Elements belonging to triads in 3-connected matroids

Manoel Lemos

Department of Mathematics, Federal University of Pernambuco, Cidade Universitária, Recife, Pernambuco 50740-540, Brazil

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Dedicated to Professor Astréa Barreto on her 60th birthday

Abstract

In this paper, we prove a conjecture proposed by Leo: a large minimally 3-connected matroid \( M \) has at least \( \left( \frac{5|E(M)| + 30}{9} \right) \) of its elements belonging to some triad. A bound on the number of elements belonging to triads of a 3-connected matroid which is close to be minimally 3-connected is also given. Both of these bounds are sharp and infinite families of matroids attaining them are constructed. A new proof of results of Lemos and Leo about triads meeting circuits with at most one removable element in 3-connected matroids is given.

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1. Introduction

Dirac [1], Halin [2] and Mader [8] showed that a minimally \( k \)-connected graph \( G \) has at least

\[
\frac{(k - 1)|V(G)| + 2k}{2k - 1}
\]

vertices of degree \( k \) for \( k = 2, k = 3 \) and \( k \geq 4 \), respectively. The bounds obtained by Dirac and Halin are sharp. The result of Mader for general \( k \) is very close to being best possible. The core of the proof of these theorems is the following: a cycle of a minimally \( k \)-connected graph has at least one vertex of degree \( k \). This result is quite important and it has been extended in [3,9,12,13] (see also [14]). Reid and Wu [16] proved edge analogs of (1.1): they got a lower bound for the number of edges meeting some vertex of degree \( k \) in terms of the total number of edges in a minimally \( k \)-connected graph. This lower bound is sharp when \( k \) is equal to 2 and 3.

Murty [10] and Oxley [11] obtained, respectively, bounds similar to (1.1) for minimally 2- and 3-connected matroids (see [4] and [5] for generalizations of these bounds). Again, the main step in the proof of these bounds is that a circuit in a minimally \( k \)-connected matroid, for \( k \in \{2, 3\} \), must meet a cocircuit with \( k \) elements. For extensions of this result, see [3,6,12,15,18]. For the first two extensions, we give new proofs of them in Section 6. Results like these are unknown for minimally \( k \)-connected matroids, with \( k \geq 4 \). For matroids, Reid and Wu [16] gave a sharp lower bound on the number of elements meeting some 2-element cocircuit in terms of the total number of elements in a minimally 2-connected matroid. But for minimally 3-connected matroids, Wu and Reid stated the following conjecture due to Leo:

**Conjecture 1.1.** Let \( M \) be a minimally 3-connected matroid with at least eight elements. Then the number of elements which meet a 3-element cocircuit is at least

\[
\frac{5|E(M)| + 30}{9}.
\]

E-mail address: manoel@dmat.ufpe.br (M. Lemos).
The proof of this conjecture can be found in Section 4 of this paper. We also give an example of an infinite family of minimally 3-connected matroids attaining the bound in this conjecture. To demonstrate it, we decompose a minimally 3-connected matroid into two minimally 3-connected matroids. This is done in Section 3. To accomplish this task, we need a lemma about 3-separations proved in Section 2.

For a 3-connected matroid $M$, we define the set of removable elements of $M$ as

$$R_0(M) = \{ e \in E(M) : M \setminus e \text{ is 3-connected} \}. $$

Note that a 3-connected matroid $M$ is minimally 3-connected if and only if $R_0(M) = \emptyset$. What happens to the number of elements belonging to triads in a 3-connected matroid $M$ such that $|R_0(M)|$ is small? We answer this question in the next theorem which is proved in Section 5:

**Theorem 1.1.** Let $M$ be a 3-connected matroid with at least five elements. Then the number of elements which meet a 3-element cocircuit is at least

$$\frac{|E(M)| + 10}{3} - |R_0(M)|.$$

Note that Conjecture 1.1 is not a consequence of this theorem. In Section 5, we describe an infinite family of matroids that attains the bound in this theorem. In Section 6, we use the decomposition defined in Section 3 to give a proof of the main results of [3,6].

2. A lemma about 3-separations

We say that a partition $\{X,Y\}$ of the ground set of a matroid $M$ is a $k$-separation, for a positive integer $k$, provided

$$r(X) + r(Y) - r(M) < k \leq \min\{|X|,|Y|\}. $$

Moreover, when $r(X) + r(Y) - r(M) = k - 1$, this $k$-separation is said to be exact. A matroid $M$ is said to be $k$-connected if $M$ does not have a $k'$-separation, for every integer $k'$ such that $0 < k' < k$.

**Lemma 2.1.** Let $\{X,Y\}$ be a 3-separation of a 3-connected matroid $M$. If $e \in X$ and $M/e$ is not 3-connected, then

(i) there is a 2-separation $\{Z,W\}$ of $M/e$ such that $Z \subseteq X$; or

(ii) for each 2-separation $\{Z,W\}$ of $M/e$, $\min\{|Z|,|W|\} = 2$ and both $Z$ and $W$ meet both $X$ and $Y$; or

(iii) for each 2-separation $\{Z,W\}$ of $M/e$, $|Z \cap X| = |W \cap X| = 1$ and $X$ is a triad of $M$.

**Proof.** Assume that (i) does not hold and let $\{Z,W\}$ be a 2-separation of $M/e$. Note that $\{Z \cup e,W\}$ is a 3-separation of $M$. Hence

$$[r(X) + r(Y) - r(M)] + [r(Z \cup e) + r(W) - r(M)] = 4. $$

By submodularity,

$$[r(X \cup Z) + r(Y \cap W) - r(M)] + [r((X \cap Z) \cup e) + r(Y \cup W) - r(M)] \leq 4. $$

(2.1)

Next, we prove that

$$|Y \cap W| \leq 1 \quad \text{or} \quad |X \cap Z| \leq 1. $$

(2.2)

If (2.2) does not hold, then $r(X \cup Z) + r(Y \cap W) - r(M) \geq 2$ and $r((X \cap Z) \cup e) + r(Y \cup W) - r(M) \geq 2$, since $M$ is 3-connected. By (2.1), we must have equality. In particular,

$$r((X \cap Z) \cup e) + r(Y \cup W) - r(M) = 2. $$

As $W$ and so $Y \cup W$ spans $e$ in $M$, it follows that $\{X \cap Z,Y \cup W\}$ is a 2-separation of $M/e$ and (i) holds; a contradiction. Thus (2.2) follows. Similarly, we have that

$$|Y \cap Z| \leq 1 \quad \text{or} \quad |X \cap W| \leq 1. $$

(2.3)
By (2.2) and (2.3), we have one of the following:
\begin{align*}
|X \cap Z| &\leq 1 \quad \text{and} \quad |X \cap W| \leq 1, \quad (2.4) \\
|Y \cap Z| &\leq 1 \quad \text{and} \quad |Y \cap W| \leq 1, \quad (2.5) \\
|Y \cap Z| &\leq 1 \quad \text{and} \quad |X \cap Z| \leq 1, \quad (2.6) \\
|Y \cap W| &\leq 1 \quad \text{and} \quad |X \cap W| \leq 1. \quad (2.7)
\end{align*}
Assume that (2.4) occurs. As $|X| \geq 3$, it follows that $|X \cap Z| = |X \cap W| = 1$ and so $|X| = 3$. Note that $r_M(X) + r_M(Y) - |X| = 2$ because $X$ is a 3-separating set of $M$. Hence $X$ is a triangle or a triad of $M$. If $X$ is a triangle of $M$, then $X - e$ is a 2-separating set of $M/e$ and so (i) happens; a contradiction. Thus $X$ is a triad of $M$. In this case, (iii) follows because any 2-separating set of $M/e$ must intersect $X - e$, by orthogonality, since it spans $e$ in $M$. We may suppose that (2.4) does not occur for every 2-separation $\{Z, W\}$ of $M/e$. Observe that (2.5) cannot happen because $|Y| \geq 3$ and $e \not\in Y$. Thus (2.6) or (2.7) occurs and so (ii) follows. 

3. A decomposition

In this section, we decompose a cominimally 3-connected matroid into two cominimally 3-connected matroids of small size. It is possible that some of the results presented here are known but we do not have a reference for them.

Let $L$ be a line of a matroid $M$. We say that a matroid $N$ is obtained from $M$ by adding $e$ freely to $L$ provided $e \not\in E(M), E(N) = E(M) \cup e$ and
\[
E(N) = \begin{cases} 
   r_M(Z) & \text{if } e \not\in Z, \\
   r_M(Z - e) & \text{if } e \in Z \text{ and } L \subseteq \text{cl}_M(Z - e), \\
   r_M(Z - e) + 1 & \text{if } e \in Z \text{ and } L \not\subseteq \text{cl}_M(Z - e). 
\end{cases}
\]
We say that a subset $X$ of the ground set of a matroid $M$ is $k$-separating (or exact $k$-separating) provided $\{X, E(M) - X\}$ is a $k$-separation of $M$ (or an exact $k$-separation of $M$).

