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Hybrid high dimensional model representation (HHDMR) on the partitioned data

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Abstract

A multivariate interpolation problem is generally constructed for appropriate determination of a multivariate function whose values are given at a finite number of nodes of a multivariate grid. One way to construct the solution of this problem is to partition the given multivariate data into low-variate data. High dimensional model representation (HDMR) and generalized high dimensional model representation (GHDMR) methods are used to make this partitioning. Using the components of the HDMR or the GHDMR expansions the multivariate data can be partitioned. When a cartesian product set in the space of the independent variables is given, the HDMR expansion is used. On the other hand, if the nodes are the elements of a random discrete data the GHDMR expansion is used instead of HDMR. These two expansions work well for the multivariate data that have the additive nature. If the data have multiplicative nature then factorized high dimensional model representation (FHDMR) is used. But in most cases the nature of the given multivariate data and the sought multivariate function have neither additive nor multiplicative nature. They have a hybrid nature. So, a new method is developed to obtain better results and it is called hybrid high dimensional model representation (HHDMR). This new expansion includes both the HDMR (or GHDMR) and the FHDMR expansions through a hybridity parameter. In this work, the general structure of this

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hybrid expansion is given. It has tried to obtain the best value for the hybridity parameter. According to this value the analytical structure of the sought multivariate function can be determined via HHDMR.

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1. Introduction

When a multivariate data is given and the analytical structure of the multivariate function is sought through this given data, a multivariate interpolation problem must be constructed. However, increasing dimensionality of the interpolation causes such problems that it becomes quite difficult to determine the analytical structure of the sought multivariate function. Hence, a divide-and-conquer algorithm is needed and high dimensional model representation (HDMR) method is developed [7,6,1,4,3]. The main purpose of the method is to partition the multivariate data into low-variate data such as constant, univariate, bivariate terms and so on. This method works through a given multivariate data constructed as a cartesian product set in the space of the independent variables.

If a random discrete data is given then another method based on HDMR called generalized high dimensional model representation (GHDMR) [8] can be used. These two methods work well when the structure of the given data, in other words, the structure of the sought multivariate function has an additive nature.

On the other hand, if the structure of the data has a multiplicative nature then another HDMR based representation model is used. This method is called factorized high dimensional model representation (FHDMR) [9,10]. This method uses the components of the HDMR expansion when the nodes of the given data are the elements of a cartesian product set. If the data is a random discrete data then this method uses the GHDMR components instead of the HDMR components.

In this work, the relations for the constant, univariate and bivariate components of the HDMR expansion are given. On the other hand, only the relations for the constant and univariate components of GHDMR are given. So, if the HDMR method is used to partition the given data then the FHDMR expansion is constructed by using the constant, univariate and bivariate terms for the partitioned data. Otherwise, if the GHDMR method is used for partitioning the FHDMR expansion includes the constant and the univariate terms only.

When the sought multivariate function has neither additive nor multiplicative nature then a hybrid based model representation is needed. Hybrid high dimensional model representation (HHDMR) method is developed for this purpose. This method uses both the HDMR (or GHDMR) expansion and the FHDMR expansion to determine the general analytical structure of the multivariate function.

The paper is organized as follows. Sections 2 and 3 involve certain details of HDMR and GHDMR, respectively. Section 4 covers the interpolation of partitioned data while Section 5 stands for some explanations to FHDMR. Section 6 gives the main topic, HHDMR. Section 7 is for hybridity optimization while Section 8 includes implementations. Finally, Section 9 presents concluding remarks.

2. High Dimensional Model Representation

The equation of the HDMR for a given multivariate function is as follows:

$$f(x_1, \ldots, x_N) = f_0 + \sum_{i_1=1}^{N} f_{i_1}(x_{i_1}) + \sum_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{N} f_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + f_{12\ldots N}(x_1, \ldots, x_N). \tag{1}$$

The HDMR terms of the given multivariate function are the right hand side terms of this expansion. These terms are the constant term, univariate terms, bivariate terms and the other high-variate terms. The following vanishing conditions are used to be able to obtain the right hand side components of the expansion.

$$\int_{a_1}^{b_1} \mathrm{d}x_1 \cdots \int_{a_N}^{b_N} \mathrm{d}x_N W(x_1, \ldots, x_N) f_i(x_i) = 0, \quad 1 \leqslant i \leqslant N. \tag{2}$$

These vanishing conditions correspond to the following orthogonality condition.

$$(f_{i_1 i_2 \ldots i_k}, f_{i_1 i_2 \ldots i_l}) = 0, \{i_1, i_2, \ldots, i_k\} \not\equiv \{i_1, i_2, \ldots, i_l\}, \quad 1 \leqslant k, l \leqslant N. \tag{3}$$

The right hand side terms of (1) are the orthogonal decomposition components of the original function. Because these components must satisfy these orthogonality conditions. These orthogonality conditions are defined over an inner product and the inner product is defined as

$$(u, v) \equiv \int_{a_1}^{b_1} \mathrm{d}x_1 W_1(x_1) \ldots \int_{a_N}^{b_N} \mathrm{d}x_N W_N(x_N) u(x_1, \ldots, x_N) v(x_1, \ldots, x_N), \tag{4}$$

where $u(x_1, \ldots, x_N)$ and $v(x_1, \ldots, x_N)$ are two arbitrary functions. According to this definition, square of both the original function and the right hand side components of the HDMR expansion are assumed to be integrable functions. These square integrals are defined over a certain interval of the independent variables and a weight function is related to each variable. The weight function appearing in the vanishing conditions is assumed as a product type of function

$$W(x_1, \ldots, x_N) \equiv \prod_{j=1}^{N} W_j(x_j), \quad x_j \in [a_j, b_j], \quad 1 \leqslant j \leqslant N. \tag{5}$$

The following normalization criteria for each univariate factor of the weight function is assumed for easy determination of the HDMR components.

$$\int_{a_j}^{b_j} \mathrm{d}x_j W_j(x_j) = 1, \quad 1 \leqslant j \leqslant N. \tag{6}$$

It is assumed that the values of a multivariate function are given at the nodes of a cartesian product set in the Euclidean space defined by the independent variables. For this purpose the data of the variable x_j can be defined as follows.

$$\kappa_j \equiv \{\xi_j^{(k_j)}\}_{k_j=1}^{k_j=n_j} = \{\xi_j^{(1)}, \ldots, \xi_j^{(n_j)}\}, \quad 1 \leqslant j \leqslant N. \tag{7}$$

The following cartesian product can be constructed from that data:

$$\kappa \equiv \{\tau | \tau = (x_1, x_2, \ldots, x_N), x_j \in \kappa_j, 1 \leqslant j \leqslant N\}. \tag{8}$$

Only the values of the sought multivariate function, $f(x_1, \ldots, x_N)$ on the points of that cartesian product set must be used in interpolation. For this purpose, the weight function must be formatted as a linear combination of several Dirac delta functions [11]. This means that the weight function can be defined as

$$W(x_1, \ldots, x_N) \equiv \prod_{j=1}^{N} \left[\sum_{k_j=1}^{n_j} \alpha_{k_j}^{(j)} \delta(x_j - \xi_j^{(k_j)}) \right], \quad x_j \in [a_j, b_j], \tag{9}$$

where

$$W_j(x_j) \equiv \sum_{k_j=1}^{n_j} \alpha_{k_j}^{(j)} \delta(x_j - \xi_j^{(k_j)}), \quad 1 \leqslant j \leqslant N. \tag{10}$$

The α constants appearing in the weight function are used to be able to give different importance to each node of the interpolation problem. When the normalization criteria given in (6) is taken into consideration and the left hand side of this relation is rewritten for the selected weight function given in (10), the following relation for these constants can be obtained by using Delta function's properties:

$$\sum_{k_j=1}^{n_j} \alpha_{k_j}^{(j)} = 1, \quad 1 \leqslant j \leqslant N. \tag{11}$$

Using the properties of the weight function and the orthogonality conditions, the right hand side terms of the HDMR expansion can be obtained. To obtain the constant term, both sides of the HDMR equation given in (1) are multiplied by the weight function, $W_1(x_1)W_2(x_2)\cdots W_N(x_N)$, and are integrated over whole Euclidean space defined by independent variables. For this purpose, the following I_0 operator can be defined. This operator can be written by using an arbitrary square integrable function, $F(x_1, \ldots, x_N)$ as follows:

$$I_0 F(x_1, \ldots, x_N) \equiv \int_{a_1}^{b_1} \mathrm{d}x_1 \cdots \int_{a_N}^{b_N} \mathrm{d}x_N W(x_1, \ldots, x_N) F(x_1, \ldots, x_N). \tag{12}$$

To obtain the univariate and the bivariate terms of the HDMR expansion the following operators can be defined in the same manner.

$$\begin{aligned} I_m F(x_1, \ldots, x_N) &\\ \equiv \int_{a_1}^{b_1} &\mathrm{d}x_1 W_1(x_1) \cdots \int_{a_{m-1}}^{b_{m-1}} \mathrm{d}x_{m-1} W_{m-1}(x_{m-1}) \\ &\times \int_{a_{m+1}}^{b_{m+1}} \mathrm{d}x_{m+1} W_{m+1}(x_{m+1}) \cdots \int_{a_N}^{b_N} \mathrm{d}x_N W_N(x_N) F(x_1, \ldots, x_N), \quad 1 \leqslant m \leqslant N. \end{aligned} \tag{13}$$

$$
\begin{aligned}
&I_{m_1m_2}F(x_1,\ldots,x_N)\\
&\quad\equiv \int_{a_1}^{b_1} \mathrm{d}x_1 W_1(x_1)\cdots\int_{a_{m_1-1}}^{b_{m_1-1}} \mathrm{d}x_{m_1-1}W_{m_1-1}(x_{m_1-1})\\
&\qquad\times\int_{a_{m_1+1}}^{b_{m_1+1}} \mathrm{d}x_{m_1+1}W_{m_1+1}(x_{m_1+1})\cdots\int_{a_{m_2-1}}^{b_{m_2-1}} \mathrm{d}x_{m_2-1}W_{m_2-1}(x_{m_2-1})\\
&\qquad\times\int_{a_{m_2+1}}^{b_{m_2+1}} \mathrm{d}x_{m_2+1}W_{m_2+1}(x_{m_2+1})\cdots\int_{a_N}^{b_N} \mathrm{d}x_N W_N(x_N)F(x_1,\ldots,x_N),\quad 1\leqslant m_1\leqslant m_2\leqslant N.
\end{aligned}
\tag{14}
$$

