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Some Nonexistence Results for Quasilinear Elliptic Problems

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1. INTRODUCTION

This paper is devoted to nonexistence of positive solutions for some quasilinear elliptic inequalities and their systems in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, without prescribing any boundary conditions.

We can formulate the typical problem that we shall study as follows: "Let $\mathbb{A}$ be a second order differential operator in divergence form and let $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a given function. What are the sufficient conditions that imply the nonexistence of positive solutions of

$$\mathbb{A}(u) \geq f(x, u) \qquad \text{in } \Omega, \tag{1.1}$$

when $u \in S$, S being a suitable functional class that depends on $\mathbb{A}$, f, and Ω?"

The origin of the problem dates back to the classical Liouville theorem for the Laplacian. In the case $\Omega = \mathbb{R}^N$, its nonlinear versions have been studied by many authors in connection with associated Dirichlet problems in bounded domains (see [1] and references therein). Most results in this direction deal with the class of radial solutions (see [10, 12]). However, methods developed by Gidas and Spruck for the Laplace equation [6] and by Mitidieri and Pohozaev for a wide class of quasilinear elliptic inequalities [7–9] allow us to obtain sharp nonexistence results without any assumptions concerning the behaviour of eventual solutions. Here we adapt for our purposes some of the techniques developed in [9].

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For a bounded Ω and $\mathbb{A} = \Delta$, the most celebrated results are due to Brezis and Cabré [2] (see also an earlier result obtained by Ni [11]). In particular, they have proved that the problem

$$\begin{cases} -\Delta u \geq |x|^{-\alpha} u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

has no solutions for $\alpha \geq 2$ and $q \geq 2$. In this paper, we generalize the method of [2] in order to extend these nonexistence results to single inequalities and systems of the form

$$\begin{cases} -\Delta_p u \geq |x|^{-\alpha} u^a v^b & \text{in } \Omega, \\ -\Delta_q v \geq |x|^{-\beta} u^c v^d & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega \end{cases} \tag{1.3}$$

with $p, q > 1$ and appropriate conditions on a, b, c, d, α, and β.

The proof is based on an argument by contradiction which involves both upper and lower a priori estimates. The lower bound follows directly from the strong comparison principle. As for the upper bound, in anti-coercive problems for the p-Laplacian it can be obtained by an appropriate change of variables and applying the weak comparison principle to the transformed problem, as in [2]. However, this method requires rather restrictive structural assumptions on the operator. Nevertheless, it is possible to obtain the necessary estimates for a much wider class of problems, for which the strong comparison principle is generally unknown. For this purpose, we use special test functions in the definition of a weak solution similarly to [7–9].

This paper is organized as follows.

Section 2 contains some auxiliary results which are used in the sequel and a nonexistence theorem for a model problem associated to the p-Laplacian operator.

Section 3 is devoted to systems of quasilinear elliptic inequalities of the form (1.3).

In Sections 4 and 5, these results are extended, respectively, to scalar and vector non-homogeneous problems including coercive ones.

2. SINGLE INEQUALITIES

We start by considering the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u \geq |x|^{-\alpha} u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \in \Omega$, and $N \geq 1$ is the dimension of the space.

We define the functional class of solutions as

$$S = \left\{ u \in C(\overline{\Omega}), |Du|^p, |x|^{-\alpha} u^q \in L^1_{loc}(\Omega) \right\}$$

which satisfy (2.1) in the sense of $\mathscr{D}'(\Omega)$, that is,

$$\int_\Omega |Du|^{p-2} (Du, Dg) \,\mathrm{d}x \geq \int_\Omega |x|^{-\alpha} u^q g \,\mathrm{d}x \tag{2.2}$$

for all $g \in \mathscr{D}(\Omega)$ such that $g \geq 0$ in Ω. Throughout this paper, $(\cdot\,,\cdot)$ will denote the standard scalar product in $\mathbb{R}^N$.

Now we are able to formulate our first result.

THEOREM 2.1. *Let $\alpha \geq \min(p, N)$ and $q > p - 1$. Then problem* (2.1) *has only a trivial solution in S.*

One of the main ingredients in the proof of this theorem is the following lemma.

LEMMA 2.2. *Let $u \in W^{1,p}_{loc}(\Omega)$ and $f \in L^1_{loc}(\Omega)$ satisfy*

$$-\Delta_p u \geq f \qquad \text{in } \mathscr{D}'(\Omega). \tag{2.3}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^1, concave function such that

$$0 \leq \phi' \leq C \qquad \text{in } \mathbb{R}$$

for some constant C. Then $(\phi'(u(x)))^{p-1} \in L^1_{loc}(\Omega)$ and

$$-\Delta_p \phi(u) \geq (\phi'(u))^{p-1} f \qquad \text{in } \mathscr{D}'(\Omega). \tag{2.4}$$

Proof. We modify for our use the method of proof of Lemma 1.7 in [2] and Lemma 2 in [3]. Indeed, let us first suppose that $\phi \in C^2(\mathbb{R})$, $u \in C^2(\overline{\Omega})$, and, respectively, $f \in C(\overline{\Omega})$. Then we have

$$\begin{aligned} -\Delta_p \phi(u) &= -(p-1)(\phi'(u))^{p-2} \cdot \phi''(u) \cdot |Du|^{p-2} \cdot \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 \\ &\quad - (\phi'(u))^{p-1} \cdot \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|Du|^{p-2} \frac{\partial u}{\partial x_i} \right) \\ &\geq (\phi'(u))^{p-1} \cdot f(x). \end{aligned} \tag{2.5}$$

It is clear that inequality (2.4) holds also for concave $\phi \in C^1(\mathbb{R})$. If $u \in W^{1,p}_{loc}(\Omega)$ is a solution to (2.3) in the sense of $\mathscr{D}'(\Omega)$, then we can take a sequence $\{u_k\} \in C^2(\overline{\Omega})$ such that $u_k \to u$ in the sense of $W^{1,p}_{loc}(\Omega)$. Inequality (2.4) holds for each u_k. Multiplying it by an arbitrary nonnegative test function $g \in \mathscr{D}(\Omega)$ and integrating its left-hand side by parts, we get

$$\int_\Omega |D\phi(u_k)|^{p-2}(D\phi(u_k), Dg)\,\mathrm{d}x \geq \int_\Omega (\phi'(u_k))^{p-1} fg\,\mathrm{d}x.$$

Passing to the limit as $k \to \infty$, we obtain the claim. ∎

For the proof of the theorem, we shall also make use of the following result. (Here and in the sequel B_η denotes $B_\eta(0)$.)

LEMMA 2.3. *Consider the problem*

$$\begin{cases} -\Delta_p w \geq c_0|x|^{-p} & \text{in } \mathscr{D}'(B_\eta), \\ w \geq 0 & \text{a.e. in } B_\eta, \end{cases} \tag{2.6}$$

where $\eta, c_0 > 0$.