Suppose that $X$ is an exact 3-separating set of a matroid $M$ such that $r(X) \geq 3$. In this paragraph, we define a factor of $M$ with respect to $X$. If $B_1$ and $B_2$ are basis for $M/X$ and $M'\setminus X$, respectively, then
\[
|B_1| + |B_2| = r(X) + r(E(M) - X) = r(M) + 2.
\]
So there are elements $a$ and $b$ of $B_2$ such that $(B_1 \cup B_2) - \{a, b\}$ is a basis of $M$. Hence $B_1$ is a basis and $\{a, b\}$ is an independent set of $M/(B_2 - \{a, b\})$. In particular, $|M/(B_2 - \{a, b\})| = |X \cup (B_2 - \{a, b\})|$. As $B_2$ spans $E(M) - X$ in $M$, it follows that $\{a, b\}$ spans $E(M) - [X \cup (B_2 - \{a, b\})]$. Thus the line $L'$ of $M/(B_2 - \{a, b\})$ that contains $\{a, b\}$ also contains $E(M) - [X \cup (B_2 - \{a, b\})]$. Hence the ground set of $(M/(B_2 - \{a, b\}))$ is the union of $X$ and $L'$. Observe that $X \cap L' \subseteq \text{cl}_M(X) \cap \text{cl}_M(E(M) - X) \subseteq L'$.

Let $A$ be a minimal set such that $A \cap E(M) = \emptyset$ and
\[
L = A \cup [\text{cl}_M(X) \cap \text{cl}_M(E(M) - X)]
\]
has at least 3 elements. Let $N'$ be the matroid obtained from $M/(B_2 - \{a, b\})$ by adding the elements belonging to $A$ freely in the line $L'$. We say that $N = N'\setminus (L' - L)$ is a factor of $M$ with respect to $X$ having $L$ as its special line.

Lemma 3.1. Suppose that $X$ is an exact 3-separating set of a matroid $M$ such that $r(X) \geq 3$. Let $N$ be a factor of $M$ with respect to $X$ having $L$ as special line. If $Y \subseteq \text{cl}_M(X)$ and $|Y \cap L| \leq 1$, then $Y$ is a line of $M$ if and only if $Y$ is a line of $N$.

Proof. Suppose $Y$ is a line of $M$. As $|Y \cap L| = 1$, it follows that $Y$ does not span any element of $A = L - E(M)$ in $N$. Moreover, $Y$ does not span any element of $\text{cl}_M(X) - Y$ in $N$, otherwise $Y$ would span an element in $M$. Hence $Y = \text{cl}_N(Y)$ and so $Y$ is a line of $N$. 

Proof. Suppose $Y$ is a line of $M$. As $|Y \cap L| = 1$, it follows that $Y$ does not span any element of $A = L - E(M)$ in $N$. Moreover, $Y$ does not span any element of $\text{cl}_M(X) - Y$ in $N$, otherwise $Y$ would span an element in $M$. Hence $Y = \text{cl}_N(Y)$ and so $Y$ is a line of $N$. 

\[\text{Proof. Suppose } Y \text{ is a line of } M. \text{ As } |Y \cap L| = 1, \text{ it follows that } Y \text{ does not span any element of } A = L - E(M) \text{ in } N. \text{ Moreover, } Y \text{ does not span any element of } \text{cl}_M(X) - Y \text{ in } N, \text{ otherwise } Y \text{ would span an element in } M. \text{ Hence } Y = \text{cl}_N(Y) \text{ and so } Y \text{ is a line of } N.\]
Now, assume $Y$ is a line of $N$. As $r_M(Y) = r_N(Y) = 2$, it follows that $\text{cl}_M(Y)$ is a line of $M$. By the previous paragraph, $\text{cl}_M(Y)$ is a line of $N$ and so $Y = \text{cl}_M(Y)$. Thus $Y$ is a line of $M$. 

We define the \textit{connectivity function} of a matroid $H$ as 

$$\xi_H(Z, W) = r_H(Z) + r_H(W) - r(H),$$

for a partition $\{Z, W\}$ of the ground set of $H$.

\textbf{Lemma 3.2.} Suppose that $X$ is an exact 3-separating set of a matroid $M$ such that $r(X) \geq 3$. Let $N$ be a factor of $M$ with respect to $X$ having $L$ as special line. Then:

(i) When $L \subseteq Y \subseteq E(N)$ and $A = L - E(M)$, then

$$r_M([Y - A] \cup [E(M) - X]) = r_N(Y) + r_M(E(M) - X) - 2.$$ 

(ii) When $Y \subseteq X - L$, $\xi_N(Y, E(N) - Y) = \xi_M(Y, E(M) - Y)$. 

(iii) When $Y \subseteq X - L$, then $Y$ is a cocircuit of $M$ if and only if $Y$ is a cocircuit of $N$.

\textbf{Proof.} In this proof, let $B_1, B_2, a, b, L'$ and $N'$ be as defined before the statement of this lemma. First, we prove (i). As $Y$ spans $a$ and $b$ in $N'$, it follows that

$$r_N(Y) = r_N(Y \cup \{a, b\})$$

and

$$= r_N([Y - A] \cup \{a, b\})$$

$$= r_M(B_2 - \{a, b\}) - [Y - A] \cup \{a, b\})$$

$$= r_M([Y - A] \cup B_2) - r_M(B_2 - \{a, b\})$$

$$= r_M([Y - A] \cup B_2) - r_M(E(M) - X) - 2].$$

As $[Y - A] \cup B_2$ spans $E(M) - X$ in $M$, it follows that

$$r_M([Y - A] \cup [E(M) - X]) = r_N([Y - A] \cup B_2) = r_N(Y) + r_M(E(M) - X) - 2$$

and (i) follows.

Now, we prove (ii). By Lemma 3.2(i) applied to $E(N) - Y$, we get:

$$r_M([E(N) - (Y \cup A)] \cup [E(M) - X]) = r_N(E(N) - Y) + r_M(E(M) - X) - 2.$$ 

Adding $r_M(Y)$ to both sides of this identity and replacing $[E(N) - (Y \cup A)] \cup [E(M) - X]$ by $E(M) - Y$, we obtain:

$$r_M(Y) + r_M(E(M) - Y) = r_M(Y) + r_N(E(N) - Y) + r_M(E(M) - X) - 2.$$ 

As $r_M(Y) = r_N(Y)$ and

$$r_N(Y) + r_N(E(N) - Y) = \xi_N(Y, E(N) - Y) + r(N) = \xi_M(Y, E(M) - Y) + r_M(X),$$

it follows that

$$r_M(Y) + r_M(E(M) - Y) = \xi_N(Y, E(N) - Y) + r_M(X) + r_M(E(M) - X) - 2.$$ 

Observe that (ii) follows because $r_M(X) + r_M(E(M) - X) = 2 = r(M)$. In the next paragraph, we prove (iii).

Applying (i) to $E(N) - Y$ and $[E(N) - Y] \cup c$, for $c \in Y$, we get, respectively:

$$r_M(E(M) - Y) = r_N(E(N) - Y) + r_M(E(M) - X) - 2,$$

$$r_M([E(M) - Y] \cup c) = r_N([E(N) - Y] \cup c) + r_M(E(M) - X) - 2.$$ 

As $r(N) = r_M(X)$ and $r_M(E(M) - X) - 2 = r(M) - r_M(X)$, it follows that

$$r(M) - r_M(E(M) - Y) = r(N) - r_N(E(N) - Y),$$

$$r(M) - r_M([E(M) - Y] \cup c) = r(N) - r_N([E(N) - Y] \cup c).$$

Hence $E(M) - Y$ is a hyperplane of $M$ if and only if $E(N) - Y$ is a hyperplane of $N$. Thus (iii) holds.
A matroid $M$ is said to be *minimally 3-connected* provided $M$ is 3-connected and, for every $e \in E(M)$, $M \setminus e$ is not 3-connected. We say that $M$ is *continually 3-connected* if $M^*$ is minimally 3-connected. For an element $e$ of a matroid $M$, we denote by $\mathrm{si}(M/e)$ the simplification of $M/e$ and by $\mathrm{co}(M\setminus e)$ the cosimplification of $M \setminus e$.

**Lemma 3.3.** Suppose that $X$ is a 3-separating set of a 3-connected matroid $M$ such that $r(X) \geq 3$. If $N$ is a factor of $M$ with respect to $X$ having $L$ as special line, then:

(i) $N$ is 3-connected.
(ii) If $r(N) = 3$, $e \in E(N) - L$ and $\mathrm{si}(M/e)$ is not 3-connected, then $\mathrm{cl}_M(X)$ is the union of two lines that contains $e$.
(iii) For $e \in E(N) \cap E(M)$, if $M/e$ is not 3-connected, then
   (a) $N/e$ is not 3-connected; or
   (b) $\mathrm{cl}_M(X)$ is a triad of $M$.
(iv) For $e \in E(N) \cap E(M)$, if $M/e$ is 3-connected, then $N/e$ is 3-connected.
(v) If $M$ is continually 3-connected, then $N$ is continually 3-connected.