If the operator, I_0, is applied on both sides of (1) and orthogonality conditions together with (9) are used then the following structure is obtained for the constant term:

$$
f_0\equiv\sum_{\tau\in\kappa}\zeta(\tau)f(\tau),\quad \tau=(\xi_1^{(k_1)},\ldots,\xi_N^{(k_N)}),\quad \zeta(\tau)=\alpha_1^{(k_1)}\cdots\alpha_N^{(k_N)},\quad 1\leqslant k_j\leqslant n_j,\quad 1\leqslant j\leqslant N.
\tag{15}
$$

The magnitude of $f_m(x_m)$ can be obtained by using the operator given in (13) in the same manner as the case of constant term determination.

$$
\begin{aligned}
&f_m(\xi_m^{(k_m)})=\sum_{\tau_m\in\kappa^{(m)}}\zeta_m(\tau_m)f(\tau_m,\xi_m^{(k_m)})-\sum_{\tau\in\kappa}\zeta(\tau)f(\tau),\\
&\kappa^{(m)}\equiv\{\tau_m|\tau_m=(x_1,\ldots,x_{m-1},x_{m+1},\ldots,x_N),x_j\in\kappa_j,1\leqslant j\leqslant N,\ j\neq m\}\\
&\tau_m=(\xi_1^{(k_1)},\ldots,\xi_{m-1}^{(k_{m-1})},\xi_{m+1}^{(k_{m+1})},\ldots,\xi_N^{(k_N)}),\\
&\zeta_m(\tau_m)=\alpha_1^{(k_1)}\cdots\alpha_{m-1}^{(k_{m-1})}\alpha_{m+1}^{(k_{m+1})}\cdots\alpha_N^{(k_N)},\\
&\xi_m^{(k_m)}\in\kappa_m,\quad 1\leqslant k_m\leqslant n_m,\quad 1\leqslant m\leqslant N
\end{aligned}
\tag{16}
$$

and the magnitude of bivariate functions, $f_{m_1m_2}(x_{m_1},x_{m_2})$, of the HDMR expansion can be evaluated by using the operator given in (14) as

$$
\begin{aligned}
&f_{m_1m_2}(\xi_{m_1}^{(k_{m_1})},\xi_{m_2}^{(k_{m_2})})\\
&\quad=\sum_{\tau_{m_1m_2}\in\kappa^{(m_1m_2)}}\zeta_{m_1m_2}(\tau_{m_1m_2})f(\tau_{m_1m_2},\xi_{m_1}^{(k_{m_1})},\xi_{m_2}^{(k_{m_2})})\\
&\qquad-\sum_{\tau_{m_1}\in\kappa^{(m_1)}}\zeta_{m_1}(\tau_{m_1})f(\tau_{m_1},\xi_{m_1}^{(k_{m_1})})-\sum_{\tau_{m_2}\in\kappa^{(m_2)}}\zeta_{m_2}(\tau_{m_2})f(\tau_{m_2},\xi_{m_2}^{(k_{m_2})})+\sum_{\tau\in\kappa}\zeta(\tau)f(\tau),\\
&\kappa^{(m_1m_2)}\equiv\{\tau_m|\tau_m=(x_1,\ldots,x_{m_1-1},x_{m_1+1},\ldots,x_{m_2-1},x_{m_2+1},\ldots,x_N),\\
&\qquad\qquad x_j\in\kappa_j,1\leqslant j\leqslant N,\ j\neq m_1,m_2\},\\
&\kappa^{(m_1)}\equiv\{\tau_{m_1}|\tau_{m_1}=(x_1,\ldots,x_{m_1-1},x_{m_1+1},\ldots,x_N),x_j\in\kappa_j,1\leqslant j\leqslant N,\ j\neq m_1\},
\end{aligned}
$$

$$\kappa^{(m_2)} \equiv \{\tau_{m_2} | \tau_{m_2} = (x_1, \ldots, x_{m_2-1}, x_{m_2+1}, \ldots, x_N), x_j \in \kappa_j, 1 \leqslant j \leqslant N, j \neq m_2\},$$
$$\xi_{m_1}^{(k_{m_1})} \in \kappa_{m_1}, \xi_{m_2}^{(k_{m_2})} \in \kappa_{m_2},$$
$$1 \leqslant k_{m_1} \leqslant n_{m_1}, \quad 1 \leqslant k_{m_2} \leqslant n_{m_2}, \quad 1 \leqslant m_1 < m_2 \leqslant N. \tag{17}$$

So, a table of pairs of data for the univariate terms, $f_m(x_m)$ and a table of triples of data for the bivariate terms, $f_{m_1 m_2}(x_{m_1}, x_{m_2})$ can be produced.

These HDMR terms correspond to the univariate and the bivariate data obtained by partitioning the given multivariate data. The next step is to fit analytical structures to these terms. Lagrange interpolation formula is used to interpolate that partitioned data. Using the analytical structures of these terms such as constant, univariate and bivariate terms, the right hand side components of the FHDMR expansion, including the constant, univariate and bivariate terms, will be obtained. By this way the HDMR and the FHDMR expansions including at most the bivariate terms can be determined. Using these two expansions the HHDMR expansion for the given multivariate data can be written. This expansion will be the analytical structure of the sought multivariate function.

This method can be used for only the data that has the nodes of a cartesian product set. If a random discrete data is given then GHDMR must be used to partition the multivariate data. In these types of cases the components of the GHDMR expansion are used to obtain the components of the FHDMR expansion.

3. Generalized High Dimensional Model Representation

To get a more general high dimensional model representation, a nonproduct type weight function is used instead of a product type one like in HDMR. Data partitioning is done with the help of this general multivariate weight function. Therefore, this nonproduct type weight can be represented by an HDMR expansion

$$W(x_1, \ldots, x_N) = W_0 + \sum_{i_1=1}^{N} W_{i_1}(x_{i_1}) + \sum_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{N} W_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + W_{12\ldots N}(x_1, \ldots, x_N) \tag{18}$$

and the components of this expansion can be used to determine the GHDMR components of the sought multivariate weight function. To obtain the right hand side components of this expansion an auxiliary product type weight function, $\Omega(x_1, \ldots, x_N)$, is defined

$$\Omega(x_1, \ldots, x_N) \equiv \prod_{j=1}^{N} \Omega_j(x_j), \tag{19}$$

where the individual multiplicants are normalized over a hyperprism whose corners are located at the points, $(a_1, b_1), \ldots, (a_N, b_N)$. That is,

$$\int_{a_j}^{b_j} \mathrm{d}x_j \Omega(x_j) = 1, \quad 1 \leqslant j \leqslant N. \tag{20}$$

The HDMR components of the weight function $W(x_1,\ldots,x_N)$ satisfy the following orthogonality conditions.

$$\int_{a_{i_j}}^{b_{i_j}} \mathrm{d}x_{i_j}\Omega_{i_j}(x_{i_j})W_{i_1\ldots i_k}(x_{i_1},\ldots,x_{i_k})=0,\quad 1\leqslant k\leqslant N,\quad 1\leqslant j\leqslant k,\quad 1\leqslant i_1<\cdots<i_k\leqslant N. \tag{21}$$

These orthogonality conditions can be rewritten as follows for the GHDMR components of the given multivariate function under the general multivariate weight function and product type of auxiliary weight function.

$$\int_{a_1}^{b_1} \mathrm{d}x_1\cdots\int_{a_N}^{b_N} \mathrm{d}x_N\Omega(x_1,\ldots,x_N)W(x_1,\ldots,x_N)f_i(x_i)=0,\quad 1\leqslant i\leqslant N. \tag{22}$$

To obtain the right hand side components of the HDMR expansion of the general multivariate weight function and the GHDMR components of the multivariate function the operators given in (12)–(14) under the product type weight function, $\Omega(x_1,\ldots,x_N)$, instead of $W(x_1,\ldots,x_N)$ can be used.

