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Exponential stability for impulsive delay differential equations by Razumikhin method [☆]

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Abstract

In this paper, we study exponential stability for impulsive delay differential equation of the form

$$\dot{x}(t) = f(t, x_t), \qquad t \neq t_k,$$
$$\Delta x(t) = I_k(t, x_{t^-}), \quad t = t_k,\ k \in N.$$

By employing the Razumikhin technique and Lyapunov functions, several exponential stability criteria are established. Some examples are also discussed to illustrate our results.

Keywords: Razumikhin technique; Lyapunov function; Impulsive delay differential equation; Exponential stability

1. Introduction

Impulsive differential equations have attracted many researchers' attention due to their wide applications in many fields such as control technology, drug administration and threshold theory in biology and the like. Many classical results have been extended to

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impulsive systems [2,7,9–15]. By Lyapunov's direct method, various stability problems have been discussed for impulsive delay differential equations [3,10,17].

On the other hand, there have been many papers and monographs recently on stability analysis of delay differential equations, see [6,8,16] and references therein. The method of Lyapunov functions and Razumikhin technique have been widely applied to stability analysis of various delay differential equations, and they have also proved to be a powerful tool in the investigation of asymptotical properties of impulsive delay differential equations (see [18–20]).

Recently, Liu et al. [16] have investigated exponential stability for singularly perturbed delay systems without impulses by using the method of inequalities. Anokhin et al. [1] and Berezansky and Idels [4] have studied exponential stability, by using fundamental function and inequalities, for linear impulsive delay differential equations. However, there is little work done on exponential stability for impulsive delay differential equations by the Lyapunov–Razumikhin method. In this paper, we shall extend Lyapunov–Razumikhin method to general impulsive delay differential equations and establish some exponential stability criteria.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. Then in Section 3, we obtain several Razumikhin-type criteria on exponential stability for impulsive delay differential equations by using Lyapunov functions, and in the last section, we discuss some examples to illustrate our results.

2. Preliminaries

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers and R^n the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. Let N denote the set of positive integers, i.e., $N = \{1, 2, \ldots\}$.

Denote $\psi(t^+) = \lim_{s\to t^+} \psi(s)$ and $\psi(t^-) = \lim_{s\to t^-} \psi(s)$. For $a, b \in R$ with $a < b$ and for $S \subset R^n$, we define the following classes of functions:

$$\begin{aligned} PC\big([a,b],S\big) = \big\{\psi:[a,b]\to S \mid\ & \psi(t)=\psi(t^+),\ \forall t\in[a,b);\\ & \psi(t^-) \text{ exists in } S,\ \forall t\in(a,b],\ \text{and}\\ & \psi(t^-)=\psi(t) \text{ for all but at most}\\ & \text{a finite number of points } t\in(a,b]\big\},\\ PC\big([a,b),S\big) = \big\{\psi:[a,b)\to S \mid\ & \psi(t)=\psi(t^+),\ \forall t\in[a,b);\\ & \psi(t^-) \text{ exists in } S,\ \forall t\in(a,b),\ \text{and}\\ & \psi(t^-)=\psi(t) \text{ for all but at most}\\ & \text{a finite number of points } t\in(a,b)\big\}, \end{aligned}$$

and

$$PC\big([a,\infty),S\big) = \big\{\psi:[a,\infty)\to S \mid \forall c>a,\ \psi|_{[a,c]}\in PC\big([a,c],S\big)\big\}.$$

Given a constant $\tau > 0$, we equip the linear space $PC([-\tau,0],R^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau\leqslant s\leqslant 0}\|\psi(s)\|$.

Consider the impulsive delay system of the form

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t \in [t_{k-1}, t_k), \\ \Delta x(t) = I_k(t, x_{t^-}), & t = t_k,\ k \in N, \\ x_{t_0} = \phi, \end{cases} \tag{2.1}$$

where $f, I_k : R_+ \times PC([-\tau, 0], R^n) \to R^n$; $\phi \in PC([-\tau, 0], R^n)$; $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$; $\Delta x(t) = x(t^+) - x(t^-)$; and $x_t, x_{t^-} \in PC([-\tau, 0], R^n)$ are defined by $x_t(s) = x(t+s)$, $x_{t^-}(s) = x(t^- + s)$ for $-\tau \leqslant s \leqslant 0$, respectively.