**Proof.** In this proof, let $B_1, B_2, a, b, L', A$ and $N'$ be as defined before the statement of Lemma 3.1. To prove (i), suppose that $N$ is not 3-connected and let $\{Z, W\}$ be a 2-separation of $N$. First, we show that

$$\min\{|Z|, |W|\} \geq 3. \tag{3.1}$$

Suppose that $|Z| \leq 2$. As $N$ is simple, it follows that $Z$ contains a cocircuit $C^*$ of $N$. As $|C^*| \leq 2$, $E(N) - L$ spans $L$, $|L| \geq 3$ and $N/L \simeq U_2(2, 4)$, it follows, by orthogonality, that $C^* \cap L = \emptyset$. By Lemma 3.2(iii), $C^*$ is a cocircuit of $M$; a contradiction and so (3.1) follows.

Choose a 2-separation $\{Z, W\}$ of $N$ such that $|L \cap Z|$ is maximum. As $|L| \geq 3$, it follows that $|L \cap Z| \geq 2$. If $e \in W \cap L$, then $Z$ spans $e$ in $N$ and so $\{Z \cup e, W - e\}$ is a 2-separation of $N$, by (3.1); a contradiction to the choice of $\{Z, W\}$. Hence $L \subseteq Z$. By Lemma 3.2(ii), $\{W, E(M - W)\}$ is a 2-separation of $M$; a contradiction and so $\{Z, W\}$ does not exist. Hence $N$ is 3-connected. In the next paragraph, we show that property (ii) holds.

Choose the elements of $\mathrm{si}(M/e)$ so that $\mathrm{cl}_M(X) \cap \mathrm{cl}_M(E(M) - X) \subseteq \mathrm{si}(M/e)$. Note that

$$L_1 = \mathrm{cl}_M(X) \cap E(\mathrm{si}(M/e)) = \mathrm{cl}_M(X - e) \cap E(\mathrm{si}(M/e))$$

is a line of $\mathrm{si}(M/e)$. Let $N_1$ be a matroid obtained from $\mathrm{si}(M/e)$ by adding freely in the line $L_1$ a minimal set of elements $A'$ such that

$$L_2 = A' \cup [\mathrm{cl}_M(X) \cap \mathrm{cl}_M(E(M) - X)]$$

has at least three elements. Note that $N_1\backslash (L_1 - L_2)$ is a factor of $M$ with respect to $E(M) - X$ and so $N_1\backslash (L_1 - L_2)$ is 3-connected, by (i). Hence $N_1$ is 3-connected because $N_1$ is simple. As $\mathrm{si}(M/e) = N_1\backslash A'$ is not 3-connected, it follows that $|L_1| = 2$, since $L_1 \cup A'$ is a line of $N_1$. So $\mathrm{cl}_M(X - e)$ is the union of two parallel classes of $M$. Hence $\mathrm{cl}_M(X)$ is the union of two lines of $M$ that contain $e$ and property (ii) holds.

Now, we prove that property (iii) holds. Assume that (a) and (b) do not hold. In particular, $N/e$ is 3-connected and so $e \in X - L$. By Lemma 2.1 applied to the 3-separation $\{X - L, \mathrm{cl}_M(E(M) - X)\}$ of $M$, we conclude that:

1. there is a 2-separation $\{Z, W\}$ of $M/e$ such that $Z \subseteq X - L$; or
2. for each 2-separation $\{Z, W\}$ of $M/e$, $\min\{|Z|, |W|\} = 2$ and both $Z$ and $W$ meet both $X - L$ and $\mathrm{cl}_M(E(M) - X)$; or
3. $X - L$ is a triad of $M$.

First, we prove that

$$r(N) \geq 4. \tag{3.2}$$

Suppose $r(N) = 3$. As each line of $N$ that contains $e$ has two elements, since $N/e$ is 3-connected, it follows, by Lemma 3.1, that every line of $M$ that contains $e$ has two elements and so $\mathrm{si}(M/e) = M/e$. By (ii), $\mathrm{cl}_M(X)$ is the union of two lines of $M$ that contain $e$. As each of these lines have two elements, it follows that $\mathrm{cl}_M(X)$ has three elements and so $\mathrm{cl}_M(X)$ is a triad of $M$. Hence case (iii) occurs; a contradiction. Thus (3.2) holds. In particular, (3) does not occur. Assume (2) happens. Let $L_1, \ldots, L_n$ be the lines of $M$ having at least 3 elements which contains $e$. So $n \geq 1, |L_1| = \cdots = |L_n| = 3$ and, for every $i \in \{1, \ldots, n\}$, $L_i = \{e, x_i, y_i\}$, where $x_i \in X - (L \cup e)$ and $y_i \in \mathrm{cl}_M(E(M) - X)$. As $L_i - y_i$ spans $y_i$ in $M$, it follows that $y_i \in \mathrm{cl}_M(X)$, for every $i \in \{1, \ldots, n\}$, and so $y_i \in L$. By Lemma 3.1(i), for $i \in \{1, \ldots, n\}$, $L_i$ is a line of $N$ and so (a) holds; a contradiction. Assume (1) holds. By (3.2), $N/e$ is a factor of $M/e$ with respect to $X - e$ having special line $L$. By Lemma 3.2(ii), $Z$ is 2-separating in $N/e$ and so (a) follows; a contradiction.

In this paragraph, we show that property (iv) occurs. If $M/e$ is 3-connected, then $e \notin L$. By Lemma 3.1, every line of $N$ that contains $e$ has just two elements. If $r(N) = 3$, then $N/e$ is 3-connected. We may suppose that $r(N) = r(X) \geq 4$. 

\[\]
As \( N/e \) is a factor of \( M \) with respect to \( X - e \) having \( L \) as special line, it follows that \( N/e \) is 3-connected, by (i), and (iv) follows.

Suppose that \( M \) is cominimally 3-connected. To conclude that \( N \) is cominimally 3-connected, for each \( e \in E(N) \), we need to prove that \( N/e \) is not 3-connected. Observe that (v) follows from (iii) unless \( \text{cl}_M(X) \) is a triad of \( M \). But in this case, by Tutte’s triangle lemma (see [17]), every element of \( \text{cl}_M(X) \) belongs to a triangle of \( M \) and so of \( N \), by Lemma 3.1. Thus (v) also follows in this case. 

\[ \square \]

4. Proving Leo’s conjecture

For a matroid \( M \), we denote by \( T(M) \) the set of elements of \( M \) belonging to some triangle of \( M \). We set \( t(M) = |T(M)| \) and \( e(M) = |E(M)| \). When \( M \) is 3-connected, we set

\[ R_i(M) = \{ e \in E(M) : \text{co}(M \setminus e) \text{ is 3-connected} \}. \]

Next, we prove the dual of Leo’s conjecture.

**Theorem 4.1.** Let \( M \) be a cominimally 3-connected matroid. If \( |E(M)| \geq 8 \), then

\[ 9t(M) \geq 5e(M) + 30. \]

Before the proof of this result, we give an example due to Leo [7] to show that this bound is sharp. For a positive integer \( n \), let \( M_0, M_1, \ldots, M_n \) be matroids isomorphic to \( U_{3,6} \) such that

\[ |E(M_i) \cap E(M_j)| = \begin{cases} 0 & \text{if } |i - j| \geq 2, \\ 2 & \text{if } |i - j| = 1. \end{cases} \]

For \( i \in \{1, 2, \ldots, n-1\} \), we define \( A_i = E(M_i) - [E(M_{i-1}) \cup E(M_{i+1})] \). Let \( A_0, A_n, A_{n+1} \) be 2-element sets which partition, respectively, \( E(M_0) - E(M_1) \) and \( E(M_n) - E(M_{n-1}) \). Let

\[ C_{-1}, C_0, C_1, C_2, \ldots, C_n, C_{n+1}, E(M_0) \cup E(M_1) \cup \cdots \cup E(M_n) \]

be a family of pairwise disjoint sets. For each \( i \in \{-1, 0, 1, 2, \ldots, n, n+1\} \), suppose that \( |C_i| = 5 \) and that \( N_i \) is a rank-3 matroid over \( C_i \cup A_i \) such that \( C_i \) is the union of two 3-point lines of \( N_i \) and the elements of \( A_i \) are freely placed in \( N_i \). Observe that the generalized parallel connection of \( M_0, M_1, \ldots, M_n, N_{-1}, N_0, N_1, \ldots, N_n, N_{n+1} \) is a cominimally 3-connected matroid \( M \) such that \( T(M) = C_{-1} \cup C_0 \cup C_1 \cup C_2 \cup \cdots \cup C_n \cup C_{n+1} \). Thus \( t(M) = 5n + 15 \) and \( e(M) = 9n + 21 \). Hence

\[ t(M) = 5n + 15 = \frac{5(9n + 21) + 30}{9} = \frac{5e(M) + 30}{9}. \]

Next, we give a picture of this extremal example where the rank three matroids are drawn as a chain of linked pages.