$$I_0F(x_1,\ldots,x_N)\equiv\int_{a_1}^{b_1} \mathrm{d}x_1\cdots\int_{a_N}^{b_N} \mathrm{d}x_N\Omega(x_1,\ldots,x_N)F(x_1,\ldots,x_N), \tag{23}$$

$$\begin{aligned} I_iF(x_1,\ldots,x_N)\equiv{}&\int_{a_1}^{b_1} \mathrm{d}x_1\cdots\int_{a_{i-1}}^{b_{i-1}} \mathrm{d}x_{i-1}\int_{a_{i+1}}^{b_{i+1}} \mathrm{d}x_{i+1}\cdots\int_{a_N}^{b_N} \mathrm{d}x_N\Omega_1(x_1)\cdots\\ &\times\Omega_{i-1}(x_{i-1})\Omega_{i+1}(x_{i+1})\cdots\Omega_N(x_N)F(x_1,\ldots,x_N),\quad 1\leqslant i\leqslant N, \end{aligned} \tag{24}$$

$$\begin{aligned} &I_{i_1i_2}F(x_1,\ldots,x_N)\\ &\quad\equiv\int_{a_1}^{b_1} \mathrm{d}x_1\Omega_1(x_1)\cdots\int_{a_{i_1-1}}^{b_{i_1-1}} \mathrm{d}x_{i_1-1}\Omega_{i_1-1}(x_{i_1-1})\\ &\quad\times\int_{a_{i_1+1}}^{b_{i_1+1}} \mathrm{d}x_{i_1+1}\Omega_{i_1+1}(x_{i_1+1})\cdots\int_{a_{i_2-1}}^{b_{i_2-1}} \mathrm{d}x_{i_2-1}\Omega_{i_2-1}(x_{i_2-1})\\ &\quad\times\int_{a_{i_2+1}}^{b_{i_2+1}} \mathrm{d}x_{i_2+1}\Omega_{i_2+1}(x_{i_2+1})\cdots\int_{a_N}^{b_N} \mathrm{d}x_N\Omega_N(x_N)F(x_1,\ldots,x_N),\quad 1\leqslant i_1\leqslant i_2\leqslant N. \end{aligned} \tag{25}$$

Using these operators the general relations for the constant and the univariate terms of the GHDMR method are obtained as

$$I_0[\,W(x_1,\ldots,x_N)f(x_1,\ldots,x_N)\,]=W_0f_0 \tag{26}$$

and

$$
\begin{aligned}
& I_i[\,W(x_1,\ldots,x_N)f(x_1,\ldots,x_N)] \\
& \quad = (1+W_i(x_i))f_0 + (1+W_i(x_i))f_i(x_i) \\
& \qquad + (1+W_i(x_i)) \sum_{\substack{i_1=1 \\ i_1\neq i}}^{N} \int_{a_{i_1}}^{b_{i_1}} \mathrm{d}x_{i_1}\Omega_{i_1}(x_{i_1})(1+W_{i_1}(x_{i_1}))f_{i_1}(x_{i_1}) \\
& \qquad + \sum_{\substack{i_1,i_2=1,i_1<i_2 \\ (i_1=i)\vee(i_2=i)}}^{N} \int_{(1-\delta_{i_1 i})a_{i_1}+\delta_{i_1 i}a_{i_2}}^{(1-\delta_{i_1 i})b_{i_1}+\delta_{i_1 i}b_{i_2}} [\,(1-\delta_{i_1 i})\,\mathrm{d}x_{i_1}\Omega_{i_1}(x_{i_1}) + \delta_{i_1 i}\,\mathrm{d}x_{i_2}\Omega_{i_2}(x_{i_2})] \\
& \qquad \times [W_{i_1 i_2}(x_{i_1},x_{i_2}) - W_{i_1}(x_{i_1})W_{i_2}(x_{i_2})][(1-\delta_{i_1 i})f_{i_1}(x_{i_1}) + \delta_{i_1 i}f_{i_2}(x_{i_2})], \quad 1\leqslant i\leqslant N,
\end{aligned} \tag{27}
$$

where $\delta_{i_1 i}$ stands for Kronecker's delta. This equation which gives the rule to obtain the univariate terms defines a set of integral equations.

When a random discrete data set is given the above GHDMR relations are used to partition this multivariate data. Assume that the following (N+1)-tuples are taken as data to describe a multivariate function $f(x_1,\ldots,x_N)$

$$d_j \equiv (x_1^{(j)},\ldots,x_N^{(j)},\varphi_j), \quad 1\leqslant j\leqslant m, \tag{28}$$

where φ_j, is the value of the sought function $f(x_1,\ldots,x_N)$ at the point described by the first N components of d_j in the N-dimensional space we are concerned. That is,

$$\varphi_j \equiv f(x_1^{(j)},\ldots,x_N^{(j)}), \quad 1\leqslant j\leqslant m. \tag{29}$$

Therefore all information about $f(x_1,\ldots,x_N)$ are these values and this means that a weight function which picks up the values of the function it multiplies, only at these points has to be used. This necessitates the use of delta function type weight. For this problem the weight function can be defined as follows

$$W(x_1,\ldots,x_N) \equiv \sum_{j=1}^{m} \alpha_j \delta(x_1 - x_1^{(j)})\cdots\delta(x_N - x_N^{(j)}), \tag{30}$$

where α_j parameters are used for making it possible to give different importance to each individual datum. The HDMR components of this weight function are used to obtain the terms of the GHDMR method. For this purpose a product type auxilliary weight function as given in (19) is used .

Additionally, the weight function $W(x_1,\ldots,x_N)$ should be chosen as normalized under the auxiliary weight function $\Omega(x_1,\ldots,x_N)$ because of the normalization criteria appearing in the HDMR method.

$$I_0 W(x_1,\ldots,x_N) = 1. \tag{31}$$

The following constraint on the previously mentioned α parameters can be written as the normalization condition of the weight function, $W(x_1,\ldots,x_N)$,

$$W_0 = I_0(W(x_1,\ldots,x_N)) = \sum_{j=1}^{m} \alpha_j \overline{\Omega}_j = 1, \tag{32}$$

where

$$\overline{\Omega}_j \equiv \prod_{k=1}^{N} \Omega_k(x_k^{(j)}), \quad 1 \leqslant j \leqslant m. \tag{33}$$

Using this result the relation for the constant term of GHDMR expansion given in (26) can be rewritten as

$$f_0 = I_0[W(x_1, \ldots, x_N) f(x_1, \ldots, x_N)]. \tag{34}$$

This equation can be more explicitly written as follows in the case of random data partitioning by taking Eqs. (29), (30) and (33) into consideration.

$$f_0 = \sum_{j=1}^{m} \alpha_j \overline{\Omega}_j \varphi_j. \tag{35}$$

It can be seen in (27) that to determine the structure of the univariate terms for the sought multivariate function, the univariate and the bivariate terms of the HDMR expansion of the general weight function are needed. The univariate and the bivariate terms can be evaluated by using the following relations:

$$W_i(x_i) = I_i\,(W(x_1, \ldots, x_N)) - W_0, \tag{36}$$

$$W_{i_1 i_2}(x_{i_1}, x_{i_2}) = I_{i_1 i_2}(W(x_1, \ldots, x_N)) - I_{i_1}(W(x_1, \ldots, x_N)) - I_{i_2}(W(x_1, \ldots, x_N)) + W_0. \tag{37}$$

Using Eq. (33) the equation below is obtained for the univariate terms,

$$W_i(x_i) = \sum_{j=1}^{m} \frac{\alpha_j \overline{\Omega}_j}{\Omega_i(x_i^{(j)})} \delta(x_i - x_i^{(j)}) - 1, \quad 1 \leqslant i \leqslant N. \tag{38}$$

There seem to be *m* linearly independent delta functions here. But this may not be always true because some $x_i^{(j)}$ values may be equivalent depending on how the original data is given. At the beginning it has been implicitly assumed that $d_1,\ldots,d_m$ tuples are all different. This is because polation methods like interpolation, curve or hyperspace fitting, extrapolation, etc. use unique data; no data repetition is allowed. Although $d_1,\ldots,d_m$ tuples are all different their components need not be unique. That is, repetition of these components is possible without destroying the uniqueness of each datum. This means that some of $x_i^{(j)}$, $1 \leqslant j \leqslant m$, values may be equal; multiplicities may arise. To take these cases into consideration we can use a little bit different notation. Assume that $x_i^{(j_{k,1})}, x_i^{(j_{k,2})},\ldots,x_i^{(j_{k,r_{i,k}})}$, $1 \leqslant j_{k,1} < j_{k,2} < \cdots < j_{k,r_{i,k}} < m$, denote *k*. set of identical x_i coordinate values. Then, the delta functions corresponding to each of these terms will become the same, and the number of the linearly independent delta functions will decrease. In that case, it is better to denote all these equivalent values just by a single symbol, say $\xi_i^{(k)}$. The multiplicity of this entity is characterized by $r_{i,k}$, $1 \leqslant k \leqslant m_i$ as we can see from the definition above. The sum of multiplicities must be equal to *m*, that is,

$$\sum_{k=1}^{m_i} r_{i,k} = m, \quad 1 \leqslant i \leqslant N. \tag{39}$$

If the following new parameters are defined:

$$\overline{\alpha}_{i,k} \equiv \sum_{j \in J_{i,k}} \frac{\alpha_j \overline{\Omega}_j}{\Omega_i(x_i^{(j)})},$$

$$J_{i,k} \equiv \{j_{k,1}, j_{k,2}, \ldots, j_{k,r_{i,k}}\}, \quad 1 \leqslant k \leqslant m_i, \quad 1 \leqslant i \leqslant N, \tag{40}$$

Eq. (38) can be rewritten as follows:

$$W_i(x_i) = \sum_{k=1}^{m_i} \overline{\alpha}_{i,k} \delta(x_i - \xi_i^{(k)}) - 1, \quad 1 \leqslant i \leqslant N. \tag{41}$$

The next step is to determine the structure of the bivariate components of the weight function $W(x_1, \ldots, x_N)$.

