{\mathbf{s}} = 0$. At $t = 7$ the beliefs of $\mathbf{s}$ are $\omega^7_{\mathbf{b}}(8) = 0.6$ and $\omega^7_{\mathbf{b}}(9) = 0.4$. Being $t = 8$ a possible $\mathbf{b}$'s deadline, at $t = 7$ agent $\iota(7) = \mathbf{s}$ chooses her optimal action between two alternatives: to offer $(x^*(8))_{\leftarrow \mathbf{b}} = 0.3$ that $\mathbf{b}$'s types with deadlines at $t = 8$ and $t = 9$ will accept or to offer $RP_{\mathbf{b}} = 1$ that will be accepted only by the type with deadline at $t = 8$ and rejected by the other type to counteroffer $RP_{\mathbf{s}}$. The first alternative has an equivalent value $e((x^*(8))_{\leftarrow \mathbf{b}}, 7) = (x^*(8))_{\leftarrow \mathbf{b}} = 0.3$, whereas the second alternative has an equivalent value $e(RP_{\mathbf{b}}, 7) = \omega^7_{\mathbf{b}}(8) \cdot (RP_{\mathbf{b}} - RP_{\mathbf{s}}) + \omega^7_{\mathbf{b}}(9) \cdot (RP_{\mathbf{s}} - RP_{\mathbf{s}})\delta_{\mathbf{s}} + RP_{\mathbf{s}} = 0.6$. The optimal offer $x^*(7)$ of $\mathbf{s}$ at $t = 7$ is therefore $RP_{\mathbf{b}}$ with $e^*(7) = 0.6$. The optimal offer $x^*(6)$ of $\mathbf{b}$ at $t = 6$ is $(e^*(7))_{\leftarrow \mathbf{s}} = 0.4$. At $t = 5$ the beliefs of $\mathbf{s}$ are $\omega^5_{\mathbf{b}}(5) = 0.5$, $\omega^5_{\mathbf{b}}(7) = 0.3$, and $\omega^5_{\mathbf{b}}(9) = 0.2$. The optimal offer $x^*(5)$ of $\mathbf{s}$ at $t = 5$ is $(x^*(6))_{\leftarrow \mathbf{b}} = (e^*(7))_{\leftarrow \mathbf{sb}} = 0.594$ and the relative equivalent, being $t = 5$ a possible deadline of $\mathbf{b}$, is $e^*(5) = \Omega^5_{\mathbf{b}}(6) \cdot (x^*(5) - RP_{\mathbf{s}}) + RP_{\mathbf{s}} = 0.297$. The optimal offer $x^*(4)$ of $\mathbf{b}$ at $t = 4$ is $(e^*(5))_{\leftarrow \mathbf{s}} = 0.2079$. From $t = 3$ to $t = 0$ the backward induction construction continues as with complete information.

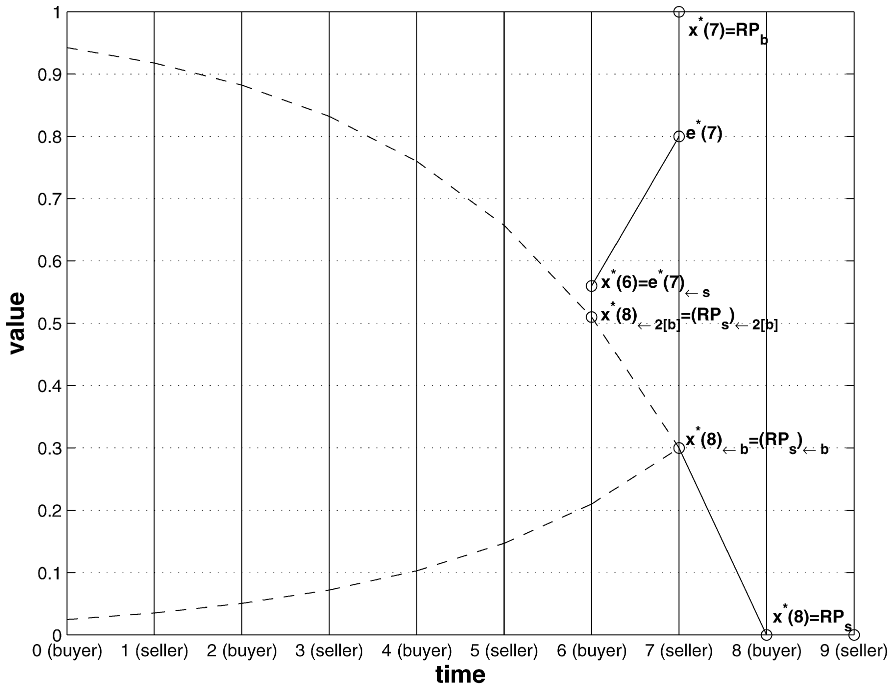

Fig. 3. Non-existence of equilibrium with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.7$, $\delta_{\mathbf{s}} = 0.7$, $\langle \mathbf{T_b} = \{5, 8, 9\}, P^0_{\mathbf{b}} = \{0.5, 0.4, 0.1\}\rangle$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$: the sequential rationality is not respected because $(e^*(7))_{\leftarrow \mathbf{s}} > (x^*(8))_{\leftarrow 2[\mathbf{b}]}$; the dashed lines are isoutility curves.

4.1.3. Anomalies in backward induction construction

The above backward induction construction provides the sequentially rational strategies relying on the system of beliefs $\overline{\mu}$ we have chosen, when they exist. However, they could not exist for some parameter settings. This can happen because one agent (precisely, the buyer) might deviate from strategy σ^* to offer something unacceptable for the opponent in order to be refused and to later be in a much stronger position and gain more. An example is depicted in Fig. 3.

The bargaining situation considered in Fig. 3 is the same considered in Fig. 2, except for the values of the probabilities: in this situation $P^0_{\mathbf{b}} = \{0.5, 0.4, 0.1\}$ instead of $P^0_{\mathbf{b}} = \{0.5, 0.3, 0.2\}$. That is, **s** believes time point $t = 8$ to be a possible **b**'s deadline with higher probability than in the previous situation. The construction from $t = 9$ to $t = 7$ is exactly the same of the previous situation, except for the value of $e^*(7)$. In this situation $e^*(7) = 0.8$. The backward induction construction imposes that at $t = 6$ an offering buyer should offer $(e^*(7))_{\leftarrow \mathbf{s}}$, but it can be easily seen that such an offer is not the optimal offer of both **b**'s types with deadline at $t = 8$ and $t = 9$. The type with deadline at $t = 8$ can actually maximize her utility offering $(e^*(7))_{\leftarrow \mathbf{s}}$. The type with deadline at $t = 9$ can instead maximize her utility offering something unacceptable (e.g., $RP_{\mathbf{s}}$), then receiving counteroffer $RP_{\mathbf{b}}$, and finally re-counteroffering $RP_{\mathbf{s}}$ at $t = 8$ (that **s** will accepted). Indeed, it can be observed in the figure that the equivalent value $(x^*(8))_{\leftarrow 2[\mathbf{b}]}$ of offering something unacceptable and reaching the agreement $(RP_{\mathbf{s}}, 9)$ is lower (and then preferable for the buyer) than the equivalent value $(e^*(7))_{\leftarrow \mathbf{s}}$ of offering $(e^*(7))_{\leftarrow \mathbf{s}}$ and reaching the agreement $((e^*(7))_{\leftarrow \mathbf{s}}, 7)$. Instead, in the bargaining situation depicted in Fig. 2 this anomaly is not present, since $(e^*(7))_{\leftarrow \mathbf{s}} < (x^*(8))_{\leftarrow 2[\mathbf{b}]}$.

This anomaly arises because different **b**'s types with $T_{\mathbf{b}_i} > t$ might have different optimal offers at time point t, violating what is prescribed by $\overline{\mu}$. This happens at any time point t where $\iota(t) = \mathbf{b}$, the time point $t + 2$ is a possible deadline of **b**, the optimal offer of **s** at time point $t + 1$ is $RP_{\mathbf{b}}$, and $U_{\mathbf{b}}(x^*(t), t) < U_{\mathbf{b}}(x^*(t + 2), t + 2)$. In order for the strategies (3) and (4) to be sequentially rational, no buyer's type must prefer making at time point t something else $x^*(t)$ to reach the agreement $(x^*(t + 2), t + 2)$. It trivially follows that the condition required for the sequential rationality of the strategies is that the following inequality holds at any time point $t \in [0, \overline{T} - 2]$ where $\iota(t) = \mathbf{b}$:

$$(e^*(t + 1))_{\leftarrow \mathbf{s}} \leqslant (x^*(t + 2))_{\leftarrow 2[\mathbf{b}]}. \tag{5}$$

The algorithm presented in [11] fails exactly when the inequality (5) does not hold. This algorithm produces by backward induction the agents' strategies keeping the initial beliefs and, once the strategies at time point $t = 0$ have been produced, the system of beliefs is designed to be consistent with the produced strategies. We discuss the application of this algorithm to a subgame of the bargaining situation depicted in Fig. 3, showing that the strategies it produces are not sequentially rational given the system of beliefs. Exactly, we apply the algorithm to the subgame starting from $t = 6$. The backward induction construction produced by [11] from $t = 9$ to $t = 7$ is the same construction above produced by our method. That is, **s** makes at $t = 7$ the offer $RP_{\mathbf{b}}$, otherwise she would accept any offer equal to or greater than $(e^*(7))_{\leftarrow \mathbf{s}}$; **b**'s type with deadline at $t = 8$ will accept $RP_{\mathbf{b}}$, instead the type with deadline at $t = 9$ will reject such an offer to make the offer $RP_{\mathbf{s}}$. The **b**'s optimal actions at $t = 6$ produced by the algorithm are: **b**'s type with deadline at $t = 8$ makes $(e^*(7))_{\leftarrow \mathbf{s}}$, whereas **b**'s type with deadline at $t = 9$ makes $(x^*(8))_{\leftarrow 2[\mathbf{b}]}$. The system of beliefs consistent with them can be easily produced: if **s** observes the offer $(e^*(7))_{\leftarrow \mathbf{s}}$, then she believes that **b**'s deadline is at $t = 8$; if **s** observes the offer $(x^*(8))_{\leftarrow 2[\mathbf{b}]}$, then she believes that **b**'s deadline is at $t = 9$; we omit the system of beliefs off the equilibrium path being unnecessary for our discussion. Given this system of beliefs we analyze the sequential rationality of the produced strategies. Consider the optimal action of **b**'s type with deadline at $t = 8$: she can improve her utility making the optimal action of **b**'s type with deadline at $t = 9$. If she offers $(x^*(8))_{\leftarrow 2[\mathbf{b}]}$, then **s** will believe that **b**'s deadline is at $t = 9$ and will accept such an offer. This is because the optimal offer **s** can make in the subgame starting from $t = 7$ where **b**'s deadline is $t = 9$ is $(x^*(8))_{\leftarrow \mathbf{b}}$ and her utility of making this offer is lower than her utility of accepting $(x^*(8))_{\leftarrow 2[\mathbf{b}]}$. Therefore, the strategies are not sequentially rational given the system of beliefs.

4.1.4. Equilibrium assessment

Collecting the system of beliefs $\overline{\mu}$, the strategies σ^*, and the sequential rationality condition presented in the previous sections, we can now state the following theorem whose proof is reported in Appendix A.1.

Theorem 2. *If for any time point $t \in [0, \overline{T} - 2]$ where $\iota(t) = \mathbf{b}$ inequality (5) holds, then the assessment $a = \langle \overline{\mu}, \sigma^* \rangle$ above described is a sequential equilibrium.*

This assessment can be computed backward from $t = \overline{T}$ on as prescribed in Section 4.1.2 until either the time point $t = 0$ is reached or the condition (5) is not satisfied. The computational time required to compute this assessment is linear in the length of the bargaining and asymptotically independent of the number of **b**'s types.

4.2. Analysis of equilibrium computation and existence

In this section we show that there is not any equilibrium assessment which can be computed backward faster than ours and that, when our assessment is not an equilibrium, there is not any equilibrium assessment in pure strategies, except for a null measure subset of the space of the parameters.

At first we characterize all the possible equilibrium assessments in terms of equilibrium strategies independently of the system of beliefs one can adopt. We show that, if a bargaining situation admits at least one equilibrium assessment a in pure strategies, then (except for a null measure subset of the space of the parameters) the equilibrium strategies must be such that at any time point t where $\iota(t) = \mathbf{b}$ the following condition holds:

$$\begin{cases} \text{for all } i, j \text{ such that } T_{\mathbf{b}_i}, T_{\mathbf{b}_j} > t \text{ it must be } \sigma^*_{\mathbf{b}_i}(t) = \sigma^*_{\mathbf{b}_j}(t) \\ \text{for all } i, j \text{ such that } T_{\mathbf{b}_i} = t, T_{\mathbf{b}_j} > t \text{ it must be } \sigma^*_{\mathbf{b}_i}(t) \neq \sigma^*_{\mathbf{b}_j}(t) \end{cases} \tag{6}$$

Notice that our $\overline{\mu}$ is a possible, but not the unique, system of beliefs consistent with (6). We base our proof on the special case with only two types of buyer, which we call, for the sake of clarity, *early buyer* $\mathbf{b}_e$ and *late buyer* $\mathbf{b}_l$ with $T_{\mathbf{b}_e} < \overline{T} \leqslant T_{\mathbf{b}_l}$. We show subsequently how the proof in the general case can be produced by iteratively applying the special case.

By considering the agents' optimal actions at any time point $t < T_{\mathbf{b}_e}$ prescribed by a generic equilibrium assessment, we state the following lemma whose proof is reported in Appendix A.2.

Lemma 3. *Given two buyer's types, at any time point $t < T_{\mathbf{b}_e}$ where $\iota(t) = \mathbf{b}$ it must be that $\sigma^*_{\mathbf{b}_e}(t) = \sigma^*_{\mathbf{b}_l}(t)$ except for a null measure subset of the space of the parameters.*

By considering the agents' optimal actions at time point $t = T_{\mathbf{b}_e}$ prescribed by a generic equilibrium assessment, we state the following lemma whose proof is reported in Appendix A.3.

Lemma 4. *Given two buyer's types, at time point $t = T_{\mathbf{b}_e}$ where $\iota(t) = \mathbf{b}$ it must be that $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e}) \neq \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$.*

By Lemmas 3 and 4 we can now state the following theorem.

Theorem 5. *Except for a null measure subset of the space of the parameters, if there is an equilibrium assessment $a = (\mu^*, \sigma^*)$, then the equilibrium strategies σ^* satisfy* (6) *at any time point t where $\iota(t) = \mathbf{b}$.*

Proof sketch. The proof in the special case with two buyer's types trivially follows from Lemmas 3 and 4. The proof in the general case with a number n of types can be easily obtained by iteratively applying the special case. Consider a set $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ of possible $\mathbf{b}$'s types where, without loss of generality, $T_{\mathbf{b}_i} < T_{\mathbf{b}_{i+1}}$ holds for any i. Consider the subgame starting from $T_{\mathbf{b}_{n-2}}$ where the possible types are $\mathbf{b}_{n-2}$, $\mathbf{b}_{n-1}$, and $\mathbf{b}_n$. Since $\mathbf{b}_{n-2}$ will make *exit* we can exclude it from our analysis and we can apply the special case of theorem deriving that $\mathbf{b}_{n-1}$ and $\mathbf{b}_n$ have the same strategies at any $t \in [T_{\mathbf{b}_{n-2}}, T_{\mathbf{b}_{n-1}})$. Therefore, we can aggregate $\mathbf{b}_{n-1}$ and $\mathbf{b}_n$ in a unique 'fictitious' type before $t = T_{\mathbf{b}_{n-1}}$ and apply the special case of theorem to the subgame starting from $T_{\mathbf{b}_{n-3}}$ where the possible types are $\mathbf{b}_{n-3}$, $\mathbf{b}_{n-2}$, and the 'fictitious' type. The iteratively application of the special case of the theorem continues until the type $\mathbf{b}_1$ has been considered. □

From Theorem 5 we can state the following corollary.

Corollary 6. *There is not any equilibrium assessment in pure strategies that can be computed backward faster than the assessment provided in Theorem* 2*.*

Proof sketch. We firstly notice that no algorithm based on backward induction can have computational complexity strictly lower than $\mathrm{O}(C \cdot \overline{T})$, where C is the computational cost of a single step of backward induction and is, in general, a function of the number of $\mathbf{b}$'s types. This is because the backward induction method requires to analyze every possible time point where the agents act. Since the complexity of our algorithm depends linearly on $\overline{T}$, we can limit our analysis to C.

By Lemmas 3 and 4, at any time point t where $\iota(t) = \mathbf{b}$ agent $\mathbf{b}$ can have two optimal actions, if time point t is a possible deadline of $\mathbf{b}$, and only one otherwise. Therefore, at any time point t where $\iota(t) = \mathbf{s}$ and time point $t + 1$ is a possible deadline of $\mathbf{b}$, $\mathbf{s}$ can compute her optimal offer searching for the offer that maximizes her expected utility in a set of at least two possible alternatives. That is, $C \geqslant 2$. Our solution, searching in a set of two possible alternatives, requires exactly the minimum computational time needed to compute a step of backward induction. In addition, our system of beliefs is the only one, among all the ones consistent with (6), that employs the same upgrading rule both on and off the equilibrium path. It follows that our assessment is the only one where the optimal strategies of the agents are the same both on and off the equilibrium path. That is, our assessment is the unique that does not require the computation of the equilibrium strategies off the equilibrium path, being these equal to the ones on the equilibrium path. □

We show now that, if there is a time point t where $\iota(t) = \mathbf{b}$ and inequality (5) does not hold, then there is not any equilibrium assessment in pure strategies except for a null measure subset of the space of the parameters. In Section 4.1.3 we showed that, if the agents reach a time point t where $\iota(t) = \mathbf{b}$ and inequality (5) does not hold relying on the initial beliefs (removed the types whose deadline has expired and normalized the residual probabilities), then at such time point t our $\overline{\mu}$ is not consistent with the optimal strategies. Here we strengthen such a statement showing that, given any $a = (\mu, \sigma)$, where μ is consistent with (6) and σ is sequentially rational given μ, if there is a time point t such that $\iota(t) = \mathbf{b}$ and inequality (5) does not hold, then μ cannot be consistent with σ. Considering the special case with two buyer's types $\mathbf{b}_e$ and $\mathbf{b}_l$, we state the following lemma whose proof is reported in Appendix A.4.

Lemma 7. *With two buyer's types, if at time point $t = T_{\mathbf{b}_e} - 2$ inequality* (5) *does not hold and $\iota(T_{\mathbf{b}_e} - 2) = \mathbf{b}$ and the agents can reach such a time relying on the initial beliefs $P^0_{\mathbf{b}}$, then it must be that $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \neq \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$.*

Obviously, no system of beliefs consistent with (6) can be consistent also with the strategies prescribed by Lemma 7. By considering Lemmas 3 and 7, we can state the following theorem whose proof is reported in Appendix A.5.