(i) *Suppose that* $p < N$ *and* $w \in W^{1,p}_{loc}(B_\eta)$ *satisfies* (2.6). *Then there exist constants* $c_1, c_2 > 0$ *depending only on* N, p, η, *and* c_0 *such that*

$$w(x) \geq c_1 - c_2 \log|x|.$$

(ii) *If* $p \geq N$, *problem* (2.6) *has no solutions in* $W^{1,p}_{loc}(B_\eta)$.

Proof. (i) First of all, let us search a radial solution $\overline{w}(x)$ of the Dirichlet problem

$$\begin{cases} -\Delta_p \overline{w} = c_0|x|^{-p} & \text{in } B_\eta, \\ \overline{w} = 0 & \text{on } \partial B_\eta \end{cases} \tag{2.7}$$

in the form $\overline{w}(x) = c_1 - c_2 \log|x|$. A straightforward formal computation yields

$$D\overline{w} = -\left(\frac{c_2 x_1}{|x|^2}, \dots, \frac{c_2 x_n}{|x|^2}\right),$$

$$-\Delta_p \overline{w} = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|D\overline{w}|^{p-2}\frac{\partial \overline{w}}{\partial x_i}\right) = (N-p)\frac{c_2^{p-1}}{|x|^p}.$$

Observe that $|D\overline{w}|^p \in L^1_{loc}(B_\eta)$ if $p < N$. Therefore, if we choose $c_2 = \sqrt[p-1]{c_0/(N-p)}$ and $c_1 = c_2 \log \eta$, then $\overline{w}$ actually satisfies (2.7) in the sense of $\mathscr{D}'(B_\eta)$, and the claim follows from the well-known weak comparison principle (see, for example, [5]).

(ii) Now suppose that $p \geq N$. Assume by contradiction that (2.6) has a solution $w \in W^{1,p}_{loc}(B_\eta)$. Then, by the Hölder inequality, for each $g \in \mathscr{D}(\Omega)$, we have

$$\int_\Omega |Dw|^{p-2}(Dw, Dg)\,\mathrm{d}x \leq \|Dw\|_p^{p-1}\|Dg\|_p. \tag{2.8}$$

Now, if we choose a test function $g \in \mathscr{D}(\Omega)$ such that $g(x) \equiv 1$ in B_η for some $\eta > 0$, $\overline{B}_\eta \subset \Omega$, then (2.2) and (2.8) imply

$$\begin{aligned}\|Dw\|_p^{p-1} &\geq \|Dg\|_p^{-1}\int_\Omega |Dw|^{p-2}(Dw, Dg)\,\mathrm{d}x\\ &\geq \|Dg\|_p^{-1}\int_\Omega |x|^{-p}g\,\mathrm{d}x \geq \|Dg\|_p^{-1}\int_{B_\eta} |x|^{-p}\,\mathrm{d}x,\end{aligned}$$

and the last integral diverges by the assumption $p \geq N$. This contradicts our hypothesis that $\|Dw\|_p < +\infty$. ▮

Proof of Theorem 2.1. Suppose for contradiction that $u \in S$ is a solution of (2.1), and that $u \not\equiv 0$, that is, there exists a point $x_0 \in \Omega \setminus \{0\}$ such that $u(x_0) > 0$. In this case we can define a function

$$f(x) = \min\{|x|^{-\alpha}u^q(x), |x_0|^{-\alpha}u^q(x_0)\}.$$

Evidently, $f \in L^\infty(\Omega)$. Therefore we can conclude that there exists a unique solution $u_1 \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} -\Delta_p u_1 = f & \text{in } \Omega,\\ u_1 = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.9}$$

By the weak comparison principle, $u \geq u_1$ in $\mathscr{D}'(\Omega)$. On the other hand, since $f \in L^\infty(\Omega)$ and $0 \leq f \not\equiv 0$ (in particular, $f(x_0) = |x_0|^{-\alpha}u^q(x_0) > 0$), we can apply to (2.9) the strong maximum principle for the Dirichlet p-Laplacian (see, for instance, [4]), which implies that there exist $\eta > 0$, $\varepsilon > 0$, and B_η with closure in Ω such that $u_1(x) \geq \varepsilon$ for all $x \in B_\eta$. Hence,

$$u \geq \varepsilon \qquad \text{in } \mathscr{D}'(B_\eta). \tag{2.10}$$

If $\alpha \geq N$, inequality (2.10) contradicts the assumption that $|x|^{-\alpha}u^q \in L^1_{loc}(\Omega)$.

In the opposite case (that is, if $N > \alpha \geq p$), we introduce the function

$$\phi(s) = \begin{cases} \dfrac{p-1}{p-1-q}\left(s^{1-q/(p-1)} - \varepsilon^{1-q/(p-1)}\right) & (s \geq \varepsilon),\\ \varepsilon^{-q/(p-1)}(s-\varepsilon) & (s \leq \varepsilon).\end{cases} \tag{2.11}$$

The assumption $q > p - 1$ implies that the function $\phi : \mathbb{R} \to \mathbb{R}$ is bounded in $\mathbb{R}_+$ and satisfies all conditions of Lemma 2.2. Hence,

$$-\Delta_p \phi(u(x)) \geq (\phi'(u))^{p-1} u^q |x|^{-\alpha} = |x|^{-\alpha}$$

in the sense of $\mathscr{D}'(B_\eta)$.

Now from the assumption $\alpha \geq p$ it follows that

$$-\Delta_p \phi(u(x)) \geq |x|^{-p} \qquad \text{in } \mathscr{D}'(B_\eta) \tag{2.12}$$

(here we assume without loss of generality that $0 < \eta \leq 1$).

Since $p \leq \alpha < N$, we can now apply Lemma 2.3(i) with $c_0 = 1$ and $w(x) = \phi(u(x))$. Thus we obtain that $\phi(u(x)) \to +\infty$ as $x \to 0$, which is incompatible with the fact that if $u \not\equiv 0$ in Ω, the function ϕ is defined and bounded on the whole $\mathbb{R}_+$, by virtue of (2.11). Thus we arrive at a contradiction which proves the theorem. ∎

Remark 2.4. If we restrict our attention to the class of *radial* solutions, then the assumption $u \in C(\overline{\Omega})$ is too strong. Indeed, if we suppose that $u = u(|x|) \in W^{1,p}(\Omega)$ and $|x|^{-\alpha} u^q \in L^1_{loc}(\Omega)$, it follows directly from the weak comparison principle that $u \geq \varepsilon$ in $\mathscr{D}'(B_\eta)$ for some $\varepsilon, \eta > 0$, $\overline{B_\eta} \subset \Omega$, and the rest of the proof remains unchanged.