$$\begin{aligned} W_{i_1 i_2}(x_{i_1}, x_{i_2}) = & \sum_{k=1}^{m_{i_1}} \sum_{\ell=1}^{m_{i_2}} \overline{\alpha}_{i_1,i_2;k,\ell} \delta(x_{i_1} - \xi_{i_1}^{(k)}) \delta(x_{i_2} - \xi_{i_2}^{(\ell)}) \\ & - \sum_{k=1}^{m_{i_1}} \overline{\alpha}_{i_1,k} \delta(x_{i_1} - \xi_{i_1}^{(k)}) - \sum_{\ell=1}^{m_{i_2}} \overline{\alpha}_{i_2,\ell} \delta(x_{i_2} - \xi_{i_2}^{(\ell)}) + 1, \quad 1 \leqslant i_1 < i_2 \leqslant N, \end{aligned} \tag{42}$$

where

$$\overline{\alpha}_{i_1,i_2;k,\ell} \equiv \sum_{\substack{j \in J_{i_1,k} \\ j \in J_{i_2,\ell}}} \frac{\alpha_j \overline{\Omega}_j}{\Omega_{i_1}(\xi_{i_1}^{(k)}) \Omega_{i_2}(\xi_{i_2}^{(\ell)})},$$

$$\begin{aligned} & J_{i_1,k} \equiv t\{j_{k,1}, j_{k,2}, \ldots, j_{k,r_{i_1,k}}\}, \quad J_{i_2,\ell} \equiv \{j_{\ell,1}, j_{\ell,2}, \ldots, j_{\ell,r_{i_2,\ell}}\}, \\ & \quad 1 \leqslant k \leqslant m_{i_1}, \quad 1 \leqslant \ell \leqslant m_{i_2}, \quad 1 \leqslant i_1 < i_2 \leqslant N. \end{aligned} \tag{43}$$

Using these components the univariant GHDMR equation given in (27) can be rewritten as follows for rather general cases:

$$\begin{aligned} & \sum_{k=1}^{m_i} \beta_{i,k} \delta(x_i - \xi_i^{(k)}) \\ & \quad = \sum_{k=1}^{m_i} \overline{\alpha}_{i,k} \delta(x_i - \xi_i^{(k)})(f_0 + f_i(\xi_i^{(k)})) + \sum_{k=1}^{m_i} \delta(x_i - \xi_i^{(k)}) \sum_{i_1=1}^{i-1} \sum_{\ell=1}^{m_{i_1}} \overline{\alpha}_{i_1,i;\ell,k} \Omega_{i_1}(\xi_{i_1}^{(l)}) f_{i_1}(\xi_{i_1}^{(l)}) \\ & \qquad + \sum_{k=1}^{m_i} \delta(x_i - \xi_i^{(k)}) \sum_{i_1=i+1}^{N} \sum_{\ell=1}^{m_{i_1}} \overline{\alpha}_{i,i_1;k,\ell} \Omega_{i_1}(\xi_{i_1}^{(l)}) f_{i_1}(\xi_{i_1}^{(l)}), \quad 1 \leqslant i \leqslant N, \end{aligned} \tag{44}$$

which implies that

$$\beta_{i,k} = \overline{\alpha}_{i,k}(f_0 + f_i(\xi_i^{(k)})) + \sum_{i_1=1}^{i-1}\sum_{\ell=1}^{m_{i_1}} \overline{\alpha}_{i_1,i;\ell,k}\Omega_{i_1}(\xi_{i_1}^{(l)}) f_{i_1}(\xi_{i_1}^{(l)})$$
$$+ \sum_{i_1=i+1}^{N}\sum_{\ell=1}^{m_{i_1}} \overline{\alpha}_{i,i_1;k,\ell}\Omega_{i_1}(\xi_{i_1}^{(l)}) f_{i_1}(\xi_{i_1}^{(l)}), \quad 1\leqslant k\leqslant m_i, \quad 1\leqslant i\leqslant N, \tag{45}$$

where

$$\beta_{i,k} \equiv \sum_{j\in J_{i,k}} \frac{\alpha_j \overline{\Omega}_j}{\Omega_i(x_i^{(j)})}\varphi_j. \tag{46}$$

This is $m_1 + \cdots + m_N$ linear equations whose unknowns are the univariate component values at the data given points of the N-dimensional space. The univariate component values satisfying the above mentioned linear equation system can be obtained by the help of the orthogonality condition given in (22). This condition can be rewritten for the univariate components as follows:

$$\int_{a_i}^{b_i} \mathrm{d}x_i \Omega_i(x_i)(1 + W_i(x_i)) f_i(x_i) = 0, \quad 1\leqslant i\leqslant N. \tag{47}$$

Finally, by using the above equation and the linear equations given in (45), the unknowns of the equation system can be found. This completes the construction of the univariate components at the given data points.

4. Interpolation

The obtained partitioned data is used to fit analytical structures of the univariate and the bivariate terms of the HDMR expansion or it is used to fit analytical structures of the univariate terms of the GHDMR expansion. By this way, multivariate interpolation, at least for these functions, can be reduced to univariate and bivariate interpolations. To determine overall structure of the function, an analytical structure should be defined or a calculation rule is needed. If the function to be determined by HDMR (or GHDMR) is smooth, then the function can be represented with the multinomial of all independent variables belonging to the continuous region. For this purpose, a polynomial for $f_m(x_m)$ should be built.

$$p_m(x_m) = \sum_{k_m=1}^{n_m} L_{k_m}(x_m) f_m(\xi_m^{(k_m)}), \quad \xi_m^{(k_m)} \in \kappa_m, \quad 1\leqslant m\leqslant N. \tag{48}$$

Here, $L_{k_m}(x_m)$s are Lagrange polynomials [2] which are independent from the structure of the function. The structure of these polynomials is given below

$$L_{k_m}(x_m) \equiv \frac{(x_m - \xi_m^{(1)})\cdots(x_m - \xi_m^{(k_m-1)})(x_m - \xi_m^{(k_m+1)})\cdots(x_m - \xi_m^{(n_m)})}{(\xi_m^{(k_m)} - \xi_m^{(1)})\cdots(\xi_m^{(k_m)} - \xi_m^{(k_m-1)})(\xi_m^{(k_m)} - \xi_m^{(k_m+1)})\ldots(\xi_m^{(k_m)} - \xi_m^{(n_m)})}$$
$$\xi_m^{(k_m)} \in \kappa_m, \quad 1\leqslant k_m\leqslant n_m, \quad 1\leqslant m\leqslant N. \tag{49}$$

As Lagrange polynomials are determined, univariate functions given in relation (48) are obtained. These functions are the ones used to appropriate multivariate functions at most univariate level by HDMR (or GHDMR). The expansion formed by the summation of these functions and the constant term provides an approximation of functions defined with the interpolation points.

This representation can be shown by the following formula:

$$f(x_1, \ldots, x_N) \approx f_0 + \sum_{m=1}^{N} p_m(x_m). \tag{50}$$

When a table of data for the bivariate function, $f_{m_1 m_2}(x_{m_1}, x_{m_2})$ is produced, to determine overall structure of the function a multinomial should be built.

$$\begin{aligned} p_{m_1 m_2}(x_{m_1}, x_{m_2}) = & \sum_{k_{m_1}=1}^{n_{m_1}} \sum_{k_{m_2}=1}^{n_{m_2}} L_{k_{m_1}}(x_{m_1}) L_{k_{m_2}}(x_{m_2}) f_{m_1 m_2}(\xi_{m_1}^{(k_{m_1})}, \xi_{m_2}^{(k_{m_2})}), \\ & \xi_{m_1}^{(k_{m_1})} \in \kappa_{m_1}, \quad \xi_{m_2}^{(k_{m_2})} \in \kappa_{m_2}, \quad 1 \leqslant m_1, m_2 \leqslant N. \end{aligned} \tag{51}$$

As these multinomials are determined, the approximate representation can be shown by the following formula:

$$f(x_1, \ldots, x_N) \approx f_0 + \sum_{m=1}^{N} p_m(x_m) + \sum_{\substack{m_1, m_2=1 \\ m_1 < m_2}}^{N} p_{m_1, m_2}(x_{m_1}, x_{m_2}). \tag{52}$$

5. Factorized High Dimensional Model Representation

The factorized form of HDMR can be obtained by using the following equation of the FHDMR expansion for a given multivariate function, $f(x_1, \ldots, x_N)$.

$$\begin{aligned} f(x_1, \ldots, x_N) = r_0 & \left[\prod_{i_1=1}^{N} (1 + r_{i_1}(x_{i_1})) \right] \left[\prod_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{N} (1 + r_{i_1 i_2}(x_{i_1}, x_{i_2})) \right] \\ & \times \cdots \times [(1 + r_{12\ldots N}(x_1, \ldots, x_N))]. \end{aligned} \tag{53}$$

The right hand side components of the above relation can be determined by making comparisons between the above relation and the relation given in (1) for HDMR. To make the comparisons, idempotent operators will be used as auxiliary tools. The properties of these operators are as follows:

$$\theta_j^{(id)} \theta_k^{(id)} \equiv \theta_k^{(id)} \theta_j^{(id)}, \quad [\theta_j^{(id)}]^2 \equiv \theta_j^{(id)}, \quad j, k = 1, \ldots, N, \tag{54}$$

HDMR and FHDMR expansions can be rewritten by using these operators and new relations are obtained as

$$S(x_1, \ldots, x_N) \equiv f_0 I + \sum_{i_1=1}^{N} f_{i_1}(x_{i_1}) \theta_{i_1}^{(id)} + \sum_{\substack{i_1, i_2=1 \\ i_1 < i_2}}^{N} f_{i_1 i_2}(x_{i_1}, x_{i_2}) \theta_{i_1}^{(id)} \theta_{i_2}^{(id)} + \cdots. \tag{55}$$

$$R(x_1,\ldots,x_N) \equiv r_0 \left[\prod_{i_1=1}^{N} (I + r_{i_1}(x_{i_1})\theta_{i_1}^{(id)})\right] \times \left[\prod_{\substack{i_1,i_2=1\\ i_1<i_2}}^{N} (I + r_{i_1 i_2}(x_{i_1},x_{i_2})\theta_{i_1}^{(id)}\theta_{i_2}^{(id)})\right] \times \cdots. \tag{56}$$

These two entities represent the same multivariate function. So, the right hand sides of these two relations must match for all idempotent operators. The constant term, the univariate terms and higher order terms of the FHDMR expansion can be determined by using this comparison.

