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NOTE

A Generalization of a Theorem of Dirac

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In this paper, we give a generalization of a well-known result of Dirac that given any k vertices in a k-connected graph where $k \geqslant 2$, there is a circuit containing all of them. We also generalize a result of Häggkvist and Thomassen. Our main result partially answers an open matroid question of Oxley. © 2001 Academic Press

1. INTRODUCTION

All graphs considered in this paper are simple. A well-known result of Dirac [2] states that, given any k vertices in a k-connected graph where $k \geqslant 2$, there is a circuit containing all of them. More generally, Dirac [2] proved the following theorem:

THEOREM 1.1. *Given any two edges and any $k-2$ vertices of a k-connected graph, where $k \geqslant 2$, there is a circuit containing all of them.*

Considering edges rather than vertices, Häggkvist and Thomassen [3] proved the following result.

THEOREM 1.2. *For any set S of independent edges of size $k-1$ in a k-connected graph, where $k \geqslant 2$, there is a circuit containing all edges of S.*

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The following corollary of Theorem 1.2 is a slight extension.

COROLLARY 1.3. *For any set S of independent paths of total length $k-1$ in a k-connected graph, where $k \geqslant 2$, there is a circuit containing all elements of S.*

The aim of this paper is to generalise both of these results, taking the following result of Oxley [7] as inspiration. A *cocircuit* of a graph is a nonempty minimal-edge cut. In a matroid, a circuit C meets a cocircuit C^* if they have at least one common element.

THEOREM 1.4. *If M is a 3-connected matroid, then, for every pair $\{a, b\}$ of distinct elements of M and every cocircuit C^* of M, there is a circuit that contains $\{a, b\}$ and meets C^*.*

Note that for a k-connected graph where $k \geqslant 2$, each vertex is associated with a cocircuit, that is, the set of edges that are incident with that vertex. Therefore, Theorem 1.4 generalizes Theorem 1.1 for the case $k=3$. The answer to the following open question of Oxley [7], if true, would be a natural extension of Dirac's results to matroids.

Question 1. Let M be a k-connected matroid where $k \geqslant 4$. Given any two edges e, f and $k-2$ cocircuits, is there a circuit of M containing e, f and meeting all the $k-2$ cocircuits?

Clearly, Question 1 is stronger than the following question:

Question 2. Given any k cocircuits of a k-connected matroid, where $k \geqslant 4$, is there a circuit of M meeting all these cocircuits?

In this paper, we show that the answers to these two questions are affirmative for graphic matroids. Our main theorem also gives a common generalization to both Theorem 1.1 of Dirac and Theorem 1.2 of Häggkvist and Thomassen. For terminology not mentioned in the paper, we follow Bondy [1]. In a graph, a circuit C meets a cocircuit C^* if $E(C) \cap E(C^*) \neq \emptyset$. Let T be a circuit or a cocircuit of a graph G, for simplicity, we may use T to denote the set of edges in T. For example, we use $G \backslash T$ to denote the graph $G \backslash E(T)$.

2. THE MAIN RESULT

Our main result generalizes both Theorem 1.1 and Theorem 1.2. It also answers Oxley's question affirmatively for graphic matroids.

THEOREM 2.1. *Let G be a k-connected graph where $k \geqslant 2$. Let S be a set of independent paths of total length s and T be a set of t cocircuits, where $s+t=k$ and $t \geqslant 1$. Then there is a circuit of G containing each path of S and meeting each cocircuit of T.*

We will use the following well-known lemma, a direct consequence of Menger's Theorem [5].

