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On the Number of Solutions of $N - p = P_3$

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A lower bound of Richert on the number of solutions of $N - p = P_3$ is improved.

1. INTRODUCTION

Let N be a large positive even number, p be a prime, and let P_r denote an almost prime with at most r prime factors counted with multiplicity. Furthermore we set

$$C_N = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p|N} \frac{p-1}{p-2}.$$

In 1969, Richert [6, 8] proved

THEOREM 1. *There exists an absolute constant N_0 such that, if $N \geqslant N_0$, then*

$$|\{p : p < N, N - p = P_3\}| \geqslant \frac{13}{3} C_N \frac{N}{\log^2 N};$$

in particular, every sufficiently large even number N can be represented in the form

$$N = p + P_3.$$

Richert's proof is based on a logarithmic weighted sieve, Jurkat–Richert's theorem, and Bombieri's theorem. Qualitatively speaking, Chen's theorem [3] is better than Theorem 1. However, as a quantitative statement about the representation of $N - p$ as a P_3, Theorem 1 is still unsurpassed. In this paper we shall prove

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THEOREM 2. *There exists an absolute constant N_1 such that, if $N \geqslant N_1$, then*

$$|\{p : p < N, N - p = P_3\}| \geqslant 6.173\, C_N \frac{N}{\log^2 N}.$$

The proof of Theorem 2 depends on a theorem of Jurkat–Richert on the linear sieve, Bombieri-type mean value theorem, and Chen's idea of switching. It is interesting to observe that we are able to show the existence of P_3 in the sequence $\{N - p : p < N\}$ without any elaborate weighted sieve. With more calculations, it is possible to improve the constant 6.173. Of course, the conjugate result for the number of representations of $p + h = P_3$ $(p \leqslant N,\ 2 \mid h)$ can be proved in precisely the same way.

2. PROOF OF THEOREM 2

Let $\mathcal{A}$ be a finite sequence of integers, $\mathcal{P}$ be a set of primes and $|\mathcal{A}|$ the number of elements in $\mathcal{A}$. Furthermore, for a positive integer d, suppose that the quantity

$$|\mathcal{A}_d| = |\{a \in \mathcal{A};\, a \equiv 0 \pmod{d}\}|$$

may be written in the form

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + r_d \qquad (\mu(d) \neq 0),$$

where $\omega(d)$ is multiplicative with $0 \leqslant \omega(p) < p$, X is a large enough parameter independent of d, r_d is considered as an error term, and $\mu(d)$ is the Möbius function.

Finally, for a given $z \geqslant 2$, we let

$$P(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} p,$$

$$S(\mathcal{A}; \mathcal{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1,$$

and

$$W(z) = \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right).$$

We now state some well-known results.

LEMMA 1 (Selberg). *If $\omega(d)$ satisfies the conditions*

$$0 \leqslant \frac{\omega(p)}{p} < 1 - \frac{1}{A_1}, \tag{1}$$

$$-A_2 \leqslant \sum_{w \leqslant p \leqslant z} \frac{\omega(p)}{p} \log p - \log \frac{z}{w} \leqslant A_3, \qquad 2 \leqslant w \leqslant z, \tag{2}$$

for some suitable constants $A_i \geqslant 1$, $i = 1, 2, 3$, then

$$S(\mathcal{A}; \mathcal{P}, z) \leqslant XW(z)e^{\gamma}\left(1 + O\left(\frac{A_2}{\log z}\right)\right) + \sum_{\substack{d < z^2 \\ d \mid P(z)}} 3^{v(d)} |r_d|.$$

Here γ is Euler's constant and $v(d)$ is the number of distinct prime factors of d. For a proof of this lemma, see [6, Chap. 5].

LEMMA 2 (Jurkat–Richert). *Suppose $\omega(d)$ satisfies conditions* (1) *and* (2). *For $\xi \geqslant z$ we have*

$$S(\mathcal{A}; \mathcal{P}, z) \leqslant XW(z)\left\{F\left(\frac{\log \xi^2}{\log z}\right) + O\left(\frac{A_2}{(\log \xi)^{1/14}}\right)\right\} + R, \tag{3}$$

$$S(\mathcal{A}; \mathcal{P}, z) \geqslant XW(z)\left\{f\left(\frac{\log \xi^2}{\log z}\right) + O\left(\frac{A_2}{(\log \xi)^{1/14}}\right)\right\} - R, \tag{4}$$

where

$$R = \sum_{\substack{d < \xi^2 \\ d \mid P(z)}} 3^{v(d)} |r_d|,$$

the functions F and f are defined by $F(u) = 2e^{\gamma}/u$, $f(u) = 0$ for $0 < u \leqslant 2$, and $(uF(u))' = f(u-1)$, $(uf(u))' = F(u-1)$ for $u \geqslant 2$.