Making comparison between these two relations through the identity operator the constant term of the FHDMR expansion can be found as

$$r_0 = f_0. \tag{57}$$

Using the operators, $\theta_{i_1}^{(id)}$, the structure of the univariate terms of the FHDMR expansion can be obtained.

$$r_{i_1}(x_{i_1}) = \frac{f_{i_1}(x_{i_1})}{f_0}, \quad 1 \leqslant i_1 \leqslant N. \tag{58}$$

The structure of the bivariate terms of the FHDMR expansion is obtained by using the operators, $\theta_{i_1}^{(id)}\theta_{i_2}^{(id)}$, as follows:

$$r_{i_1 i_2}(x_{i_1},x_{i_2}) = \frac{f_0 f_{i_1 i_2}(x_{i_1},x_{i_2}) - f_{i_1}(x_{i_1}) f_{i_2}(x_{i_2})}{(f_0 + f_{i_1}(x_{i_1}))(f_0 + f_{i_2}(x_{i_2}))}, \quad 1 \leqslant i_1 \leqslant i_2 \leqslant N. \tag{59}$$

The functions f_0, f_{i_1} and $f_{i_1 i_2}$ may be the components of either HDMR or GHDMR. The use of HDMR or GHDMR depends on the structure of the given data set. Somehow, when these components are used in the FHDMR expansion as a truncated expression then an approximate analytical structure is obtained for representing the multivariate function.

Two different expansions for an analytical structure of a multivariate function can be constructed when a multivariate data set is given. The HDMR or GHDMR and the FHDMR expansions can be written finally. But when the given data has neither an additive nor a multiplicative nature then these expansions do not individually give the best results. A new representation method is needed. This method will have a hybrid structure that includes all these two expansions.

6. Hybrid High Dimensional Model Representation

In this section a new algorithm is given for the high dimensional model representations to be used in multivariate interpolation problems. This new method can be used when the given multivariate data have an intermediate structure that means the sought multivariate function has neither additive nor multiplicative nature. For this purpose, this new representation includes the HDMR (or GHDMR) and the FHDMR expansions in its structure through a hybridity parameter. This expansion is written as

$$f(x_1,\ldots,x_N) = \gamma\left(f_0 + \sum_{i_1=1}^{N} f_{i_1}(x_{i_1}) + \cdots\right) + (1-\gamma)\left(r_0\left[\prod_{i_1=1}^{N}(1 + r_{i_1}(x_{i_1}))\right] \times \cdots\right), \tag{60}$$

where γ is the hybridity parameter. The terms f_0, f_{i_1}, and so on are the components of either HDMR or GHDMR. If the given data are from a cartesian product set in the space of independent variables then HDMR is used. On the other hand, if a random discrete data is given GHDMR is used. So, these components may be obtained by using one of these two algorithms depending on the structure of the given data set in the problems. The terms r_0, r_{i_1}, and so on are the components of the FHDMR expansion. These components are obtained by using the HDMR (or GHDMR) components.

Using the equation given in (60) an HHDMR approximant can be defined as

$$f(x_1,\ldots,x_N) \approx h_{jk}(x_1,\ldots,x_N;\gamma) \equiv \gamma S_j(x_1,\ldots,x_N) + (1-\gamma)P_k(x_1,\ldots,x_N), \tag{61}$$

where

$$\begin{aligned}
S_0(x_1,\ldots,x_N) &= f_0,\\
S_1(x_1,\ldots,x_N) &= S_0(x_1,\ldots,x_N) + \sum_{i=1}^{N} f_i(x_i),\\
&\vdots\\
S_k(x_1,\ldots,x_N) &= S_{k-1}(x_1,\ldots,x_N) + \sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\cdots<i_k}}^{N} f_{i_1\ldots i_k}(x_{i_1},\ldots,x_{i_k}), \quad 1\leqslant k\leqslant N
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
P_0(x_1,\ldots,x_N) &= r_0,\\
P_1(x_1,\ldots,x_N) &= P_0(x_1,\ldots,x_N)\prod_{i=1}^{N}(1+r_i(x_i))\\
&\vdots\\
P_k(x_1,\ldots,x_N) &= P_{k-1}(x_1,\ldots,x_N)\prod_{\substack{i_1,\ldots,i_k=1\\ i_1<\cdots<i_k}}^{N}(1+r_{i_1\ldots i_k}(x_{i_1},\ldots,x_{i_k})), \quad 1\leqslant k\leqslant N.
\end{aligned} \tag{63}$$

This approximant is called as (jk)th order HHDMR approximant. It is composed of the HDMR (or GHDMR) approximants, S_j, and the FDHMR approximants, P_k. To list these approximants a table like that of Padé Approximants can be formed.

$$\begin{matrix}
h_{00} & h_{01} & \cdots & h_{0N}\\
h_{10} & h_{11} & \cdots & h_{1N}\\
\vdots & \vdots & \cdots & \vdots\\
h_{N0} & h_{N1} & \cdots & h_{NN}
\end{matrix} \tag{64}$$

The most important step here is to determine the hybridity parameter, γ. For this purpose, a functional is defined as

$$F(x_1,\ldots,x_N;\gamma) \equiv \| f_{\mathrm{org}}(x_1,\ldots,x_N) - f_{\mathrm{HHDMR}}(x_1,\ldots,x_N;\gamma)\|^2, \tag{65}$$

where $f_{\text{org}}(x_1, \ldots, x_N)$ and $f_{\text{HHDMR}}(x_1, \ldots, x_N)$ stand for the original function and the function obtained from an HHDMR approximant, respectively. We need to obtain the γ value that minimizes the value of this norm. This minimization criteria can be written as

$$\frac{\partial F}{\partial \gamma} = 0. \tag{66}$$

By this way the best representation for the sought multivariate function can be determined via HHDMR. To evaluate this norm we need a weight function. At this point the structure of the weight function depends on the algorithm we use. It may be either HDMR algorithm or GHDMR algorithm. If the given problem needs HDMR algorithm then the weight function given in (9) is used. Taking this weight function into consideration the result of the above norm is obtained as

$$\begin{aligned} F(x_1, \ldots, x_N; \gamma) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) & [f_{\text{org}}(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) \\ & - \gamma S_j(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) - (1-\gamma) P_k(x_1^{(j_1)}, \ldots, x_N^{(j_N)})]^2, \quad 1 \leqslant j, k \leqslant N. \end{aligned} \tag{67}$$

When we take the partial differentiation of this result over γ and set it equal to zero, the following result is obtained:

$$\gamma = \frac{A_2 + A_3 - A_4 - A_5}{A_1 + A_2 - 2A_5}, \tag{68}$$

where

$$\begin{aligned} A_1 &= \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) S_j(x_1^{(j_1)}, \ldots, x_N^{(j_N)})^2, \\ A_2 &= \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) P_k(x_1^{(j_1)}, \ldots, x_N^{(j_N)})^2, \\ A_3 &= \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) f_{\text{org}}(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) S_j(x_1^{(j_1)}, \ldots, x_N^{(j_N)}), \\ A_4 &= \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) f_{\text{org}}(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) P_k(x_1^{(j_1)}, \ldots, x_N^{(j_N)}), \\ A_5 &= \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) S_j(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) P_k(x_1^{(j_1)}, \ldots, x_N^{(j_N)}). \end{aligned} \tag{69}$$

On the other hand if the GHDMR algorithm is used then the weight function is chosen as given in (30). Additionally, an auxiliary weight function is needed in GHDMR in the product type of structure that is given in (19). In this work, this product type of auxiliary weight function is chosen as

$$\Omega(x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{1}{b_i - a_i}, \tag{70}$$

where

$$a_i = \min(\{x_i^{(j_i)}\}), \quad b_i = \max(\{x_i^{(j_i)}\}), \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant j_i \leqslant m_i. \tag{71}$$

The result of the norm given in (65) is obtained under these weight and auxiliary weight functions as

$$\begin{aligned} &F(x_1, \ldots, x_N; \gamma) \\ &\quad = \left(\prod_{i=1}^{N} \frac{1}{b_i - a_i} \right) \sum_{\ell=1}^{m} \alpha_\ell [f_{\text{org}}(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) \\ &\qquad - \gamma S_j(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) - (1-\gamma) P_k(x_1^{(\ell)}, \ldots, x_N^{(\ell)})]^2, \quad 1 \leqslant j, k \leqslant N. \end{aligned} \tag{72}$$

When the minimization criteria over γ given in (66) is used the optimum value for γ is obtained as

$$\gamma = \frac{A_2 + A_3 - A_4 - A_5}{A_1 + A_2 - 2A_5}, \tag{73}$$

where

$$\begin{aligned} A_1 &= \sum_{\ell=1}^{m} \alpha_\ell S_j(x_1^{(\ell)}, \ldots, x_N^{(\ell)})^2, \\ A_2 &= \sum_{\ell=1}^{m} \alpha_\ell P_k(x_1^{(\ell)}, \ldots, x_N^{(\ell)})^2, \\ A_3 &= \sum_{\ell=1}^{m} \alpha_\ell f_{\text{org}}(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) S_j(x_1^{(\ell)}, \ldots, x_N^{(\ell)}), \\ A_4 &= \sum_{\ell=1}^{m} \alpha_\ell f_{\text{org}}(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) P_k(x_1^{(\ell)}, \ldots, x_N^{(\ell)}), \\ A_4 &= \sum_{\ell=1}^{m} \alpha_\ell S_j(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) P_k(x_1^{(\ell)}, \ldots, x_N^{(\ell)}). \end{aligned} \tag{74}$$

Using the obtained γ value for the HHDMR expansion constructed from either HDMR and FHDMR or GHDMR and FHDMR, the final HHDMR expansion for the sought multivariate function is determined within truncation approximation.