Remark 2.5. For the proof of the theorem in case $p \geq N$, we can also make use of Lemma 2.3(ii) which implies that (2.12) has no solution $\phi \in W^{1,p}_{loc}(B_\eta)$. On the other hand, it is clear that if $u \in S \subset W^{1,p}_{loc}(\Omega)$ is a solution of (2.1) and $u \not\equiv 0$, then the function $\phi(u(x))$ defined by (2.11) belongs to $W^{1,p}_{loc}(B_\eta)$ (and satisfies (2.12) by Lemma 2.2). This leads to a contradiction, too.

Remark 2.6. The above result can be easily generalized to a wider class of nonlinear inequalities of the form

$$-\Delta_p u \geq a(x) f(u) \qquad (x \in \Omega),$$

where the function $f : \mathbb{R} \to [0, +\infty)$ is continuous, nondecreasing on $[0, +\infty)$, strictly positive for $s > 0$, and

$$\int_1^{+\infty} f^{1/(1-p)}(s)\, \mathrm{d}s < +\infty,$$

while the function $a : \Omega \backslash \{0\} \to \mathbb{R}_+$ is continuous, and there exist constants $\eta > 0$, $c_0 > 0$, and $\alpha \geq \min(p, N)$ such that

$$a(x) \geq c_0 |x|^{-\alpha} \qquad \text{for every } x \in B_\eta \backslash \{0\}. \tag{2.13}$$

For this purpose, it suffices to repeat the proof of Theorem 2.1 using the function $\phi(s)$ defined by the formula

$$\phi(s) = \begin{cases} \int_{\varepsilon}^{s} f^{1/(1-p)}(s)\,\mathrm{d}s & (s \geq \varepsilon), \\ f^{1/(1-p)}(\varepsilon)(s-\varepsilon) & (s \leq \varepsilon) \end{cases}$$

instead of (2.11).

3. SYSTEMS OF QUASILINEAR ELLIPTIC INEQUALITIES

In this section we shall study the quasilinear system of elliptic inequalities

$$\begin{cases} -\Delta_p u \geq |x|^{-\alpha} u^a v^b & \text{in } \Omega, \\ -\Delta_q v \geq |x|^{-\beta} u^c v^d & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega \end{cases} \tag{3.1}$$

with p, q, and Ω as above.

We suppose that the following hypotheses hold.

(i) $a > p-1$, $d > q-1$.

(ii) $\min(b, c) > 0$.

(iii) $\alpha \geq \min(p, N)$, or $\beta \geq \min(q, N)$.

We define the space of solutions as

$$S = \Big\{(u,v) : u, v \in C(\overline{\Omega}), \\ |Du|^{p-1}, |Dv|^{q-1}, |x|^{-\alpha} u^a v^b, |x|^{-\beta} u^c v^d \in L^1_{loc}(\Omega)\Big\}$$

which satisfy (3.1) in the sense of $\mathscr{D}'(\Omega)$, that is,

$$\int_\Omega |Du|^{p-2}(Du, Dg)\,\mathrm{d}x \geq \int_\Omega |x|^{-\alpha} u^a v^b g\,\mathrm{d}x,$$

$$\int_\Omega |Dv|^{q-2}(Dv, Dg)\,\mathrm{d}x \geq \int_\Omega |x|^{-\beta} u^c v^d g\,\mathrm{d}x$$

for all $g \in \mathscr{D}(\Omega)$ such that $g \geq 0$ in Ω.

Our first result in this section will be a generalization of Theorem 2.1.

THEOREM 3.1. *Assume that hypotheses* (i)–(iii) *are fulfilled.*

Let $u, v \in S$ *be a solution of system* (3.1) *in the sense of* $\mathscr{D}'(\Omega)$. *Then* $u \equiv 0$ *or* $v \equiv 0$.

Proof. Suppose to the contrary that u and v verify our hypotheses, and that $u, v \not\equiv 0$. Similar to the beginning of the proof of Theorem 2.1, we obtain that

$$u \geq \varepsilon_1, \qquad v \geq \varepsilon_2 \text{ in } \mathscr{D}'(B_\eta) \tag{3.2}$$

for some $\varepsilon_1, \varepsilon_2 > 0$ and B_η with closure in Ω. If $\alpha \geq N$ (resp. $\beta \geq N$), then inequality (3.2) contradicts the assumption that $|x|^{-\alpha} u^a v^b \in L^1_{loc}(\Omega)$ (resp. $|x|^{-\beta} u^c v^d \in L^1_{loc}(\Omega)$).

Denote $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. By (ii), we have

$$\begin{cases} -\Delta_p u \geq |x|^{-\alpha} u^a \varepsilon^b, \\ -\Delta_q v \geq |x|^{-\beta} \varepsilon^c v^d \end{cases} \tag{3.3}$$

in $\mathscr{D}'(B_\eta)$.

Now, we can define functions ϕ and ψ as

$$\phi(s) = \begin{cases} \dfrac{p-1}{p-1-a}\left(s^{1-a/(p-1)} - \varepsilon^{1-a/(p-1)}\right) & (s \geq \varepsilon), \\ \varepsilon^{-a/(p-1)}(s - \varepsilon) & (s \leq \varepsilon), \end{cases} \tag{3.4}$$

$$\psi(s) = \begin{cases} \dfrac{q-1}{q-1-d}\left(s^{1-d/(q-1)} - \varepsilon^{1-d/(q-1)}\right) & (s \geq \varepsilon), \\ \varepsilon^{-d/(q-1)}(s - \varepsilon) & (s \leq \varepsilon). \end{cases} \tag{3.5}$$

By (i), both functions ϕ and ψ are bounded on $\mathbb{R}_+$ and satisfy all assumptions of Lemma 2.2. Hence,

$$\begin{cases} -\Delta_p \phi(u(x)) \geq |x|^{-\alpha} u^a v^b (\phi'(u))^{p-1} = |x|^{-\alpha} v^b \geq \varepsilon^b |x|^{-\alpha}, \\ -\Delta_q \psi(v(x)) \geq |x|^{-\beta} u^c v^d (\psi'(v))^{q-1} = |x|^{-\beta} u^c \geq \varepsilon^c |x|^{-\beta} \end{cases} \tag{3.6}$$

in $\mathscr{D}'(B_\eta)$.