Definition 2.1. Function $V : R_+ \times R^n \to R_+$ is said to belong to the class ν_0 if

(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$ and for each $x \in R^n$, $t \in [t_{k-1}, t_k)$, $k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists; and
(ii) $V(t, x)$ is locally Lipschitzian in all $x \in R^n$, and for all $t \geqslant t_0$, $V(t, 0) \equiv 0$.

Definition 2.2. Given a function $V : R_+ \times R^n \to R_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by

$$D^+V\big(t, \psi(0)\big) = \limsup_{h\to 0^+} \frac{1}{h}\Big[V\big(t+h, \psi(0) + hf(t, \psi)\big) - V\big(t, \psi(0)\big)\Big],$$

for $(t, \psi) \in R_+ \times PC([-\tau, 0], R^n)$.

Definition 2.3. The trivial solution of system (2.1) is said to be exponentially stable if, for any initial data $x_{t_0} = \phi$, there exist constants $\alpha > 0$, $M \geqslant 1$ such that

$$\big\|x(t, t_0, \phi)\big\| \leqslant M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \geqslant t_0. \tag{2.2}$$

We shall make the following assumptions.

(H1) $f(t, \psi)$ is *composite-PC*, i.e., if for each $t_0 \in R_+$ and $\alpha > 0$, where $[t_0, t_0+\alpha] \in R_+$, if $x \in PC([t_0 - \tau, t_0 + \alpha], R^n)$ and x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \alpha]$, then the composite function g defined by $g(t) = f(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \alpha], R^n)$.
(H2) $f(t, \psi)$ is *quasi-bounded*, i.e., if for each $t_0 \in R_+$ and $\alpha > 0$, where $[t_0, t_0+\alpha] \in R_+$, and for each compact set $F \in R^n$ there exists some $M > 0$ such that $\|f(t, \psi)\| \leqslant M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-\tau, 0], F)$.
(H3) For each fixed $t \in R_+$, $f(t, \psi)$ is a continuous function of ψ on $PC([-\tau, 0], R^n)$.
(H4) $f(t, 0) = 0$, and $I_k(t, 0) = 0$ for all $t \in R_+$, $k \in N$.

It is shown in [2] that under assumptions (H1)–(H3), the initial value problem (2.1) has a solution $x(t, t_0, \phi) \triangleq x(t)$ existing in a maximal interval I. If, in addition, $f(t, \psi)$ is locally Lipschitz in ψ, then the solution is unique. Assumption (H4) enables that system (2.1) admits a trivial solution $x(t) \equiv 0$.

3. Main results

In this section, we shall develop Lyapunov–Razumikhin methods and establish two theorems which provide sufficient conditions for exponential stability of the trivial solution of system (2.1).

Theorem 3.1. *Assume that hypotheses* (H1)–(H4) *are satisfied and there exist a function* $V \in \nu_0$*, and constants* $p > 0$, $c_1 > 0$, $c_2 > 0$, $\lambda > 0$, $d_k \geqslant 0$, $k \in N$*, such that the following conditions hold*:

(i) $c_1\|x\|^p \leqslant V(t,x) \leqslant c_2\|x\|^p$;
(ii) $D^+V(t,\varphi(0)) \leqslant -m(t)V(t,\varphi(0))$*, for all* $t \neq t_k$ *in* R_+ *whenever* $V(t,\varphi(0)) \geqslant V(t+s,\varphi(s))e^{-\int_{t-\tau}^{t} m(s)\,ds}$ *for* $s \in [-\tau,0]$*, where* $m(t) \in PC([t_0-\tau,\infty),R_+)$ *and* $\inf_{t\geqslant t_0-\tau} m(t) \geqslant \lambda$;
(iii) $V(t_k,\varphi(0)+I_k(t_k,\varphi)) \leqslant (1+d_k)V(t_k^-,\varphi(0))$*, with* $\sum_{k=1}^{\infty} d_k < \infty$*, and* $\varphi(0^-) = \varphi(0)$.