In this work, for HDMR, the constant, univariate and bivariate terms are tried to be obtained. On the other hand, for GHDMR, the constant and the univariate terms are obtained for simplicity. Hence, only the corresponding FHDMR components can be evaluated. So, we can construct a table like

$$\begin{matrix} h_{00}(x_1, \ldots, x_N), & h_{01}(x_1, \ldots, x_N), & h_{02}(x_1, \ldots, x_N), \\ h_{10}(x_1, \ldots, x_N), & h_{11}(x_1, \ldots, x_N), & h_{12}(x_1, \ldots, x_N), \\ h_{20}(x_1, \ldots, x_N), & h_{21}(x_1, \ldots, x_N), & h_{22}(x_1, \ldots, x_N), \end{matrix} \tag{75}$$

when we use HDMR and FHDMR in the HHDMR expansion. If we use GHDMR and FHDMR in the expansion then the following table is obtained.

$$\begin{matrix} h_{00}(x_1, \ldots, x_N), & h_{01}(x_1, \ldots, x_N), \\ h_{10}(x_1, \ldots, x_N), & h_{11}(x_1, \ldots, x_N). \end{matrix} \tag{76}$$

7. Best Representation Determination

According to the above mentioned methods, HDMR or GHDMR, FHDMR and HHDMR, several representations can be obtained approximately by using the constant, univariate and bivariate terms of the mentioned truncated expansions like S_0, S_1, S_2, P_0, P_1, P_2, h_{00}, h_{01}, h_{02}, h_{10}, h_{11}, h_{12}, h_{20}, h_{21}, h_{22}.

As obtaining these several representations there exists a new question, that is, how to find the best approximate representation for the sought multivariate function. For this purpose, the norm

$$\mathcal{N} = \| f_{\mathrm{org}}(x_1, \ldots, x_N) - f_{\mathrm{new}}(x_1, \ldots, x_N) \| \tag{77}$$

will be evaluated. Here, $f_{\mathrm{new}}(x_1, \ldots, x_N)$ stands for the multivariate function obtained via a high dimensional model representation expansion. The determination of this norm depends on the method that is used. If HDMR expansion is used in the method then the following result is obtained:

$$\begin{aligned}\mathcal{N} &= \| f_{\mathrm{org}}(x_1, \ldots, x_N) - f_{\mathrm{new}}(x_1, \ldots, x_N) \| \\ &= \left[\sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{i=1}^{N} \alpha_{j_i}^{(i)} \right) \times [f_{\mathrm{org}}(x_1^{(j_1)}, \ldots, x_N^{(j_N)}) - f_{\mathrm{new}}(x_1^{(j_1)}, \ldots, x_N^{(j_N)})]^2 \right]^{1/2}.\end{aligned} \tag{78}$$

This relation is used in a problem when the HDMR expansion or the FHDMR expansion including HDMR or the HHDMR expansion including HDMR and FHDMR is chosen. On the other hand if GHDMR expansion is used in the method then the result is obtained as

$$\begin{aligned}\mathcal{N} &= \| f_{\mathrm{org}}(x_1, \ldots, x_N) - f_{\mathrm{new}}(x_1, \ldots, x_N) \| \\ &= \left[\left(\prod_{i=1}^{N} \frac{1}{b_i - a_i} \right) \times \sum_{\ell=1}^{m} \alpha_\ell [f_{\mathrm{org}}(x_1^{(\ell)}, \ldots, x_N^{(\ell)}) - f_{\mathrm{new}}(x_1^{(\ell)}, \ldots, x_N^{(\ell)})]^2 \right]^{1/2}\end{aligned} \tag{79}$$

and this relation is used when the high dimensional model representation method includes GHDMR expansion.

The minimum norm value obtained either in the first type (in which the relation given in (78) is used) or in the second type (in which the relation given in (79) is used) through all the evaluated norm values will show the best representation for the sought multivariate function. This result is assumed to be the best representation for the multivariate function.

8. Implementations

In this section there exist two main parts. In the first part the examples in which the given multivariate data set is a cartesian product set in the space of independent variables are given. In the second part of this section the examples have multivariate data sets whose nodes are the elements of random discrete data. The data sets given in the examples are constructed from the known functions for testing.

8.1. First type HHDMR implementations

The elements of the cartesian product set is constructed through the multivariate function

$$f(x_1, x_2, x_3, x_4, x_5) = 3^{(x_1+x_2+x_3+x_4+x_5)} + (x_1 + x_2 + x_3 + x_4 + x_5) \tag{80}$$

which has five independent variables. This function does not have neither an exactly additive nature nor an exactly multiplicative nature. So, it can be said that the HHDMR approximant is the best high dimensional model representation for this function. This best approximant will be h_{11} in this work. Whenever the higher level approximants are evaluated the efficiency of the HHDMR approximants increases.

A cartesian product set is constructed by using this function. The elements of the space of each independent variable are chosen as

$$\begin{aligned}
\xi_1 &= \{0.242, 0.387, 0.451, 0.473\},\\
\xi_2 &= \{0.056, 0.082, 0.122, 0.291, 0.302\},\\
\xi_3 &= \{0.219, 0.345\},\\
\xi_4 &= \{0.032, 0.045, 0.134, 0.256\},\\
\xi_5 &= \{0.139, 0.118, 0.223, 0.389, 0.405\},
\end{aligned} \tag{81}$$

where ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 stand for the points of the independent variables, x_1, x_2, x_3, x_4 and x_5 respectively. As a result, a cartesian product set is obtained with 800 nodes. We try to represent the sought multivariate function by using the elements of this cartesian product set via HHDMR approximants including the HDMR and the FHDMR expansions.

In this subsection we use the constant, univariate and bivariate terms of the HDMR and the FHDMR expansions and we try to determine the structures of the HHDMR approximants related to these terms as given in (75). These structures are not given here because of their lengths. However, the norm values of these structures are as

$$\begin{aligned}
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - S_1(x_1, \ldots, x_N)\| &= 0.13524499453977703533,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - S_2(x_1, \ldots, x_N)\| &= 0.01364771657059033 3214,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - P_1(x_1, \ldots, x_N)\| &= 0.02149689711443302713,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - P_2(x_1, \ldots, x_N)\| &= 0.00494390805395597012 79,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{01}(x_1, \ldots, x_N)\| &= 0.02129544761833787891 1,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{02}(x_1, \ldots, x_N)\| &= 0.00494382900012345308 08,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{10}(x_1, \ldots, x_N)\| &= 0.13524499453977703533,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{11}(x_1, \ldots, x_N)\| &= 0.00146326838505963439 9,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{12}(x_1, \ldots, x_N)\| &= 0.00493824466793513770 83,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{20}(x_1, \ldots, x_N)\| &= 0.01364771657059033321 4,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{21}(x_1, \ldots, x_N)\| &= 0.01121670490533706852 9,\\
\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{22}(x_1, \ldots, x_N)\| &= 0.00069521416178464045 601.
\end{aligned} \tag{82}$$

These results do not include h_{00} in which only the constant terms of the HDMR and the FHDMR expansions are used. Because this approximant cannot represent a multivariate function.

As it is estimated the best result is obtained via an HHDMR approximant, h_{22}. This approximant includes the constant, univariate and the bivariate terms of HDMR and FHDMR. This representation

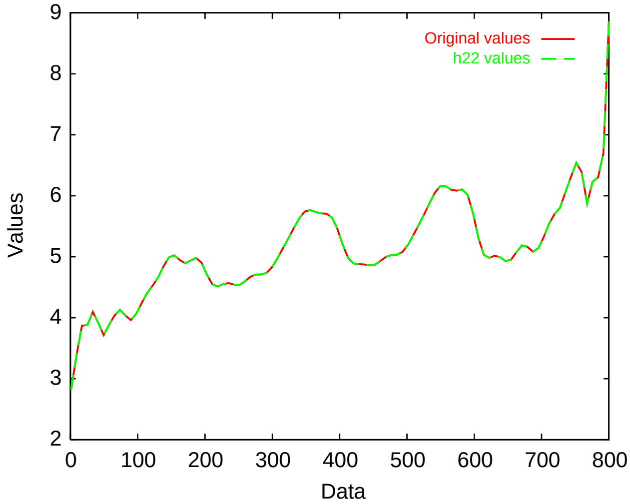

Fig. 1. Comparison between $f_{\mathrm{org}}(x_1, \ldots, x_N)$ and $h_{22}(x_1, \ldots, x_N)$.

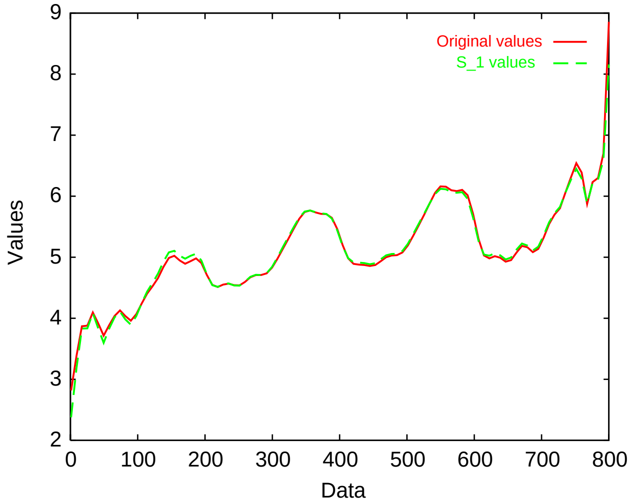

Fig. 2. Comparison between $f_{\mathrm{org}}(x_1, \ldots, x_N)$ and $S_1(x_1, \ldots, x_N)$.

obtained from the given multivariate data is the best approximation to the sought multivariate function. In Fig. 1 the comparison between the original function and the function obtained via this HHDMR approximant is shown. This comparison is done between the values of the original function and $h_{22}(x_1, \ldots, x_N)$ at the given nodes.