Theorem 8. *Except for a null measure subset of the space of the parameters, if there exists a time point t where $\iota(t) = \mathbf{b}$ and inequality* (5) *does not hold, then there is not any equilibrium assessment in pure strategies.*

Summarily, bargaining with uncertain deadlines can admit more sequential equilibria in pure strategies. Our assessment has two properties: there is not any other equilibrium assessment that can be computed backward faster and it is an equilibrium whenever there is an equilibrium assessment in pure strategies.

5. Equilibrium analysis in mixed strategies

In this section we present our assessment in mixed strategies. At first we analyze (Section 5.1) the situation where the buyer's types are two, and then we provide (Section 5.2) the solution in presence of a generic number of buyer's types.

5.1. Mixed strategy in presence of two types

We consider a bargaining situation with two buyer's types, $\mathbf{b}_e$ and $\mathbf{b}_l$, and such that condition (5) does not hold at time point $t = T_{\mathbf{b}_e} - 2$. For the sake of brevity we use $\omega_{\mathbf{b}_e}^{t+1}$ in the place of $\omega_{\mathbf{b}}^{t+1}(T_{\mathbf{b}_e})$, and analogously $\omega_{\mathbf{b}_l}^{t+1}$ in the place of $\omega_{\mathbf{b}}^{t+1}(T_{\mathbf{b}_l})$. Condition (5) can be expressed in terms of beliefs as follows: there is a threshold $\overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} = \frac{(x^*(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{b}]} - (x^*(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{s}]}}{(RP_{\mathbf{b}})_{\leftarrow \mathbf{s}} - (x^*(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{s}]}}$ such that, if the agents reach time point $t = T_{\mathbf{b}_e} - 2$ with $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} \leqslant \overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}$, then the equilibrium in the continuation game starting from $t = T_{\mathbf{b}_e} - 2$ can be in pure strategies.[5] As we showed in the previous section, since every equilibrium assessment in pure strategies allows the agents to reach time point $t = T_{\mathbf{b}_e} - 2$ with initial beliefs on the equilibrium path, then, if $\omega_{\mathbf{b}_e}^{0} > \overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}$, no equilibrium assessment in pure strategies exists. In order to overtake such a problem, we introduce an assessment in mixed strategies which does not allow the agents to reach time point $t = T_{\mathbf{b}_e} - 2$ with $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} > \overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}$ on the equilibrium path. Specifically, in our assessment the probability $\omega_{\mathbf{b}_e}^{t}$ monotonically decreases on the equilibrium path and is equal to $\overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}$ at time point $t = T_{\mathbf{b}_e} - 2$. The principle we exploit to make $\omega_{\mathbf{b}_e}$ decreasing on the equilibrium path is: the strategy of $\mathbf{b}_l$ is pure and prescribes a single action, whereas the strategy of $\mathbf{b}_e$ is mixed and prescribes the randomization between the optimal action of $\mathbf{b}_l$ and another action. The optimal action of $\mathbf{b}_l$ is the unique action that allows the game to continue on the equilibrium path. Therefore, once the optimal action of $\mathbf{b}_l$ is observed, the probability $\omega_{\mathbf{b}_e}$ is reduced according to the probability whereby $\mathbf{b}_e$ makes the observed action. In order to be an equilibrium it is also necessary that $\mathbf{s}$ randomizes.

According to this principle, several equilibrium assessments in mixed strategies can be found when condition (5) does not hold. We provide an assessment with the property that it is an equilibrium if and only if condition (5) does not hold. We preliminarily introduce the construction of the assessment and we subsequently discuss the details.

We start backward from time point $t = \overline{T}$ (with two $\mathbf{b}$'s types $\overline{T} = \min\{T_{\mathbf{b}_l}, T_{\mathbf{s}}\}$). In the continuation game starting from $t = T_{\mathbf{b}_e}$ the equilibrium strategy must be pure. Easily, $\mathbf{b}_e$'s dominant action is *exit*, whereas $\mathbf{b}_l$ will never make *exit* before $t = \overline{T}$. Thus $\mathbf{b}_e$ and $\mathbf{b}_l$ have strictly different optimal actions and then $\mathbf{b}_e$ will never make the optimal action of $\mathbf{b}_l$. In the interval $[T_{\mathbf{b}_e}, \overline{T}]$ we use the complete information construction presented in Section 2.2, where the unique possible $\mathbf{b}$'s type is $\mathbf{b}_l$. We denote such a construction by $x_{\mathbf{b}_l}^{*}(t)$. When condition (5) does not hold at $t = T_{\mathbf{b}_e} - 2$, the backward induction construction discussed in Section 4.1.2 is altered introducing a mixed strategy from $T_{\mathbf{b}_e}$ on, until the mixed strategy leads to an equilibrium. We denote with $[\overset{<}{t}, \overset{>}{t}]$ the interval of time wherein the assessment is in mixed strategies. It is such that $\overset{>}{t} = T_{\mathbf{b}_e}$ and $\overset{<}{t}$ is the first time point before $\overset{>}{t}$ where the assessment cannot be in mixed strategies. The time point $\overset{<}{t}$ will be determined during the backward induction construction checking some conditions.

[5] A continuation game is defined as a subgame, but its root node can be a non-singleton information set.

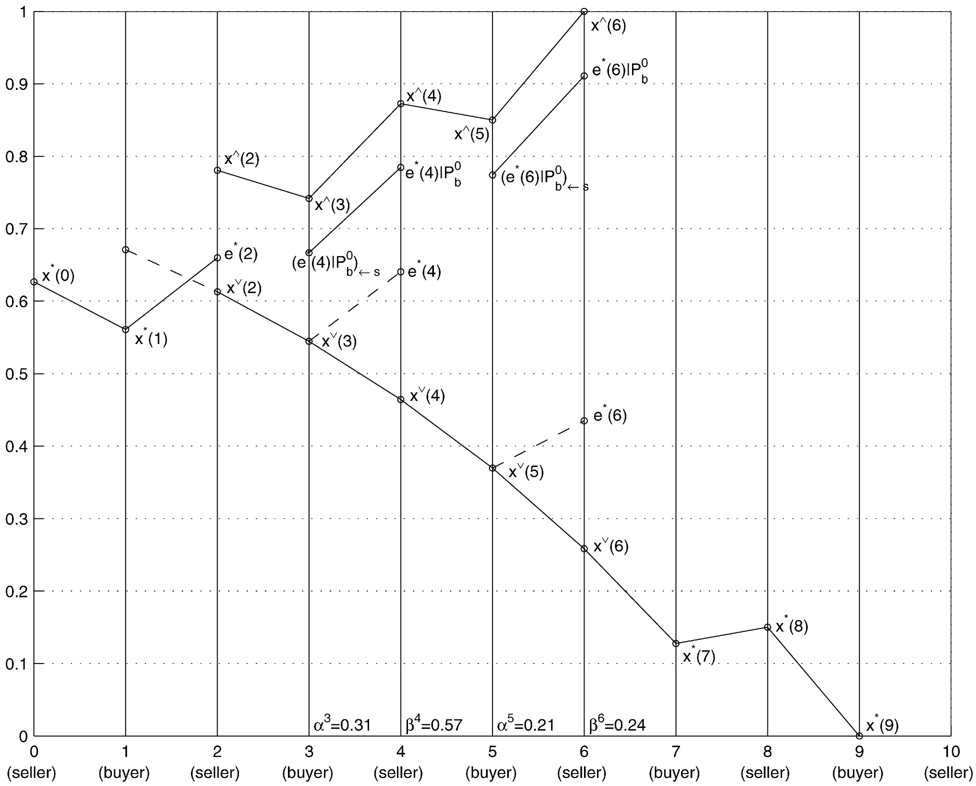

Fig. 4. Backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.85$, $\delta_{\mathbf{s}} = 0.85$, $\langle T_{\mathbf{b}} = \{7, 10\}, P^0_{\mathbf{b}} = \{0.9, 0.1\}\rangle$, $T_{\mathbf{s}} = 11$, $\iota(0) = \mathbf{s}$.

If these conditions are satisfied before reaching $t = 0$, then $\overset{<}{t} > 0$ and the construction follows from $t = \overset{<}{t}$ to $t = 0$ in pure strategies as discussed in Section 2.2, otherwise $\overset{<}{t} = 0$.

The construction in mixed strategies employed in $[\overset{<}{t}, \overset{>}{t}]$ is based on two sequences of offers: a "high" sequence denoted by $\hat{x}(t)$, and a "low" sequence denoted by $\check{x}(t)$ (see Fig. 4 for an example; the details of the construction will be discussed later). Basically, at each time point t where they act, $\mathbf{b}_e$ and $\mathbf{s}$ mix between offering the values given by the two sequences $\hat{x}(t)$ and $\check{x}(t)$ with probabilities α^t and β^t, respectively, whereas $\mathbf{b}_l$ acts a pure strategy offering $\check{x}(t)$. The probabilities α^t and β^t will be computed such that the mixed strategy is sequentially rational. We present our assessment in mixed strategies as follows:

(i) *a priori* fix two sequences $\hat{x}(t)$ and $\check{x}(t)$ whereby agents' optimal actions will be based and derive agents' strategies σ^*,
(ii) determine the probabilities α^t and β^t whereby agents will randomize over their optimal actions,
(iii) produce the system of beliefs to be Bayes consistent with the strategies σ^* on the equilibrium path,
(iv) *a posteriori* prove the sequential rationality of the strategies and the consistency of the system of beliefs off the equilibrium path.

We discuss in what follows the application of our method.

5.1.1. Mixed strategies

We fix the two sequences of offers $\hat{x}(t)$s and $\check{x}(t)$s as follows:

- $\hat{x}(t) = x^*_{\mathbf{b}_e}(t)$, i.e., the optimal offer at time point t in the complete information game where $\mathbf{b}$ is of the type $\mathbf{b}_e$, calculated as prescribed in Section 2.2,
- $\check{x}(t) = (x^*(T_{\mathbf{b}_e}))_{\leftarrow(T_{\mathbf{b}_e}-t)[\mathbf{b}]}$, i.e., the backward propagation of $x^*(T_{\mathbf{b}_e})$ with respect to $\mathbf{b}$'s utility.

The equilibrium strategies are built backward as follows:

- $T_{\mathbf{b}_e} \leqslant t$: the agents' strategies are pure and equal to those provided in Section 4.1, i.e. $\mathbf{b}_l$ and $\mathbf{s}$ base their optimal actions on $x^*(t) \equiv x^*_{\mathbf{b}_l}(t)$;
- $\overset{<}{t}+2 \leqslant t < T_{\mathbf{b}_e}$ and $\omega^t_{\mathbf{b}_l} \neq 0$: the optimal actions at $t = T_{\mathbf{b}_e} - 2$ cannot be pure since inequality (5) does not hold. Moreover, by construction, $\mathbf{b}$ cannot have pure optimal actions at any time point $t \in [\overset{<}{t}+2, T_{\mathbf{b}_e})$. The condition required to have pure optimal actions at time point t when the construction from time point $t+1$ on follows in mixed strategies is a generalization of inequality (5). Specifically, such a condition requires that $\iota(t) = \mathbf{b}$ and that inequality
$$(e^*(t+1)|P^0_{\mathbf{b}})_{\leftarrow \mathbf{s}} \leqslant (\check{x}(t+1))_{\leftarrow \mathbf{b}} \tag{7}$$
holds, where $e^*(t+1)|P^0_{\mathbf{b}}$ denotes the optimal equivalent of $\mathbf{s}$ at time point $t+1$ calculated with initial beliefs $P^0_{\mathbf{b}}$ after removing the types whose deadline has expired and normalizing the residual probabilities. (In absence of mixed strategies at t it holds $e^*(t+1)|P^0_{\mathbf{b}} = e^*(t+1)$.) The agents employ mixed strategies as follows:
  - $\mathbf{b}_l$ employs a pure strategy based on $\check{x}(t)$: if $\sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1)$, then she accepts, otherwise she offers $\check{x}(t)$;
  - $\mathbf{b}_e$ employs a mixed strategy based on $\check{x}(t)$ and $\hat{x}(t)$: if $\sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1)$, then she accepts; if $\check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant \hat{x}(t-1)$, then she accepts with probability $1-\alpha^t$ and rejects to offer $\check{x}(t)$ with probability α^t; otherwise she offers $\hat{x}(t)$ with probability $1-\alpha^t$ and offers $\check{x}(t)$ with probability α^t;
  - $\mathbf{s}$ employs a mixed strategy based on $\check{x}(t)$ and $\hat{x}(t)$: if $\sigma_{\mathbf{b}}(t-1) \geqslant \hat{x}(t-1)$, then she accepts; if $\hat{x}(t-1) > \sigma_{\mathbf{b}}(t-1) \geqslant \check{x}(t-1)$, then she accepts with probability $\beta^t(\sigma_{\mathbf{s}}(t-2))$ and rejects to offer $\hat{x}(t)$ with probability $1-\beta^t(\sigma_{\mathbf{s}}(t-2))$;[6] otherwise she offers $\hat{x}(t)$;
- $\overset{<}{t}+2 \leqslant t < T_{\mathbf{b}_e}$ and $\omega^t_{\mathbf{b}_l} = 0$: the optimal actions of $\mathbf{s}$ and of $\mathbf{b}_e$ are those in the corresponding complete information game, whereas the optimal action of $\mathbf{b}_l$ is the one she employs when $\overset{<}{t}+2 \leqslant t \leqslant T_{\mathbf{b}_e}$ and $\omega^t_{\mathbf{b}_l} \neq 0$;
- $t = \overset{<}{t}+1$: $\mathbf{s}$ employs a pure strategy maximizing her equivalent value between offering $\check{x}(\overset{<}{t}+1)$ and $\hat{x}(\overset{<}{t}+1)$;
- $t \leqslant \overset{<}{t}$: being $x^*(t) < \check{x}(t)$, inequality (7) holds and then the optimal actions are pure and are those provided in Section 2.2 where $x^*(\overset{<}{t}) = (e^*(\overset{<}{t}+1))_{\leftarrow \mathbf{s}}$.

By construction, time point $t = \overset{<}{t}$ is the first time point back from $T_{\mathbf{b}_e}$ where inequality (7) holds and the equilibrium can be in pure strategies. Obviously, when the backward construction reaches the initial time point without holding inequality (7), part of the above strategy does not appear.

The equivalent value $e(x,t)$ of $\mathbf{s}$ for any t such that $\overset{<}{t} < t < T_{\mathbf{b}_e}$ can be calculated as follows (we recall that $EU_{\mathbf{s}}(\mathit{offer}(x), t) = U_{\mathbf{s}}(e(x,t), t)$):
$$e(x,t) = \begin{cases} x > \hat{x}(t) & \omega^t_{\mathbf{b}_e} \cdot (1-\alpha^{t+1}) \cdot (\hat{x}(t+1))_{\leftarrow \mathbf{s}} + (1 - \omega^t_{\mathbf{b}_e} \cdot (1-\alpha^{t+1})) \cdot (\check{x}(t+1))_{\leftarrow \mathbf{s}} \\ \hat{x}(t) \geqslant x > \check{x}(t) & \omega^t_{\mathbf{b}_e} \cdot (1-\alpha^{t+1}) \cdot x + (1 - \omega^t_{\mathbf{b}_e} \cdot (1-\alpha^{t+1})) \cdot (\check{x}(t+1))_{\leftarrow \mathbf{s}} \\ \check{x}(t) \geqslant x & x \end{cases}$$

Formally, the equilibrium strategies can be defined by specifying the agents' choice rules on the basis of $x^*(t)$, $\hat{x}(t)$, $\check{x}(t)$, α^t, and β^t as extension of the ones reported in (3) and (4) as follows:
$$\sigma^*_{\mathbf{b}_l}(t) = \begin{cases} t = 0 & \mathit{offer}(x^*(0)) \\ 0 < t \leqslant \overset{<}{t} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant x^*(t-1) \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overset{<}{t} < t \leqslant T_{\mathbf{b}_e} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \mathit{offer}(\check{x}(t)) & \text{otherwise} \end{cases} \\ T_{\mathbf{b}_e} < t < \overline{T} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant x^*(t-1) \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{b}_l} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant RP_{\mathbf{b}} \\ \mathit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{b}_l} < t & \mathit{exit} \end{cases}$$

[6] The dependency of β^t on $\sigma_{\mathbf{s}}(t-2)$ is needed to grant the existence of the equilibrium also when $\mathbf{s}$ acts at time point $t-2$ off the equilibrium.

$$\sigma^*_{\mathbf{b}_e}(t) = \begin{cases} t = 0 & \mathit{offer}(x^*(0)) \\ 0 < t \leqslant \overset{<}{t} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant x^*(t-1) \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overset{<}{t} < t < T_{\mathbf{b}_e} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \begin{cases} \mathit{accept} & 1-\alpha^t \\ \mathit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{if } \check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant \hat{x}(t-1) \\ \begin{cases} \mathit{offer}(\hat{x}(t)) & 1-\alpha^t \\ \mathit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{otherwise} \end{cases} \\ t = T_{\mathbf{b}_e} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant RP_{\mathbf{b}} \\ \mathit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{b}_e} < t & \mathit{exit} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t) = \begin{cases} t = 0 & \mathit{offer}(x^*(0)) \\ 0 < t < \overset{<}{t} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant x^*(t-1) \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ t = \overset{<}{t}+1 & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(\overset{<}{t}) \geqslant x^*(\overset{<}{t}) \\ \mathit{offer}(x) x = \arg\max_{x \in \{\check{x}(\overset{<}{t}+1), \hat{x}(\overset{<}{t}+1)\}} e(x, \overset{<}{t}+1) & \text{otherwise} \end{cases} \\ \overset{<}{t}+3 \leqslant t < T_{\mathbf{b}_e} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant \hat{x}(t-1) \\ \begin{cases} \mathit{accept} & \beta^t(\sigma_{\mathbf{s}}(t-2)) \\ \mathit{offer}(\hat{x}(t)) & 1-\beta^t(\sigma_{\mathbf{s}}(t-2)) \end{cases} & \text{if } \hat{x}(t-1) > \sigma_{\mathbf{b}}(t-1) \geqslant \check{x}(t-1) \\ \mathit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases} \\ T_{\mathbf{b}_e} < t < \overline{T} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant x^*(t-1) \\ \mathit{offer}(x^*(t)) & \text{otherwise} \end{cases} \\ \overline{T} \leqslant t \leqslant T_{\mathbf{s}} & \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t-1) \geqslant RP_{\mathbf{s}} \\ \mathit{exit} & \text{otherwise} \end{cases} \\ T_{\mathbf{s}} < t & \mathit{exit} \end{cases}$$

5.1.2. Mixing probabilities

The values of the probabilities α^t and $\beta^t(\sigma_{\mathbf{s}}(t-2))$ must be such that the following conditions hold:

- α^t is such that $\mathbf{s}$, once she has observed the offer $\check{x}(t)$, is indifferent between accepting such an offer and rejecting it to offer $\hat{x}(t+1)$ at time point $t+1$. Formally, $U_{\mathbf{s}}(\check{x}(t), t+1) = EU_{\mathbf{s}}(\mathit{offer}(\hat{x}(t+1)), t+2)$ given that $\sigma_{\mathbf{b}}(t) = \mathit{offer}(\check{x}(t))$;
- $\beta^t(\sigma_{\mathbf{s}}(t-2))$ is such that $\mathbf{b}_e$, if $\sigma_{\mathbf{s}}(t-2) \in (\check{x}(t-2), \hat{x}(t-2)]$, is indifferent between accepting $\sigma_{\mathbf{s}}(t-2)$ and rejecting it to offer $\check{x}(t-1)$ at time point $t-1$, and, if $\sigma_{\mathbf{s}}(t-2) \notin (\check{x}(t-2), \hat{x}(t-2)]$, is indifferent between offering $\hat{x}(t-1)$ and offering $\check{x}(t-1)$ at time point $t-1$. Formally, $U_{\mathbf{b}}(\sigma_{\mathbf{s}}(t-2), t-1) = EU_{\mathbf{b}}(\mathit{offer}(\check{x}(t-1)), t)$ and $U_{\mathbf{b}}(\hat{x}(t-1), t) = EU_{\mathbf{b}}(\mathit{offer}(\check{x}(t-1)), t)$, respectively.[7]

The formulas to compute the values of α^t and $\beta^t(\sigma_{\mathbf{s}}(t-2))$ are

$$\alpha^t = \frac{\omega^{*,t+1}_{\mathbf{b}_e} \cdot (1-\omega^{*,t}_{\mathbf{b}_e})}{\omega^{*,t}_{\mathbf{b}_e} \cdot (1-\omega^{*,t+1}_{\mathbf{b}_e})}, \qquad \beta^t(\sigma_{\mathbf{s}}(t-2)) = \begin{cases} \sigma_{\mathbf{s}}(t-2) \in (\check{x}(t-2), \hat{x}(t-2)] & \frac{(\hat{x}(t))_{\leftarrow 2[\mathbf{b}]} - \sigma_{\mathbf{s}}(t-2)}{(\hat{x}(t))_{\leftarrow 2[\mathbf{b}]} - \check{x}(t-2)} \\ \sigma_{\mathbf{s}}(t-2) \notin (\check{x}(t-2), \hat{x}(t-2)] & \frac{(\hat{x}(t))_{\leftarrow 2[\mathbf{b}]} - \hat{x}(t-2)}{(\hat{x}(t))_{\leftarrow 2[\mathbf{b}]} - \check{x}(t-2)} \end{cases} \tag{8}$$

where

$$\omega^{*,t}_{\mathbf{b}_e} = \begin{cases} t > T_{\mathbf{b}_e} & 0 \\ \overset{<}{t}+1 < t \leqslant T_{\mathbf{b}_e} & \begin{cases} \iota(t) = \mathbf{b} & \omega^{*,t+1}_{\mathbf{b}_e} \\ \iota(t) = \mathbf{s} & \frac{(1-\omega^{*,t+2}_{\mathbf{b}_e}) \cdot \check{x}(t-1) + \omega^{*,t+2}_{\mathbf{b}_e} \cdot \hat{x}(t-1) - (\check{x}(t+1))_{\leftarrow 2[\mathbf{s}]}}{\hat{x}(t-1) - (\check{x}(t+1))_{\leftarrow 2[\mathbf{s}]}} \end{cases} \\ t \leqslant \overset{<}{t}+1 & \omega^0_{\mathbf{b}_e} \end{cases} \tag{9}$$

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[7] The expected utility $EU_{\mathbf{b}}$ can be defined similarly to $EU_{\mathbf{s}}$.

The sequence of $\omega_{\mathbf{b}_e}^{*,t}$s constitutes the beliefs of **s** as forced by Bayes' rule when she observes the optimal offer $\check{x}(t-1)$.

It can be easily showed that the values of $\beta^t(\sigma_{\mathbf{s}}(t-2))$ are always well defined, i.e. $\in (0,1)$ in $[\overset{<}{t}+2, \overset{>}{t}]$. This is because at any time point t belonging to this interval $\hat{x}(t) > \check{x}(t)$. We now show that α^t and $\omega_{\mathbf{b}_e}^{*,t}$ have well defined values of probability, i.e., $\alpha^t \in (0,1)$ and $\omega_{\mathbf{b}_e}^{*,t} \in [\overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}, \omega_{\mathbf{b}_e}^{0}]$, only in the interval $[\overset{<}{t}+2, \overset{>}{t}]$.

By considering time point $t = T_{\mathbf{b}_e} - 2$ and formulas (8) and (9), we can state the following lemma whose proof is reported in Appendix A.6.

Lemma 9. *When at time point $t = T_{\mathbf{b}_e} - 2$ condition* (5) *does not hold, then $\alpha^{T_{\mathbf{b}_e}-2}$ and $\omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$ have well defined probability values and* vice versa.

By considering any time point t such that $\overset{<}{t}+2 < t < T_{\mathbf{b}_e}$, we can now state the following lemma whose proof is reported in Appendix A.7.

Lemma 10. *Consider a time point t such that the construction from time point $t+2$ to $T_{\mathbf{b}_e}$ is based on the above mixed strategies: when at such a time point t condition* (7) *does not hold, then α^t and $\omega_{\mathbf{b}_e}^{*,t+1}$ have well defined probability values and* vice versa.

By the inductively applying Lemma 10 we have the following theorem whose proof is trivial and then omitted.

Theorem 11. *When condition* (5) *does not hold at $t = T_{\mathbf{b}_e} - 2$, then exclusively in the interval $[\overset{<}{t}, \overset{>}{t}]$ the values of the probabilities α^t and $\omega_{\mathbf{b}_e}^{*,t+1}$ are well defined and* vice versa.

This means that the values of the probabilities α^t and $\omega_{\mathbf{b}_e}^{*,t+1}$ in our mixed strategy are well defined at any time point t where agents cannot employ pure strategies. Therefore, the use of our mixed strategy is perfectly complementary to the use of our pure strategy.

5.1.3. System of beliefs

Now we specify the system of beliefs μ. For any t such that $t \leqslant \overset{<}{t}+1$ or $T_{\mathbf{b}_e} \leqslant t$ μ is the system of beliefs employed in Section 4.1.1. For any t such that $\overset{<}{t}+2 \leqslant t \leqslant T_{\mathbf{b}_e}$, on the equilibrium path, μ is the system of beliefs induced by the Bayes' rule according to the above equilibrium strategies, while off the equilibrium path it is such that **s**, once she has observed $\sigma_{\mathbf{b}}(t-1) \in [\check{x}(t-1), \hat{x}(t-1))$, is indifferent between accepting it or rejecting it to offer $\hat{x}(t)$. Specifically μ is:

$$\mu(t) = \begin{cases} \sigma_{\mathbf{b}}(t-1) > \hat{x}(t-1) & \omega_{\mathbf{b}_e}^{t} = 1 \\ \hat{x}(t-1) \geqslant \sigma_{\mathbf{b}}(t-1) > \check{x}(t-1) & \omega_{\mathbf{b}_e}^{t} = \omega_{\mathbf{b}_e}^{*,t} + \frac{(1-\omega_{\mathbf{b}_e}^{*,t})\cdot(\sigma_{\mathbf{b}}(t-1)-\check{x}(t-1))}{(\hat{x}(t-1)-\check{x}(t-1))} \\ \check{x}(t-1) \geqslant \sigma_{\mathbf{b}}(t-1) & \omega_{\mathbf{b}_e}^{t} = \omega_{\mathbf{b}_e}^{*,t} \end{cases}$$

We can now illustrate the bargaining situation depicted in Fig. 4, where $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.85$, $\delta_{\mathbf{s}} = 0.85$, $\langle T_{\mathbf{b}} = \{7, 10\}, P_{\mathbf{b}}^0 = \{0.9, 0.1\}\rangle$, $T_{\mathbf{s}} = 11$, and $\iota(0) = \mathbf{s}$. Easily, the backward induction construction starts from time point $\overline{T} = 10$ where **s** acts. Since at any time point from $t = 7$ to $t = 10$ **s** believes **b**'s deadline to be $t = 10$, the construction in these time points is the one with complete information where **b**'s deadline is $t = 10$. On the equilibrium path $\mathbf{b}_e$ makes *exit* in this interval of time. The optimal offer $x^*(9)$ of $\mathbf{b}_l$ at $t = 9$ is $RP_{\mathbf{s}} = 0$, the optimal offer $x^*(8)$ of **s** at $t = 8$ is $(x^*(9))_{\leftarrow \mathbf{b}} = 0.15$, and the optimal offer $x^*(7)$ of $\mathbf{b}_l$ at $t = 7$ is $(x^*(8))_{\leftarrow \mathbf{s}} \simeq 0.13$.

Consider time point $t = 6$, the strategy of **s** cannot be pure. We build the pure strategy construction as prescribed by Section 4.1 and we show that the condition that must be satisfied to have pure strategies, i.e. condition (5), does not hold. According to the pure strategy construction presented in Section 4.1, since $t = 7$ is a possible **b**'s deadline, at time point $t = 6$ **s** chooses the offer that maximizes her expected utility between $RP_{\mathbf{b}} = 1$ and $(x^*(7))_{\leftarrow \mathbf{b}} \simeq 0.26$. Since $e(RP_{\mathbf{b}}, 6) \simeq 0.91$ and $e((x^*(7))_{\leftarrow \mathbf{b}}, 6) \simeq 0.26$, the optimal offer $x^*(6)$ of **s** is $RP_{\mathbf{b}}$ and the relative optimal equivalent value $e^*(6)$ is 0.91. In the figure, this value of $e^*(6)$ is labeled with $e^*(6)|P_{\mathbf{b}}^0$ (this is because in what follows we introduce a mixed strategy and then the value of $e^*(6)$ will be modified). At time point $t = 5$ the condition (5) does not hold, being $(e^*(6)|P^0_{\mathbf{b}})_{\leftarrow \mathbf{s}} \simeq 0.78 > (x^*(7))_{\leftarrow 2[\mathbf{b}]} \simeq 0.37$.

Therefore, we introduce a mixed strategy at time points $t = 5$ and $t = 6$ as prescribed by Section 5.1.1. The "high" sequence of offers $\hat{x}(t)$s is: $\hat{x}(6) = RP_{\mathbf{b}} = 1$, and $\hat{x}(5) = (\hat{x}(6))_{\leftarrow \mathbf{s}} = 0.85$. The "low" sequence of offers $\check{x}(t)$s is: $\check{x}(7) = x^*(7) \simeq 0.13$, $\check{x}(6) = (\check{x}(7))_{\leftarrow \mathbf{b}} \simeq 0.37$, and $\check{x}(5) = (\check{x}(6))_{\leftarrow \mathbf{b}} \simeq 0.46$. The optimal actions at time point $t = 5$ are: $\mathbf{b}_e$ randomizes between offering $\hat{x}(5)$ with probability $1 - \alpha^5$ and offering $\check{x}(5)$ with probability α^5, $\mathbf{b}_l$ offers $\check{x}(5)$; at time point $t = 6$ the optimal actions of $\mathbf{s}$ are to accept $\hat{x}(5)$ and to randomize between accepting $\check{x}(5)$ with probability β^6 and rejecting it to offer $\hat{x}(6)$ with probability $1 - \beta^6$. The value of α^5 is calculated imposing that, when $\mathbf{b}$ offers $\check{x}(5)$, the expected utilities of $\mathbf{s}$ from accepting it and offering $\hat{x}(6)$ are the same: $EU_{\mathbf{s}}(accept, 6) = U_{\mathbf{s}}(\check{x}(5), 6)$ and $EU_{\mathbf{s}}(offer(\hat{x}(6)), 6) = \omega^{*,6}_{\mathbf{b}_e} \cdot U_{\mathbf{s}}(\hat{x}(6), 7) + (1 - \omega^{*,6}_{\mathbf{b}_e}) \cdot U_{\mathbf{s}}(\check{x}(7), 7)$ where $\omega^{*,6}_{\mathbf{b}_e}$ is the probability that $\mathbf{b}$ is of the type $\mathbf{b}_e$, once she has offered $\check{x}(5)$ and $\mathbf{s}$ has upgraded her beliefs according to μ. The value of α^5 is $\simeq 0.21$. Similarly, the value of β^6 is calculated imposing that the expected utilities of $\mathbf{b}_e$ from offering $\hat{x}(5)$ and $\check{x}(5)$ are equal. The value of β^6 is $\simeq 0.24$. Due to the presence of the mixed strategy, the optimal equivalent value $e^*(6)$ is lower than $e^*(6)|P^0_{\mathbf{b}}$ and it is $(e^*(6))_{\leftarrow \mathbf{s}} = \check{x}(5)$.

Given the optimal actions of the agents from $t = 5$ to $t = 10$, $\mathbf{s}$ chooses at time point $t = 4$ her optimal action between offering $\hat{x}(4)$ and $\check{x}(4)$. In the figure it can be observed that her optimal offer is $\hat{x}(4)$ and the relative equivalent value $e^*(4)|P^0_{\mathbf{s}}$ is such that at time point $t = 3$ condition (7) does not hold. Thus, it is necessary to introduce a mixed strategy also at time points $t = 4$ and $t = 5$. The process continues to the initial time point $t = 0$. Notice that at time point $t = 1$ condition (7) holds and the construction follows from $t = 2$ to $t = 0$ in pure strategies.

5.1.4. Equilibrium assessment

Collecting the strategies, the mixing probabilities, and the system of beliefs presented in the previous sections, we can state the following theorem whose proof is reported in Appendix A.8.

Theorem 12. *The above assessment is a sequential equilibrium if and only if condition* (5) *does not hold at time point* $t = T_{\mathbf{b}_e} - 2$.

The calculation of the equilibrium can be accomplished step by step from the deadline of the bargaining back to the initial time, determining, in addition to the sequences of the optimal offers $x^*(t)$s and of the optimal equivalent values $e^*(t)$s computed in the solution with pure strategies, the sequences $\hat{x}(t)$, $\check{x}(t)$, α^t, $\omega^{*,t}_{\mathbf{b}_e}$, and β^t. The computational time required to find the equilibrium depends linearly on the length of the bargaining and requires only a maximization between two possible offers at the time point $t = T_{\mathbf{b}_e} - 1$ (exactly as the equilibrium in pure strategies). Analogously to what showed in Section 4.2, it can be showed that there is not any equilibrium assessment in mixed strategies that can be computed backward faster than ours.

5.2. Mixed strategy in presence of more than two types

We show here how the mixed strategy presented in the previous section can be generalized to the case where the buyer's types are more than two. We omit in this section the formulas to compute the equilibrium, because they are long and unnecessary for the comprehension of the result; we report them in Appendix B. The presence of many types raises several complications in the calculation of the equilibrium strategies. Nevertheless, the principle whereby the mixed strategy is based still holds: there are two sequences of offers, a "high" sequence $\hat{x}(t)$ and a "low" sequence $\check{x}(t)$, and the agents base their strategies (pure or mixed) on these. The equilibrium strategies can be calculated backward from time point $t = \overline{T}$ to the initial time point $t = 0$ employing pure strategies and inserting, when needed, mixed strategies. Therefore, the equilibrium path will be composed of intervals of time points where the agents have a single optimal actions and of intervals of time points where the agents randomize over two optimal actions (see, e.g., Fig. 6). We consider the most general case where all the time points before $\overline{T}$ are possible deadlines of $\mathbf{b}$ and there exists a time point $\overset{>}{t}$ such that $\iota(\overset{>}{t}) = \mathbf{b}$ and condition (5) does not hold at $t = \overset{>}{t} - 2$. We need to build the equilibrium in mixed strategies from $\overset{>}{t}$ back to $\overset{<}{t}$ where $\overset{<}{t}$ is such that $\iota(\overset{<}{t}) = \mathbf{b}$ and the equilibrium strategy can be pure. Exactly as with two types, the construction can be produced backward checking at each time point t where $\iota(t) = \mathbf{b}$ whether condition (7) is satisfied or not. In the affirmative case, the construction can be in pure strategies, otherwise, the construction must be in mixed strategies.

Although **b**'s types can be many, we have found an equilibrium assessment where different **b**'s types can have at any time point t at most three different optimal actions. This means that the buyer's types can be grouped in three distinct sets and we can limit our analysis to them without searching for the optimal actions of each single buyer's type. This leads to an efficient computation of the equilibrium: at any time point t no more than three different optimal actions must be computed. Denoting by $\theta_{\mathbf{b}}(t)$ the **b**'s type whose deadline is at t, at any time point t the buyer's types can be grouped as follows:

- $\hat{\Theta}_{\mathbf{b}}(t)$: all the types belonging to $\hat{\Theta}_{\mathbf{b}}(t)$ have pure strategies accepting any offer lower than or equal to $(\hat{x}(t))_{\leftarrow\mathbf{b}}$ and otherwise offering $\hat{x}(t)$; let be $\hat{\Theta}_{\mathbf{b}}(\overset{>}{t}-1)=\emptyset$ the initialization set of $\hat{\Theta}_{\mathbf{b}}$, and $\hat{\Omega}^{t_1}_{\mathbf{b}}(t_2)=\sum_{\theta_{\mathbf{b}}(t)\in\hat{\Theta}_{\mathbf{b}}(t_2)}\omega^{t_1}_{\mathbf{b}}(t)$ the cumulative probability of $\hat{\Theta}_{\mathbf{b}}(t_2)$ at time point t_1;
- $\tilde{\Theta}_{\mathbf{b}}(t)$: all the types belonging to $\tilde{\Theta}_{\mathbf{b}}(t)$ have mixed strategies accepting any offer lower than or equal to $(\hat{x}(t))_{\leftarrow\mathbf{b}}$ with probability $1-\alpha^t$ and rejecting it to offer $\check{x}(t)$ with probability α^t; let be $\tilde{\Theta}_{\mathbf{b}}(\overset{>}{t}-1)=\emptyset$ the initialization set of $\tilde{\Theta}_{\mathbf{b}}$, and $\tilde{\Omega}^{t_1}_{\mathbf{b}}(t_2)=\sum_{\theta_{\mathbf{b}}(t)\in\tilde{\Theta}_{\mathbf{b}}(t_2)}\omega^{t_1}_{\mathbf{b}}(t)$ the cumulative probability of $\tilde{\Theta}_{\mathbf{b}}(t_2)$ at time point t_1;
- $\check{\Theta}_{\mathbf{b}}(t)$: all the types belonging to $\check{\Theta}_{\mathbf{b}}(t)$ have pure strategies accepting any offer lower than or equal to $\check{x}(t-1)$ and otherwise offering $\check{x}(t)$; let be $\check{\Theta}_{\mathbf{b}}(\overset{>}{t}-1)=\{\theta_{\mathbf{b}}(t)\colon\ t\geqslant\overset{>}{t}\}$ the initialization set of $\check{\Theta}_{\mathbf{b}}$, and $\check{\Omega}^{t_1}_{\mathbf{b}}(t_2)=\sum_{\theta_{\mathbf{b}}(t)\in\check{\Theta}_{\mathbf{b}}(t_2)}\omega^{t_1}_{\mathbf{b}}(t)$ the cumulative probability of $\check{\Theta}_{\mathbf{b}}(t_2)$ at time point t_1.