**Proof of Theorem 4.1.** Suppose this result is not true and choose a counter-example \( M \) so that \( (t(M), e(M)) \) is minimum in the lexicographic order. To simplify the notation, we set

\[ \alpha = \frac{5}{9} \text{ and } \beta = \frac{30}{9}. \]

If \( t(M) = e(M) \), then \( e(M) < \alpha e(M) + \beta \) and so

\[ (1 - \alpha)8 \leq (1 - \alpha)e(M) < \beta, \]

\[ (4.1) \]
there are 3-point lines

Hence the result does not hold for

We arrive at a contradiction because

Moreover, \( X \) and \( Y \) are both 3-separating sets of \( M \). For \( Z \subseteq \{X, Y\} \), let \( M_Z \) be a factor of \( M \) with respect to \( Z \) having \( L_Z \) as special line. By Lemma 3.3(v), \( M_Z \) is a minimally 3-connected matroid. As \( L \) is spanned in \( M \) by both \( Z \) and \( E(M) - Z \), it follows that \( L \subseteq L_Z \). So \( e \in L_X \cap L_Y \). If \( |L_Z \cap E(M)| \geq 3 \), then \( L_X = L_Y \) is a line of \( M \) that contains \( e \); a contradiction because \( e \) does not belong to a triangle of \( M \). Thus \( 1 \leq |L_Z \cap E(M)| \leq 2 \). In particular, \( |L_X| = |L_Y| = 3 \). We may assume that \( L_X = L_Y \), say \( L = L_X \).

Now, we prove that there is \( X_e \in \{X - L, Y - L\} \) such that:

1. \( e \in L_e \), where \( L_e = cl_e(X_e) - X_e \).
2. \( |L_e| \leq 2 \).
3. \( X_e \subseteq R_1(M^*) \).
4. \( L_e = cl_e(E(M) - X_e) \cap cl_e(X_e) \).
5. there are 3-point lines \( L_{1e} \) and \( L_{2e} \) of \( M \) such that

   \[ X_e \cup [L_e \cap T(M)] = L_{1e} \cup L_{2e} \]

   and \( L_{1e} \cap L_{2e} = \{x_e\} \), for some \( x_e \in X_e \cup [L_e \cap T(M)] \).

To prove the existence of \( X_e \) we deal with two cases.

**Case 1:** \( [L \cap E(M)] \cap T(M) \neq \emptyset \).

Hence \( L \cap E(M) = \{e, f\} \), for some \( f \in T(M) \). As \( \{E(M_X) - L, E(M_Y) - L, L \cap E(M)\} \) is a partition of \( E(M) \), it follows that:

\[ e(M) = e(M_X) + e(M_Y) - 4. \] \hspace{1cm} (4.2)

By Lemma 3.1, for \( Z \subseteq \{X, Y\} \), an element of \( Z - L \) belongs to a triangle of \( M_Z \) if and only if it belongs to a triangle of \( M \). As \( f \) belongs to a triangle of \( M \), it follows that:

\[ t(M) = t(M_X) + t(M_Y) - 5. \] \hspace{1cm} (4.3)

Assume the result holds for both \( M_X \) and \( M_Y \). Hence

\[ t(M) \geq z e(M) + \beta, \]

for \( Z \subseteq \{X, Y\} \). Adding these inequalities, we obtain:

\[ t(M_X) + t(M_Y) \geq z [e(M_X) + e(M_Y)] + 2 \beta. \]

Replacing (4.2) and (4.3), we get:

\[ t(M) + 5 \geq z [e(M) + 4] + 2 \beta. \]

This inequality can be reordered as

\[ t(M) - z e(M) - \beta \geq 4 \alpha + \beta - 5. \] \hspace{1cm} (4.4)

We arrive at a contradiction because \( \alpha \) and \( \beta \) satisfy:

\[ 4 \alpha + \beta - 5 \geq 0. \] \hspace{1cm} (4.5)

Hence the result does not hold for \( M_X \) or \( M_Y \), say \( M_Y \).

As \( r(M_Y) < r(M) \), it follows that \( e(M_Y) \leq 7 \) and so \( |Y - L| \leq 4 \). Now, we prove that \( r(M_Y) = 3 \). If \( r(M_Y) > 3 \), then \( r(M_Y) = 4 \) and so \( T^*_e = Y - (L \cup c) \) is a triad of \( M_Y \), for each \( c \in Y - L \). Hence \( M_Y \) is 3-connected. We arrive at a contradiction because \( M_Y/c \) is 3-connected, for every \( c \in Y - L \). Thus \( r(M_Y) = 3 \).

So each element of \( M_Y \) belongs to a line having at least three elements, since \( M_Y \) is minimally 3-connected. Let \( L_1, \ldots, L_n \) be the lines of \( M_Y \) having at least three elements which are not equal to \( L \). Observe that \( L_e \subseteq E(M_e) - (L - f) \) because (a) the elements belonging to \( L - E(M) \) a freely placed in \( L \); and (b) \( e \) does not belong to a triangle of \( M \), by hypothesis, and so, by Lemma 3.1, to a triangle of \( M_Y \). Hence \( L_1 \cup \cdots \cup L_n \in \{Y - L, (Y \cup f) - (L - f)\} \). As \( L_1 \cup f \) is properly contained in \( Y - L \) because \( M \) is 3-connected, it follows that \( n \geq 2 \). Thus \( n = 2 \) and \( |L_1| = |L_2| = 3 \). Observe that \( L_1 \) or \( L_2 \) contains \( f \) (the possible
geometric representations of $M_Y$ can be found in the next figure). In this case, we take $X_e$ to be $Y - L$. Note that (X3) holds by Lemma 3.3(ii).

Case 2: $[L \cap E(M)] \cap T(M) = \emptyset$.

For $Z \in \{X, Y\}$, let $H_Z$ be a matroid (see its geometric representation in the next figure) such that:

(i) $L = E(H_X) \cap E(H_Y)$;  
(ii) $E(H_X) - L, E(H_Y) - L$ and $E(M)$ are pairwise disjoint;  
(iii) $|E(H_X)| = |E(H_Y)| = 8$;  
(iv) $r(H_X) = r(H_Y) = 3$; and  
(v) for $Z \in \{X, Y\}$, $H_Z$ has only three lines $L$, $L_{1Z}$ and $L_{2Z}$ having at least three elements and $L_{1Z} \cup L_{2Z} = E(H_Z) - L$.

By Lemma 3.3(v), for $Z \in \{X, Y\}$, $M_Z$ is cominimally 3-connected and so $K_Z$, the generalized parallel connection across $Z$ of $M_Z$ and $H_Z$, is cominimally 3-connected. Choose $c \in L - E(M)$. Now, we prove that $N_Z = K_Z \setminus c$ is 3-connected. If $W$ is a 2-separating set for $N_Z$, then $|W| > 2$, otherwise $T^* = W \cup c$ is a triad of $K_Z$ which is contrary to orthogonality ($|T^* \cap L| \geq 2$ and $T^* \cap L_{id} \neq \emptyset$, for some $i \in \{1, 2\}$, and so $|T^* \cap L_{id}| \geq 2$). Hence $W - d$ is a 2-separating set for $K_Z \setminus c$, for $d \in L_{1Z} \cap L_{2Z}$. But the only separating sets of $K_Z \setminus c$ are $L_{1Z} - d$, $L_{2Z} - d$ and theirs complements. We have a contradiction because $L_{1Z}$ or $L_{2Z}$ is not a 2-separating set of $N_Z$. Hence $N_Z$ is 3-connected. As $K_Z$ is cominimally 3-connected and $c$ belongs to just one line of $K_Z$ having at least three points, namely $L$, it follows that $N_Z$ is comiminally 3-connected because $N_Z/g$ is not 3-connected, for every $g \in L - c$.