We can assume without loss of generality that $0 < \eta \leq 1$. Therefore, if $\alpha \geq p$ (recall that either this assumption, or $\beta \geq q$ holds by (iii)), the first inequality in (3.6) implies

$$-\Delta_p \phi(u(x)) \geq \varepsilon^b |x|^{-p} \qquad \text{in } \mathscr{D}'(B_\eta). \tag{3.7}$$

If $N > p$, we can apply to (3.7) Lemma 2.3(i) with $c_0 = \varepsilon^b$ and $w(x) = \phi(u(x))$. Thus we can obtain that $\phi(u(x)) \to +\infty$ as $x \to 0$. On the other hand, if $p \geq N$, then (3.7) has no solutions in $W^{1,p}_{loc}(B_\eta)$ by Lemma 2.3(ii). Similarly, if $\beta \geq q$, we can apply the same argument to the second inequality in (3.6) and come to a conclusion that $\psi(v(x)) \to +\infty$ as $x \to 0$ (or doesn't exist in $W^{1,q}_{loc}(B_\eta)$). This conclusion contradicts the fact that if $u \not\equiv 0$ (resp. $v \not\equiv 0$) in Ω satisfies (3.1), the function ϕ (resp. ψ) is defined and bounded on $\mathbb{R}_+$ by virtue of (3.4) (resp. (3.5)), and moreover, if $u \in W^{1,p}_{loc}(B_\eta)$ (resp. $v \in W^{1,q}_{loc}(B_\eta)$), the same property must hold for $\phi(u)$ (resp. $\psi(v)$). This completes the proof of the theorem. ∎

Remark 3.2. From the proof of Theorem 3.1 it is clear that in order to establish nonexistence of solutions of the first inequality in (3.1), it is sufficient to assume that

$$a > p - 1, \qquad b > 0, \qquad \alpha \geq \min(p, N),$$

and for the second inequality of (3.1), respectively

$$d > q - 1, \qquad c > 0, \qquad \beta \geq \min(q, N).$$

Moreover, similarly to Section 2, power functions at the right-hand side can be replaced by more general nonlinearities. In fact, we can consider the system of quasilinear elliptic inequalities

$$\begin{cases} -\Delta_p u \geq a(x) g_1(u) g_2(v) & \text{in } \Omega, \\ -\Delta_q v \geq b(x) g_3(u) g_4(v) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

where the assumptions on a, b, and g_i $(i = 1, \ldots, 4)$ are specified below. Then the following assertion holds.

THEOREM 3.3. *Let* $p, q > 1$.

Suppose that there exist constants $c_1, c_2, \eta > 0$, $\alpha \geq \min(p, N)$, *and* $\beta \geq \min(q, N)$ *such that* $\overline{B}_\eta \subset \Omega$ *and the functions* $a, b \in C(\Omega \setminus \{0\}; \mathbb{R}_+)$ *satisfy the inequalities*

$$a(x) \geq c_1 |x|^{-\alpha}, \qquad b(x) \geq c_2 |x|^{-\beta} \text{ in } B_\eta \setminus \{0\}.$$

Assume also that the nonlinearities $g_i : \mathbb{R} \to [0, +\infty)$ $(i = 1, \ldots, 4)$ *are continuous, nondecreasing on* $[0, +\infty)$, $g_i(s) > 0$ *if* $s > 0$, *and*

$$\int_1^{+\infty} g_1^{1/(1-p)}(s)\, \mathrm{d}s < +\infty$$

or, alternatively,

$$\int_1^{+\infty} g_4^{1/(1-q)}(s)\,\mathrm{d}s < +\infty.$$

Then any solution $(u, v) \in S$ *of problem* (3.8) *is such that* $u \equiv 0$ *or* $v \equiv 0$.

4. NON-HOMOGENEOUS SCALAR PROBLEMS

In order to prove the previous results, we needed Lemma 2.2 which was based, in its turn, on the homogeneity of the left-hand side of the inequalities in question. Moreover, certain assumptions on the structure of the operator and/or behaviour of the solutions were essential in order to apply the strong maximum principle. However, these assumptions are rather restrictive. In the two remaining sections we shall show that an appropriate modification of the methods developed in [7–9] for quasilinear elliptic inequalities and systems in $\mathbb{R}^N$ enables us to prove nonexistence results for a certain class of non-homogeneous problems.

Here and in the sequel we shall assume that $A : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function such that for any $(x, s, t) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ we have

$$c_1 |t|^{p-2} \le A(x, s, t) \le c_2 |t|^{p-2}, \tag{4.1}$$

where $c_1, c_2 > 0$ and $p > 1$. Denote $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.

Consider the problem

$$\begin{cases} -\mathrm{div}(A(x, u, |Du|) Du) \ge a(x) u^q & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

where the function a satisfies the inequality

$$a(x) \ge c_0 \rho^{-\alpha}(x) \qquad (x \in \Omega, \rho(x) \le \eta_0) \tag{4.3}$$

for some $\eta_0 > 0$, $c_0 > 0$, and $\alpha > p$.

We define the functional class of solutions as

$$X_{q,\beta}^{a,p}(\Omega) = \{u : \Omega \to \mathbb{R}_+, a(x) u^{q+\beta}, a(x) u^q, |Du|^p u^{\beta-1} \in L_{loc}^1(\Omega)\},$$

where $\beta \in \mathbb{R}$ is a suitable negative number, $p > 1$, and $q \ge 0$.

THEOREM 4.1. *Let* $A : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *satisfy inequalities* (4.1) *with* $1 < p < q + 1$, $p \le N - 1$.

Suppose that the function $a : \Omega \to \mathbb{R}_+$ *is continuous and that there exist constants* $\eta_0 > 0$, $c_0 > 0$, *and* $\alpha > p$ *such that* $a(x)$ *enjoys property* (4.3).

Then problem (4.2) *has only a trivial solution in* $X^{a,p}_{q,\beta}(\Omega)$ *for any* β *such that* $1 - p < \beta < 0$.

Proof. We shall strictly follow the scheme of the proof of Theorems 2.1 and 4.1 in [9]. Let $1 - p < \beta < 0$ and suppose that u is a solution of (4.2). We also introduce a cut-off function $\varphi \in C_0^\infty(\overline{\Omega}; \mathbb{R}_+)$ whose concrete form will be chosen later.