Analogously to the situation with two types, **s** has mixed equilibrium strategies based on $\hat{x}(t)$ and $\check{x}(t)$ with a $\beta^t(\sigma_{\mathbf{s}}(t-2))$.

The construction can be done backward from $\overset{>}{t}$ on as follows. The "low" sequence of offers $\check{x}(t)$s is easily determinable, being $\check{x}(t)=(x^*(\overset{>}{t}))_{\leftarrow(\overset{>}{t}-t)[\mathbf{b}]}$. The "high" sequence of offers $\hat{x}(t)$s is harder to find and depends on the probability values of the **s**'s beliefs. When $\iota(t)=\mathbf{s}$ the value of $\hat{x}(t)$ is the optimal offer that **s** would make at such time point relying on the initial beliefs after removing the types whose deadline has expired and normalizing the residual probabilities. Precisely, at any time point t such that $\iota(t)=\mathbf{s}$, **s** maximizes her expected utility given her beliefs. If $t=\overset{>}{t}-1$ the equivalent value of **s** is:

$$e(x,t)=\begin{cases} x>RP_{\mathbf{b}} & \check{\Omega}^t_{\mathbf{b}}(t+1)\cdot((\check{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+RP_{\mathbf{s}}\\ RP_{\mathbf{b}}\geqslant x>\check{x}(t) & \omega^t_{\mathbf{b}}(t+1)\cdot(x-RP_{\mathbf{s}})+\check{\Omega}^t_{\mathbf{b}}(t+1)\cdot((\check{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+RP_{\mathbf{s}}\\ \check{x}(t)\geqslant x & \Omega^t_{\mathbf{b}}(t+1)\cdot(x-RP_{\mathbf{s}})+RP_{\mathbf{s}}\end{cases}$$

whereas, if $t<\overset{>}{t}-1$, the equivalent value of **s** is:

$$e(x,t)=\begin{cases} x>RP_{\mathbf{b}} & (\hat{\Omega}^t_{\mathbf{b}}(t+1)+(1-\alpha^{t+1})\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1))\cdot((\hat{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})\\ & \quad+(\alpha^{t+1}\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1)+\check{\Omega}^t_{\mathbf{b}}(t+1))\cdot((\check{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+RP_{\mathbf{s}}\\ RP_{\mathbf{b}}\geqslant x>(\hat{x}(t+1))_{\leftarrow\mathbf{b}} & \omega^t_{\mathbf{b}}(t+1)\cdot(x-RP_{\mathbf{s}})+(\hat{\Omega}^t_{\mathbf{b}}(t+1)+(1-\alpha^{t+1})\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1))\\ & \quad\times((\hat{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+(\alpha^{t+1}\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1)+\check{\Omega}^t_{\mathbf{b}}(t+1))\\ & \quad\times((\check{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+RP_{\mathbf{s}}\\ (\hat{x}(t+1))_{\leftarrow\mathbf{b}}\geqslant x>\check{x}(t) & (1-\check{\Omega}^t_{\mathbf{b}}(t+1)-\alpha^{t+1}\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1))\cdot(x-RP_{\mathbf{s}})+(\check{\Omega}^t_{\mathbf{b}}(t+1)\\ & \quad+\alpha^{t+1}\cdot\tilde{\Omega}^t_{\mathbf{b}}(t+1))\cdot((\check{x}(t+1))_{\leftarrow\mathbf{s}}-RP_{\mathbf{s}})+RP_{\mathbf{s}}\\ \check{x}(t)\geqslant x & \Omega^t_{\mathbf{b}}(t+1)\cdot(x-RP_{\mathbf{s}})+RP_{\mathbf{s}}\end{cases}$$

At time point $t=\overset{>}{t}-1$ the value that maximizes the equivalent value $e(x,t)$ is $RP_{\mathbf{b}}$. At each time point $t<\overset{>}{t}-1$ only three possible offers can maximize $e(x,t)$: $RP_{\mathbf{b}}$, $(\hat{x}(t+1))_{\leftarrow\mathbf{b}}$, or $\check{x}(t)$. The value x that maximizes $e(x,t)$ is the value to be assigned to $\hat{x}(t)$.

The determination of $\hat{x}(t)$ when $\iota(t)=\mathbf{b}$ is more complicated than when $\iota(t)=\mathbf{s}$. Indeed, in presence of more than two **b**'s types, the choice rules that each **b**'s type employs to choose her optimal actions can be of four different forms. Each choice rule has mutually exclusive conditions, i.e., when a specific choice rule can be employed to produce an equilibrium the other three cannot, and *vice versa*. When the condition of a specific choice rule is satisfied, it holds that the strategies it prescribes are sequentially rational given the system of beliefs, the mixing probabilities are well defined, and the system of beliefs is consistent with the strategies (the proof is omitted, being similar to the proof of the case with two types, but long). Furthermore, for each possible setting of the parameters and for each time point t, at least one choice rule can be employed. Each choice rule is characterized by:

- a specific composition of the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, and $\check{\Theta}_{\mathbf{b}}(t)$,
- a specific strategy $\langle \sigma^*_{\hat{\Theta}_{\mathbf{b}}(t)}(t), \sigma^*_{\tilde{\Theta}_{\mathbf{b}}(t)}(t), \sigma^*_{\check{\Theta}_{\mathbf{b}}(t)}(t), \sigma^*_{\mathbf{s}}(t+1)\rangle$ with specific formulas to compute $\hat{x}(t)$ and α^t,
- a specific system of beliefs $\mu(t)$.

In what follows we just overview the four choice rules and their construction; we report in Appendix B the formulas to compute them. Given the construction from time point $\overline{T}$ to time point $t+1$, it is possible to find the choice rule that can be employed at a time point t when $\iota(t) = \mathbf{b}$ to produce the equilibrium. The principal condition that characterizes the four choice rules is the value of $\hat{x}(t+1)$ computed in the construction from time point $\overline{T}$ to time point $t+1$. We call *choice rule 1* the choice rule that can be employed when $\hat{x}(t+1) = (\hat{x}(t+2))_{\leftarrow \mathbf{b}}$. If $\hat{x}(t+1) = RP_{\mathbf{b}}$ three possible choice rules can be employed, we call them *choice rules 2.1*, *2.2*, and *2.3*. The conditions that discriminate their employment are mathematically complicated, and then omitted here. Once the conditions have been checked and the choice rule to employ has been found, it is necessary to compose the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, and $\check{\Theta}_{\mathbf{b}}(t)$, and to calculate $\hat{x}(t)$, α^t, and β^{t+1}.

Summarily, once the optimal action of $\mathbf{s}$ has been found at time point t given the construction from time point $\overline{T}$ to $t+1$, the choice rule to employ at time point $t-1$ can be found by checking some conditions. The equilibrium can be found in time linear in the length of the bargaining and asymptotically independent of the number of buyer's types.

6. Solving algorithm and extension to the multiple issue setting

In this section we initially present (Section 6.1) our solving algorithm and then we show (Section 6.2) how our algorithm can be extend to address the situation where the issues to negotiate are more than one.

6.1. Solving algorithm

We collect the results presented in Sections 4 and 5 and we provide an algorithm to compute the equilibrium assessment. Specifically, in the two previous sections we have provided the equilibrium assessment in function of some parameters, here we provide an algorithm to compute such parameters. The inputs of the algorithm are $\delta_{\mathbf{b}}$, $\delta_{\mathbf{s}}$, $\langle \mathbf{T_b}, \omega^0_{\mathbf{b}}(t)\rangle$, $T_{\mathbf{s}}$, $RP_{\mathbf{b}}$, $RP_{\mathbf{s}}$, and $\iota(t)$. The output of the algorithm is the set of the parameters needed to specify the equilibrium assessment: $x^*(t)$, $e^*(t)$, $\hat{x}(t)$, $\check{x}(t)$, α^t, β^t, $\tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)$, $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, and $\check{\Theta}_{\mathbf{b}}(t)$.

The algorithm is simple (see Algorithm 1) and works iteratively from time point $t = \overline{T}$ to time point $t = 0$. At each iteration t the algorithm adds a time point backward to the current partial solution according to the agent that acts at

$\overline{T} \leftarrow \min\{T_{\mathbf{s}}, \max_i\{T_{\mathbf{b}_i}\}\}$
initialize variables, i.e., x^*, e^*, $\hat{\Theta}_{\mathbf{b}}$, $\tilde{\Theta}_{\mathbf{b}}$, $\check{\Theta}_{\mathbf{b}}$, at time point $\overline{T}-1$
for $(t = \overline{T}-2, t \geqslant 0, t = t-1)$ **do**
  if $\iota(t) = \mathbf{s}$
    compute $x^*(t)$ as the value that maximizes $e(x,t)$ relying on the initial beliefs $P^0_{\mathbf{b}}$, after removing the types whose deadline has expired and normalizing the residual probabilities, as prescribed in Section 5.2
  else
    if condition (7) holds **then**
      compute $x^*(t)$ as prescribed in Section 4.1
    else
      determine the choice rule in mixed strategies whose conditions are satisfied, compose the sets $\hat{\Theta}_{\mathbf{b}}$, $\tilde{\Theta}_{\mathbf{b}}$, $\check{\Theta}_{\mathbf{b}}$, and compute $\hat{x}(t)$, $\check{x}(t)$, α^t, β^{t+1} as prescribed in Section 5.2 and Appendix B
    end if
  end if
end for
return $[x^*(t), e^*(t), \hat{x}(t), \check{x}(t), \alpha^t, \beta^t, \hat{\Theta}_{\mathbf{b}}(t), \tilde{\Theta}_{\mathbf{b}}(t), \check{\Theta}_{\mathbf{b}}(t), \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)]$

Algorithm 1. EQUILIBRIUM_FINDER $(\delta_{\mathbf{b}}, \delta_{\mathbf{s}}, \langle \mathbf{T_b}, \omega^0_{\mathbf{b}}(t)\rangle, T_{\mathbf{s}}, RP_{\mathbf{b}}, RP_{\mathbf{s}}, \iota(t))$.

time point t. At each time point t where $\iota(t) = \mathbf{s}$, the optimal offer of $\mathbf{s}$ is computed as a maximization relying on the initial beliefs, after removing the types whose deadline has expired and normalizing the residual probabilities, among $RP_{\mathbf{b}}$, $(\hat{x}(t+1))_{\leftarrow\mathbf{b}}$, and $(\check{x}(t+1))_{\leftarrow\mathbf{b}}$, as prescribed in Section 5.2. At each time point where $\iota(t) = \mathbf{b}$, we need to check condition (7): if it holds then $\mathbf{b}$'s strategy is pure, otherwise it is mixed. If it is pure, then the optimal offer at time point t can be simply computed in closed form, otherwise a mixed strategy is introduced at time point t and consequently at time point $t+1$. In this latter case it is necessary to determine what choice rule to employ checking the conditions reported in Appendix B. Subsequently, the sets $\hat{\Theta}_{\mathbf{b}}$, $\tilde{\Theta}_{\mathbf{b}}$, and $\check{\Theta}_{\mathbf{b}}$ are composed and the values of $\hat{x}(t)$, $\check{x}(t)$, α^t, and β^{t+1} are computed as prescribed in Appendix B.

The algorithm has a computational complexity that is linear in $\overline{T}$ and asymptotically independent of the number of buyer's types. In the worst case it requires to check three conditions at any time point: when $\iota(t) = \mathbf{s}$, three values can maximize $e(x,t)$, whereas, when $\iota(t) = \mathbf{b}$, at most three conditions must be checked to determine the choice to be employed.

We show two examples of backward induction construction produced with our algorithm, reporting also, for any time point wherein the strategies are mixed, the form of the employed choice rule, the composition of the sets $\hat{\Theta}_{\mathbf{b}}$, $\tilde{\Theta}_{\mathbf{b}}$, and $\check{\Theta}_{\mathbf{b}}$, and the values of probabilities of α^t and β^t. In Fig. 5 we report an example of construction where the probability distribution is uniform in $[1, \max_i\{T_{\mathbf{b}_i}\}]$. In Fig. 6 we report an example of construction where the probability distribution is based on two Gaussian-like distributions with means at $t = 7$ and $t = 18$.

We discuss some details of the bargaining situation depicted in Fig. 5. The time point from which we apply the algorithm is $\overline{T} = \min\{\max\{\mathbf{T_b}\}, T_{\mathbf{s}}\} = 20$. At $t = 20$ $\mathbf{s}$ acts and she accepts any offer equal to or greater than $RP_{\mathbf{s}} = 0$, otherwise she makes *exit*. At $t = 19$, the optimal offer $x^*(19)$ of $\theta_{\mathbf{b}}(20)$ is $RP_{\mathbf{s}} = 0$, instead $\theta_{\mathbf{b}}(19)$ does not make any offer, but makes *exit*.

Consider time point $t = 18$. The beliefs of $\mathbf{s}$, obtained from $P^0_{\mathbf{b}}$ after removing the types whose deadline has expired and normalizing the residual probabilities, are $\omega^{18}_{\mathbf{b}}(t) = 0.\bar{3}$ for any $t \in [18, 20]$. Being $t = 19$ a possible deadline of $\mathbf{b}$, at $t = 18$ $\mathbf{s}$ chooses the offer between $RP_{\mathbf{b}}$ and $(x^*(19))_{\leftarrow\mathbf{b}}$ that maximizes her expected utility. The equivalent value $e(RP_{\mathbf{b}}, 18)$ is $e(RP_{\mathbf{b}}, 18) = \omega^{18}_{\mathbf{b}}(19) \cdot (RP_{\mathbf{b}} - RP_{\mathbf{s}}) + \Omega^{18}_{\mathbf{b}}(20) \cdot (RP_{\mathbf{s}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}} = 0.5$, instead the equivalent value $e((x^*(19))_{\leftarrow\mathbf{b}}, 18)$ is $e((x^*(19))_{\leftarrow\mathbf{b}}, 18) = \Omega^{18}_{\mathbf{b}}(19) \cdot ((x^*(19))_{\leftarrow\mathbf{b}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}} = 0.01\bar{6}$. Therefore, the optimal offer $x^*(18)$ of $\mathbf{s}$ is $RP_{\mathbf{b}} = 1$ with $e^*(18) = 0.01\bar{6}$. The value of $e^*(18)$ is labeled in figure with $e^*(18)|P^0_{\mathbf{b}}$ (this is because in what follows we introduce a mixed strategy and then the value of $e^*(18)$ will be modified).

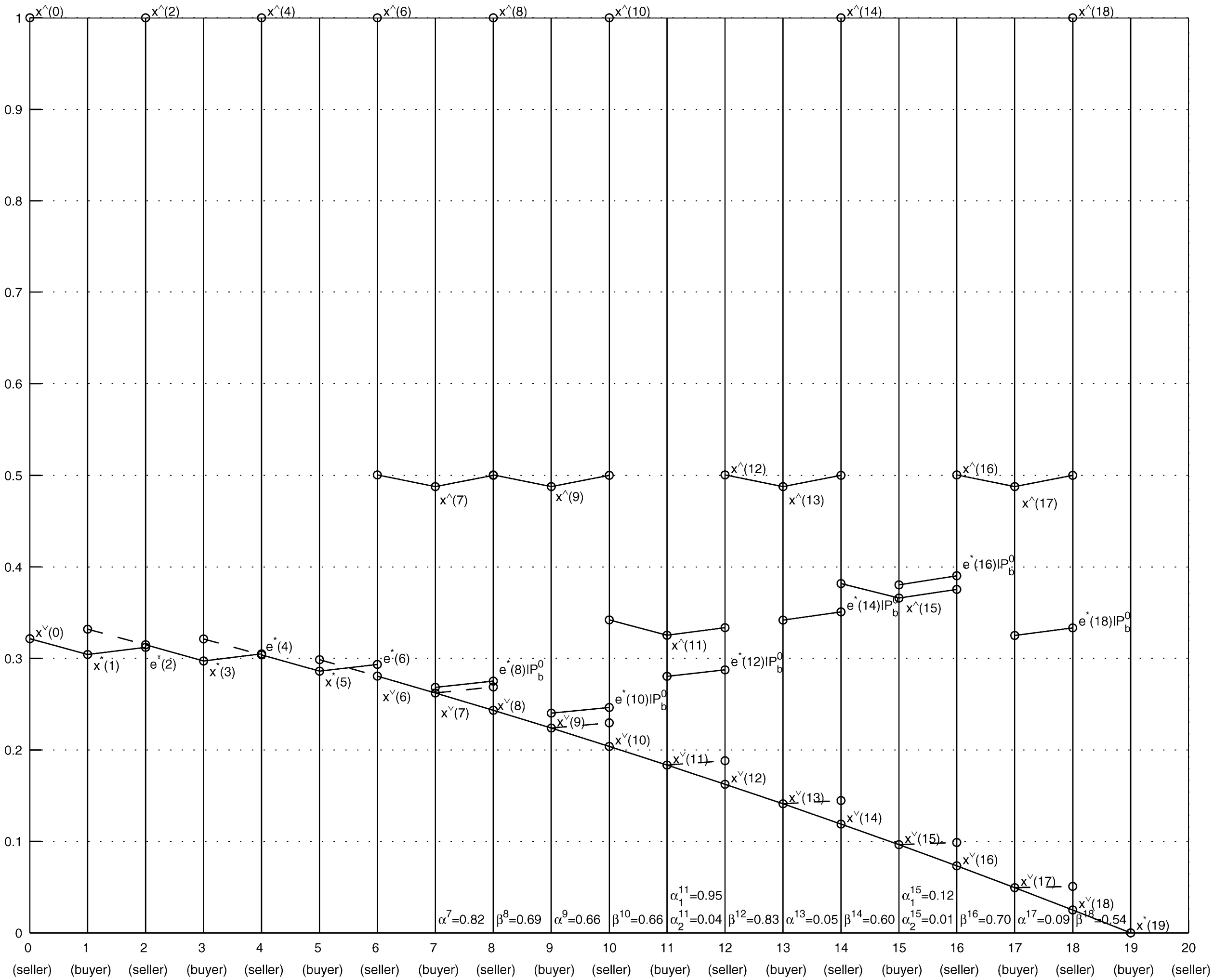
Consider time point $t = 17$. Condition (7) does not hold, being $(e^*(18))_{\leftarrow\mathbf{s}} > (x^*(19))_{\leftarrow 2[\mathbf{b}]}$. We need therefore to introduce a mixed strategy at $t = 17$ and $t = 18$. Checking the conditions reported in Appendix B, we see that the choice rule to be employed is the choice rule 1. It prescribes that at $t = 17$ the types $\theta_{\mathbf{b}}(18)$ and $\theta_{\mathbf{b}}(19)$ randomize between offering $\check{x}(18)$ and $\hat{x}(18)$ with $\alpha^{17} = 0.09$ and $1 - \alpha^{17}$, respectively, whereas the type $\theta_{\mathbf{b}}(20)$ offers $\check{x}(17)$. The value of $\check{x}(17)$ can be easily calculated being $(x^*(19))_{\leftarrow 2[\mathbf{b}]} = 0.049$; the value of $\hat{x}(18)$, calculated as prescribed by Appendix B, is $(\frac{\omega^0_{\mathbf{b}}(19)}{\omega^0_{\mathbf{b}}(17)+\omega^0_{\mathbf{b}}(18)}(RP_{\mathbf{b}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}})\delta_{\mathbf{s}} = 0.5$. The introduction of a mixed strategy at $t = 17$ induces a mixed strategy also for $\mathbf{s}$ at $t = 18$. Exactly, at $t = 18$, if $\mathbf{s}$ receives $\hat{x}(17)$, she accepts it and, if she receives the offer $\check{x}(17)$, she randomizes between accepting it and rejecting it to offer $RP_{\mathbf{s}}$.