Also, the graph for the worst representation is given in Fig. 2.

Another example can be given for the following multivariate function:

$$f(x_1, x_2, x_3, x_4, x_5) = 3^{(x_1+x_2+x_3+x_4+x_5)} + (x_1 + x_2 + x_3 + x_4 + x_5)^5 \tag{83}$$

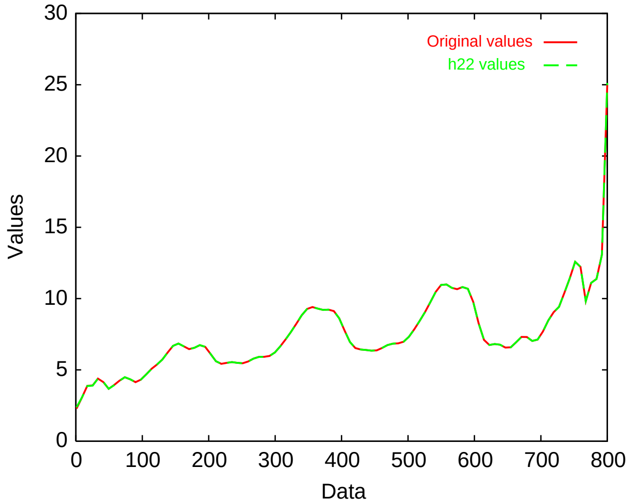

Fig. 3. Comparison between $f_{\mathrm{org}}(x_1,\ldots,x_N)$ and $h_{22}(x_1,\ldots,x_N)$.

and the elements of the cartesian product set can be constructed from this function again according to the points given in (81). The multiplicative nature of this function is more powerful than the first one.

The norm values obtained for the high dimensional model representations by using the elements of the given cartesian product set are as

$$\begin{aligned}
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-S_1(x_1,\ldots,x_N)\| &= 1.279622230216363416,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-S_2(x_1,\ldots,x_N)\| &= 0.25655195337274256609,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-P_1(x_1,\ldots,x_N)\| &= 0.1208121815594374236,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-P_2(x_1,\ldots,x_N)\| &= 0.071503347228574939441,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{01}(x_1,\ldots,x_N)\| &= 0.11551890836967593447,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{02}(x_1,\ldots,x_N)\| &= 0.071234334554449252377,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{10}(x_1,\ldots,x_N)\| &= 1.279622230216363416,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{11}(x_1,\ldots,x_N)\| &= 0.028330471813608913224,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{12}(x_1,\ldots,x_N)\| &= 0.069029107823516765199,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{20}(x_1,\ldots,x_N)\| &= 0.25655195337274256609,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{21}(x_1,\ldots,x_N)\| &= 0.11330787977165316297,\\
\|f_{\mathrm{org}}(x_1,\ldots,x_N)-h_{22}(x_1,\ldots,x_N)\| &= 0.022688420249987844076.
\end{aligned} \tag{84}$$

According to these results the graph for the best representation is obtained as in Fig. 3 and the graph for the worst representation is obtained as in Fig. 4.

To test the method presented here for the multivariate functions having larger numbers of independent variables following multivariate function is chosen

$$f(x_1,\ldots,x_{10}) = 3^{\left(\sum_{i=1}^{10} x_i\right)} + \sum_{i=1}^{10} x_i, \tag{85}$$

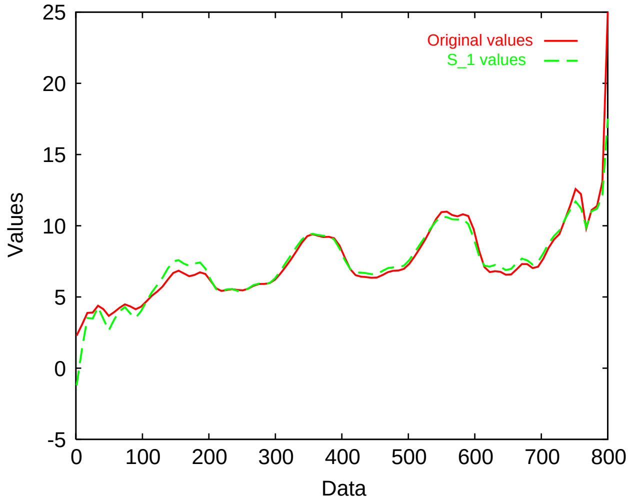

Fig. 4. Comparison between $f_{\mathrm{org}}(x_1, \ldots, x_N)$ and $S_1(x_1, \ldots, x_N)$.

which has 10 independent variables and the elements of the space of each independent variable are as

$$
\begin{aligned}
\xi_1 &= \{0.242, 0.387, 0.451, 0.473\},\\
\xi_2 &= \{0.045, 0.072\},\\
\xi_3 &= \{0.219, 0.345\},\\
\xi_4 &= \{0.032, 0.045, 0.134, 0.256\},\\
\xi_5 &= \{0.028, 0.105\},\\
\xi_6 &= \{0.147, 0.301\},\\
\xi_7 &= \{0.067, 0.1, 0.156, 0.205\},\\
\xi_8 &= \{0.391, 0.412, 0.504, 0.597\},\\
\xi_9 &= \{0.2, 0.3\},\\
\xi_{10} &= \{0.02, 0.03\},
\end{aligned}
\tag{86}
$$

where ξ sets stand for the points of the independent variables of the selected function given in (85). Using these points a cartesian product set having 16384 nodes is constructed. Nature of this example is very similar to the first example of this subsection. This time, number of independent variables, thus the number of nodes appearing in the cartesian product set, is quite greater than the first example.

When we represent the sought multivariate function by using this cartesian product set via HHDMR method, following norm values are obtained:

$$
\begin{aligned}
\| f_{\text{org}}(x_1,\ldots,x_N) - S_1(x_1,\ldots,x_N)\| &= 0.29526876087691317882,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - S_2(x_1,\ldots,x_N)\| &= 0.03115760100782516281 3,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - P_1(x_1,\ldots,x_N)\| &= 0.0045368835831433528207,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - P_2(x_1,\ldots,x_N)\| &= 0.0024944809740598579755,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{01}(x_1,\ldots,x_N)\| &= 0.0045185481593857157829,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{02}(x_1,\ldots,x_N)\| &= 0.0024942864956462731098,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{10}(x_1,\ldots,x_N)\| &= 0.29526876087691317882,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{11}(x_1,\ldots,x_N)\| &= 0.0032362299190889442295,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{12}(x_1,\ldots,x_N)\| &= 0.0024828436593431108486,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{20}(x_1,\ldots,x_N)\| &= 0.031157601007825162813,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{21}(x_1,\ldots,x_N)\| &= 0.0034424195018858646235,\\
\| f_{\text{org}}(x_1,\ldots,x_N) - h_{22}(x_1,\ldots,x_N)\| &= 0.00022271477719217785418.
\end{aligned} \tag{87}
$$

If we compare the norm values above and the ones given in (82), then we see that the convergence rate of the HHDMR method becomes better when the number of independent variables of the sought function increases. This result was observed via several number of other numerical tests done during the preparation of this paper. This is because of the nature of the method. The purpose of the method is to obtain better representations for the multivariate functions having large number of independent variables. However, of course, increase in the number of independent variables causes an increase in the number of nodes appearing in the cartesian product set and an increase in the CPU time needed for the computer programs written to obtain numerical results.

8.2. *Second type HHDMR implementations*

In this part we try to construct a multivariate function by using the given random data. This random data set is constructed via a MuPAD [5] program and the values of the multivariate function at those points are evaluated by using a known function to test the efficiency of the method.

In this implementation the elements of the space of each independent variable are chosen in the intervals of

$$
0.1 \leqslant \xi_1 \leqslant 0.5, \quad 0.3 \leqslant \xi_2 \leqslant 0.7, \quad 0.5 \leqslant \xi_3 \leqslant 0.8, \quad 0.6 \leqslant \xi_4 \leqslant 0.9, \quad 0.2 \leqslant \xi_5 \leqslant 0.6, \tag{88}
$$

where ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5 stand for the points of the independent variables, x_1, x_2, x_3, x_4 and x_5 respectively. There are 400 nodes in this random data set.

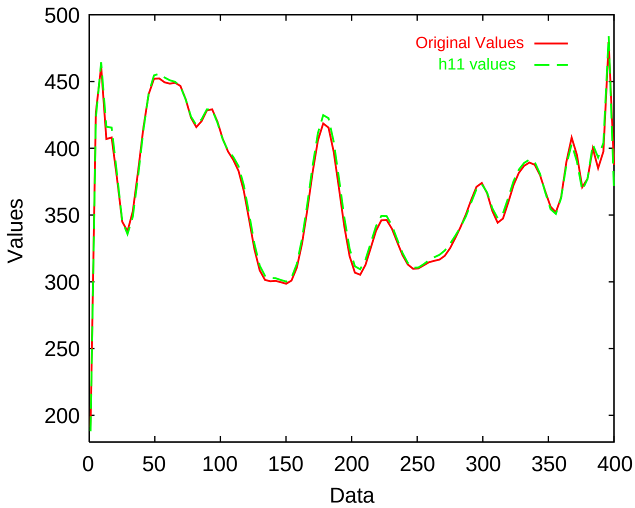

Fig. 5. Comparison between $f_{\text{org}}(x_1, \ldots, x_N)$ and $h_{11}(x_1, \ldots, x_N)$.