As $\{E(N_X) - E(H_X), E(N_Y) - E(H_Y), L \cap E(M)\}$ is a partition of $E(M)$, it follows that

$$e(M) = e(N_X) + e(N_Y) + |L \cap E(M)| - 14. \quad (4.6)$$

By Lemma 3.1, for $Z \in \{X, Y\}$, an element of $cl_M(Z) = Z \cup [L \cap E(M)]$ belongs to a triangle of $N_Z$ if and only if it belongs to a triangle of $M$. Hence

$$t(M) = t(N_X) + t(N_Y) - 10. \quad (4.7)$$

Assume the result holds for both $N_X$ and $N_Y$. Hence

$$t(N_Z) \geq xe(N_Z) + \beta,$$

for $Z \in \{X, Y\}$. Adding this inequality for $Z \in \{X, Y\}$, we obtain:

$$t(N_X) + t(N_Y) \geq xe(N_X) + e(N_Y) + 2\beta.$$

Replacing (4.6) and (4.7), we get:

$$t(M) + 10 \geq xe(M) + 14 - |L \cap E(M)| + 2\beta.$$

This inequality can be reordered as

$$t(M) - xe(M) - \beta \geq [14 - |L \cap E(M)|]x + \beta - 10. \quad (4.8)$$

We arrive at a contradiction because $1 \leq |L \cap E(M)| \leq 2$ and $x$ and $\beta$ satisfy

$$12x + \beta - 10 \geq 0. \quad (4.9)$$

Hence the result does not hold for $N_X$ or $N_Y$, say $N_X$. 

or
By the choice of $M$, it follows that $r(Y) \leq r(H_Y) = 3$. As $r(Y) \geq 3$, it follows that $r(Y) = 3$ and so $r(M) = r(N_X)$. Again, by the choice of $M$, we conclude that $|Y - L| \leq 5$. As $r(M_Y) = r(Y) = 3$ and $M_Y$ is co-minimally 3-connected, it follows that each element of $M_Y$ belongs to a line having at least three elements. Let $L_1, \ldots, L_n$ be the lines of $M_Y$ having at least three elements which are not equal to $L$. Observe that $L_1 \subseteq E(M_Y) - L$ because (a) the elements belonging to $L - E(M)$ is freely placed in $L$; and (b) the elements belonging to $L \cap E(M)$ do not belong to a triangle of $M$, by hypothesis, and so of $M_Y$, by Lemma 3.1. Hence $L_1 \cup \cdots \cup L_n = Y - L$. As $L_1$ is properly contained in $Y - L$ because $2 = r(L_1) < r(Y - L) = r(Y) = 3$, it follows that $n \geq 2$. Thus $n = 2$ and $|L_1| = |L_2| = 3$. We take $X_e$ to be equal to $Y - L$ in this case.

Next, we prove that

$$X_e = X_f \text{ or } X_e \cap X_f = \emptyset,$$

when $\{e, f\}$ is a 2-subset of $E(M) - T(M)$. (4.10)

Moreover, when the first case happens $L_e = L_f = \{e, f\}$. Suppose that (4.10) does not hold for some $e$ and $f$. In particular,

$$X_e \cap X_f \neq \emptyset.$$ (4.11)

As $E(M) - X_e$ does not span any element of $X_e$, by (X1) and (X4), and $X_f \cup [L_f \cap T(M)] = L_1f \cup L_2f$, by (X5), it follows that:

$$|L_i - X_e| \neq 2, \quad \text{for } i \in \{1, 2\}.$$ (4.12)

Next, we prove that

$$(X_f \cup L_f) \cap (X_e \cup L_e) \text{ does not contain a basis of } X_f \cup L_f.$$ (4.13)

If $(X_f \cup L_f) \cap (X_e \cup L_e)$ contains a basis of $X_f \cup L_f$, then $X_e \cup L_e$ spans $X_f \cup L_f$. Hence $X_f \cup L_f \subseteq X_e \cup L_e$. As $X_e \subseteq R_1(M^*)$, by (X3), it follows that $f \in L_e$ and so $L_e = \{e, f\}$, by (X1) and (X2). Thus $X_e = X_f$ and $L_e = L_f$, by (X2) and (X5); a contradiction. Thus $(X_f \cup L_f) \cap (X_e \cup L_e)$ does not contain a basis of $X_f \cup L_f$ and (4.13) holds. Hence, by (4.11), (4.12) and (4.13), $L_e = \{e, x_e\}$ and $L_{xf} = X_e \cap X_f$, for some $i \in \{1, 2\}$, say $i = 1$, and $L_{2f} - x_f = X_f - (X_e \cup L_e)$. Thus $L_{1f}$ is a 3-point line contained in $X_e \cup [L_e \cap T(M)]$. Hence $L_{1f} = L_{xe}$, for some $j \in \{1, 2\}$, say $j = 1$. Similarly, $L_f = \{f, x_e\}$ and $X_e \subseteq L_{1f} = L_{xe}$. As $\{x_e, x_f\} \subseteq L_{xe}$ and $|L_{xe} \cap X_e| \geq 2$, it follows, by (X3) and (X5), that $x_e = x_f$. As $X_e$ and $X_f$ are 3-separating sets of $M$ such that $X_e \cap X_f = L_{xf} - x_e$, it follows, by submodularity, that $X_e \cup X_f$ is a 3-separating set of $M$. Observe that $e, f$ and $x_e$ are elements of $M$ not belonging to $X_e \cup X_f$. As $\{e, f, x_e\} \subseteq \text{cl}_M(X_e \cup X_f)$, it follows that $\{e, f, x_e\}$ is contained in a line of $M$; a contradiction. Thus (4.10) holds.

There are elements $e_1, \ldots, e_n$ belonging to $E(M) - T(M)$ such that $L_{e_1} - T(M), \ldots, L_{e_n} - T(M)$ partition $E(M) - T(M)$. Hence

$$|E(M) - T(M)| = n + r,$$ (4.14)

where $r$ is the number of indices $i \in \{1, \ldots, n\}$ such that $|L_{e_i} - T(M)| = 2$. As $X_{e_1}, \ldots, X_{e_n}$ are pairwise disjoint, by (4.10), it follows that $t(M) \geq 4n + r$ and so

$$t(M) = 4n + r + \gamma,$$ (4.15)

for some non-negative integer $\gamma$. As $t(M) < \omega(M) + \beta$, it follows that:

$$4n + r + \gamma < 3(5n + 2r + \gamma) + \beta.$$ This inequality can be rewritten as

$$\gamma(1 - \alpha) < n(5\alpha - 4) + r(2\alpha - 1) + \beta.$$ Replacing the values of $\alpha$ and $\beta$, we obtain:

$$\gamma \leq 4; 30 + r - 11n.$$ (4.16)

As $0 \leq r \leq n$, we arrive at a contradiction unless $n \leq 2$.

To conclude the proof, we divide it in some cases. If $n = 1$ and $r = 1$, then $X_{e_1}$ and $E(M) - (X_{e_1} \cup L_{e_1})$ have both at least five elements (if $|E(M) - (X_{e_1} \cup L_{e_1})| \leq 4$, then we can take $E(M) - (X_{e_1} \cup L_{e_1})$ for $X_{e_1}$ in Case 2 and we arrive at a contradiction). Hence $\gamma \geq 5$ and we have a contradiction to (4.16). If $n = 1$ and $r = 0$, then $X_{e_1}$ and $E(M) - (X_{e_1} \cup L_{e_1})$ have both at least four elements (if $|E(M) - (X_{e_1} \cup L_{e_1})| \leq 3$, then we can take $E(M) - (X_{e_1} \cup L_{e_1})$ for $X_{e_1}$ in Case 1 and we arrive at a contradiction). Hence $\gamma \geq 5$ and again, we have a contradiction to (4.16). Hence $n = 2$. If $c \in T(M) - [X_{e_1} \cup X_{e_2}]$, then there is a triangle $T$ of $M$ such that $c \in T$. By (X1) and (X4), $T \cap X_{e_1} = \emptyset$ or $|T \cap X_{e_1}| = 2$, for $i \in \{1, 2\}$. If $T \cap X_{e_1} = T \cap X_{e_2} = \emptyset$, then $\gamma \geq 3$; a contradiction to (4.16). Thus $|T \cap X_{e_i}| = 2$ and $c \in L_{e_i}$, for some $i \in \{1, 2\}$. Hence $E(M) = X_{e_1} \cup X_{e_2} \cup L_{e_1} \cup L_{e_2}$. Let $M_1$ be a factor of $M$ with respect to $X_{e_2} \cup L_{e_2} \cup L_{e_1}$ having special line $L_1$. Note that

5. Proof of Theorem 1.1

For a 3-connected matroid $M$, let $r_0(M) = |R_0(M)|$. In this section, we prove the dual of Theorem 1.1, namely:

**Theorem 5.1.** Let $M$ be a 3-connected matroid. If $|E(M)| \geq 5$, then

$$3(t(M) + r_0(M^*)) \geq e(M) + 10.$$  

Before the proof of this result, we give an example to show that this bound is sharp. For a positive integer $n$, let $A_0, B_1, A_1, B_2, A_2, B_3, A_3, \ldots, A_{n-1}, B_n, A_n$ be pairwise disjoint sets such that, for each $i \in \{0, 1, 2, \ldots, n\}$, $|A_i| = 2$ and, for each $j \in \{1, 2, \ldots, n\}$, $|B_j| = 1$. For $i \in \{1, 2, \ldots, n\}$, let $M_i$ be a matroid isomorphic to $\mathbb{U}_{3,5}$ over $A_i \cup B_i \cup A_i$. Let $M$ be the generalized parallel connection of $M_1, M_2, \ldots, M_n$. Note that $M$ is 3-connected. Moreover, $M/e$ is not 3-connected if and only if $e \in A_1 \cup A_2 \cup \cdots \cup A_{n-1}$. As $M$ does not have any triangle, it follows that:

$$t(M) + r_0(M^*) = n + 4 = \frac{(3n + 2) + 10}{3} = \frac{e(M) + 10}{3}.$$  

Next, we give a picture of this extremal example where the rank three matroids are drawn as a chain of linked pages.