Consider time point $t = 16$. The beliefs of $\mathbf{s}$, obtained from $P^0_{\mathbf{b}}$ after removing the types whose deadline has expired and normalizing the residual probabilities, are $\omega^{16}_{\mathbf{b}}(t) = 0.2$ for any $t \in [16, 20]$. At $t = 16$ $\mathbf{s}$ chooses the offer among $RP_{\mathbf{b}}$, $(\hat{x}(17))_{\leftarrow\mathbf{b}} \simeq 0.5$, and $(\check{x}(17))_{\leftarrow\mathbf{b}} \simeq 0.073$ that maximizes her expected utility. The equivalent values calculated as prescribed in Section 5.2 are: $e(RP_{\mathbf{b}}, 16) \simeq 0.394$, $e((\hat{x}(17))_{\leftarrow\mathbf{b}}, 16) \simeq 0.392$, and $e((\check{x}(17))_{\leftarrow\mathbf{b}}, 16) \simeq 0.058$. Therefore, the optimal action of $\mathbf{s}$ is to offer $RP_{\mathbf{b}}$ with $e^*(16) = 0.394$. The value $e^*(16)$ is labeled in figure with $e^*(16)|P^0_{\mathbf{b}}$ (this is because in what follows we introduce a mixed strategy and then the value of $e^*(16)$ will be modified).

At $t = 15$, exactly as it happens at time point $t = 17$, condition (7) does not hold. Therefore a mixed strategy must be introduced at time points $t = 15$ and $t = 16$. In this case the choice rule to employ is the choice rule 2.3. The construction continues up to the initial time point as illustrated in figure.

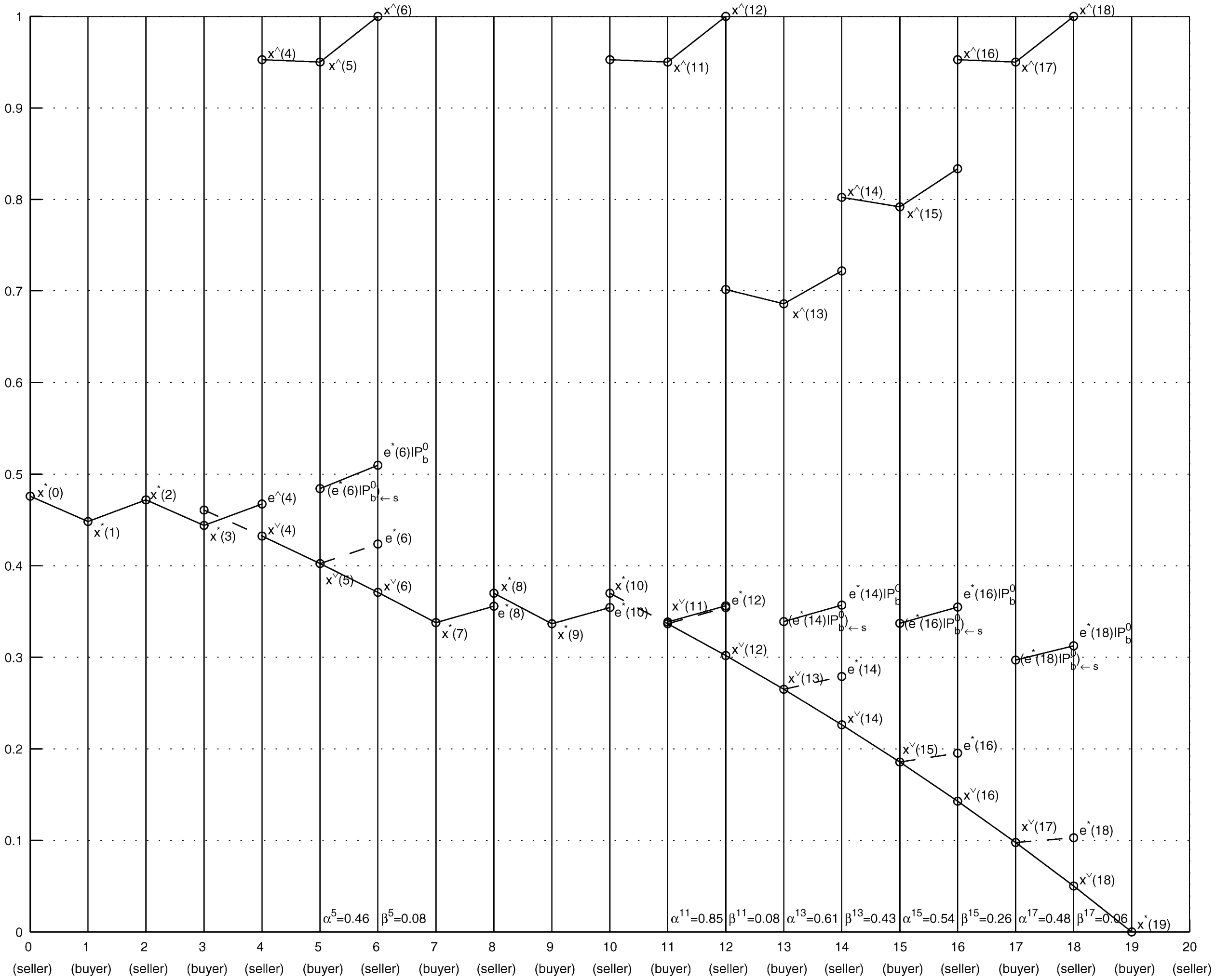
6.2. Extension to the multiple issue setting

Multiple issue bargaining captures situations where agents can find agreements over the values of several parameters. A common example is the negotiation between a buyer and a seller over several attributes of a good, such as



$\mathbf{t = 7}$ **(choice rule 2.1)**
$\hat{\Theta}_{\mathbf{b}}(7) = \emptyset$
$\tilde{\Theta}_{\mathbf{b}}(7) = \{\theta_{\mathbf{b}}(8), \theta_{\mathbf{b}}(9)\}$
$\check{\Theta}_{\mathbf{b}}(7) = \{\theta_{\mathbf{b}}(t) : t \in [10, 20]\}$
$\mathbf{t = 9}$ **(choice rule 2.1)**
$\hat{\Theta}_{\mathbf{b}}(9) = \emptyset$
$\tilde{\Theta}_{\mathbf{b}}(9) = \{\theta_{\mathbf{b}}(10), \theta_{\mathbf{b}}(11)\}$
$\check{\Theta}_{\mathbf{b}}(9) = \{\theta_{\mathbf{b}}(t), t \in [12, 20]\}$
$\mathbf{t = 11}$ **(choice rule 2.3)**
$\hat{\Theta}_{\mathbf{b}}(11) = \{\theta_{\mathbf{b}}(12)\}$
$\tilde{\Theta}_{\mathbf{b}}(11) = \{\theta_{\mathbf{b}}(t), , t \in [13, 15]\}$
$\check{\Theta}_{\mathbf{b}}(11) = \{\theta_{\mathbf{b}}(t), t \in [16, 20]\}$
$\mathbf{t = 13}$ **(choice rule 2.2)**
$\hat{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(14)\}$
$\tilde{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(15)\}$
$\check{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(t), t \in [16, 20]\}$
$\mathbf{t = 15}$ **(choice rule 2.3)**
$\hat{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(16)\}$
$\tilde{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(t), t \in [17, 19]\}$
$\check{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(20)\}$
$\mathbf{t = 17}$ **(choice rule 2.1)**
$\hat{\Theta}_{\mathbf{b}}(17) = \emptyset$
$\tilde{\Theta}_{\mathbf{b}}(17) = \{\theta_{\mathbf{b}}(18), \theta_{\mathbf{b}}(19)\}$
$\check{\Theta}_{\mathbf{b}}(17) = \{\theta_{\mathbf{b}}(20)\}$

Fig. 5. Backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.975$, $\delta_{\mathbf{s}} = 0.975$, $\langle T_{\mathbf{b}} = [0, 20]$, $P_{\mathbf{b}}^0 = \{0, 0.05\}\rangle$, $T_{\mathbf{s}} = 28$, $\iota(0) = \mathbf{s}$; we report also the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, and $\check{\Theta}_{\mathbf{b}}(t)$ and the choice rule in mixed strategies whose conditions are satisfied at time point t.



t = 5 (choice rule 2.1)
$\hat{\Theta}_{\mathbf{b}}(5) = \emptyset$
$\tilde{\Theta}_{\mathbf{b}}(5) = \{\theta_{\mathbf{b}}(6), \theta_{\mathbf{b}}(7)\}$
$\check{\Theta}_{\mathbf{b}}(5) = \{\theta_{\mathbf{b}}(t) : t \in [8, 20]\}$
t = 13 (choice rule 1)
$\hat{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(14)\}$
$\tilde{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(t), t \in [15, 17]\}$
$\check{\Theta}_{\mathbf{b}}(13) = \{\theta_{\mathbf{b}}(t), t \in [18, 20]\}$
t = 15 (choice rule 2.2)
$\hat{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(16)\}$
$\tilde{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(17)\}$
$\check{\Theta}_{\mathbf{b}}(15) = \{\theta_{\mathbf{b}}(t), t \in [18, 20]\}$
t = 17 (choice rule 2.1)
$\hat{\Theta}_{\mathbf{b}}(17) = \emptyset$
$\tilde{\Theta}_{\mathbf{b}}(17) = \{\theta_{\mathbf{b}}(18), \theta_{\mathbf{b}}(19)\}$
$\check{\Theta}_{\mathbf{b}}(17) = \{\theta_{\mathbf{b}}(20)\}$

Fig. 6. Backward induction construction with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} = 0.96$, $\delta_{\mathbf{s}} = 0.96$, $\langle T_{\mathbf{b}} = [0, 20]$, $P^0_{\mathbf{b}} = \{0, 0, 0, 0, 0, 0.02, 0.11, 0.24, 0.11, 0.02, 0, 0, 0, 0, 0, 0, 0.02, 0.11, 0.24, 0.11, 0.02\}\rangle$, $T_{\mathbf{s}} = 28$, $\iota(0) = \mathbf{s}$; we report also the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, and $\check{\Theta}_{\mathbf{b}}(t)$ and the choice rule in mixed strategies whose conditions are satisfied at time point t.

the price, the level of quality, the guarantee expiration, the delivery time, and so on. The alternating-offers protocol can be easily extended to capture this situation: the utility functions, the offers, and the acceptance of the agents are defined on tuples of values, one for each issue (see [7] for more details). For each issue the agents have a specific reservation value and a specific discount factor. However, each agent has usually a single deadline for all the issues, being the deadline relative to the good to negotiate and not to the single issues that characterize the good [6,12] (a model where agents have different deadlines over different issues is discussed in [7]). Multiple issue bargaining can be implemented according to different procedures, e.g. the issues can be negotiated sequentially or concurrently. In what follows we consider the in-bundle procedure, where the issues are negotiated concurrently. This is because, as showed in [11], the in-bundle procedure is the unique procedure that allows one to produce efficient agreements.

We denote by m the number of issues to negotiate. In presence of complete information the solution of the multiple issue situation [7] is similar to the solution of the single issue situation described in Section 2.2, but two differences there are. First, the optimal offers $\mathbf{x}^*(t)$s are tuples of values that specify a value for each single issue. Second, with a single issue the offers to accept can be easily expressed specifying a threshold on the value of the received offer, e.g. $\mathbf{s}$ accepts at t any offer $\mathbf{y}$ such that $y \geqslant (x^*(t))_{\leftarrow \mathbf{s}}$, with multiple issues instead the threshold is on the utility of the received offer, e.g. $\mathbf{s}$ accepts at t any offer $\mathbf{y}$ such that $U_{\mathbf{s}}(\mathbf{y}, t) \geqslant U_{\mathbf{s}}(\mathbf{x}^*(t), t)$. The sequence of the optimal offers $\mathbf{x}^*(t)$s can be found by backward induction by extending the backward propagation of an offer we defined in Definition 1. Essentially, the backward induction construction is the same: at each time point t the optimal offer $\mathbf{x}^*(t)$ of agent $\iota(t)$ is the offer such that agent $\iota(t+1)$ is indifferent at $t+1$ between accepting it and making her optimal offer $\mathbf{x}^*(t+1)$. Formally, $U_{\iota(t+1)}(\mathbf{x}^*(t), t+1) = U_{\iota(t+1)}(\mathbf{x}^*(t+1), t+2)$. The difference between the multiple issue situation and the single issue situation lays on how $\mathbf{x}^*(t)$ can be computed given $\mathbf{x}^*(t+1)$. With a single issue, the backward propagation $y = x_{\leftarrow i}$ is a function that maps an offer $x \in \mathbb{R}$ at t to an offer $y \in \mathbb{R}$ at $t-1$ (keeping constant the utility of agent i) and the calculation of y can be easily accomplished in closed form. With multiple issues, the backward propagation $\mathbf{y} = \mathbf{x}_{\leftarrow i}$ maps an offer $\mathbf{x} \in \mathbb{R}^m$ at t to an offer $\mathbf{y} \in \mathbb{R}^m$ at $t-1$ (keeping constant the utility of agent i) and the calculation of $\mathbf{y}$ requires the solution of a convex programming problem [6]. In the most common situations where the utility functions are linear, this problem can be solved in time linear in the number of issues, being a fractional knapsack problem (a proof is provided in [11]). Summarily, the redefinition of the backward propagation of an offer in presence of multiple issues allows one to treat the settings with single issue and with multiple issue in the same way.

The extension of our work in presence of multiple issues can be achieved exactly as in the case with complete information. Specifically, for each time point of the bargaining the algorithm produces the sequences $\mathbf{x}^*(t)$, $\mathbf{e}^*(t)$, $\hat{\mathbf{x}}(t)$, and $\check{\mathbf{x}}(t)$ solving for each single element of the sequences a fractional knapsack problem. Therefore, the computational complexity of the algorithm is linear in the length of the bargaining, linear in the number of the issues, and asymptotically independent of the number of types of the agent whose deadline is uncertain, i.e. $\mathrm{O}(m \cdot \overline{T})$.

7. Conclusions and future works

The game theoretical study of bargaining situations is a prominent issue in computer science, since it allows one to prescribe the behavior of rational agents. Nevertheless literature lacks of solutions when information is incomplete and the actions available to the agents are infinite. On the one hand, game theory provides an appropriate solution concept for extensive-form games with incomplete information, but no solving technique to find it. On the other hand, algorithms available in computer science literature work only with finite games and do not produce systems of beliefs. This pushes researchers to analyze each setting independently and to develop *ad hoc* specific algorithms.

In this paper we have focused on the alternating-offers bargaining with one-sided uncertainty on the deadlines, and we have game theoretically studied it to provide a solution. We have found a couple of agents' choice rules that apply an action and a probability distribution over the actions, respectively, and we showed that it is always possible to produce an equilibrium where the actions at any single time point are those prescribed either by the pure strategy choice rule or by the mixed strategy choice rule. We have showed in addition that our solution is such that there is not any other solution that can be computed backward faster. The solving algorithm we provided is simple, being instead complicated the game theoretical analysis that led to the development of the algorithm. Furthermore, the solving algorithm is efficient, being its computational complexity asymptotically independent of the number of types of the player whose deadline is uncertain. Exactly, its computational complexity is $\mathrm{O}(m \cdot \overline{T})$ where m is the number of issues and $\overline{T}$ is the deadline of the bargaining.

Our work paves the way to a novel approach for employing game theoretical tools in the study of sequential games with incomplete information. Schematically, the approach we have followed is organized in the following steps: analysis of the game in the attempt to purge the set of the choice rules to employ to produce an equilibrium, finding the conditions such that a choice rule can be employed, studying the existence of the equilibrium for all the values of the parameters, and, finally, building the solution employing at each time point the choice rule whose conditions are satisfied.

Our intention is to complete our work along two main directions. The first one concerns the extension of our results in bargaining situations where other parameters are uncertain (i.e., δ_i, RP_i) and where the uncertainty over the deadlines is two-sided. The second one concerns the automatization of the procedure we have followed in our game theoretical analysis in order to develop efficient algorithms to solve extensive-form games with incomplete information.

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Appendix A. Proofs of the main theoretical results

We report in this section the proofs of the main results provided in the paper.

A.1. *Proof of Theorem 2*

The sequential rationality can be easily proved by mathematical induction. Consistency can be proved by the assessment sequence $a_n = \langle \mu_n, \sigma_n \rangle$ where:

- σ_n is the fully mixed strategy profile such that before the real deadline of an agent there is probability $1 - \frac{1}{n}$ of performing the action prescribed by σ^* and the remaining probability $\frac{1}{n}$ is uniformly distributed among the other allowed actions; while at the deadline or after it there is probability $1 - \frac{1}{n^2}$ of performing the action prescribed by σ^* and the remaining probability $\frac{1}{n^2}$ is uniformly distributed among the other allowed actions.
- μ_n is the system of beliefs obtained applying Bayes' rule starting from the same *a priori* probability distribution $P^0_{\mathbf{b}}$ as in $\overline{\mu}$.