Multiplying (4.2) by $H_\beta = u_\varepsilon^\beta \varphi$, where $u_\varepsilon = u + \varepsilon$ with $\varepsilon > 0$, and integrating by parts we obtain

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, dx \le \beta \int A(x, u, |Du|) |Du|^2 u_\varepsilon^{\beta-1} \varphi \, dx + \int A(x, u, |Du|) u_\varepsilon^\beta (Du, D\varphi) \, dx$$

which by (4.1) implies

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, dx \le c_2 \left(\beta \int |Du|^p u_\varepsilon^{\beta-1} \varphi \, dx + \int |Du|^{p-2} (Du, D\varphi) u_\varepsilon^\beta \, dx \right)$$
$$\le c_2 \left(\beta \int |Du|^p u_\varepsilon^{\beta-1} \varphi \, dx + \int |Du|^{p-1} |D\varphi| u_\varepsilon^\beta \, dx \right).$$

By the Young inequality with parameter $\delta > 0$ we have

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, dx + c_2 |\beta| \int |Du|^p u_\varepsilon^{\beta-1} \varphi \, dx \le \frac{c_2 \delta^p (p-1)}{p} \int |Du|^p u_\varepsilon^{\beta-1} \varphi \, dx + \frac{c_2}{p\delta^p} \int u_\varepsilon^{\beta+p-1} \frac{|D\varphi|^p}{\varphi^{p-1}} \, dx. \quad (4.4)$$

By putting $\theta_\delta = c_2(|\beta| - \delta^p (p-1)/p)$ (note that $\theta(\delta) > 0$ if δ is chosen sufficiently small) and $c_\delta = c_2/p\delta^p$ we obtain

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, dx + \theta_\delta \int |Du|^p u_\varepsilon^{\beta-1} \varphi \, dx \le c_\delta \int u_\varepsilon^{\beta+p-1} \frac{|D\varphi|^p}{\varphi^{p-1}} \, dx$$

and then

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, dx \le c_\delta \int u_\varepsilon^{\beta+p-1} a(x)^{1/y} \frac{|D\varphi|^p}{\varphi^{p-1}} a(x)^{-1/y} \, dx, \quad (4.5)$$

where $y > 1$ has to be chosen. Next from (4.5) it follows that

$$\int a(x) u^q u_\varepsilon^\beta \varphi \, \mathrm{d}x \le c_\delta \left(\int a(x) u_\varepsilon^{(\beta+p-1)y} \varphi \, \mathrm{d}x \right)^{1/y} \cdot \left(\int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}} a(x)^{-y'/y} \, \mathrm{d}x \right)^{1/y'} \tag{4.6}$$

with $1/y + 1/y' = 1$.

Next we can choose y satisfying

$$q + \beta = (\beta + p - 1) y,$$

i.e., $y = \frac{q+\beta}{\beta+p-1}$. We observe that if $\beta > 1 - p$ then $y > 1$ since, by assumption, $q > p - 1 > -\beta$.

Using now (4.3) and passing to the limit as $\varepsilon \to 0$, from (4.6) we get

$$\int a(x) u^{q+\beta} \varphi \, \mathrm{d}x \le c_0 c_\delta \int \frac{|D\varphi|^{py'}}{\varphi^{py'-1}} \rho^{\alpha y'/y}(x) \, \mathrm{d}x. \tag{4.7}$$

Now choose the cut-off function φ. Let $\chi_\eta \in C_0^\infty(\overline{\Omega}; [0,1])$ be such that

$$\chi_\eta(x) = \begin{cases} 1 & (\rho(x) \ge 2\eta), \\ 0 & (\rho(x) \le \eta), \end{cases}$$

and $|D\chi_\eta| \le c_3/\eta$ with $c_3 > 0$ independent of η.

We define φ by $\varphi = \varphi_\eta := \chi_\eta^\lambda$ with λ chosen sufficiently large.

By this choice of φ, using a standard change of variables (see [9]), we finally obtain

$$\int_{\Omega_\eta} a(x) u^{q+\beta} \, \mathrm{d}x \le k \eta^\sigma, \tag{4.8}$$

where $\Omega_\eta = \{x \in \Omega : \varphi_\eta(x) = 1\}$, $k > 0$ does not depend on η, and

$$\sigma = N - 1 - \frac{p(q+\beta)}{q-p+1} + \alpha \left(\frac{\beta + p - 1}{q - p + 1} \right). \tag{4.9}$$

Observe that for $\beta > 1 - p > -q$, we have

$$\sigma = N - 1 - \alpha - \frac{(p-\alpha)(q+\beta)}{q-p+1} > N - 1 - \alpha - (p - \alpha)$$
$$= N - 1 - p \ge 0. \tag{4.10}$$

(We have made use of the assumptions $p < \min(\alpha, q+1)$ and $p \le N - 1$.)

Passing to the limit as $\eta \downarrow 0$, we obtain

$$\int_\Omega a(x) u^{q+\beta}\,\mathrm{d}x \le 0. \tag{4.11}$$

This completes the proof. ∎

If we choose $\beta > 0$, the same technique allows us to prove a similar result for coercive problems. Indeed, the following theorem is valid.

THEOREM 4.2. *Let* $1 < p \le N - 1$ *and* $q > p - 1$. *Suppose that the function* $a : \Omega \setminus \{0\} \to \mathbb{R}_+$ *is continuous and that there exist constants* $\eta_0 > 0$, $c_0 > 0$, *and* $\alpha > p$ *such that a satisfies inequality* (4.3).

Then the problem

$$\begin{cases} \operatorname{div}(A(x,u,|Du|)Du) \ge a(x)u^q & \text{in } \Omega, \\ u \ge 0 & \text{in } \Omega, \\ u \in X_{q,\beta}^{a,p}(\Omega) \end{cases} \tag{4.12}$$

has only a trivial solution for any $0 < \beta < p - 1$.

Another important class of operators that can be studied are the so-called "mean curvature type" operators.

DEFINITION 4.3. Let $A : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Suppose that there exists $C > 0$ such that for any $t \ge 0$ we have

$$0 < A(t) \le C. \tag{4.13}$$

Then the operator T defined by

$$Tu = \operatorname{div}(A(|Du|)Du) \qquad \text{for } u \in X_{q,\beta}^{a,2}(\Omega) \tag{4.14}$$

is called a "mean curvature type operator" associated to the function A.

Some important examples are "mean curvature operator"

$$A(t) = \frac{1}{\sqrt{1+t^2}} \tag{4.15}$$

and the "generalized mean curvature operator"

$$A(t) = \frac{1}{(1+|t|^k)^s}, \qquad k, s > 0. \tag{4.16}$$

A slight modification of the proof of Theorem 4.1 gives the following result:

THEOREM 4.4. *Let $N \geq 2$. Assume that the function $A : \mathbb{R}_+ \to \mathbb{R}$ satisfies inequality* (4.13).

Suppose also that the function $a : \Omega \setminus \{0\} \to \mathbb{R}_+$ is continuous and that there exist constants $\eta > 0$, $c_0 > 0$, and $\alpha > 2$ such that a enjoys property (4.3).

Then problem (4.2) (*resp.* (4.12)) *with $q > 1$ has only a trivial solution in $X^{a,2}_{q,\beta}(\Omega)$ for any β such that $-1 < \beta < 0$ (resp. for any $0 < \beta < 1$).*

Finally, suppose that instead of (4.3) the function $a \in C(\overline{\Omega} \setminus \{0\}; \mathbb{R}_+)$ satisfies the inequality

$$a(x) \geq c|x|^{-\alpha} \qquad \left(x \in B_{\eta_0}\right) \tag{4.17}$$

for some $\eta_0, c > 0$ and $\alpha \geq p$, $\overline{B_{\eta_0}} \subset \Omega$. Then the following result holds.

