Comp. & Maths. with Appls. Vol. 9, No. 3, pp. 323–325, 1983
Printed in Great Britain.
0097-4943/83/030323-03$03.00/0
Pergamon Press Ltd.

EDITOR'S REMARKS

HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS: A FEW OPENING COMMENTS

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(*Received May* 1982)

1. INTRODUCTION

"Early deterministic models of biological populations where necessarily highly simplified in nature. The usual simplifying assumptions were that changes in the environment, including those brought about by the increasing population itself, all took place instantaneously, all of the terms were linear, and the only variables in the equations were the population numbers or population densities. Although analytic solutions to the simplified equations have been demonstrated, the situations described by these early equations are so far from those observed in natural populations, and in natural systems in general, that the validity of the whole approach has been questioned quite often. In many cases, the behavior of the laboratory populations-grown under strictly controlled conditions-has been quite different from that predicted by the theoretical models that were proposed to describe the specific behavior."[1]. Let us very briefly review some of the literature to see how various models arise.

2. DISCRETE AGE STRUCTURE

While homogeneous age structure population models offer a wide variety of interesting mathematical problems and a "horn of plenty" when it comes to dynamical behaviors[2], in biological populations it is natural to consider the question of how many people of age a there will be at time t. This problem may also be phrased in terms of the density of individuals of age a at time t. These questions involve the inclusion of age-structure in the population model. One can introduce age structure in a variety of ways.

Assuming a discrete time and age structure, letting n_t be a column vector containing m age groupings of a population at time t, letting $M(\mathbf{n}_t)$ be the classical Lewis–Leslie matrix, and letting $\mathbf{b}(\mathbf{n}_t)$ be an immigration/emigration vector, one can write the following classical Lewis–Leslie model[3–6].

$$\mathbf{n}_{t+1} = M(\mathbf{n}_t)\mathbf{n}_t + \mathbf{b}(\mathbf{n}_t) \tag{2.1}$$

Discrete models are not only found in demography, and ecology (see Refs.[2, 9] for extensive bibliographies), but also in cell kinetics and cellular aging[10, 11].

The continuous time-discrete age models are essentially of the same form as equation (2.1) where t is considered to be a continuous variable. In this case, we arrive at a system of differential equations having the form

$$\frac{\mathrm{d}\mathbf{n}(t)}{\mathrm{d}t} = M(\mathbf{n}, t)\mathbf{n}(t) + \mathbf{b}(\mathbf{n}, t) \tag{2}$$

An elegant discussion on this type of model may be found in Ref.[7].

3. CONTINUOUS AGE-TIME STRUCTURE: HYPERBOLIC SYSTEMS

In passing to the limit of continuous age-time structure in biological populations, one is lead most naturally to the partial differential equation as a means of describing the density of

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individuals of age a at time t, denoted $n(t, a)$. The most common among these models is the Von Foerster age-time model of population dynamics. Here, one assumes that if $\mu(t, a)$ is the age specific death rate, Δh is a small time increment, and the number of deaths in the population is proportional to the product $\mu(t, a)n(t, a)\Delta h$ then the net change in the population over the time increment Δh is given by the equation

$$n(t+\Delta h, a+\Delta h) - n(t, a) = -\mu(t, a)n(t, a)\Delta h \tag{3.1}$$

expanding $n(t+\Delta h, a+\Delta h)$ in a Taylor series and letting $\Delta h \to 0$ we obtain the classical linear nonconservative hyperbolic partial differential equation

$$\frac{\partial n(t, a)}{\partial t} + \frac{\partial n(t, a)}{\partial a} = -\mu(t, a)n(t, a) \tag{3.2a}$$

with the given initial population distribution

$$n(0, a) = n_0(a). \tag{3.2b}$$

Finally, at any given time, the number of newborns (births) in this population (which is always assumed to be female in these types of model) is given by

$$n(t, 0) = \int_0^\infty \lambda(t, a)n(t, a)\,\mathrm{d}a \equiv B(t) \tag{3.2c}$$

Equations (3.2a, b, c) constitute the McKendrick/Von Foerster model of population growth. One can demonstrate that these equations may be solved to obtain a general solution for the system via the method of characteristics. For details see Refs. [1, 8].

4. CLOSING COMMENTS

Since the appearance of the McKendrick/Von Foerster system much work has been done studying their solution under a variety of assumptions. Many of the papers in this issue contain references to a number of classic papers in this area. It is beyond the scope of these opening comments to even begin a comprehensive literature bibliography. We will leave this to the various authors in this journal.

This particular special topic journal issue arose out of the need to fuse together research in analytic and numerical solution of hyperbolic partial differential equations, as well as a personal desire to ascertain what the areas of current application were.

For example, equations (3.2a, b, c) describe a non-environmentally influenced population. If we were to extend this model to a population with logistic density dependent growth, migration, and density dependent deathrate, one would arrive at a nonlinear hyperbolic partial differential equation of the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(t, a, n)n\left[1 - \frac{n}{K(t)}\right] + m(t, a, n) \tag{4.1}$$

Numerical methods for equations of this type are sparse in the literature. Yet, there is good reason for having to deal with such equations, both from a numerical as well as an analytic point of view.

One might also wish consider the following question. Given a model of the form of (4.1) in which the analytic solution is not available, how might we fit that model to known experimental data. Given a set of data, how might we estimate the birthrate and deathrate functions. Further, how might we choose these functions so that we obtain the best fit of the experimental data to the model. For more details see Ref. [9]. Some of these questions will be addressed in this issue, other questions will, no doubt be posed, and many shall remain unanswered.

It is hoped that this special issue will provide the reader with an insight into the wide variety of real world applications of hyperbolic partial differential equations.

The contributors to this special edition consist of guest speakers at the 1982 IMACS Conference Symposium Session on Hyperbolic Partial Differential Equations with Integral Boundary Conditions, as well as those solicited from experts working in the field of hyperbolic partial differential equations. To those who were missed, I must claim sole responsibility for this oversight.

Acknowledgements—Many people have contributed to the completion of this special issue. I would first like to thank all of the contributors, without whom there would be no issue. I would also like to thank those who did not contribute but who took the time to suggest possible contributors.

Without supportive surroundings no edition can survive. This work was performed while I was on leave at the Department of Mathematics, UC Santa Barbara during the year 1981–1982. Consequently, I would like to thank John Ernest for his support of interdisciplinary mathematics, the typing and administrative staff for their diligence and efforts, and I would like to acknowledge financial support under grants UCSB 8-584527-19900-7 and UCSB 8-5976699-04727-3.

Finally, I would like to thank K. L. Cooke, S. Busenberg, R. Kalaba and R. Bellman for their stimulating conservations and for their time. Each of them, in his own way, has contributed towards my advancement in this area.

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