Each assessment a_n is "Bayes consistent" by construction. The convergence of σ_n to σ^* is trivial. As to μ_n, given any arbitrary legal sequence S of actions (such that the bargaining does not conclude at the end of S and such that $\mathbf{s}$ is the agent acting after S), call $P^S_n = \langle \omega^S_{n,\mathbf{b}_1}, \omega^S_{n,\mathbf{b}_2}, \dots, \omega^S_{n,\mathbf{b}_m} \rangle$ the probabilities that $\mathbf{s}$ assigns to $\mathbf{b}$'s deadlines after sequence S according to μ_n. Sequence S might contain actions that could be interpreted as being in accordance to the strategies σ^* (i.e. actions that are the actions prescribed by strategies σ^* for some deadline $T_{\mathbf{b}_i}$); let τ the time of the latest of such actions in S (if there are no such actions, set $\tau = -1$ by convention). Let $t = |S|$. Some calculation shows that, if $t \leqslant \overline{T}$, then

$$\omega^S_{n,\mathbf{b}_i} = \begin{cases} 0 & \text{if } T_{\mathbf{b}_i} \leqslant \tau \\ \dfrac{\frac{1}{n^{t-T_{\mathbf{b}_i}}} \cdot \omega^0_{\mathbf{b}_i}}{\sum_{T_{\mathbf{b}_h} \geqslant t} \omega^0_{\mathbf{b}_h} + \sum_{\tau < T_{\mathbf{b}_h} \leqslant t} \frac{1}{n^{t-T_{\mathbf{b}_h}}} \cdot \omega^0_{\mathbf{b}_h}} & \text{if } \tau < T_{\mathbf{b}_i} \leqslant t \\ \dfrac{\omega^0_{\mathbf{b}_i}}{\sum_{T_{\mathbf{b}_h} > t} \omega^0_{\mathbf{b}_h} + \sum_{\tau < T_{\mathbf{b}_h} < t} \frac{1}{n^{t-T_{\mathbf{b}_h}}} \cdot \omega^0_{\mathbf{b}_h}} & \text{if } t < T_{\mathbf{b}_i} \end{cases}$$

Therefore

$$\lim_{n \to +\infty} \omega^S_{n,\mathbf{b}_i} = \begin{cases} 0 & \text{if } T_{\mathbf{b}_i} \leqslant t \\ \dfrac{\omega^0_{\mathbf{b}_i}}{\sum_{T_{\mathbf{b}_h} \geqslant t} \omega^0_{\mathbf{b}_h}} & \text{if } t < T_{\mathbf{b}_i} \end{cases}$$

Therefore P_n^S converges to the beliefs prescribed by $\overline{\mu}$ in S and, consequently, μ_n converges to $\overline{\mu}$. □

A.2. Proof of Lemma 3

Let $a^* = (\mu^*, \sigma^*)$ an equilibrium assessment in pure strategies. By contradiction, let $\sigma^*_{\mathbf{b}_e}(t) \neq \sigma^*_{\mathbf{b}_l}(t)$. Therefore by Bayes' rule the beliefs of $\mathbf{s}$ at time point $t+1$ are: if action $\sigma^*_{\mathbf{b}_e}(t)$ is observed, then $\langle \omega^{t+1}_{\mathbf{b}_e} = 1, \omega^{t+1}_{\mathbf{b}_l} = 0\rangle$ (for the sake of brevity we use $\omega^{t+1}_{\mathbf{b}_e}$ in the place of $\omega^{t+1}_{\mathbf{b}}(T_{\mathbf{b}_e})$, and analogously $\omega^{t+1}_{\mathbf{b}_l}$); if action $\sigma^*_{\mathbf{b}_l}(t)$ is observed, then $\langle \omega^{t+1}_{\mathbf{b}_e} = 0, \omega^{t+1}_{\mathbf{b}_l} = 1\rangle$.

We consider at first the case where $t \leqslant T_{\mathbf{b}_e} - 2$. If at time point t $\mathbf{b}$ acts on the equilibrium path, then at time point $t+1$ the optimal strategy of $\mathbf{s}$ is:

- if $\sigma^*_{\mathbf{b}_e}(t)$ is observed, then accept any offer greater than or equal to $x^*_{\mathbf{b}_e}(t)$ (i.e., the optimal offer at time point t in the complete information game where $\mathbf{b}_e$ is the type of $\mathbf{b}$) and otherwise offer $x^*_{\mathbf{b}_e}(t+1)$ that on the equilibrium path will be accepted by $\mathbf{b}$ at time point $t+2$;
- if $\sigma^*_{\mathbf{b}_l}(t)$ is observed, then accept any offer greater than or equal to $x^*_{\mathbf{b}_l}(t)$ (i.e., the optimal offer at time point t in the complete information game where $\mathbf{b}_l$ is the type of $\mathbf{b}$) and otherwise offer $x^*_{\mathbf{b}_l}(t+1)$ that on the equilibrium path will be accepted by $\mathbf{b}$ at time point $t+2$.

We notice that for a generic time point t where $\iota(t) = \mathbf{b}$ the two possible equilibrium agreements will be reached no later than the time point $t+2$. Then, since we are considering the case where $t \leqslant T_{\mathbf{b}_e} - 2$, the agreements are reached no later than $T_{\mathbf{b}_e}$. We notice also that $\mathbf{b}_e$ and $\mathbf{b}_l$ have, for any $t \leqslant T_{\mathbf{b}_e}$, the same utility function. Then, in order to be an equilibrium, $\sigma^*_{\mathbf{b}_e}(t)$ and $\sigma^*_{\mathbf{b}_l}(t)$ must be such that the two corresponding equilibrium agreements give the same utility to $\mathbf{b}_e$ and $\mathbf{b}_l$, otherwise one type at least deviates. We consider all the possible cases.

- If $\sigma^*_{\mathbf{b}_e}(t)$ and $\sigma^*_{\mathbf{b}_l}(t)$ are offers that $\mathbf{s}$ will accept at time point $t+1$, then, since the outcomes are different by hypothesis, they will give different utilities. As a result, one type at least will deviate from her action to make the action of the other type.
- If $\sigma^*_{\mathbf{b}_e}(t)$ and $\sigma^*_{\mathbf{b}_l}(t)$ are offers that will not be accepted at time point $t+1$, then $\mathbf{s}$ will make different offers at time point $t+1$ according to the observed $\mathbf{b}$'s offer and such offers will be accepted at time point $t+2$, giving thus different utilities. Precisely, being $x^*_{\mathbf{b}_e}(t+1) > x^*_{\mathbf{b}_l}(t+1)$, it follows that $U_{\mathbf{b}}(x^*_{\mathbf{b}_e}(t+1), t+1) < U_{\mathbf{b}}(x^*_{\mathbf{b}_l}(t+1), t+1)$. As a result, $\mathbf{b}_e$ will deviate from her strategy $\sigma^*_{\mathbf{b}_e}(t)$ to make $\sigma^*_{\mathbf{b}_l}(t)$.
- If $\sigma^*_{\mathbf{b}_l}(t)$ is an offer that will be accepted at time point $t+1$ by $\mathbf{s}$, whereas $\sigma^*_{\mathbf{b}_e}(t)$ is an offer that will not be accepted at time point $t+1$ by $\mathbf{s}$ to offer $x^*_{\mathbf{b}_e}(t+1)$, then, by hypothesis, it should be $\sigma^*_{\mathbf{b}_l}(t) = (x^*_{\mathbf{b}_e}(t+1))_{\leftarrow \mathbf{b}}$, but $\mathbf{b}_l$ can maximize her utility offering $(x^*_{\mathbf{b}_e}(t+1))_{\leftarrow \mathbf{s}} = x^*_{\mathbf{b}_e}(t)$ given that $\mathbf{s}$ will accept it. As a result, $\mathbf{b}_l$ will deviate from her strategy $\sigma^*_{\mathbf{b}_l}(t)$ to make $\sigma^*_{\mathbf{b}_e}(t)$.
- If $\sigma^*_{\mathbf{b}_e}(t)$ is an offer that will be accepted at time point $t+1$ by $\mathbf{s}$, whereas $\sigma^*_{\mathbf{b}_l}(t)$ is an offer that will not be accepted at the time point $t+1$ by $\mathbf{s}$ to offer $x^*_{\mathbf{b}_l}(t+1)$, then, by hypothesis, it should be $\sigma^*_{\mathbf{b}_e}(t) = (x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}}$, but if $(x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}} < x^*_{\mathbf{b}_e}(t)$ or $(x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}} > x^*_{\mathbf{b}_e}(t)$, $\mathbf{b}_e$ can maximize her utility offering $x^*_{\mathbf{b}_e}(t)$ that will be accepted.[8] As a result, $\mathbf{b}_e$ will deviate from her strategy $\sigma^*_{\mathbf{b}_e}(t)$, except for a null measure subset of the space of the parameters, exactly when $x^*_{\mathbf{b}_e}(t) \equiv (x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}}$, to make $\sigma^*_{\mathbf{b}_l}(t)$. (For the sake of clarity, we report in Fig. A.1 an example of construction wherein $\mathbf{b}_e$ and $\mathbf{b}_l$ have different equilibrium strategies at time point $t = 0$.)

Thus, it follows by contradiction that $\sigma^*_{\mathbf{b}_e}(t) = \sigma^*_{\mathbf{b}_l}(t)$, except for a null measure subset of the space of the parameters.

We consider now the case where $t = T_{\mathbf{b}_e} - 1$. If at time point t $\mathbf{b}$ acts on the equilibrium path, then at the time point $t = T_{\mathbf{b}_e}$ the optimal strategy of $\mathbf{s}$ is:

[8] Notice that if $(x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}} < x^*_{\mathbf{b}_e}(t)$, then $(x^*_{\mathbf{b}_l}(t+1))_{\leftarrow \mathbf{b}}$ will be rejected by $\mathbf{s}$ to offer $x^*_{\mathbf{b}_e}(t+1)$. And $x^*_{\mathbf{b}_e}(t)$ is better for $\mathbf{b}_e$ with respect to $x^*_{\mathbf{b}_e}(t+1)$.

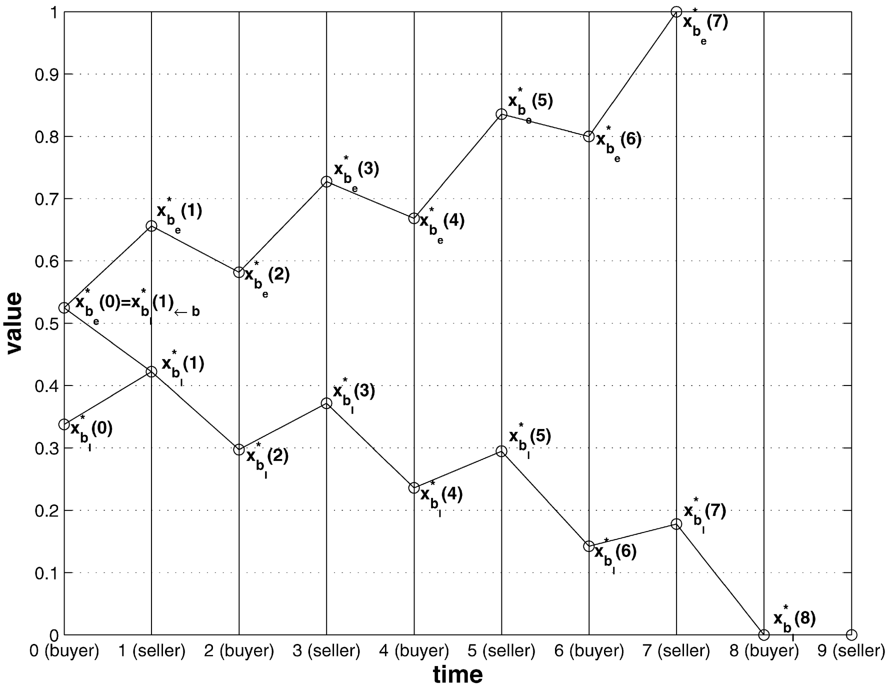

Fig. A.1. Equilibrium with $RP_{\mathbf{b}} = 1$, $RP_{\mathbf{s}} = 0$, $\delta_{\mathbf{b}} \simeq 0.8222$, $\delta_{\mathbf{s}} = 0.8$, $\langle \mathbf{T_b} = \{8, 9\}, P^0_{\mathbf{b}} = \{0.8, 0.2\}\rangle$, $T_{\mathbf{s}} = 10$, $\iota(0) = \mathbf{b}$ where at time point $t = 0$ the two types have different pure equilibrium strategies; the equilibrium existence is granted by the singularity $(RP_{\mathbf{s}})_{\leftarrow 3[\mathbf{bs}]2[\mathbf{b}]} \equiv (RP_{\mathbf{b}})_{\leftarrow 3[\mathbf{sb}]\mathbf{s}}$.

- if $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ is observed, then accept any offer greater than or equal to $RP_{\mathbf{s}}$ and otherwise exit (on the equilibrium path $\mathbf{b}_e$ will not accept anything beyond $T_{\mathbf{b}_e}$);
- if $\sigma^*_{\mathbf{b}_l}(t)$ is observed, then accept any offer greater than or equal to $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ and otherwise offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$ that on the equilibrium path will be accepted at time point $T_{\mathbf{b}_e} + 1$.

The proof is analogous to the case where $t \leqslant T_{\mathbf{b}_e} - 2$. Notice that $\mathbf{b}_e$ and $\mathbf{b}_l$ have the same utility function at time point $t = T_{\mathbf{b}_e}$. We consider all the possible cases.

- If both $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ and $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ are offers that would be accepted by $\mathbf{s}$ at $t = T_{\mathbf{b}_e}$, then one between $\mathbf{b}_e$ and $\mathbf{b}_l$ would deviate from her strategy to make the strategy of the other type.
- If $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ and $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ are offers that will not be accepted at time point $t + 1$, then both $\mathbf{b}_e$ and $\mathbf{b}_l$ deviate from their strategies to offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ that will be accepted by $\mathbf{s}$.
- If $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ is an offer that will be accepted at time point $T_{\mathbf{b}_e}$ by $\mathbf{s}$, whereas $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ is an offer that will not be accepted at time point $T_{\mathbf{b}_e}$ by $\mathbf{s}$ to offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$, then, by hypothesis, it should be $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1) < x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$, but, independently of the conjectures off the equilibrium path, the offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ will be accepted by $\mathbf{s}$. Thus, both $\mathbf{b}_e$ and $\mathbf{b}_l$ would offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$.
- If $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$ is an offer that will be accepted at time point $T_{\mathbf{b}_e}$ by $\mathbf{s}$, whereas $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ is an offer that will not be accepted at time point $T_{\mathbf{b}_e}$ by $\mathbf{s}$ to offer $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$, then, by hypothesis, it should be $x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1) \leqslant \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1) \leqslant (x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e}))_{\leftarrow \mathbf{b}}$. Since type $\mathbf{b}_e$ will gain utility equal to 0 (being $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 1)$ unacceptable, $\mathbf{s}$ will exit), type $\mathbf{b}_e$ can gain more offering $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 1)$.

Thus, it follows by contradiction that $\sigma^*_{\mathbf{b}_e}(t) = \sigma^*_{\mathbf{b}_l}(t)$. □

A.3. Proof of Lemma 4

Let $a^* = (\mu^*, \sigma^*)$ be an equilibrium assessment in pure strategies. By contradiction, let $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e}) = \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$. Independently of the system of beliefs, the strategy $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e})$ is to accept anything lower than $RP_{\mathbf{b}}$, otherwise exit.

However, $\mathbf{b}_l$, independently of the system of beliefs, does not accept anything greater than $(x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e}))_{\leftarrow\mathbf{b}}$ that is strictly lower than $RP_{\mathbf{b}}$. Thus, by contradiction $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e}) \neq \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$. □

A.4. Proof of Lemma 7

Let $a^* = (\mu^*, \sigma^*)$ an equilibrium assessment in pure strategies. We consider at first the case where $\overline{T} = T_{\mathbf{b}_e} + 1$. Let $(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}} > (x^*(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{b}]}$ (i.e., inequality (5) is violated at $t = T_{\mathbf{b}_e} - 2$). Assume that the agents have reached the time point $T_{\mathbf{b}_e} - 2$ relying on the initial beliefs. In this situation the equilibrium strategies at time points $t = T_{\mathbf{b}_e}$ and $t = T_{\mathbf{b}_l}$ do not depend on the system of beliefs, being:

$$\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e}) = \begin{cases} accept & \text{if } \sigma_{\mathbf{s}}(T_{\mathbf{b}_e} - 1) \leqslant RP_{\mathbf{b}} \\ exit & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e}) = \begin{cases} accept & \text{if } \sigma_{\mathbf{s}}(T_{\mathbf{b}_e} - 1) \leqslant (RP_{\mathbf{s}})_{\leftarrow\mathbf{b}} \\ offer(RP_{\mathbf{s}}) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(\overline{T}) = \begin{cases} accept & \text{if } \sigma_{\mathbf{b}}(T_{\mathbf{b}_e}) \geqslant RP_{\mathbf{s}} \\ exit & \text{otherwise} \end{cases}$$

whereas at time points $t = T_{\mathbf{b}_e} - 2$ and $t = T_{\mathbf{b}_e} - 1$ the equilibrium strategies depend on the system of beliefs of $\mathbf{s}$. At $t = T_{\mathbf{b}_e} - 1$ the beliefs of $\mathbf{s}$ on the equilibrium path depend on the equilibrium actions of $\mathbf{b}$'s types at $t = T_{\mathbf{b}_e} - 2$. By contradiction, let $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) = \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$. By Bayes' rule (implied by Kreps and Wilson's consistency) the beliefs of $\mathbf{s}$ at time point $t = T_{\mathbf{b}_e} - 1$, once observed $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) = \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$, are the initial ones. Therefore, if at time point $t = T_{\mathbf{b}_e} - 2$ $\mathbf{b}$ acts on the equilibrium path, then at time point $t = T_{\mathbf{b}_e} - 1$ the optimal strategy of $\mathbf{s}$ is to accept any offer greater than or equal to $(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}}$ and otherwise offer $RP_{\mathbf{b}}$. It must be $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \in [(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}}, (RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}]$, otherwise $\mathbf{b}_e$ would deviate from offering $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2)$ independently of μ^*.[9] Analogously it must be $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2) \leqslant (RP_{\mathbf{s}})_{\leftarrow 2[\mathbf{b}]}$, otherwise $\mathbf{b}_l$ would deviate from $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$ independently of μ^*.[10] Since, by hypothesis, $(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}} > (RP_{\mathbf{s}})_{\leftarrow 2[\mathbf{b}]}$, if $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) = \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$, then at least one type between $\mathbf{b}_e$ and $\mathbf{b}_l$ deviates. Therefore, it follows by contradiction that $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \neq \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$.