This lemma is also true if $1 < \xi < z$ but $z \ll \xi^{\lambda}$ with a positive constant λ, in which case the O constant in (3) depends also on λ. The proof of this lemma can be found in [6, Chap. 8].

It is well known that

$$F(u) = \frac{2e^{\gamma}}{u}, \qquad 0 < u \leqslant 3, \tag{5}$$

$$F(u) = \frac{2e^{\gamma}}{u}\left(1 + \int_2^{u-1} \log(t-1)\frac{dt}{t}\right), \qquad 3 \leqslant u \leqslant 5, \tag{6}$$

and

$$f(u) = \frac{2e^{\gamma}}{u} \log(u-1), \qquad 2 \leqslant u \leqslant 4. \tag{7}$$

LEMMA 3 (Bombieri). *Given any positive constant* A, *there exists a positive constant* $B_1 = B_1(A)$ *such that*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B_1}} \max_{y \leqslant x} \max_{(l,d)=1} \left| \pi(y; d, l) - \frac{\operatorname{li} y}{\varphi(d)} \right| \ll \frac{x}{(\log x)^A},$$

where $\varphi(d)$ *is the Euler function and* $\operatorname{li} y$ *is the logarithmic integral of* y,

$$\pi(y; d, l) = \sum_{\substack{p \leqslant y \\ p \equiv l (\bmod d)}} 1 \qquad \text{and} \qquad \operatorname{li} y = \int_2^y \frac{dt}{\log t}.$$

For a proof, see [1, 4]. Simplified proofs have been given by Gallagher and Vaughan.

LEMMA 4 (Ding and Pan). *Let*

$$\pi(y; a, d, l) = \sum_{\substack{ap \leqslant y \\ ap \equiv l (\bmod d)}} 1$$

and let $f(a)$ *be a real function,* $f(a) \ll 1$; *then, for any given* $A > 0$, *we have*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B_2}} \max_{y \leqslant x} \max_{(l,d)=1} \left| \sum_{\substack{a \leqslant x^{1-\varepsilon} \\ (a,d)=1}} f(a) \left(\pi(y; a, d, l) - \frac{\operatorname{li} y/a}{\varphi(d)} \right) \right| \ll \frac{x}{(\log x)^A},$$

where $B_2 = B_2(A) = \frac{3}{2}A + 17$ *and* $0 < \varepsilon \leqslant 1$.

The proof of this mean value theorem can be found in [7].

Using familiar methods, we deduce from Lemmas 3 and 4, respectively,

LEMMA 5. *Given any positive constant* A, *there exists a positive constant* $B_3 = B_3(A)$ *such that*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B_3}} \mu^2(d) 3^{\nu(d)} \max_{y \leqslant x} \max_{(l,d)=1} \left| \pi(y; d, l) - \frac{\operatorname{li} y}{\varphi(d)} \right| \ll \frac{x}{(\log x)^A}.$$

LEMMA 6. *Assume the hypotheses of Lemma* 4. *Given any positive constant* A, *there exists a positive constant* $B_4 = B_4(A)$ *such that*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B_4}} \mu^2(d) 3^{\nu(d)} \max_{y \leqslant x} \max_{(l,d)=1} \left| \sum_{\substack{a \leqslant x^{1-\varepsilon} \\ (a,d)=1}} f(a) \left(\pi(y; a, d, l) - \frac{\operatorname{li} y/a}{\varphi(d)} \right) \right|$$

$$\ll \frac{x}{(\log x)^A}.$$

To proceed further, we take

$$\mathcal{A}=\{N-p \quad : p \leqslant N\}, \qquad \mathcal{P}=\{p : p \nmid N\}.$$

Consider the quantity

$$|\{N-p : p \leqslant N, (N-p, P(N^{1/8}))=1\}|=S(\mathcal{A}, \mathcal{P}, N^{1/8}).$$

Since $\sum_{p \mid N} 1=O(\log N)$, clearly we have

LEMMA 7.

$$|\{N-p : p \leqslant N, N-p=P_3\}|$$
$$\geqslant S(\mathcal{A}; \mathcal{P}, N^{1/8})-S_1-S_2-S_3-S_4+O(\log N),$$

where

$$S_1=|\{N-p : p \leqslant N, N-p=p_1 p_2 p_3 p_4, p_i \geqslant N^{1/8}, i=1,..., 4\}|,$$
$$S_2=|\{N-p : p \leqslant N, N-p=p_1 p_2 \cdots p_5, p_i \geqslant N^{1/8}, i=1,..., 5\}|,$$
$$S_3=|\{N-p : p \leqslant N, N-p=p_1 p_2 \cdots p_6, p_i \geqslant N^{1/8}, i=1,..., 6\}|,$$
$$S_4=|\{N-p : p \leqslant N, N-p=p_1 p_2 \cdots p_7, p_i \geqslant N^{1/8}, i=1,..., 7\}|,$$

and p_i stands for a prime, $p_i \nmid N$.