The following multivariate function is used for this numerical implementation:

$$f(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_3 + x_4 + x_5)^6. \tag{89}$$

Using this given information the norm values are obtained for the high dimensional model representations as follows.

$$\begin{aligned}
&\| f_{\text{org}}(x_1, \ldots, x_N) - S_1(x_1, \ldots, x_N)\| = 80.523419697190321995,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - P_1(x_1, \ldots, x_N)\| = 30.077784150567727218,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{01}(x_1, \ldots, x_N)\| = 27.038305263583560 32,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{10}(x_1, \ldots, x_N)\| = 80.523419697190321995,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{11}(x_1, \ldots, x_N)\| = 19.305513430561362063.
\end{aligned} \tag{90}$$

The nature of the given multivariate function is neither additive nor multiplicative. So we need to use both GHDMR and FHDMR for this function. According to these results it is easily seen that the best result is obtained for h_{11}. This is also seen in the given graphs. Fig. 5 shows that the best result is obtained when HHDMR expansion is used. The convergence of the other representations of which GHDMR and FHDMR are seen in Figs. 6 and 7.

The method works well when the number of independent variables of the sought function increase and this result is shown in the previous subsection. However, the main purpose of the GHDMR method is to use less number of nodes. All nodes appearing in the given cartesian product set are not used, instead certain nodes which are selected randomly will be used to determine an approximate analytical structure for the sought multivariate function. Hence, the number of these randomly selected nodes affects the efficiency of the method. To show this effect, the multivariate function whose analytical structure given in (89) is used again. But, this time there are 800 nodes in the given set instead of 2000 nodes which

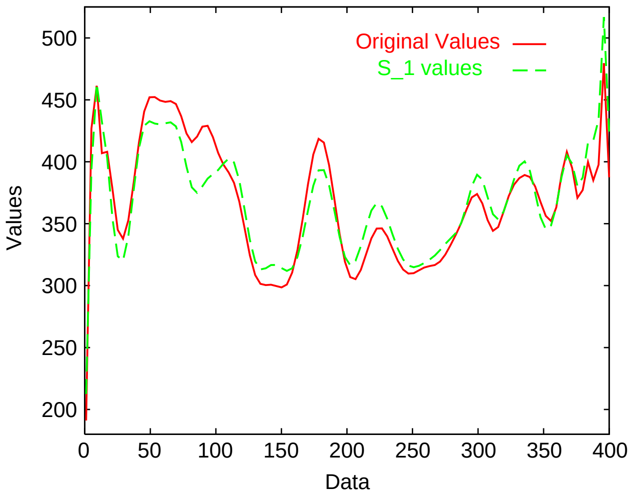

Fig. 6. Comparison between $f_{\mathrm{org}}(x_1, \ldots, x_N)$ and $S_1(x_1, \ldots, x_N)$.

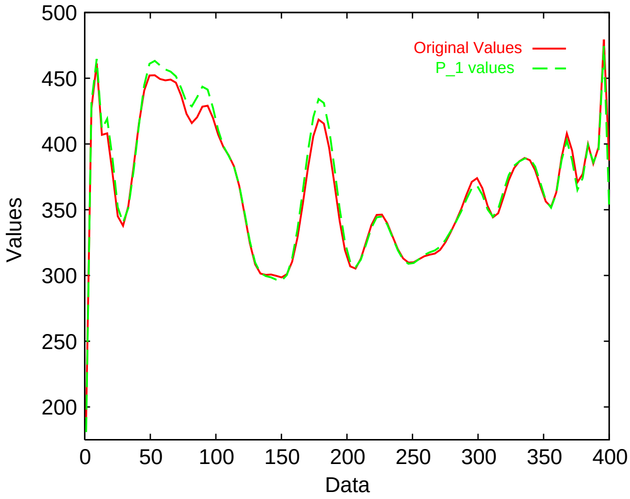

Fig. 7. Comparison between $f_{\mathrm{org}}(x_1, \ldots, x_N)$ and $P_1(x_1, \ldots, x_N)$.

correspond to the whole mesh. Norm values are obtained as follows for this case.

$$\begin{aligned}
&\| f_{\mathrm{org}}(x_1, \ldots, x_N) - S_1(x_1, \ldots, x_N)\| = 78.262873999161280334,\\
&\| f_{\mathrm{org}}(x_1, \ldots, x_N) - P_1(x_1, \ldots, x_N)\| = 28.025500630630085936,\\
&\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{01}(x_1, \ldots, x_N)\| = 24.738907968576798583,\\
&\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{10}(x_1, \ldots, x_N)\| = 78.262873999161280334,\\
&\| f_{\mathrm{org}}(x_1, \ldots, x_N) - h_{11}(x_1, \ldots, x_N)\| = 16.793878521419317341.
\end{aligned} \tag{91}$$

A last case can be given for the same multivariate function where there are 1600 nodes in the set. This time, following norm values are evaluated:

$$\begin{aligned}
&\| f_{\text{org}}(x_1, \ldots, x_N) - S_1(x_1, \ldots, x_N)\| = 74.141285261016663574,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - P_1(x_1, \ldots, x_N)\| = 21.134001763000061555,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{01}(x_1, \ldots, x_N)\| = 20.472502150402939149,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{10}(x_1, \ldots, x_N)\| = 74.141285261016663574,\\
&\| f_{\text{org}}(x_1, \ldots, x_N) - h_{11}(x_1, \ldots, x_N)\| = 11.793369068976120446.
\end{aligned} \tag{92}$$

It can be seen that increase in the number of nodes randomly selected from the whole mesh permits us to obtain better results through the HHDMR method. Only disadvantage here is to need more CPU time when larger numbers of nodes are selected to determine an approximate structure for the sought multivariate function. However, by this way better approximations to the sought functions are obtained.

9. Conclusion

When a data set is given and the analytical structure of the multivariate function which passes through the nodes of this data set is asked then an interpolation method is used to determine this structure. If the sought function is a multivariate function and the given data set is constructed from so many nodes then it may be difficult to obtain the resulting structure by using the existing interpolation methods. One way to reach the solution is to use a divide-and-conquer method like high dimensional model representation (HDMR). This method approximately partitions the multivariate data into less-variate data. After making tests for this method in data partitioning problems it is seen that the HDMR expansion can be used for the given data whose nodes are the elements of a cartesian product set. When a random discrete data set is given then it is needed a general method. This method is generalized HDMR (GHDMR). These two methods are efficient for the multivariate functions which have additive natures as seen in the numerical implementations given in the previous section. Another method is factorized HDMR (FHDMR) and this method is useful for the multiplicative type of functions.

When a data set is given and it has neither exactly additive nor exactly multiplicative nature then a new method is developed and is called hybrid HDMR. This method uses both the HDMR (or GHDMR) and the FHDMR expansions. In the implementations the representations obtained via HDMR, GHDMR, FHDMR and HHDMR are examined through the given nodes and it is seen that if the nature of the multivariate function is neither exactly additive nor exactly multiplicative then the HHDMR approximant gives the best results. This can be observed also in the given graphs.

Implementations given in the paper also show that the performance of the HHDMR method becomes better when the number of independent variables increases. This is a result of the structure of the HDMR which is one of the components of this new method. The other components GHDMR and FHDMR methods have also the same philosophy as HDMR. These methods were designed for the multivariate functions having large numbers of independent variables. Hence, better approximations for the sought functions are obtained through HHDMR method when the number of independent variable increase to large values. In HDMR based HHDMR applications all nodes of the given cartesian product set are used. We partition this multivariate data into at most bivariate data. Hence, Lagrange interpolation formula is used for only at most these bivariate data instead of the original multivariate data. As a result the computational complexity of the given multivariate interpolation problem and the CPU time needed for the computer programs reduce. On the other hand, the accuracy of the obtained approximations seems to be acceptable for the engineering problems if the norm values obtained for the numerical implementations are examined carefully. Hence, it can be said that there is no need to use the multivariate data through Lagrange interpolation formula (one N-dimensional interpolation problem) to determine an analytical structure for the sought function. Instead, the partitioned data can be used in the Lagrange interpolation formula. This means that there exist N number of one-dimensional interpolation problems when we use only the univariate data or, in addition to this, $N(N-1)/2$ number of two-dimensional interpolation problems when we use both univariate and bivariate data.

In GHDMR based HHDMR applications randomly selected data set is used. The nodes of this data set are selected from all nodes of the whole mesh randomly. At this point, behaviour of the accuracy of the approximations obtained through GHDMR based HHDMR algorithm must be examined when different numbers of nodes appearing in the given data set are tested in the same interpolation problem. Results obtained in Section 8.2 show that when larger number of nodes are taken into consideration, better approximations are obtained for the sought function. At this point, needed CPU time increases. One may decide either to have better approximation by using larger data or to have less CPU time by using less data. However, it can be said that the performance of the method is also acceptable for the interpolation problems that have less data. As we do not use all the nodes of the given mesh, Lagrange multivariate interpolation formula cannot be used for these types of data. In this work, first we partition the given random discrete data into univariate data by using the GHDMR method and then one-dimensional Lagrange interpolation formula is used and required expansions are obtained for the FHDMR method and finally for the HHDMR method. Perhaps the most unpleasnt aspect of GHDMR is the necessity of solving a set of linear equations whose coefficient matrix may have quite large condition number in certain circumstances. We will focus on this point and try to find a way to get rid of this necessity in our coming studies.

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