Then the trivial solution of system (2.1) *is exponentially stable.*

Proof. Let $x(t) = x(t,t_0,\phi)$ be a solution of system (2.1) and $V(t) = V(t,x(t))$. We shall show

$$V(t) \leqslant c_2 \prod_{i=0}^{k-1}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)\,ds}, \quad t \in [t_{k-1},t_k),\ k \in N,$$

where $d_0 = 0$. Let

$$Q(t) = \begin{cases} V(t) - c_2\prod_{i=0}^{k-1}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)\,ds}, & t \in [t_{k-1},t_k),\ k \in N,\\ V(t) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)\,ds}, & t \in [t_0-\tau,t_0]. \end{cases}$$

We need to show that $Q(t) \leqslant 0$ for all $t \geqslant t_0$. It is clear that $Q(t) \leqslant 0$ for $t \in [t_0-\tau,t_0]$ since $Q(t) \leqslant v(t) - c_2\|\phi\|_\tau^p \leqslant 0$ by condition (i).

Take $k = 1$, we shall show $Q(t) \leqslant 0$ for $t \in [t_0,t_1)$. In order to do this we let $\alpha > 0$ be arbitrary and show that $Q(t) \leqslant \alpha$ for $t \in [t_0,t_1)$. Suppose not, then there exists some $t \in [t_0,t_1)$ so that $Q(t) > \alpha$. Let $t^* = \inf\{t \in [t_0,t_1):\ Q(t) > \alpha\}$, since $Q(t) \leqslant 0 < \alpha$ for $t \in [t_0-\tau,t_0]$, we know $t^* \in (t_0,t_1)$. Note that $Q(t)$ is continuous on $[t_0,t_1)$, then $Q(t^*) = \alpha$ and $Q(t) \leqslant \alpha$ for $t \in [t_0-\tau,t^*]$.

Notice $V(t^*) = Q(t^*) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}$; and for $s \in [-\tau,0]$, we have

$$\begin{aligned} V(t^*+s) &= Q(t^*+s) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*+s} m(s)\,ds}\\ &\leqslant \alpha + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*-\tau} m(s)\,ds}\\ &\leqslant \left(\alpha + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}\right)e^{-\int_{t^*}^{t^*-\tau} m(s)\,ds}\\ &= V(t^*)e^{\int_{t^*-\tau}^{t^*} m(s)\,ds}. \end{aligned}$$

So by condition (ii), we have $D^+V(t^*) \leqslant -m(t^*)V(t^*)$, then we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + m(t^*)c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds} \\ &\leqslant -m(t^*)\big(V(t^*) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}\big) \\ &= -m(t^*)\alpha < 0, \end{aligned}$$

which contradicts the definition of t^*, so we get $Q(t) \leqslant \alpha$ for all $t \in [t_0, t_1)$. Let $\alpha \to 0^+$, we have $Q(t) \leqslant 0$ for $t \in [t_0, t_1)$.

Now we assume that $Q(t) \leqslant 0$ for $t \in [t_0, t_m)$, $m \geqslant 1$. We shall show that $Q(t) \leqslant 0$ for $t \in [t_0, t_{m+1})$.

By condition (iii), we have

$$\begin{aligned} Q(t_m) &= V(t_m) - c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t_m} m(s)\,ds} \\ &\leqslant (1+d_m)V\big(t_m^-\big) - c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t_m} m(s)\,ds} \\ &= (1+d_m)Q\big(t_m^-\big) \leqslant 0. \end{aligned}$$

Let $\alpha > 0$ be arbitrary, we need to show $Q(t) \leqslant \alpha$ for $t \in (t_m, t_{m+1})$. Suppose not, let $t^* = \inf\{t \in [t_m, t_{m+1})\colon\ Q(t) > \alpha\}$. Since $Q(t_m) \leqslant 0 < \alpha$, by the continuity of $Q(t)$, we get, $t^* > t_m$, $Q(t^*) = \alpha$ and $Q(t) \leqslant \alpha$ for $t \in [t_0, t^*]$.