In what follows, $\varepsilon_1, \varepsilon_2,...$ denote small positive numbers, tending to 0 as N approaches infinity.

LEMMA 8.

$$S(\mathcal{A}; \mathcal{P}, N^{1/8}) \geqslant 8.7888\, C_N \frac{N}{\log^2 N}.$$

Proof. We apply (4) of Lemma 2 with

$$X=\operatorname{li} N, \qquad \omega(p)=\frac{p}{p-1}, \qquad p \nmid N,$$
$$z=N^{1/8}, \qquad \xi^2=N^{1/2} \log^{-c} N,$$

where c is a suitable positive constant. Combining with Lemma 5 and (7), we find

$$S(\mathcal{A}; \mathcal{P}, N^{1/8}) \geqslant (1-\varepsilon_1) 8 \log 3 \cdot C_N \frac{N}{\log^2 N}$$
$$\geqslant 8.7888\, C_N \frac{N}{\log^2 N},$$

provided that we choose N large enough. This completes the proof of Lemma 8.

It remains to estimate S_1, S_2, S_3, S_4 from above. This is accomplished by using Chen's idea of switching, Lemma 1 and Lemma 6.

LEMMA 9. $S_1 \leqslant 2.3952\, C_N\, N/\log^2 N$.

Proof. According to the definition,

$$S_1 = \frac{1}{4!} \sum_{\substack{p_i \geqslant N^{1/8}, p_i \nmid N \\ i=1,\ldots,4}} \sum_{\substack{N-p=p_1p_2p_3p_4 \\ p \leqslant N}} 1$$

$$= \frac{1}{4!} \sum_{\substack{p_i \geqslant N^{1/8}, p_i \nmid N \\ i=1,\ldots,4}} \sum_{\substack{p=N-p_1p_2p_3p_4 \\ p \leqslant N}} 1$$

$$\leqslant \frac{1}{4!} \sum_{\substack{p_1p_2p_3 \leqslant N^{7/8} \\ p_i \geqslant N^{1/8}, i=1,2,3 \\ (p_1p_2p_3, N)=1}} \sum_{\substack{p=N-p_1p_2p_3p_4 \\ p_4 < N/p_1p_2p_3,\, p_4 \nmid N \\ p \leqslant N}} = \frac{1}{4!} S_1', \quad \text{say.}$$

We now consider the sets

$$\mathscr{B} = \{b : b = p_1p_2p_3,\, b \leqslant N^{7/8},\, (b, N) = 1,\, p_i \geqslant N^{1/8},\, i = 1, 2, 3\}$$

and

$$\mathscr{L} = \{l : l = N - bp,\, b \in \mathscr{B},\, bp \leqslant N\}.$$

Clearly,

$$|\mathscr{B}| \ll N^{7/8} \qquad \text{and} \qquad b \geqslant N^{3/8}, \quad b \in \mathscr{B}.$$

Moreover,

$$|\{l : l \in \mathscr{L},\, l \leqslant N^{3/8}\}| \ll N^{7/8}.$$

It follows that if we choose

$$\mathscr{P} = \{p : p \nmid N\},$$

then

$$S_1' \leqslant S(\mathscr{L}; \mathscr{P}, z) + O(N^{7/8}), \qquad z \leqslant N^{3/8}.$$

To estimate $S(\mathscr{L};\mathscr{P},z)$ from above, we apply Lemma 1 with

$$X=\sum_{b\in\mathscr{B}}\operatorname{li}\frac{N}{b},$$

$$\omega(d)=\frac{d}{\varphi(d)},\qquad \mu(d)\neq 0,\qquad (d,N)=1.$$

Conditions (1) and (2) are met. Taking $z^2=D=N^{1/2}\log^{-c}N$, where c is a suitable positive constant, we get

$$S(\mathscr{L};\mathscr{P},D^{1/2})\leqslant 8(1+\varepsilon_2)C_N\frac{X}{\log N}+R_1+R_2,$$

where

$$R_1=\sum_{\substack{d\leqslant D\\(d,N)=1}}\mu^2(d)3^{v(d)}\left|\sum_{\substack{b\in\mathscr{B}\\(b,d)=1}}\left(\sum_{\substack{bp\leqslant N\\bp\equiv N(d)}}1-\frac{1}{\varphi(d)}\operatorname{li}\frac{N}{b}\right)\right|$$

and

$$R_2=\sum_{\substack{d\leqslant D\\(d,N)=1}}\frac{\mu^2(d)3^{v(d)}}{\varphi(d)}\sum_{\substack{b\in\mathscr{B}\\(b,d)>1}}\operatorname{li}\frac{N}{b}.$$