All the cases where $\overline{T} > T_{\mathbf{b}_e} + 1$ and $(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}} > (x^*(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{b}]}$ (i.e., violation of inequality (5) at $t = T_{\mathbf{b}_e} - 2$) can be analyzed similarly, because, independently of μ^*, $\mathbf{s}$ will accept at time point $T_{\mathbf{b}_e} + 1$ any offer x such that $x \geqslant x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$ (i.e., the equilibrium offer at time point $T_{\mathbf{b}_e}$ when $\mathbf{s}$ believes $\mathbf{b}$ to be of type $\mathbf{b}_l$). It must be $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \in [(e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}}, (RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}]$, otherwise $\mathbf{b}_e$ would deviate from offering $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2)$ independently of μ^*. Analogously it must be $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2) \leqslant (x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{b}]}$, otherwise $\mathbf{b}_l$ would deviate from $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$ independently of μ^*. By construction, $x^*(T_{\mathbf{b}_e}) \equiv x^*_{\mathbf{b}_l}(T_{\mathbf{b}_e})$, then it follows that $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \neq \sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2)$. □

A.5. Proof of Theorem 8

We start considering the special case with two buyer's types. By Lemma 3 it is almost always possible to reach time point $t = T_{\mathbf{b}_e} - 2$ relying on the initial beliefs (if $\mathbf{b}$ makes the equilibrium offers, which are the same for both types, and $\mathbf{s}$ refuses them, it can be seen that Kreps and Wilson's consistency forces μ^* to be such that the beliefs remain the initial ones). In this situation, if inequality (5) does not hold at $t = T_{\mathbf{b}_e} - 2$, then from Lemma 7 it follows $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) \neq \sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_l} - 2)$, that contradicts Lemma 3. Therefore, if inequality (5) does not hold at $t = T_{\mathbf{b}_e} - 2$, then the equilibrium assessment a^* does not exist, except for a null measure subset of the space of the parameters.

The proof in the general case where the number of types of $\mathbf{b}$ is n can be easily obtained by iteratively applying the special case where the types are two as suggested in the sketch of the proof of Theorem 5. □

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[9] We consider $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) < (e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}}$: if $\mathbf{b}_e$ makes such an offer, $\mathbf{s}$ would reject it to offer $RP_{\mathbf{b}}$ at $t = T_{\mathbf{b}_e} - 1$ that will be accepted by $\mathbf{b}_e$ at time point $t = T_{\mathbf{b}_e}$; however, if $\mathbf{b}_e$ makes $(RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}$, such an offer would be accepted by $\mathbf{s}$ independently of her beliefs, giving to $\mathbf{b}_e$ more utility than offering $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2)$, i.e., $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) < (e^*(T_{\mathbf{b}_e} - 1))_{\leftarrow\mathbf{s}}$ cannot be an equilibrium offer for $\mathbf{b}_e$. We consider $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) > (RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}$: if $\mathbf{b}_e$ makes such an offer, $\mathbf{s}$ would accept it; however, if $\mathbf{b}_e$ makes $(RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}$, such an offer would be accepted by $\mathbf{s}$ at time point $t = T_{\mathbf{b}_e} - 1$ independently of her beliefs, giving to $\mathbf{b}_e$ more utility than offering $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2)$, i.e., $\sigma^*_{\mathbf{b}_e}(T_{\mathbf{b}_e} - 2) > (RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}$ cannot be an equilibrium offer for $\mathbf{b}_e$.

[10] Whatever the system of beliefs of $\mathbf{s}$ is, $\mathbf{s}$ would accept $(RP_{\mathbf{b}})_{\leftarrow\mathbf{s}}$ at time point $t = T_{\mathbf{b}_l}$, that is equivalent for $\mathbf{b}_l$ to the offer $(RP_{\mathbf{s}})_{\leftarrow 2[\mathbf{b}]}$ accepted at time point $t = T_{\mathbf{b}_e} - 1$. Thus offering $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2) > (RP_{\mathbf{s}})_{\leftarrow 2[\mathbf{b}]}$ is dominated for $\mathbf{b}_l$ by making any offer unacceptable by $\mathbf{s}$ at $t = T_{\mathbf{b}_e} - 1$, and then, at time point $t = T_{\mathbf{b}_e}$ offer $RP_{\mathbf{s}}$ that will be accepted, i.e., $\sigma^*_{\mathbf{b}_l}(T_{\mathbf{b}_e} - 2) \leqslant (RP_{\mathbf{s}})_{\leftarrow 2[\mathbf{b}]}$.

A.6. Proof of Lemma 9

At time point $t = T_{\mathbf{b}_e} - 1$, being $\omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}} = 0$, we have $\omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1} = \frac{\check{x}(T_{\mathbf{b}_e}-2)-(\check{x}(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{s}]}}{\hat{x}(T_{\mathbf{b}_e}-2)-(\check{x}(T_{\mathbf{b}_e}))_{\leftarrow 2[\mathbf{s}]}} = \overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2}$. Being for any t $\hat{x}(t) > \check{x}(t)$ and $\check{x}(t) > (\check{x}(t+2))_{\leftarrow 2[\mathbf{s}]}$, it follows that $1 > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1} > 0$. If inequality (5) does not hold, then $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$ and, *vice versa*, if $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$, then inequality (5) does not hold. If $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$, then $1 > \alpha^{T_{\mathbf{b}_e}-2} > 0$ and, *vice versa*, if $1 > \alpha^{T_{\mathbf{b}_e}-2} > 0$, then $\omega_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-2} > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$. Thus the probability values of $\omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1}$ and $\alpha^{T_{\mathbf{b}_e}-2}$ are well defined if and only if inequality (5) does not hold. □

A.7. Proof of Lemma 10

The proof is by mathematical induction. The base of the induction is given by Lemma 9, exactly it is $1 > \omega_{\mathbf{b}_e}^{*,T_{\mathbf{b}_e}-1} = \overline{\omega}_{\mathbf{b}_e}^{T_{\mathbf{b}_e}-1}$ if and only if condition (5) does not hold. The inductive step can be proved as follows. The probability $\omega_{\mathbf{b}_e}^{*,t}$ can be written as:

$$\omega_{\mathbf{b}_e}^{*,t} = \omega_{\mathbf{b}_e}^{*,t+2} \cdot \frac{a}{a+b} + \frac{b}{a+b}$$

where $a = \hat{x}(t-1) - \check{x}(t-1)$ and $b = \check{x}(t-1) - (\check{x}(t+1))_{\leftarrow 2[\mathbf{s}]}$. The values a and b are such that $a, b \in (0,1)$ for all $t \in [\overset{<}{t}, \overset{>}{t}]$. Thus for any time $t \in [\overset{<}{t}, \overset{>}{t}]$ where $\iota(t) = \mathbf{b}$, if $1 > \omega_{\mathbf{b}_e}^{*,t+2} > 0$, then $1 > \omega_{\mathbf{b}_e}^{*,t} > \omega_{\mathbf{b}_e}^{*,t+2} > 0$. If inequality (7) does not hold at time point t, then it is $\omega_{\mathbf{b}_e}^{t} > \omega_{\mathbf{b}_e}^{*,t+2}$ and, *vice versa*, if $\omega_{\mathbf{b}_e}^{t} > \omega_{\mathbf{b}_e}^{*,t+2}$, then inequality (7) does not hold at time point t. If $\omega_{\mathbf{b}_e}^{t} > \omega_{\mathbf{b}_e}^{*,t+2}$, then it is $1 > \alpha^t > 0$ and, *vice versa*, if $1 > \alpha^t > 0$, then $\omega_{\mathbf{b}_e}^{t} > \omega_{\mathbf{b}_e}^{*,t+2}$. Thus the probability values of $\omega_{\mathbf{b}_e}^{*,t+2}$ and α^t are well defined if and only if inequality (7) does not hold. □

A.8. Proof of Theorem 12

We at first provide the proof of sequential rationality of the strategies given the system of beliefs.

At any time point $t \geqslant T_{\mathbf{b}_e}$ the proof of sequential rationality of the strategies is trivial, since in these time points the game is with complete information. At any time point $t < T_{\mathbf{b}_e}$ the proof is by mathematical induction, where the base of the induction is given by the analysis of time points $t = T_{\mathbf{b}_e} - 1$ and $t = T_{\mathbf{b}_e} - 2$.

Consider time point $t = T_{\mathbf{b}_e} - 1$. If **s** receives any offer equal to or greater than $\hat{x}(T_{\mathbf{b}_e} - 2)$, she believes her opponent to be of type $\mathbf{b}_e$ and her optimal action is to accept. Therefore she cannot gain more by deviating from σ^*. If **s** receives any offer belonging to the interval $[\check{x}(T_{\mathbf{b}_e} - 2), \hat{x}(T_{\mathbf{b}_e} - 2))$, the system of beliefs is such that it is indifferent for **s** to accept such an offer or reject it to offer $\hat{x}(T_{\mathbf{b}_e} - 1)$ and no action is better than these. Therefore she cannot gain more by deviating from σ^*. Being by construction $\hat{x}(T_{\mathbf{b}_e} - 1) > \check{x}(T_{\mathbf{b}_e} - 1)$, it holds $\beta^{T_{\mathbf{b}_e}-1} \in (0,1)$ and then **s** can actually randomize. If **s** receives any offer lower than $\hat{x}(T_{\mathbf{b}_e} - 2)$, the system of beliefs is such that the expected utility of offering $\hat{x}(T_{\mathbf{b}_e} - 1)$ is greater than the utility of accepting the received offer and no action is better than making such an offer. Therefore **s** cannot gain more by deviating from σ^*.

Consider time point $t = T_{\mathbf{b}_e} - 2$. Consider $\mathbf{b}_e$. If $\mathbf{b}_e$ receives any offer greater than $\hat{x}(T_{\mathbf{b}_e} - 3)$, her optimal action is to reject it and the value of $\beta^{T_{\mathbf{b}_e}-1}$ is such that offering $\hat{x}(T_{\mathbf{b}_e} - 2)$ and offering $\check{x}(T_{\mathbf{b}_e} - 2)$ give the same expected utility to $\mathbf{b}_e$. Therefore $\mathbf{b}_e$ does not deviate from σ^*. If $\mathbf{b}_e$ receives any offer belonging to the interval $(\check{x}(T_{\mathbf{b}_e} - 3), \hat{x}(T_{\mathbf{b}_e} - 3)]$, the value of $\beta^{T_{\mathbf{b}_e}-1}$ is such that accepting the received offer and rejecting it to offer $\check{x}(T_{\mathbf{b}_e} - 2)$ give the same expected utility to $\mathbf{b}_e$ and no action is better than these. Therefore $\mathbf{b}_e$ does not deviate from σ^*. Not holding by construction condition (5), it holds $\alpha^{T_{\mathbf{b}_e}-2} \in (0,1)$ and then $\mathbf{b}_e$ can actually randomize. If $\mathbf{b}_e$ receives any offer equal to or lower than $\hat{x}(T_{\mathbf{b}_e} - 3)$, her optimal action is to accept the received offer. Therefore $\mathbf{b}_e$ cannot gain more by deviating from σ^*. Consider $\mathbf{b}_l$. If $\mathbf{b}_l$ receives any offer greater than $\check{x}(T_{\mathbf{b}_e} - 3)$, her optimal action is to reject it and offer $\check{x}(T_{\mathbf{b}_e} - 2)$. Therefore $\mathbf{b}_l$ does not deviate from σ^*. If $\mathbf{b}_l$ receives any offer equal to or lower than $\hat{x}(T_{\mathbf{b}_e} - 3)$, her optimal action is to accept the received offer. Therefore $\mathbf{b}_l$ cannot gain more by deviating from σ^*.

The two steps above can be inductively applied until α^t or β^t have not well defined values of probability. When condition (7) holds, α^t and β^t have not well defined values of probabilities and the construction can continue in pure strategies as prescribed in Section 4.1.

The consistency can be proved with the following fully mixed strategy. For any time point t such that $t \leqslant \overset{<}{t}$ or $t \geqslant T_{\mathbf{b}_e}$, the fully mixed strategy to employ is the one used in [5], i.e. for any $t < T_{\mathbf{b}_i}$ $\mathbf{b}_i$ puts $1 - \frac{1}{n}$ on the equilibrium strategy and $\frac{1}{n}$ distributed over all the other strategies and for $t \geqslant T_{\mathbf{b}_i}$ $\mathbf{b}_i$ puts $1 - \frac{1}{n^2}$ on the equilibrium strategy and $\frac{1}{n^2}$ distributed over all the other strategies. For all ts such that $\overset{<}{t} < t < \overset{>}{t}$ $\mathbf{s}$ puts $\beta^t(\sigma_{\mathbf{s}}(t-2)) \cdot (1 - \frac{1}{n})$ and $(1 - \beta^t(\sigma_{\mathbf{s}} - 2)) \cdot (1 - \frac{1}{n})$ on the equilibrium strategies and $\frac{1}{n}$ distributed over all the other strategies. The fully mixed strategy of $\mathbf{b}$ is more complicated (for the sake of simplicity, we report non-normalized probabilities):

- $\mathbf{b}_e$: she puts $\alpha^t \cdot (1 - \frac{1}{n})$ and $(1 - \alpha^t) \cdot (1 - \frac{1}{n})$ on the equilibrium strategies, $\frac{1}{n}$ distributed over all the offers $> \hat{x}(t)$, $\alpha^t \cdot \frac{1}{n}$ distributed over all the offers $< \check{x}(t)$, and $C_{\mathbf{b}_e} \cdot \frac{1}{n}$ distributed over all the offers belonging to $(\check{x}(t), \hat{x}(t))$;
- $\mathbf{b}_l$: she puts $(1 - \frac{1}{n} - \frac{1}{n^{2T_{\mathbf{b}_l}}})$ on the equilibrium strategy, $\frac{1}{n^{2T_{\mathbf{b}_l}}}$ distributed over all the offers $> \hat{x}(t)$, $\frac{1}{n}$ distributed over all the offers $< \check{x}(t)$, and $C_{\mathbf{b}_l} \cdot \frac{1}{n}$ distributed over all the offers belonging to $(\check{x}(t), \hat{x}(t))$.

The coefficients $C_{\mathbf{b}_e}$ and $C_{\mathbf{b}_l}$ are computed such that the consistency holds also in the interval $(\check{x}(t), \hat{x}(t))$. □

Appendix B. Formulas to compute mixed strategy with many types

Four choice rules can be employed at time point t when $\iota(t) = \mathbf{b}$ and the equilibrium assessment cannot be in pure strategies. However, given a setting, only one choice rule can be employed at a single time point t to produce an equilibrium, whereas different choice rules can be employed within a construction at different time points. The principal condition that discriminates the use of the choice rules is the value of $\hat{x}(t+1)$ computed in the backward construction from time point $\overline{T}$ to time point $t+1$. When $\hat{x}(t+1) = (\hat{x}(t+2))_{\leftarrow \mathbf{b}}$, only one choice rule can be employed. We call it choice rule 1, and we describe it in Appendix B.1. When $\hat{x}(t+1) = RP_{\mathbf{b}}$, it is necessary to check other conditions, being three the possible choice rules that can be employed. We call them choice rules 2.1, 2.2, and 2.3, and we discuss them in Appendix B.2.

B.1. Choice rule 1

It can be employed at time point t when $\sigma^*_{\mathbf{s}}(t+1) = (\hat{x}(t+2))_{\leftarrow \mathbf{b}}$. In this situation it can be easily seen that it is always possible to increase the value of α^t to have $U_{\mathbf{s}}(\check{x}(t), t+1) = EU_{\mathbf{s}}(\mathit{offer}(\hat{x}(t+1)), t+1)$ preserving that $\mathbf{s}$ prefers at time point $t+1$ offering $(\hat{x}(t+2))_{\leftarrow \mathbf{b}}$ to offering $RP_{\mathbf{b}}$. The rule to use in order to compose the sets $\hat{\Theta}_{\mathbf{b}}(t), \tilde{\Theta}_{\mathbf{b}}(t), \check{\Theta}_{\mathbf{b}}(t)$ is:

$$\begin{cases} \hat{\Theta}_{\mathbf{b}}(t) = \{\theta_{\mathbf{b}}(t+1)\} \\ \tilde{\Theta}_{\mathbf{b}}(t) = \hat{\Theta}_{\mathbf{b}}(t+2) \cup \tilde{\Theta}_{\mathbf{b}}(t+2) \cup \{\theta_{\mathbf{b}}(t+2)\} \\ \check{\Theta}_{\mathbf{b}}(t) = \check{\Theta}_{\mathbf{b}}(t+2) \end{cases}$$

The parameters whereby the equilibrium strategy and the system of beliefs are based can be computed as follows:

$$\hat{x}(t) = \frac{\tilde{\Omega}^0_{\mathbf{b}}(t)}{\hat{\Omega}^0_{\mathbf{b}}(t) + \tilde{\Omega}^0_{\mathbf{b}}(t)} \cdot [(\hat{x}(t+1))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}}] + RP_{\mathbf{s}}$$

$$\tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) = \frac{\check{x}(t) \cdot (1 - \tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2)) + \tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2) \cdot (\hat{x}(t+1))_{\leftarrow \mathbf{s}} - (\check{x}(t+2))_{\leftarrow 2[\mathbf{s}]}}{(\hat{x}(t+1))_{\leftarrow \mathbf{s}} - (\check{x}(t+2))_{\leftarrow 2[\mathbf{s}]}}$$

$$\alpha^t = \frac{\tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \cdot \check{\Omega}^0_{\mathbf{b}}(t)}{(1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot \tilde{\Omega}^0_{\mathbf{b}}(t)}$$

$$\alpha^{t+2} = \frac{\tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2) \cdot (1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot (\omega^0_{\mathbf{b}}(t+2) + \hat{\Omega}^0_{\mathbf{b}}(t+2) + \tilde{\Omega}^0_{\mathbf{b}}(t+2))}{(1 - \tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2)) \cdot \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \cdot \tilde{\Omega}^0_{\mathbf{b}}(t+2)}$$

$$\beta^{t+1} = \frac{RP_{\mathbf{b}} - \hat{x}(t)}{RP_{\mathbf{b}} - \check{x}(t)}$$

The optimal actions of **b** at time point t and of **s** at time point $t+1$ can be defined specifying the choice rules employed by the agents:

$$\sigma^*_{\hat{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (\hat{x}(t))_{\leftarrow \mathbf{b}} \\ \textit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\tilde{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \begin{cases} \textit{accept} & 1-\alpha^t \\ \textit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{if } \check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant (\hat{x}(t))_{\leftarrow \mathbf{b}} \\ \begin{cases} \textit{offer}(\hat{x}(t)) & 1-\alpha^t \\ \textit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{otherwise} \end{cases}$$

$$\sigma^*_{\check{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (\check{x}(t))_{\leftarrow \mathbf{b}} \\ \textit{offer}(\check{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t+1) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{b}}(t) \geqslant (\check{x}(t+1))_{\leftarrow \mathbf{s}} \text{ or } \sigma_{\mathbf{b}}(t) = \hat{x}(t) \\ \begin{cases} \textit{accept} & \beta^t \\ \textit{offer}(\hat{x}(t)) & 1-\beta^t \end{cases} & \text{if } (\hat{x}(t+1))_{\leftarrow \mathbf{s}} > \sigma_{\mathbf{b}}(t) > \hat{x}(t) \text{ or } \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) \\ \textit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