Since $V(t^*) = Q(t^*) + c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}$, then for any $s \in [-\tau, 0]$, we have

$$\begin{aligned} V(t^*+s) &\leqslant Q(t^*+s) + c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*+s} m(s)\,ds} \\ &\leqslant \alpha + c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*-\tau} m(s)\,ds} \\ &\leqslant \left(\alpha + c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}\right) e^{-\int_{t^*}^{t^*-\tau} m(s)\,ds} \\ &= V(t^*) e^{\int_{t^*-\tau}^{t^*} m(s)\,ds}. \end{aligned}$$

Thus by condition (ii), we have $D^+V(t^*) \leqslant -m(t^*)V(t^*)$, and then we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + m(t^*)c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds} \\ &\leqslant -m(t^*)\left(V(t^*) - c_2 \prod_{i=0}^{m}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)\,ds}\right) \\ &= -m(t^*)\alpha < 0. \end{aligned}$$

Again this contradicts the definition of t^*, which implies $Q(t) \leqslant \alpha$ for all $t \in [t_m, t_{m+1})$. Let $\alpha \to 0^+$, we have $Q(t) \leqslant 0$ for all $t \in [t_m, t_{m+1})$. So $Q(t) \leqslant 0$ for all $t \in [t_0, t_{m+1})$. Thus by the method of induction, we get

$$V(t) \leqslant c_2 \prod_{i=0}^{k-1} (1 + d_i) \|\phi\|_\tau^p e^{-\int_{t_0}^t m(s)\,ds}, \quad t \in [t_{k-1}, t_k),\ k \in N.$$

By condition (i)–(iii), we have

$$c_1 \|x\|^p \leqslant V(t) \leqslant c_2 \prod_{i=0}^{k-1} (1 + d_i) \|\phi\|_\tau^p e^{-\int_{t_0}^t m(s)\,ds} \leqslant c_2 M \|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \geqslant t_0,$$

which yields

$$\|x\| \leqslant \left(\frac{c_2 M}{c_1}\right)^{\frac{1}{p}} \|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \geqslant t_0,$$

where $M = \prod_{i=1}^{\infty} (1 + d_i) < \infty$ since $\sum_{k=1}^{\infty} d_k < \infty$. Thus the proof is complete. □

Theorem 3.2. *Assume that hypotheses* (H1)–(H4) *are satisfied and there exist a function* $V \in \nu_0$, *and constants* $p > 0$, $q > 1$, $c_1 > 0$, $c_2 > 0$ *and* $\eta \geqslant \frac{\ln q}{\tau}$ *such that*:

(i) $c_1 \|x\|^p \leqslant V(t, x) \leqslant c_2 \|x\|^p$;
(ii) $D^+ V(t, \varphi(0)) \leqslant -\eta V(t, \varphi(0))$, *for all* $t \neq t_k$ *in* R_+ *whenever* $q V(t, \varphi(0)) \geqslant V(t + s, \varphi(s))$ *for* $s \in [-\tau, 0]$;
(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \leqslant \psi_k(V(t_k^-, \varphi(0)))$, *where* $\varphi(0^-) = \varphi(0)$, *and* $\psi_k(s)$ *is continuous*, $0 \leqslant \psi_k(as) \leqslant a\psi_k(s)$ *and* $\psi_k(s) \geqslant s$ *hold for any* $a \geqslant 0$ *and* $s \geqslant 0$, *and there exists* $H \geqslant 1$ *such that*

$$\psi_k\big(\psi_{k-1}\big(\ldots\big(\psi_1(s)\big)\ldots\big)\big)/s \leqslant H, \quad s > 0,\ k \in N.$$

Then the trivial solution of system (2.1) *is exponentially stable.*

Proof. Choose $q = e^{\lambda\tau} > 1$ for some $\lambda > 0$, we shall show

$$V(t) \leqslant \psi_{k-1}\big(\psi_{k-2}\big(\ldots\big(\psi_1\big(\psi_0\big(V(t_0)\big)\big)\big)\ldots\big)\big) e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in N,$$

where $\psi_0(s) = s$ for any $s \in R$. Let

$$Q(t) = \begin{cases} V(t) - \psi_{k-1}(\psi_{k-2}(\ldots(\psi_0(V(t_0)))\ldots)) e^{-\lambda(t-t_0)}, & t \in [t_{k-1}, t_k),\ k \in N, \\ Q(t_0), & t \in [t_0 - \tau, t_0]. \end{cases}$$