LEMMA 2.2. *Let G be a k-connected graph, where $k \geqslant 1$, let x be a vertex of G, and let Y be a set of vertices of G, where $x \notin Y$. Then there exist distinct vertices $y_1, y_2, ..., y_m$ in Y, where $m = \min\{k, |Y|\}$ and internally-disjoint paths $P_1, P_2, ..., P_m$, such that* (i) *P_i is an xy_i-path, and* (ii) *$V(P_i) \cap Y = \{y_i\}$ for all $1 \leqslant i \leqslant m$.*

Proof of Theorem 2.1. We use induction. Suppose that $T = \{C_1^*, C_2^*, ..., C_t^*\}$. Clearly, the theorem is true for 2-connected graphs. Suppose that the theorem is true for all $(k-1)$-connected graphs, where $k \geqslant 3$. Now suppose that G is a k-connected graph ($k \geqslant 3$). If $t=1$, then by Corollary 1.3, there is a circuit C containing all paths of S. If $t \geqslant 2$, by induction, there is a circuit C containing each path of S and meeting all except possibly one cocircuit, say C_1^* in T. If C meets C_1^* also then we are done. Thus we may assume that $E(C) \cap E(C_1^*) = \emptyset$. As C_1^* is a cocircuit, $G \backslash C_1^*$ has exactly two components, denoted by H and T. Clearly, either C is a circuit of H or a circuit of T. Without loss of generality we may assume that $V(C) \subseteq V(H)$. Let $A = V(H) \cap V(C_1^*)$ and $B = V(T) \cap V(C_1^*)$. Suppose that $C = v_0 v_1 \cdots v_p v_0$ where $v_{p+1} = v_0$ and $p \geqslant 2$.

Case 1. Suppose that $|A| \leqslant k-1$. Then as G has no vertex-cut with fewer than k elements, we deduce that $V(H) = A$ and $V(C) \subseteq A$. As S is a set of independent paths and each is contained in C, there is some i, $0 \leqslant i \leqslant p$, such that $v_i v_{i+1}$ is not in any path of S. As both v_i and v_{i+1} are elements of A, v_i and v_{i+1} are end-vertices of some edges a and b of C_1^*, respectively. Assume that $a = v_i u$ and $b = v_{i+1} w$. As T is connected, there is a path P connecting u and w in T (if $u = w$, take P as the trivial path). Thus $C_1 = C \cup \{a, b\} \cup P \backslash v_i v_{i+1}$ is a circuit containing all paths of S and meeting C_1^*. We claim that C_1 is a desired circuit. If $t=1$, then it is clearly true. Suppose $t \geqslant 2$. For each $j = 2, 3, ..., t$, we know that $E(C) \cap E(C_j^*) \neq \emptyset$. Since a cocircuit and a circuit cannot have exactly one element in common, we conclude that $(E(C) \backslash v_i v_{i+1}) \cap E(C_j^*) \neq \emptyset$. As $E(C) \backslash v_i v_{i+1} \subseteq E(C_1)$, it follows that C_1 is a circuit with the required properties.

Case 2. Suppose that $|A| \geqslant k$. Add a new vertex x and for each vertex y in A add an edge xy. We denote the new graph by G_1. Since $|A| \geqslant k$, it follows that G_1 is also k-connected. By Lemma 2.2, there exist distinct vertices $y_1, y_2, ..., y_m$ in $V(C)$, where $m = \min\{k, |V(C)|\}$ and internally-disjoint paths $P[y_1, x]$, $P[y_2, x]$, ..., $P[y_m, x]$, such that (i) $P[y_i, x]$ is a $y_i x$-path, and (ii) $V(P[y_i, x]) \cap V(C) = \{y_i\}$ for all $1 \leqslant i \leqslant m$.

Suppose that $|V(C)| \geqslant k$. Then $m = k$ and $y_1, y_2, ..., y_k$ induce a partition of C into k segments. Now S contains a total of s edges which belong to at most s segments. Each cocircuit of $T - \{C_1^*\}$ meets C also. As $s + t - 1 < k$, by the pigeonhole principle, we can find a segment L such that $C \backslash L$ still contains all paths of S and meets each of $C_2^*, C_3^*, ..., C_t^*$. Suppose that $L = C[y_i, y_j]$. Traverse from y_i to x in the path $P[y_i, x]$ and let u_1 be the first vertex in the path such that $u_1 \in A$. Similarly choose w_1 in the path $P[y_j, x]$. By the choice of u_1 and w_1 and that C_1^* is a cocircuit of G_1, we deduce that $V(P[y_i, u_1]) \subseteq V(H)$ and $V(P[y_j, v_1]) \subseteq V(H)$, where $P[y_i, u_1]$ is the subpath of $P[y_i, x]$ from y_i to u_1 and $P[y_j, w_1]$ is the subpath of $P[y_j, x]$ from y_j to w_1. As $u_1, w_1 \in A$, we deduce that there exist u_2, w_2 in B such that $\{u_1 u_2, w_1 w_2\} \subseteq C_1^*$. As T is connected, there is a path Q connecting u_2 and w_2 (if $u_2 = w_2$, take Q as the trivial path). Then $(C \backslash L) \cup P[y_i, u_1] \cup P[y_j, w_1] \cup \{u_1 u_2, w_1 w_2\} \cup Q$ is the required circuit.