THEOREM 4.5. *Let $A : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfy inequalities* (4.1) *with $1 < p < q + 1$, $p \leq N$, and let $a \in C(\overline{\Omega} \setminus \{0\}; \mathbb{R}_+)$ enjoy property* (4.17) *for some $\eta_0, c > 0$ and $\alpha > p$, $\overline{B_{\eta_0}} \subset \Omega$.*

Then for $1 - p < \beta < 0$ problem (4.2) *has no solutions $u \in X^{a,p}_{q,\beta}(\Omega)$ such that*

$$u(x) \geq \varepsilon > 0 \qquad a.e. \text{ on } \partial B_{\eta_0}. \tag{4.18}$$

Proof. From the weak comparison principle and (4.18) it follows that $u \geq \varepsilon$ in B_{η_0}. Hence, using (4.17), we obtain

$$\int_{B_\eta} u^{q+\beta}(x)\, \mathrm{d}x \geq k_1 \varepsilon^{q+\beta} \eta^{N-\alpha} \qquad \text{for all } 0 < \eta < \eta_0, \tag{4.19}$$

where $k_1 > 0$ does not depend on η and ε.

On the other hand, inequality (4.7) with a cut-off function $\varphi_\eta = \chi_\eta^\lambda$, $\chi_\eta \in C_0^\infty(\overline{\Omega}; [0,1])$ such that

$$\chi_\eta(x) = \begin{cases} 1 & \left(x \in \overline{B_\eta}\right), \\ 0 & \left(x \in \overline{\Omega} \setminus B_{2\eta}\right) \end{cases}$$

and λ sufficiently large implies

$$\int_{B_\eta} u^{q+\beta}(x)\, \mathrm{d}x \leq k_2 \eta^{\sigma+1} \tag{4.20}$$

with σ defined as in (4.8) and $k_2 > 0$ independent of η.

Comparing (4.19) and (4.20) and taking $\eta \downarrow 0$, we come to a contradiction which proves the theorem. ∎

Remark 4.6. In particular, Theorem 4.5 implies that if the function a satisfies (4.17), problem (4.2) has no nontrivial radial solutions in $X^{a,p}_{q,\beta}(\Omega)$. Moreover, if the function A is such that the strong maximum principle holds for (4.2), then from Theorem 4.5 it follows that this problem has no nontrivial solutions in $X^{a,p}_{q,\beta}(\Omega)$ at all.

5. NON-HOMOGENEOUS SYSTEMS

In this section, we extend the results of Section 4 to systems of quasilinear elliptic inequalities of second order.

In particular, consider the problem

$$\begin{cases} -\operatorname{div}(A(x,u,|Du|)Du) \geq a(x)v^{q_1} & \text{in } \Omega, \\ -\operatorname{div}(B(x,v,|Dv|)Dv) \geq b(x)u^{p_1} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

assuming that $A, B : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are Carathéodory functions such that for any $(x,s,t) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ we have

$$\begin{aligned} c_1|t|^{p-2} \leq A(x,s,t) \leq c_2|t|^{p-2}, \\ c_3|t|^{q-2} \leq B(x,s,t) \leq c_4|t|^{q-2}, \end{aligned} \tag{5.2}$$

where $c_i > 0$ $(i = 1, \ldots, 4)$.

We define the class of solutions as

$$S_\gamma = \{(u,v) : \Omega \to \mathbb{R}_+ \times \mathbb{R}_+, a(x)v^{q_1}u^\gamma, a(x)v^{q_1}, b(x)u^{p_1}v^\gamma, b(x)u^{p_1}, |Du|^p u^{\gamma-1}, |Dv|^q v^{\gamma-1} \in L^1_{loc}(\Omega)\}.$$

for some $\gamma \in \mathbb{R}$.

Then the following result holds.

THEOREM 5.1. *Let $N \geq 2$. Assume that the functions $A, B : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfy inequalities* (5.2) *with* $\max(p,q) < N - 1$, $\min(p,q) > 1$. *Suppose also that*

(i) $p - 1 < p_1$, $q - 1 < q_1$.

If $a, b \in C(\Omega)$ are non-negative functions such that

(ii) $a(x) \geq c\rho^{-\alpha}(x)$, $b(x) \geq d\rho^{-\beta}(x)$ *in* $\{x \in \Omega : \rho(x) \leq \eta_0\}$ *for some $c, d, \eta_0 > 0$ and α, β satisfying the assumption*

$$\alpha \geq p - \frac{q_1}{q-1}(q - \beta) \tag{5.3}$$

or, *alternatively*,

$$\beta \geq q - \frac{p_1}{p-1}(p-\alpha), \tag{5.4}$$

then problem (5.1) *has no nontrivial solutions* $(u, v) \in S_\gamma$ *for sufficiently small* $\gamma < 0$.

Proof. To prove this theorem, we adapt a method developed in [7–9] for quasilinear elliptic systems in $\mathbb{R}^N$ (see Theorem 5.1 of [9]).

Let $(u, v) \in W^{1,p}_{loc}(\Omega) \times W^{1,q}_{loc}(\Omega)$ be a solution to (5.1), and let $\varphi \in C_0^\infty(\overline{\Omega}; \mathbb{R}_+)$ be a standard cut-off function chosen as in Section 4.

By multiplying the first and second equation of (5.1) by $u_\varepsilon^\gamma \varphi$ and respectively by $v_\varepsilon^\gamma \varphi$, where $u_\varepsilon = u + \varepsilon$, $v_\varepsilon = v + \varepsilon$ with $\varepsilon > 0$, and integrating by parts we find

$$\int a(x) v^{q_1} u_\varepsilon^\gamma \varphi \, dx \leq \gamma \int A(x, u, |Du|) |Du|^2 u_\varepsilon^{\gamma-1} \varphi \, dx + \int A(x, u, |Du|)(Du, D\varphi) u_\varepsilon^\gamma \, dx,$$

$$\int b(x) u^{p_1} v_\varepsilon^\gamma \varphi \, dx \leq \gamma \int B(x, v, |Dv|) |Dv|^2 v_\varepsilon^{\gamma-1} \varphi \, dx + \int B(x, v, |Dv|)(Dv, D\varphi) v_\varepsilon^\gamma \, dx.$$