![Diagram](https://example.com/diagram.png)

**Proof of Theorem 5.1.** We define

$$\delta(M) = \begin{cases} 1 & \text{if $M$ has different triangles with non-empty intersection,} \\ 0 & \text{otherwise.} \end{cases}$$

Instead of Theorem (5.1), we show that

$$3(t(M) + r_0(M^*) - \delta(M)) \geq e(M) + 10$$  

provided $|E(M)| \geq 7$. Theorem 5.1 is a consequence of (5.1) unless $|E(M)| \in \{5, 6\}$. But in this case $E(M) \subseteq T(M) \cup R_0(M^*)$ and Theorem 5.1 follows. Suppose (5.1) is not true and choose a counter-example $M$ such that $(r(M), e(M))$ is minimum in the lexicographic order. If $t(M) + r_0(M^*) = e(M)$, then $3e(M) < e(M) + 3\delta(M) + 10$ and $e(M) < \frac{2}{3}$; a contradiction. Thus $E(M) - [T(M) \cup R_0(M^*)] \neq \emptyset$. Choose $e \in E(M) - [T(M) \cup R_0(M^*)]$. Let $\{X, Y\}$ be a 2-separation of $M/e$. Observe that $\min\{r_{M/e}(X), r_{M/e}(Y)\} \geq 2$ because $e$ does not belong to a triangle of $M$. As $X$ and $Y$ spans $e$ in $M$, it follows that

$$\min\{r_{M/e}(X), r_{M/e}(Y)\} \geq 3.$$  

Moreover, $X$ and $Y$ are both 3-separating sets of $M$. For $Z \subseteq \{X, Y\}$, let $M_Z$ be a factor of $M$ with respect to $Z$ having $L_Z$ as special line. As $e$ is spanned in $M$ by both $Z$ and $E(M) - Z$, it follows that $e \in L_Z$. So $e \in L_X \cap L_Y$. If $|L_X \cap E(M)| \geq 3$, then $L_X = L_Y$ is a line of $M$ that contains $e$; a contradiction because $e$ does not belong to a triangle of $M$. Thus $1 \leq |L_X \cap E(M)| \leq 2$. In particular, $|L_X| = |L_Y| = 3$. We may assume that $L_X = L_Y$, say $L = L_X$. Observe that $\text{cl}_M(Z)$ is not a triad of $M$ because $|Z| \geq 3$ and $e \in \text{cl}_M(Z) \cap L$. By Lemma 3.3(iii) and (iv), $M/f$ is 3-connected if and only if $M_Z/f$ is 3-connected, for every $f \in \text{cl}_M(Z)$. Thus

$$R_0(M^*) = R_0(M^*_X) \cup R_0(M^*_Y).$$  

(5.2)
We prove that there is \( X_e \in \{X - L, Y - L\} \) such that

(X1) \( X_e \) is a triad of \( M' \);

(X2) \( X_e \cup e \) is a circuit of \( M' \);

(X3) \( e \in L_e \), where \( L_e = \text{cl}_M(X_e) - X_e \);

(X4) \( L_e = \text{cl}_M(X_e) \cap \text{cl}_M(E(M) - X_e) \) has at most two elements; and

(X5) \( X_e \subseteq T(M) \cup R_0(M^*) \).

To prove the existence of \( X_e \) we deal with two cases.

Case 1: There is \( f \in [L \cap E(M)] \cap T(M) \) such that, for each \( Z \in \{X, Y\} \), there is a line \( L' \) such that \( f \in L' \subseteq \text{cl}_M(Z) \) and \( |L'| \geq 3 \).

Hence \( L \cap E(M) = \{e, f\} \). As \( \{E(M_X) - L, E(M_Y) - L, L \cap E(M)\} \) is a partition of \( E(M) \), it follows that:

\[
e(M) = e(M_X) + e(M_Y) - 4.
\]

(5.3)

By Lemma 3.1, for \( Z \in \{X, Y\} \), an element of \( Z - L \) belongs to a triangle of \( M_Z \) if and only if it belongs to a triangle of \( M \). As \( f \) belongs to a triangle of \( M \), it follows that:

\[
t(M) = t(M_X) + t(M_Y) - 5.
\]

(5.4)

By (5.2), we have that

\[
r_0(M^*) = r_0(M^*_X) + r_0(M^*_Y).
\]

(5.5)

Assume the result holds for both \( M_X \) and \( M_Y \). Hence

\[3[t(M_Z) + r_0(M^*_Z) - \delta(M^*_Z)] \geq e(M_Z) + 10,
\]

for \( Z \in \{X, Y\} \). Adding these inequalities, we obtain:

\[3[t(M) + r_0(M^*)] + 3[r_0(M^*_X) + r_0(M^*_Y)] - 3[\delta(M^*_X) + \delta(M^*_Y)] \geq [e(M_X) + e(M_Y)] + 20.
\]

Replacing (5.3), (5.4) and (5.5), we get:

\[3[t(M) + 5] + 3[r_0(M^*)] - 3[\delta(M^*_X) + \delta(M^*_Y)] \geq [e(M) + 4] + 20.
\]

This inequality can be reordered as

\[3[t(M) + r_0(M^*) - \delta(M)] \geq [e(M) + 10] + 3[\delta(M_X) + \delta(M_Y) - \delta(M)] - 1.
\]

(5.6)

For \( Z \in \{X, Y\} \), \( \delta(M_Z) = 1 \) because \( L_Z \) and \( L'_Z \) are different lines of \( M_Z \) each with at least three points meeting at \( f \); a contradiction by (5.6). Hence the result does not hold for \( M_X \) or \( M_Y \), say \( M_Y \).

As \( r(M_Y) < r(M) \), it follows that \( e(M_Y) \leq 6 \) and so \( |Y - L| \leq 3 \). Thus \( r(M_Y) = 3 \) and so \( r_0(M^*_Y) + t(M_Y) = e(M_Y) \).

Again, we have a contradiction, unless \( \delta(M_Y) = 1 \) and \( e(M_Y) = 6 \). Observe that \( X_e = Y - L \) satisfies (X1), (X3) and (X4).

Note that (X2) holds because \( X_e \) spans \( e \) in \( M \) and so there is a circuit \( C \) of \( M \) such that \( e \in C \subseteq X_e \cup e \). As \( C \) is not a triangle of \( M \), it follows that \( C = X_e \cup e \). To conclude (X5), it is enough to prove that \( \text{si}(M/g) \) is 3-connected, for each \( g \in X_e \).

By Lemma 3.3(ii), this is the case. With this, we conclude the existence of \( X_e \).

Case 2: If \( f \in [L \cap E(M)] \cap T(M) \), then there is \( Z \in \{X, Y\} \) such that \( f \) does not belong to a triangle of \( M \) contained in \( \text{cl}_M(Z) \).

For \( Z \in \{X, Y\} \), let \( H_Z \) be a matroid isomorphic to \( P_6 \) such that \( L \) is a line of \( H_Z \) and \( H_X - L, H_Y - L, E(M) \) are pairwise disjoint sets. (The matroid \( P_6 \) has \( \{1, 2, 3, 4, 5, 6\} \) as ground set and \( \{X \subseteq \{1, 2, 3, 4, 5, 6\} : |X| = 3 \text{ and } X \neq \{1, 2, 3\}\} \) as set of bases.) Let \( K_Z \) be the generalized parallel connection of \( M_Z \) and \( H_Z \). Choose \( c \in L \cap E(M) \). As in the proof of Theorem 4.1, \( N_Z = K_Z \setminus c \) is 3-connected. Note that \( R_0(N_Z^c) = R_0(M^*_Z) \cup [E(H_Z) - L] \) and so, by (5.2),

\[
r_0(M^*) = r_0(M^*_Z) + r_0(M^*_Y) = r_0(N^*_Y) + r_0(N^*_Y) - 6.
\]

(5.7)

As \( \{E(N_Z) - E(H_X), E(N_Y) - E(H_Y), L \cap E(M)\} \) is a partition of \( E(M) \), it follows that

\[
e(M) = e(N_X) + e(N_Y) + |L \cap E(M)| - 10.
\]

(5.8)