Suppose that $|V(C)| \leqslant k - 1$. Then $m = |V(C)|$ and $V(C) = \{y_1, y_2, ..., y_m\}$. Relabeling if necessary, we may assume that $C = y_1 y_2, ..., y_m y_1$ (assume that $y_{m+1} = y_1$). As S is a set of independent paths and each is contained in C, there is some j, $1 \leqslant j \leqslant m$, such that $v_j v_{j+1}$ is not in any path of S. Now we show that $E(C) \backslash v_j v_{j+1}$ meets all cocircuits of T except C_1^*. This is clearly true for $t = 1$. Suppose that $t \geqslant 2$. Since a cocircuit and a circuit cannot have exactly one common element, we conclude that $(E(C) \backslash y_j y_{j+1}) \cap E(C_i^*) \neq \emptyset$ for all $2 \leqslant i \leqslant t$. By a similar argument to that in the previous paragraph, we can find a circuit containing all paths of S and meeting all $C_1^*, C_2^*, ..., C_t^*$. This completes the proof of the theorem. ∎

3. CONSEQUENCES

If we take each cocircuit in Theorem 2.1 as a cocircuit associated with some vertex, we get the following result immediately.

Corollary 3.1. *Let G be a k-connected graph where $k \geqslant 2$. Let S be a set of independent paths with a total of s edges and T be a set of t vertices, where $s + t = k$ and $t \geqslant 1$. Then there is a circuit of G containing each path of S and each vertex of T.*

The above corollary generalizes both Theorem 1.1 (when $k \geqslant 3$) and Theorem 1.2. Indeed, if $s = k - 1$, the above corollary generalizes Theorem 1.2. When we take $s = 2$, we get the following result immediately, which answers Oxley's Question 1 for graphic matroids affirmatively (note that when $k=2$, the result is well-known).

COROLLARY 3.2. *Let G be a k-connected graph where $k \geqslant 2$. Given any two edges e, f and $k-2$ cocircuits $C_1^*, C_2^*, ..., C_{k-2}^*$, there is a circuit of G containing $\{e, f\}$ and meeting all of $C_1^*, C_2^*, ..., C_{k-2}^*$.*

When $s=0$ in Theorem 2.1, we get the following result, which answers Question 2 for graphic matroids. It also generalizes the well-known result of Dirac that given any k vertices in a k-connected graph where $k \geqslant 2$, there is a circuit containing all of them.

COROLLARY 3.3. *Let G be a k-connected graph where $k \geqslant 2$. For any set S of k cocircuits, there is a circuit of G meeting each member of S.*

The following immediate consequence of Theorem 2.1 generalizes Theorem 1.2 of Häggkvist and Thomassen [3].

COROLLARY 3.4. *Let G be a k-connected graph, where $k \geqslant 2$. Then for any set S of independent edges of size $k-1$ and a cocircuit C^*, there is a circuit containing all edges of S and meeting S^*.*

When $t=0$ in Theorem 2.1, the result may not be true anymore. However, Lovasz [4] conjectured that for any set L of independent edges of size k in a k-connected graph, if k is even or $G-L$ is connected, then G has a circuit containing all edges of L. Our theorem shows that for any set L of independent edges of size k, and for any cocircuit C^* containing an element $e \in L$, there is a circuit containing $L \backslash e$ and meeting C^*.

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