Using inequality (5.2), we obtain

$$\int a(x) v^{q_1} u_\varepsilon^\gamma \varphi \, dx \leq c_2 \left(\gamma \int |Du|^p u_\varepsilon^{\gamma-1} \varphi \, dx + \int |Du|^{p-1} |D\varphi| u_\varepsilon^\gamma \, dx \right),$$

$$\int b(x) u^{p_1} v_\varepsilon^\gamma \varphi \, dx \leq c_4 \left(\gamma \int |Dv|^q v_\varepsilon^{\gamma-1} \varphi \, dx + \int |Dv|^{q-1} |D\varphi| v_\varepsilon^\gamma \, dx \right).$$

Further, by the Young inequality with parameter $\delta > 0$ we have

$$\int a(x) v^{q_1} u_\varepsilon^\gamma \varphi \, dx + d''_\delta \int |Du|^p u_\varepsilon^{\gamma-1} \varphi \, dx \leq d'_\delta \int \frac{|D\varphi|^p}{\varphi^{p-1}} u_\varepsilon^{\gamma+p-1} \, dx, \tag{5.5}$$

$$\int b(x) u^{p_1} v_\varepsilon^\gamma \varphi \, dx + c''_\delta \int |Dv|^q v_\varepsilon^{\gamma-1} \varphi \, dx \leq c'_\delta \int \frac{|D\varphi|^q}{\varphi^{q-1}} v_\varepsilon^{\gamma+q-1} \, dx, \tag{5.6}$$

where the constants $c'_\delta, c''_\delta, d'_\delta, d''_\delta > 0$ depend only on c_2, c_4, p, q, γ, and $\delta > 0$. Next we multiply (5.1) by φ and obtain by the Hölder inequality

$$\int a(x) v^{q_1} \varphi \, \mathrm{d}x \le \left(\int |Du|^p u^{\gamma-1} \varphi \, \mathrm{d}x \right)^{(p-1)/p} \left(\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{(1-\gamma)(p-1)} \, \mathrm{d}x \right)^{1/p}, \tag{5.7}$$

$$\int b(x) u^{p_1} \varphi \, \mathrm{d}x \le \left(\int |Dv|^q v^{\gamma-1} \varphi \, \mathrm{d}x \right)^{(q-1)/q} \left(\int \frac{|D\varphi|^q}{\varphi^{q-1}} v^{(1-\gamma)(q-1)} \, \mathrm{d}x \right)^{1/q}. \tag{5.8}$$

By using (5.5) and (5.6) and passing to the limit as $\varepsilon \to 0$, this last estimate implies that

$$\int a(x) v^{q_1} \varphi \, \mathrm{d}x \le D_\delta \left(\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{\gamma+p-1} \, \mathrm{d}x \right)^{(p-1)/p} \cdot \left(\int \frac{|D\varphi|^p u^{(1-\gamma)(p-1)}}{\varphi^{p-1}} \, \mathrm{d}x \right)^{1/p}, \tag{5.9}$$

$$\int b(x) u^{p_1} \varphi \, \mathrm{d}x \le E_\delta \left(\int \frac{|D\varphi|^q}{\varphi^{q-1}} v^{\gamma+q-1} \, \mathrm{d}x \right)^{(q-1)/q} \cdot \left(\int \frac{|D\varphi|^q v^{(1-\gamma)(q-1)}}{\varphi^{q-1}} \, \mathrm{d}x \right)^{1/q}, \tag{5.10}$$

where E_δ and $D_\delta > 0$ depend only on c_2, c_4, p, q, γ, and $\delta > 0$.

Now we apply the Hölder inequality with parameters a, a' to the first integral on the right-hand side of (5.9) and we get

$$\left(\int \frac{|D\varphi|^p}{\varphi^{p-1}} u^{\gamma+p-1} \, \mathrm{d}x \right)^{(p-1)/p} \le \left(\int b(x) u^{(\gamma+p-1)a} \varphi \, \mathrm{d}x \right)^{(p-1)/pa} \left(\int b(x)^{-(a'/a)} \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}} \, \mathrm{d}x \right)^{(p-1)/pa'} \tag{5.11}$$

with $1/a + 1/a' = 1$. By choosing the parameter a so that

$$(\gamma + p - 1)a = p_1$$

from (5.9) and (5.11) we have

$$\int a(x)v^{q_1}\varphi\,\mathrm{d}x \le D_\delta\left(\int b(x)u^{p_1}\varphi\,\mathrm{d}x\right)^{(p-1)/pa} \cdot\left(\int b(x)^{-a'/a}\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,\mathrm{d}x\right)^{(p-1)/pa'} \cdot\left(\int \frac{|D\varphi|^{p}}{\varphi^{p-1}}u^{(1-\gamma)(p-1)}\,\mathrm{d}x\right)^{1/p}. \tag{5.12}$$

By repeating this procedure with parameters $y, y' > 1$ on the third integral of (5.12) we obtain

$$\int \frac{|D\varphi|^{p}}{\varphi^{p-1}}u^{(1-\gamma)(p-1)}\,\mathrm{d}x \le \left(\int b(x)u^{(1-\gamma)(p-1)y}\varphi\,\mathrm{d}x\right)^{1/y}\left(\int b(x)^{-y'/y}\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,\mathrm{d}x\right)^{1/y'}, \tag{5.13}$$

where $1/y + 1/y' = 1$. By choosing y so that $(1-\gamma)(p-1)y = p_1$ in (5.13) and taking into account (5.12) we deduce that

$$\int a(x)v^{q_1}\varphi\,\mathrm{d}x \le D_\delta\left(\int b(x)u^{p_1}\varphi\,\mathrm{d}x\right)^{(p-1)/pa} \cdot\left(\int b(x)^{-a'/a}\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,\mathrm{d}x\right)^{(p-1)/pa'} \cdot\left(\int b(x)u^{p_1}\varphi\,\mathrm{d}x\right)^{1/py}\left(\int b(x)^{-y'/y}\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,\mathrm{d}x\right)^{1/py'},$$

that is,

$$\int a(x)v^{q_1}\varphi\,\mathrm{d}x \le D_\delta\left(\int b(x)u^{p_1}\varphi\,\mathrm{d}x\right)^{(p-1)/pa+1/py} \cdot\left(\int b(x)^{-a'/a}\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,\mathrm{d}x\right)^{(p-1)/pa'} \cdot\left(\int b(x)^{-y'/y}\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,\mathrm{d}x\right)^{1/py'}, \tag{5.14}$$

where we have chosen the parameters a, y such that

$$\begin{cases} \dfrac{1}{y} + \dfrac{1}{y'} = 1, & (1-\gamma)(p-1)y = p_1, \\ \dfrac{1}{a} + \dfrac{1}{a'} = 1, & (\gamma + p - 1)a = p_1. \end{cases} \tag{5.15}$$