By Lemma 3.1, for \( Z \in \{X, Y\} \), an element of \( \text{cl}_M(Z) \cup [L \cap E(M)] \) belongs to a triangle of \( N_Z \) if and only if it belongs to a triangle of \( M' \mid \text{cl}_M(Z) \). By hypothesis, no element of \( L \cap E(M) \) belongs to triangles of both \( N_X \) and \( N_Y \). Hence

\[
t(M) = t(N_X) + t(N_Y).
\]

(5.9)
Assume (5.1) holds for both $N_X$ and $N_Y$. Hence

$$3[t(N_X) + r_0(M^*_X) - \delta(M_X)] \geq e(N_X) + 10,$$

for $Z \subseteq \{X, Y\}$. Adding this inequality for $Z \subseteq \{X, Y\}$, we obtain:

$$3[t(N_X) + t(N_Y)] + 3[r_0(N_X^*) + r_0(N_Y^*)] - 3[\delta(N_X) + \delta(N_Y)] \geq [e(N_X) + e(N_Y)] + 20.$$

Replacing (5.7), (5.8) and (5.9), we get:

$$3[t(M)] + 3[r_0(M^*) + 6] - 3[\delta(N_X) + \delta(N_Y)] \geq [e(M) + 10 - |L \cap E(M)|] + 20.$$

This inequality can be reordered as

$$3[t(M) + r_0(M^*) - \delta(M)] \geq [e(M) + 10] + [2 - |L \cap E(M)|] + 3[\delta(N_X) + \delta(N_Y) - \delta(M)].$$

We have a contradiction unless

$$2 - |L \cap E(M)| + 3[\delta(N_X) + \delta(N_Y) - \delta(M)] < 0.$$

As $|L \cap E(M)| \leq 2$, it follows that

$$\delta(N_X) + \delta(N_Y) < \delta(M).$$

Hence $\delta(N_X) = \delta(N_Y) = 0$. So $\delta(M) = 1$. We have a contradiction because, in this case,

$$\delta(M) = \max\{\delta(N_X), \delta(N_Y)\},$$

since no element of $L$ belongs to triangles contained in both $cl_M(X)$ and $cl_M(Y)$. Hence (5.1) does not hold for $N_X$ or $N_Y$, say $N_Y$.

By the choice of $M$, it follows that $r(Y) \leq r(H_Y) = 3$. As $r(Y) \geq 3$, it follows that $r(Y) = 3$ and so $r(M) = r(N_X)$. Again, by the choice of $M$, we conclude that $|Y - L| \leq 3$. Thus, as in the previous case, we can take $X_e$ to be equal to $Y - L$.

Now, we prove that

$$X_e = X_f \text{ or } X_e \cap X_f = \emptyset,$$

for every 2-subset \{$e, f$\} of $E(M) - [T(M) \cup R_0(M^*)]$. (5.10)

Suppose (5.10) fails. In particular, $X_e \cap X_f \neq \emptyset$. As $X_e$ and $X_f$ are triads of $M$, by (X1), and $X_e \cup e$ and $X_f \cup f$ are circuits of $M$, it follows, by orthogonality, that

$$|X_e \cap X_f| = 2 \text{ or } e \in X_f \text{ and } f \in X_e.$$

By (X5), we must have $|X_e \cap X_f| = 2$. By submodularity, $X_e \cup X_f$ is a 3-separating set of $M$ because $|E(M) - (X_e \cup X_f)| \geq 3$. As $X_e \cup X_f$ spans $e$ and $f$, by (X2), it follows that $cl_M(X_e \cup X_f) = X_e \cup X_f \cup \{e, f\}$ because $\{e, f\}$ is not contained in a triangle of $M$. Moreover, $X_e \cup X_f$ is a 2-separating set of $M/e$ and so

$$T^* = E(M) - [X_e \cup X_f \cup \{e, f\}]$$

is a triad of $M$, since we can replace $X_e$ by a set in $\{X_e \cup X_f, T^*\}$ which must be a triad of $M$, by (X1), and so it must be equal to $T^*$. In particular, $e(M) = 9$. As $X_e \cup X_f$ is contained in a coline of $M$, it follows that $X_e \cup X_f \subseteq R_0(M^*)$. Observe that $M$ does not have a triangle, otherwise it would be contained in $E(M) - (X_e \cup X_f \cup \{e, f\})$. Thus, by (X5),

$$T^* \subseteq R_0(M^*).$$

We arrive at a contradiction because

$$3[t(M) + r_0(M^*) - \delta(M)] = 21 > 19 = |E(M)| + 10.$$

Hence (5.10) follows.

There are elements $e_1, \ldots, e_n$ belonging to $E(M) - [T(M) \cup R_0(M^*)]$ such that $L_{e_i} = [T(M) \cup R_0(M^*)], \ldots, L_{e_n} = [T(M) \cup R_0(M^*)]$ partition $E(M) - [T(M) \cup R_0(M^*)]$. Hence

$$|E(M) - [T(M) \cup R_0(M^*)]| = n + r,$$

where $r$ is the number of indices $i \in \{1, \ldots, n\}$ such that $|L_{e_i} - [T(M) \cup R_0(M^*)]| = 2$. As $X_{e_1}, \ldots, X_{e_n}$ are pairwise disjoint, by (4.10), it follows that $|T(M) \cup R_0(M^*)| \geq 3n$ and so

$$|T(M) \cup R_0(M^*)| = 3n + \gamma,$$

for some non-negative integer $\gamma$. As $3[t(M) + r_0(M^*) - \delta(M)] < e(M) + 10$, it follows that

$$3[3n + \gamma - \delta(M)] < 4n + r + \gamma + 10.$$

This inequality can be rewritten as

$$0 \leq 2\gamma < 10 + r + 3\delta(M) - 5n.$$

As $0 \leq r \leq n$, we arrive at a contradiction unless $n \leq 3$. 

\[\text{(5.13)}\]
If \( n = 3 \), then \( \gamma = 0 \), \( r = 3 \) and \( \delta(M) = 1 \). In particular, \( T(M) \cup R_0(M^*) = X_{i_1} \cup X_{i_2} \cup X_{i_3} \) and \( E(M) - [T(M) \cup R_0(M^*)] = L_{i_1} \cup L_{i_2} \cup L_{i_3} \). As \( \delta(M) = 1 \), it follows that \( M \) has a triangle \( T \). Thus \( T \cap X_{i_1} \neq \emptyset \), for some \( i \in \{1,2,3\} \). By (X1) and orthogonality, \( |T \cap X_{i_1}| = 2 \). We have a contradiction because \( T \subseteq cl_M(X_{i_1}) = X_{i_1} \cup L_{i_1} \). Thus \( n \leq 2 \).

If \( n = 1 \), then \( |E(M) - (X_{i_1} \cup L_{i_1})| \geq 3 \) and so \( \gamma \geq 3 \). By (5.13), \( \delta(M) = 1 \). Let \( T_1 \) and \( T_2 \) be a triangles of \( M \). By orthogonality, \( T_1 \cap X_{i_1} = \emptyset \) or \( T_2 \subseteq X_{i_1} \cup L_{i_1} \), for each \( i \in \{1,2\} \). Hence there is an \( i \in \{1,2\} \) such that \( T_1 \cap X_{i_1} = \emptyset \), say \( i = 1 \), because every triangle of \( M \) contained in \( X_{i_1} \cup L_{i_1} \) contains the element belonging to \( L_{i_1} - e_1 \). As \( M \) is 3-connected, it follows that \( |E(M) - (X_{i_1} \cup L_{i_1} \cup T_1)| \geq 1 \). Thus \( \gamma \geq 4 \). By (5.13), \( \gamma = 4 \) and \( r = 1 \). In particular, \( e(M) = 9 \) and, since \( r = 1 \), \( M \) does not have a triangle; a contradiction. Hence \( n = 2 \). Now, we show that each triangle \( T \) of \( M \) is contained in \( X_{i_1} \cup L_{i_1} \), for some \( i \in \{1,2\} \). If \( T \) is not contained in \( X_{i_1} \cup L_{i_1} \), then, by orthogonality, \( T \cap (X_{i_1} \cup X_{i_2}) = \emptyset \) and so \( \gamma \geq 3 \); a contradiction to (5.13). Next, we prove that \( \delta(M) = 0 \). Suppose \( \delta(M) = 1 \). As \( X_{i_1} \cup L_{i_1} \) contains at most one triangle, for \( i \in \{1,2\} \), it follows that \( X_{i_2} \cup L_{i_2} \) contains a triangle \( T_i \) of \( M \), for every \( i \in \{1,2\} \). Thus \( T_1 \cap L_{i_2} = \{e_i\} \) and so \( r = 0 \) and \( \gamma \geq 2 \). We have a contradiction. Hence \( \delta(M) = 0 \) and, by (5.13), \( \gamma = 0 \) and \( r \geq 1 \). As \( \gamma = 0 \), it follows that \( E(M) = X_{i_1} \cup X_{i_2} \cup L_{i_1} \cup L_{i_2} \). Let \( M_1 \) be a factor of \( M \) with respect to \( X_{i_2} \cup L_{i_2} \cup L_{i_1} \) having special line \( L_1 \). Note that \( L_{i_1} \subseteq L_1 \). By Lemma 3.3(i), \( M_1 \) is 3-connected. Note that \( X_{i_2} \) is a 3-separating set of \( M_1 \), by Lemma 3.2(ii). Let \( M_2 \) be a factor of \( M_1 \) with respect to \( L_{i_2} \cup L_1 \) having special line \( L_2 \). Note that \( L_{i_2} \subseteq L_2 \). By Lemma 3.3(i), \( M_2 \) is 3-connected. But \( E(M_2) \) is the union of two 3-elements lines, namely \( L_1 \) and \( L_2 \); a contradiction because \( M_2 \) cannot be 3-connected. Thus Theorem 5.1 follows.