The system of beliefs is:

$$\mu(t+1) = \begin{cases} \sigma_{\mathbf{b}}(t) \geqslant (\hat{x}(t+1))_{\leftarrow \mathbf{s}} & \begin{cases} \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \\ \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 1 \\ \check{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \end{cases} \\ (\hat{x}(t+1))_{\leftarrow \mathbf{s}} > \sigma_{\mathbf{b}}(t) > \hat{x}(t) & \begin{cases} \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \\ \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) = \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) + (1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot \frac{\sigma_{\mathbf{b}}(t) - \check{x}(t)}{\hat{x}(t) - \check{x}(t)} \\ \check{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 1 - \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) \end{cases} \\ \sigma_{\mathbf{b}}(t) = \hat{x}(t) & \begin{cases} \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) = (1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot \tilde{\Omega}^{t}_{\mathbf{b}}(t) \cdot (1 + \frac{\hat{\Omega}^0_{\mathbf{b}}(t)}{\tilde{\Omega}^0_{\mathbf{b}}(t)}) \\ \qquad - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \cdot \check{\Omega}^{t}_{\mathbf{b}}(t) \\ \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 1 - \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) \\ \check{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \end{cases} \\ \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) & \begin{cases} \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \\ \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) = \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) + (1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot \frac{\sigma_{\mathbf{b}}(t) - \check{x}(t)}{\hat{x}(t) - \check{x}(t)} \\ \check{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 1 - \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) \end{cases} \\ \check{x}(t) \geqslant \sigma_{\mathbf{b}}(t+1) & \begin{cases} \hat{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 0 \\ \tilde{\Omega}^{t+1}_{\mathbf{b}}(t+1) = \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \\ \check{\Omega}^{t+1}_{\mathbf{b}}(t+1) = 1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \end{cases} \end{cases}$$

B.2. Choice rule 2

It requires that $\sigma^*_{\mathbf{s}}(t+1) = RP_{\mathbf{b}}$ and there are three possible forms of choice rules. In this situation it is not always possible to increase the value of α^t to have $U_{\mathbf{s}}(\check{x}(t), t+1) = EU_{\mathbf{s}}(\textit{offer}(\hat{x}(t+1)), t+1)$ preserving that **s** prefers at time point $t+1$ offering $RP_{\mathbf{b}}$ to offering $(\hat{x}(t+2))_{\leftarrow \mathbf{b}}$. For high values of α^t **s** could prefer at time point $t+1$ offering $(\hat{x}(t+2))_{\leftarrow \mathbf{b}}$ to offering $RP_{\mathbf{b}}$. In order to introduce the conditions of the choice rules, we need to compute the following values:

$$A = \left(\frac{\hat{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} + \frac{\tilde{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} \cdot (1 - \alpha^{t+2})\right) \cdot ((\hat{x}(t+2))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}})$$
$$+ \left(\alpha^{t+2} \cdot \frac{\tilde{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} + \frac{\check{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)}\right) \cdot ((\check{x}(t+2))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}}$$

$$B = \left(\frac{\hat{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} + \frac{\tilde{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} \cdot (1-\alpha^{t+2})\right) \cdot ((\hat{x}(t+2))_{\leftarrow \mathbf{b}} - RP_{\mathbf{s}})$$
$$+ \left(\alpha^{t+2} \cdot \frac{\tilde{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)} + \frac{\check{\Omega}^0_{\mathbf{b}}(t+2)}{\Omega^0_{\mathbf{b}}(t+3)}\right) \cdot ((\check{x}(t+2))_{\leftarrow \mathbf{s}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}}$$

$$D = \left(\frac{\omega^0_{\mathbf{b}}(t+2)}{\tilde{\Omega}^0_{\mathbf{b}}(t)} \cdot (RP_{\mathbf{b}} - RP_{\mathbf{s}}) + RP_{\mathbf{s}}\right) \cdot \delta_{\mathbf{s}}$$

$$C = \frac{\frac{\check{x}(t)}{\delta_{\mathbf{s}}} - A}{\frac{\omega^0_{\mathbf{b}}(t+2)}{\tilde{\Omega}^0_{\mathbf{b}}(t)} - A}$$

$$\overline{\Omega}^* = \frac{B - A}{(\frac{\omega^0_{\mathbf{b}}(t+2)}{\tilde{\Omega}^0_{\mathbf{b}}(t)} - A) - (\frac{\omega^0_{\mathbf{b}}(t+2)}{\tilde{\Omega}^0_{\mathbf{b}}(t)} \cdot ((\hat{x}(t+2))_{\leftarrow \mathbf{b}} - RP_{\mathbf{s}}) - B)}$$

B.2.1. Choice rule 2.1

It can be employed when $C \geqslant \overline{\Omega}^*$. In this situation the value of α^t is such that **s** keeps to prefer at time point $t+1$ offering $RP_{\mathbf{b}}$ to offering $(\hat{x}(t+2))_{\leftarrow \mathbf{b}}$. The rule to use in order to compose the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, $\check{\Theta}_{\mathbf{b}}(t)$ is:

$$\begin{cases} \hat{\Theta}_{\mathbf{b}}(t) = \emptyset \\ \tilde{\Theta}_{\mathbf{b}}(t) = \{\theta_{\mathbf{b}}(t+1), \theta_{\mathbf{b}}(t+2)\} \\ \check{\Theta}_{\mathbf{b}}(t) = \hat{\Theta}_{\mathbf{b}}(t+2) \cup \tilde{\Theta}_{\mathbf{b}}(t+2) \cup \check{\Theta}_{\mathbf{b}}(t+2) \end{cases} \tag{B.1}$$

The parameters whereby the equilibrium strategy and the system of beliefs are based can be computed as follows:

$$\hat{x}(t) = D$$
$$\tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) = C$$
$$\alpha^t = \frac{\tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t) \cdot \check{\Omega}^0_{\mathbf{b}}(t)}{(1 - \tilde{\Omega}^{*,t+1}_{\mathbf{b}}(t)) \cdot \tilde{\Omega}^0_{\mathbf{b}}(t)}$$
$$\alpha^{t+2} = \frac{\tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2) \cdot \check{\Omega}^0_{\mathbf{b}}(t+2)}{(1 - \tilde{\Omega}^{*,t+3}_{\mathbf{b}}(t+2)) \cdot \tilde{\Omega}^0_{\mathbf{b}}(t+2)}$$
$$\beta^{t+1} = \frac{RP_{\mathbf{b}} - \hat{x}(t)}{RP_{\mathbf{b}} - \check{x}(t)}$$

The optimal actions of **b** at time point t and of **s** at time point $t+1$ are:

$$\sigma^*_{\tilde{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \begin{cases} \textit{accept} & 1-\alpha^t \\ \textit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{if } \check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant (\hat{x}(t))_{\leftarrow \mathbf{b}} \\ \begin{cases} \textit{offer}(\hat{x}(t)) & 1-\alpha^t \\ \textit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{otherwise} \end{cases}$$

$$\sigma^*_{\check{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (\check{x}(t))_{\leftarrow \mathbf{b}} \\ \textit{offer}(\check{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t+1) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{b}}(t) \geqslant \hat{x}(t) \\ \begin{cases} \textit{accept} & \beta^t \\ \textit{offer}(\hat{x}(t)) & 1-\beta^t \end{cases} & \text{if } \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) \\ \textit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

The system of beliefs is:

$$\mu(t+1) = \begin{cases} \sigma_{\mathbf{b}}(t) \geqslant \hat{x}(t) & \begin{cases} \tilde{\Omega}_{\mathbf{b}}^{t+1}(t+1) = 1 \\ \check{\Omega}_{\mathbf{b}}^{t+1}(t+1) = 0 \end{cases} \\ \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) & \begin{cases} \tilde{\Omega}_{\mathbf{b}}^{t+1}(t+1) = \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) + (1 - \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t)) \cdot \frac{\sigma_{\mathbf{b}}(t) - \check{x}(t)}{\hat{x}(t) - \check{x}(t)} \\ \check{\Omega}_{\mathbf{b}}^{t+1}(t+1) = 1 - \tilde{\Omega}_{\mathbf{b}}^{t+1}(t+1) \end{cases} \\ \check{x}(t) \geqslant \sigma_{\mathbf{b}}(t+1) & \begin{cases} \tilde{\Omega}_{\mathbf{b}}^{t+1}(t+1) = \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) \\ \check{\Omega}_{\mathbf{b}}^{t+1}(t+1) = 1 - \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) \end{cases} \end{cases}$$

B.2.2. Choice rule 2.2

It can be employed when $C < \overline{\Omega}^*$ and $D \geqslant (\hat{x}(t+2))_{\leftarrow 2[\mathbf{b}]}$. In this situation the value of α^t would be such that $\mathbf{s}$ prefers at time point $t+1$ offering $(\hat{x}(t+2))_{\leftarrow \mathbf{b}}$ to offering $RP_{\mathbf{b}}$, thus a slight variation of the choice rule 2.1 is needed. Choice rule 2.2 and choice rule 2.3 are complementary when $C < \overline{\Omega}^*$. The rule to use in order to compose the sets $\hat{\Theta}_{\mathbf{b}}(t)$, $\tilde{\Theta}_{\mathbf{b}}(t)$, $\check{\Theta}_{\mathbf{b}}(t)$ is:

$$\begin{cases} \hat{\Theta}_{\mathbf{b}}(t) = \{\theta_{\mathbf{b}}(t+1)\} \\ \tilde{\Theta}_{\mathbf{b}}(t) = \{\theta_{\mathbf{b}}(t+2)\} \\ \check{\Theta}_{\mathbf{b}}(t) = \hat{\Theta}_{\mathbf{b}}(t+2) \cup \tilde{\Theta}_{\mathbf{b}}(t+2) \cup \check{\Theta}_{\mathbf{b}}(t+2) \end{cases} \tag{B.2}$$

The parameters whereby the optimal actions and the system of beliefs are based can be computed as follows:

$$\hat{x}(t) = D$$
$$\tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) = C$$
$$\alpha^t = \frac{\tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) \cdot \check{\Omega}_{\mathbf{b}}^{0}(t)}{(1 - \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t)) \cdot \tilde{\Omega}_{\mathbf{b}}^{0}(t)}$$
$$\alpha^{t+2} = \frac{\tilde{\Omega}_{\mathbf{b}}^{*,t+3}(t+2) \cdot \check{\Omega}_{\mathbf{b}}^{0}(t+2)}{(1 - \tilde{\Omega}_{\mathbf{b}}^{*,t+3}(t+2)) \cdot \tilde{\Omega}_{\mathbf{b}}^{0}(t+2)}$$
$$\beta^{t+1} = \frac{RP_{\mathbf{b}} - \hat{x}(t)}{RP_{\mathbf{b}} - (\hat{x}(t+2))_{\leftarrow 2[\mathbf{b}]}}$$

The optimal actions of $\mathbf{b}$ at time point t and of $\mathbf{s}$ at time point $t+1$ are:

$$\sigma^*_{\hat{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \hat{x}(t-1) \\ \mathit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\tilde{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \begin{cases} \mathit{accept} & 1-\alpha^t \\ \mathit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{if } \check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant (\hat{x}(t))_{\leftarrow \mathbf{b}} \\ \begin{cases} \mathit{offer}(\hat{x}(t)) & 1-\alpha^t \\ \mathit{offer}(\check{x}(t)) & \alpha^t \end{cases} & \text{otherwise} \end{cases}$$

$$\sigma^*_{\check{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (\check{x}(t))_{\leftarrow \mathbf{b}} \\ \mathit{offer}(\check{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t+1) = \begin{cases} \mathit{accept} & \text{if } \sigma_{\mathbf{b}}(t) \geqslant \hat{x}(t) \\ \begin{cases} \mathit{offer}((\hat{x}(t+2))_{\leftarrow \mathbf{b}}) & \beta^t \\ \mathit{offer}(\hat{x}(t)) & 1-\beta^t \end{cases} & \text{if } \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) \\ \mathit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

The system of beliefs is exactly the same of that for choice rule 1.

B.2.3. Choice rule 2.3

It can be employed when $C < \overline{\Omega}^*$ and $D < (\hat{x}(t+2))_{\leftarrow 2[\mathbf{b}]}$. The rule to use in order to compose the sets $\hat{\Theta}_{\mathbf{b}}(t), \tilde{\Theta}_{\mathbf{b}}(t), \check{\Theta}_{\mathbf{b}}(t)$ is:

$$\begin{cases} \hat{\Theta}_{\mathbf{b}}(t) = \{\theta_{\mathbf{b}}(t+1)\} \\ \tilde{\Theta}_{\mathbf{b}}(t) = \hat{\Theta}_{\mathbf{b}}(t+2) \cup \tilde{\Theta}_{\mathbf{b}}(t+2) \cup \{\theta_{\mathbf{b}}(t+2)\} \\ \check{\Theta}_{\mathbf{b}}(t) = \check{\Theta}_{\mathbf{b}}(t+2) \end{cases}$$

This assessment form presents two anomalies with respect to the others. The first one is that the types that randomize, i.e, $\tilde{\Theta}_{\mathbf{b}}(t)$, do it with different probabilities. Specifically, all the types belonging to the set $\hat{\Theta}_{\mathbf{b}}(t+2) \cup \tilde{\Theta}_{\mathbf{b}}(t+2)$ randomize with probability α_1^t, whereas the type $\theta_{\mathbf{b}}(t+2)$ randomizes with probability α_2^t. The second one is that this choice rule modifies the construction from time point $\overline{T}$ to time point $t+1$. Specifically, it modifies the values of $\hat{x}(t+1)$. In order to compute the parameters whereby the equilibrium strategies are based, we need to compute the following values:

$$E = \frac{\tilde{\Omega}_{\mathbf{b}}^{*,t+3}(t+2)}{1 - \tilde{\Omega}_{\mathbf{b}}^{*,t+3}(t+2)}$$

$$F = RP_{\mathbf{b}} - (\hat{x}(t+2))_{\leftarrow \mathbf{b}} - E \cdot ((\hat{x}(t+2))_{\leftarrow \mathbf{b}} - (\hat{x}(t+2))_{\leftarrow \mathbf{s}})$$

$$G = (\hat{x}(t+2))_{\leftarrow \mathbf{b}} - (\hat{x}(t+2))_{\leftarrow \mathbf{s}}$$

$$H = (1+E) \cdot G_{\leftarrow \mathbf{s}}$$

The parameters whereby the optimal actions and the system of beliefs are based can be computed as follows:

$$\hat{x}(t) = \frac{\tilde{\Omega}_{\mathbf{b}}^{0}(t)}{\hat{\Omega}_{\mathbf{b}}^{0}(t) + \tilde{\Omega}_{\mathbf{b}}^{0}(t)} \cdot [(\hat{x}(t+2))_{\leftarrow \mathbf{bs}} - RP_{\mathbf{s}}] + RP_{\mathbf{s}}$$

$$\hat{x}(t+1) = (\hat{x}(t+2))_{\leftarrow \mathbf{b}}$$

$$\tilde{\Omega}_{\mathbf{b},1}^{*,t+1}(t) = \frac{\check{x}(t) + (\hat{x}(t))_{\leftarrow \mathbf{s}} \cdot E - (1+E) \cdot (\check{x}(t+2))_{\leftarrow 2[\mathbf{s}]} + E \cdot \frac{G}{F} \cdot H}{(1 + (1+E) \cdot \frac{G}{F}) \cdot H}$$

$$\tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t) = \tilde{\Omega}_{\mathbf{b}}^{*,t+1}(t) \cdot (1+E) \cdot \frac{G}{F} - E \cdot \frac{G}{F}$$

$$\alpha_1^t = \frac{\tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t) \cdot \check{\Omega}_{\mathbf{b}}^{0}(t) \cdot (1+E)}{(1 - \tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t)) \cdot \tilde{\Omega}_{\mathbf{b}}^{0}(t+2) - \tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t) \cdot \tilde{\Omega}_{\mathbf{b}}^{0}(t+2) \cdot E}$$

$$\alpha_2^t = \frac{\tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t) \cdot \check{\Omega}_{\mathbf{b}}^{0}(t) \cdot (1+E)}{(1 - \tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t)) \cdot \omega_{\mathbf{b}}^{0}(t+2) - \tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t) \cdot \omega_{\mathbf{b}}^{0}(t+2) \cdot E}$$

$$\alpha^t = \frac{\alpha_2^t \cdot \omega_{\mathbf{b}}^{0}(t+2) + \alpha_1^t \cdot \tilde{\Omega}_{\mathbf{b}}^{0}(t+2)}{\tilde{\Omega}_{\mathbf{b}}^{0}(t)}$$

$$\alpha^{t+2} = E \cdot (\hat{\Omega}_{\mathbf{b}}^{0}(t+2) + \tilde{\Omega}_{\mathbf{b}}^{0}(t+2)) \cdot \frac{1 - \tilde{\Omega}_{\mathbf{b},1}^{*,t+1}(t) - \tilde{\Omega}_{\mathbf{b},2}^{*,t+1}(t)}{\tilde{\Omega}_{\mathbf{b}}^{0}(t+2)}$$

$$\beta^t = \frac{RP_{\mathbf{b}} - \hat{x}(t)}{RP_{\mathbf{b}} - \check{x}(t)}$$

The optimal actions of **b** at time point t and of **s** at time point $t+1$ are:

$$\sigma^*_{\hat{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} \textit{accept} & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \hat{x}(t-1) \\ \textit{offer}(\hat{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\check{\Theta}_{\mathbf{b},i}(t)}(t) = \begin{cases} accept & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant \check{x}(t-1) \\ \begin{cases} accept & 1-\alpha_i^t \\ offer(\check{x}(t)) & \alpha_i^t \end{cases} & \text{if } \check{x}(t-1) < \sigma_{\mathbf{s}}(t-1) \leqslant (\hat{x}(t))_{\leftarrow \mathbf{b}} \\ \begin{cases} offer(\hat{x}(t)) & 1-\alpha_i^t \\ offer(\check{x}(t)) & \alpha_i^t \end{cases} & \text{otherwise} \end{cases}$$

$$\sigma^*_{\check{\Theta}_{\mathbf{b}}(t)}(t) = \begin{cases} accept & \text{if } \sigma_{\mathbf{s}}(t-1) \leqslant (\check{x}(t))_{\leftarrow \mathbf{b}} \\ offer(\check{x}(t)) & \text{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{s}}(t+1) = \begin{cases} accept & \text{if } \sigma_{\mathbf{b}}(t) \geqslant \hat{x}(t) \\ \begin{cases} offer((\hat{x}(t+2))_{\leftarrow \mathbf{b}}) & \beta^t \\ offer(\hat{x}(t)) & 1-\beta^t \end{cases} & \text{if } \hat{x}(t) > \sigma_{\mathbf{b}}(t) > \check{x}(t) \\ offer(\hat{x}(t)) & \text{otherwise} \end{cases}$$

The system of beliefs can be trivially obtained extending that of choice rule 1.

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