If $b\in\mathscr{B}$, then $N^{3/8}\leqslant b\leqslant N^{7/8}$. Thus

$$R_1=\sum_{\substack{d\leqslant D\\(d,N)=1}}\mu^2(d)3^{v(d)}\left|\sum_{\substack{N^{3/8}\leqslant a\leqslant N^{7/8}\\(a,d)=1}}f(a)\left(\sum_{\substack{ap\leqslant N\\ap\equiv N(d)}}1-\frac{1}{\varphi(d)}\operatorname{li}\frac{N}{a}\right)\right|,$$

where

$$f(a)=\sum_{\substack{b=a\\b\in\mathscr{B}}}1\leqslant 6.$$

Using Lemma 6 and with an appropriate choice of c, we obtain

$$R_1\ll\frac{N}{\log^3 N}.$$

Next we turn to R_2. Changing the variable of summation from d to q and using the fact that $\mu^2(q)3^{v(q)}<d^2(q)$, where $d(q)$ is the divisor function, we find

$$R_2 \leqslant \sum_{q \leqslant D} \frac{d^2(q)}{\varphi(q)} \sum_{\substack{a \leqslant N^{7/8} \\ (a,q) \geqslant N^{1/8}}} f(a) \operatorname{li} \frac{N}{a}$$

$$\leqslant N \sum_{q \leqslant D} \frac{d^2(q)}{\varphi(q)} \sum_{\substack{a \leqslant N^{7/8} \\ (a,q) \geqslant N^{1/8}}} \frac{1}{a}$$

$$\ll N^{1+\varepsilon} \sum_{q \leqslant D} \frac{1}{q} \sum_{\substack{a \leqslant N^{7/8} \\ (a,q) \geqslant N^{1/8}}} \frac{1}{a}$$

$$= N^{1+\varepsilon} \sum_{q \leqslant D} \frac{1}{q} \sum_{\substack{m \mid q \\ m \geqslant N^{1/8}}} \sum_{\substack{a \leqslant N^{7/8} \\ (a,q) = m}} \frac{1}{a}$$

$$\ll N^{1+2\varepsilon} \sum_{q \leqslant D} \frac{1}{q} \sum_{\substack{m \mid q \\ m \geqslant N^{1/8}}} \frac{1}{m}$$

$$\ll N^{(7/8)+3\varepsilon}.$$

Finally, observe that

$$X \leqslant \sum_{N^{1/8} \leqslant p_1 \leqslant N^{5/8}} \sum_{N^{1/8} \leqslant p_2 \leqslant N^{6/8}/p_1} \sum_{N^{1/8} \leqslant p_3 \leqslant N^{7/8}/p_1 p_2} \operatorname{li} \frac{N}{p_1 p_2 p_3}.$$

Using Stieltjes integration, we get

$$X \leqslant (1+\varepsilon_3) \frac{N}{\log N} \int_{1/8}^{5/8} \int_{1/8}^{(6/8)-a} \int_{1/8}^{(7/8)-a-b} \frac{1}{abc(1-a-b-c)}\, dc\, db\, ba$$

$$= 2(1+\varepsilon_3) \frac{N}{\log N} \int_{1/8}^{5/8} \int_{1/8}^{(6/8)-a} \frac{\log(7-8a-8b)}{ab(1-a-b)}\, db\, da.$$

Putting all these estimates together, we have

$$S_1 \leqslant \frac{(1+\varepsilon_4)}{4!}\, 16\, C_N \frac{N}{\log^2 N} \int_{1/8}^{5/8} \int_{1/8}^{(6/8)-a} \frac{\log(7-8a-8b)}{ab(1-a-b)}\, db\, da$$

$$\leqslant 2.3952\, C_N \frac{N}{\log^2 N},$$

provided that N is sufficiently large. This completes the proof of Lemma 9.

Remark. The evaluation of the above double integral is done by a computer, using Simpson's rule.

Similarly, we have

LEMMA 10.

$$S_2 \leqslant 0.2140\, C_N \frac{N}{\log^2 N},$$

$$S_3 \leqslant 0.0052\, C_N \frac{N}{\log^2 N},$$

and

$$S_4 \leqslant 0.0007\, C_N \frac{N}{\log^2 N}.$$

Proof. We follow closely the proof of Lemma 9 and use more numerical integrations, again done by the computer.

Finally, we can now prove Theorem 2. This follows from Lemmas 7–10.

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