6. On Lemos's and Leo's theorems

In this section, we prove the main results of Lemos [3] and Leo [6]. First, we study the cocircuits of a 3-connected matroid that meets both sets of a vertical 3-separation.

**Lemma 6.1.** Suppose that \( X \) is a 3-separating set of a 3-connected matroid \( M \) such that \( \min \{r(X), r(E(M) - X)\} \geq 3 \). Let \( N \) be a factor of \( M \) with respect to \( X \) having \( L \) as special line. If \( L - E(M) \neq \emptyset \) and \( C^* \) is a cocircuit of \( M \) such that \( C^* \) meets both \( X \cup L \) and \( [E(M) - X] \cup L \), then there is \( Y \subseteq L \) such that \( |Y| \leq 1 \) and \( [(L - Y) \cap C^*] \cup (Y - L) \) is a cocircuit of \( N \).

**Proof.** We set \( N_1 = N \). Let \( N_2 \) be a factor of \( M \) with respect to \( E(M) - X \) having \( L_2 \) as a special line. We may label the elements of \( L_2 \) so that \( L_2 = L \). Let \( H = E(M) - C^* \). For \( i \in \{1,2\} \), we set \( C^*_i = E(M_i) \cap C^* \) and \( H_i = E(M_i) \cap H \). As \( H \) is a hyperplane of \( M \) and \( C^*_i \neq \emptyset \), by hypothesis, it follows that \( H_i \) does not span \( M_i \). Thus

\[
r(H_i) = r(M_i) - 1 - \delta_i,
\]

for some \( \delta_i \geq 0 \). Hence

\[
r(H_1) + r(H_2) = r(M_1) + r(M_2) - 2 - \delta_1 - \delta_2 = r(M) - \delta_1 - \delta_2.
\]

As

\[
r(H_1) + r(H_2) = r(H) + \delta = r(M) - 1 + \delta,
\]

for some \( \delta \geq 0 \), it follows that

\[
\delta + \delta_1 + \delta_2 = 1.
\]  \hspace{1cm} (6.1)

In particular \( \{\delta, \delta_1, \delta_2\} \subseteq \{0, 1\} \). If \( \delta_1 = 0 \), then \( cl_H(H_1) \) is a hyperplane of \( M_1 \). Now, we prove that \( L \not\subseteq cl_H(H_1) \). If \( L \subseteq cl_H(H_1) \), then \( C^*_i - L \) contains a cocircuit \( D^* \) of \( M_i \). As \( C^*_i - L \) is contained in \( E(M_i) - L \), it follows that \( D^* \) is a cocircuit of \( M_i \), by Lemma 3.2(iii); a contradiction and so \( L \not\subseteq cl_H(H_i) \). Hence \( |cl_H(H_i) \cap L| \leq 1 \). Thus

\[
C^*_i \cup L \quad \text{or} \quad C^*_i \cup (L - t)
\]

is a cocircuit of \( M_i \), for some \( t \in L \). The result follows in this case. So, we may assume that \( \delta_1 = 1 \). Hence \( \delta_2 = \delta = 0 \), by (6.1). As \( \delta = 0 \), it follows that \( L \cap H = \emptyset \) and so \( L \cap E(M) \subseteq C^* \). Choose \( e \in C^*_i - L \) (\( e \) exists, by orthogonality, since \( X - L \) spans \( L \) in \( M \)). If \( H_1 \cup e \) spans \( L \) in \( M_1 \), then \( C^*_i - L \) contains a cocircuit of \( M_1 \) which is a cocircuit of \( M \), by Lemma 3.2(ii); a contradiction. Hence \( H_1 \cup e \) does not span \( L \) in \( M_1 \). Thus \( H_1 \cup e \) does not span any element belonging to \( A = L \cup E(M) \) and so, when \( c \in A \), \( H_1 \cup c \) does not span any element of \( C^*_i - L \). As \( C^*_i - L \) cannot be a cocircuit of \( M \), it follows, by Lemma 3.2(ii), that \( C^*_i - L \) is not a circuit of \( M_1 \) and so \( H_1 \cup c \) does not span any element of \( L - c \). Hence \( (C^*_i - L) \cup (L - c) \) is a cocircuit of \( M_i \); the result also follows in this case.
Now, we prove an extension of the main result of Lemos [3]:

**Theorem 6.1.** Suppose that $C^*$ is a cocircuit of a 3-connected matroid $M$ such that $r(M) \geq 3$. If $M \setminus e$ is not 3-connected, for every $e \in C^*$, then $C^*$ meets different lines of $M$ each with at least three elements.

Lemos’s Theorem says only that $C^*$ meets two triangles of $M$. It may be possible that these triangles belongs to the same line.

**Proof of Theorem 6.1.** Suppose this result is not true and choose a counter-example $M$ so that $r(M)$ is minimum. First, we show that there is $e \in C^*$ such that $e$ does not belong to a triangle of $M$. Suppose that, for each $e \in C^*$, there is a line $L_e$ of $M$ such that $e \in L_e$ and $|L_e| \geq 3$. By the choice of $M$, $L_e = L_f$ for every 2-subset $\{e, f\}$ of $C^*$. Hence $C^* \subseteq L_e$, for every $e \in C^*$. Thus

$$r(C^*) + r^*(C^*) - |C^*| = 2 + (|C^*| - 1) - |C^*| = 1$$

and so $|E(M) - C^*| \leq 1$. Hence $r(M) = r(C^*) = 2$; a contradiction. Thus there is $e \in C^*$ such that $e$ does not belong to a triangle of $M$.

Let $\{X, Y\}$ be a 2-separation of $M/e$. As $e$ does not belong to a triangle of $M$, it follows that

$$\min\{rw_e(X), rw_e(Y)\} \geq 2.$$ 

For $Z \in \{X, Y\}$, let $M'_Z$ be a factor of $M$ with respect to $Z$ having $L_Z$ as special line. We may choose the elements of $L_X$ and $L_Y$ so that $L_X = L_Y$, say $L = L_X$. As $e$ does not belong to a triangle of $M$, it follows that $L - E(M) \neq \emptyset$ and so $C_Z = (Z - L) \cap C^* \cup A_Z$ is a cocircuit of $M_Z$, by Lemma 6.1, for some $A_Z \subseteq L$ such that $|L - A_Z| \leq 1$. By Lemma 3.3(iii), $M_Z/f$ is not 3-connected, for each $f \in C_Z - L$ and so $M_Z/f$ is not 3-connected, for every $f \in C_Z$, since $M_Z/f$ is not 3-connected, for every $f \in L$. By the choice of $M$, it follows that $C_Z$ meets a line $L'_Z$ of $M_Z$ such that $L'_Z \neq L_Z$ and $|L'_Z| \geq 3$. Note that $L'_Z \subseteq cl_M(Z)$ and so $L'_X \neq L'_Y$. By Lemma 3.1, $L'_X$ and $L'_Y$ are different lines of $M$ both meeting $C^*$; a contradiction and the result follows. \(\square\)

Now, we prove the main result of Leo [6]:

**Theorem 6.2.** Suppose that $C^*$ is a cocircuit of a 3-connected matroid $M$ such that $|E(M)| \geq 4$. Let $f$ be an element of $C^*$. If $M/e$ is not 3-connected, for every $e \in C^* - f$, then $C^*$ meets at least one triangle of $M$.

**Proof.** Suppose this result is not true and let $M$ be a counter-example so that $r(M)$ is minimum. Thus $C^*$ does not meet any triangle of $M$. By Theorem 6.1, $M/f$ is 3-connected. Choose $e \in C^* - f$. Let $\{X, Y\}$ be a 2-separation for $M/e$ such that $f \notin Y$. Let $N$ be a factor of $M$ with respect to $X$ having special line $L$. Note that $f \notin L$. As in the last paragraph of the proof of Theorem 6.1, we conclude that $C^*$ meets a line $L'$ of $M$ such that $L' \subseteq cl_M(X)$, $L' \neq L$ and $|L'| \geq 3$. We arrive at a contradiction. \(\square\)

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**References**