We observe that this choice of a and y is admissible by our assumption (i) provided $\gamma < 0$ is chosen sufficiently small. Introducing the new parameters b and z such that

$$\begin{cases} \dfrac{1}{z} + \dfrac{1}{z'} = 1, & (1-\gamma)(q-1)z = q_1, \\ \dfrac{1}{b} + \dfrac{1}{b'} = 1, & (\gamma + q - 1)b = q_1 \end{cases} \tag{5.16}$$

and estimating now the right-hand side of (5.10), we obtain

$$\begin{aligned} \int b(x) u^{p_1} \varphi \, \mathrm{d}x \le E_\delta \left(\int a(x) v^{q_1} \varphi \, \mathrm{d}x \right)^{(q-1)/qb + 1/qz} & \\ \cdot \left(\int a(x)^{-b'/b} \frac{|D\varphi|^{qb'}}{\varphi^{qb'-1}} \, \mathrm{d}x \right)^{(q-1)/qb'} & \\ \cdot \left(\int a(x)^{-z'/z} \frac{|D\varphi|^{qz'}}{\varphi^{qz'-1}} \, \mathrm{d}x \right)^{1/qz'} & . \end{aligned} \tag{5.17}$$

Combining (5.14) and (5.17), we finally get

$$\begin{aligned} \left(\int a(x) v^{q_1} \varphi \, \mathrm{d}x \right)^{1-mn} \le D_\delta E_\delta^n \left(\int a(x)^{-b'/b} \frac{|D\varphi|^{qb'}}{\varphi^{qb'-1}} \, \mathrm{d}x \right)^{n(q-1)/qb'} & \\ \cdot \left(\int a(x)^{-z'/z} \frac{|D\varphi|^{qz'}}{\varphi^{qz'-1}} \, \mathrm{d}x \right)^{n/qz'} & \\ \cdot \left(\int b(x)^{-a'/a} \frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}} \, \mathrm{d}x \right)^{(p-1)/pa'} & \\ \cdot \left(\int b(x)^{-y'/y} \frac{|D\varphi|^{py'}}{\varphi^{py'-1}} \, \mathrm{d}x \right)^{1/py'} & \end{aligned} \tag{5.18}$$

and

$$\left(\int b(x)u^{p_1}\varphi\,\mathrm{d}x\right)^{1-mn} \le E_\delta D_\delta^m\left(\int b(x)^{-a'/a}\frac{|D\varphi|^{pa'}}{\varphi^{pa'-1}}\,\mathrm{d}x\right)^{m(p-1)/pa'}$$

$$\cdot\left(\int b(x)^{-y'/y}\frac{|D\varphi|^{py'}}{\varphi^{py'-1}}\,\mathrm{d}x\right)^{m/py'}$$

$$\cdot\left(\int a(x)^{-b'/b}\frac{|D\varphi|^{qb'}}{\varphi^{qb'-1}}\,\mathrm{d}x\right)^{(q-1)/qb'}$$

$$\cdot\left(\int a(x)^{-z'/z}\frac{|D\varphi|^{qz'}}{\varphi^{qz'-1}}\,\mathrm{d}x\right)^{1/qz'}, \tag{5.19}$$

where

$$\begin{cases} n := \dfrac{p-1}{pa} + \dfrac{1}{py}, \\ m := \dfrac{q-1}{qb} + \dfrac{1}{qz}. \end{cases} \tag{5.20}$$

An easy computation, by taking into account (5.15) and (5.16), gives the explicit values of m and n, that is,

$$m = \frac{q-1}{q_1}, \qquad n = \frac{p-1}{p_1}. \tag{5.21}$$

Consequently from assumption (i) it follows that the exponent appearing on the left-hand side of (5.18)–(5.19) is such that

$$1 - mn = \frac{p_1q_1 - (p-1)(q-1)}{p_1q_1} > 0.$$

Next by the same change of variables as in Section 4 it follows that for any $0 < \eta < \eta_0$

$$\int_{\Omega_\eta} v^{q_1}\,\mathrm{d}x \le F_\delta \eta^{\sigma_1+\tau_1} \tag{5.22}$$

and

$$\int_{\Omega_\eta} u^{p_1}\,\mathrm{d}x \le G_\delta \eta^{\sigma_2+\tau_2}, \tag{5.23}$$

where F_δ and G_δ are positive and independent of η and

$$\sigma_1 = \frac{1}{1-mn}(nz_1 + z_2), \qquad \sigma_2 = \frac{1}{1-mn}(z_1 + mz_2),$$
$$\tau_1 = \frac{n}{1-mn}(\alpha m + \beta), \qquad \tau_2 = \frac{m}{1-mn}(\alpha + \beta n),$$
$$z_1 := \frac{N-1-qb'}{qb'}(q-1) + \frac{N-1-qz'}{qz'},$$
$$z_2 := \frac{N-1-pa'}{pa'}(p-1) + \frac{N-1-py'}{py'}.$$

By our choice of parameters (5.15)–(5.16) the explicit values of σ_1, σ_2, τ_1, and τ_2 are given by

$$\begin{cases} \sigma_1 = \sigma_v := N-1-p-(p-1)\left\{\dfrac{p(q-1)+qq_1}{p_1q_1-(p-1)(q-1)}\right\}, \\ \sigma_2 = \sigma_u := N-1-q-(q-1)\left\{\dfrac{q(p-1)+pp_1}{p_1q_1-(p-1)(q-1)}\right\}, \\ \tau_1 = \tau_v := \dfrac{(\alpha(q-1)+\beta q_1)(p-1)}{p_1q_1-(p-1)(q-1)}, \\ \tau_2 = \tau_u := \dfrac{(\alpha p_1 + \beta(p-1))(q-1)}{p_1q_1-(p-1)(q-1)}. \end{cases} \tag{5.24}$$

From assumptions (i), (ii) it follows that either $\sigma_1 + \tau_1 > 0$, or $\sigma_2 + \tau_2 > 0$. Letting $\eta \downarrow 0$, we accomplish the proof similarly to that of Theorem 4.1. ∎

Remark 5.2. If inequality (5.3) (resp. (5.4)) is strict, the same result holds for $p = \max(p,q) = N-1$ (resp. for $q = \max(p,q) = N-1$).

Remark 5.3. Similarly to the previous section, this result can be extended to coercive problems, to quasilinear elliptic systems containing operators of mean curvature type, and to those of the form

$$\begin{cases} -\operatorname{div}(A(x,u,|Du|)Du) \geq |x|^{-\alpha}u^a v^b & \text{in } \Omega, \\ -\operatorname{div}(A(x,v,|Dv|)Dv) \geq |x|^{-\beta}u^c v^d & \text{in } \Omega, \\ u, v \geq 0 & \text{in } \Omega \end{cases}$$

under appropriate assumptions on a, b, c, d, α, and β.

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