When $k = 1$, we shall show $Q(t) \leqslant 0$ for all $t \in [t_0, t_1)$. In order to do this, we shall show that $Q(t) \leqslant \alpha$ for any arbitrarily given $\alpha > 0$. Suppose that there exists some $t \in [t_0, t_1)$ so that $Q(t) > \alpha$. Let $t^* = \inf\{t \in [t_0, t_1)\colon\ Q(t) > \alpha\}$, since $Q(t_0) \leqslant V(t_0) - V(t_0) = 0 < \alpha$ and hence $Q(t) \leqslant \alpha$ for $t \in [t_0 - \tau, t_0]$, we know $t^* \in (t_0, t_1)$. Note that $Q(t)$ is continuous on $[t_0, t_1)$, then $Q(t^*) = \alpha$ and $Q(t) \leqslant \alpha$ for $t \in [t_0 - \tau, t^*]$.

Since $V(t^*) = Q(t^*) + V(t_0) e^{-\lambda(t^*-t_0)}$, then for $s \in [-\tau, 0]$, we have

$$V(t^*+s) = Q(t^*+s) + V(t_0)e^{-\lambda(t^*+s-t_0)} \leqslant \alpha + V(t_0)e^{-\lambda(t^*-t_0)}e^{\lambda\tau}$$
$$\leqslant \bigl(\alpha + V(t_0)e^{-\lambda(t^*-t_0)}\bigr)e^{\lambda\tau} = V(t^*)e^{\lambda\tau} \leqslant qV(t^*).$$

So by condition (ii), we have $D^+V(t^*) \leqslant -\eta V(t^*)$, then we have

$$D^+Q(t^*) = D^+V(t^*) + \lambda V(t_0)e^{-\lambda(t^*-t_0)} \leqslant -\eta V(t^*) + \lambda V(t_0)e^{-\lambda(t^*-t_0)}$$
$$\leqslant -\lambda\bigl(V(t^*) - V(t_0)e^{-\lambda(t^*-t_0)}\bigr) = -\lambda\alpha < 0,$$

which contradicts the definition of t^*, so we get $Q(t) \leqslant \alpha$ for all $t \in [t_0, t_1)$. Let $\alpha \to 0^+$, we have $Q(t) \leqslant 0$ for $t \in [t_0, t_1)$.

Now we assume that $Q(t) \leqslant 0$ for $t \in [t_0, t_k)$, $k \geqslant 1$. Then we shall show $Q(t) \leqslant 0$ for $t \in [t_0, t_{k+1})$. Let $\alpha > 0$ be arbitrary, we shall show $Q(t) \leqslant \alpha$ for $t \in (t_k, t_{k+1})$. Suppose not, let $t^* = \inf\{t \in [t_k, t_{k+1}):\ Q(t) > \alpha\}$.

By condition (iii), we have

$$\begin{aligned}
Q(t_k) &= V(t_k) - \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t_k-t_0)} \\
&\leqslant \psi_k\bigl(V\bigl(t_k^-\bigr)\bigr) - \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t_k-t_0)} \\
&\leqslant \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)e^{-\lambda(t_k-t_0)}\bigr) \\
&\quad - \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t_k-t_0)} \\
&\leqslant e^{-\lambda(t_k-t_0)}\psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr) \\
&\quad - \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t_k-t_0)} \\
&\leqslant 0.
\end{aligned}$$

Since $Q(t_k) \leqslant 0 < \alpha$, by the continuity of $Q(t)$, we have $t^* > t_k$, $Q(t^*) = \alpha$ and $Q(t) \leqslant \alpha$ for $t \in [t_0 - \tau, t^*]$.

Since $V(t^*) = Q(t^*) + \psi_k(\psi_{k-1}(\ldots(\psi_0(V(t_0)))\ldots))e^{-\lambda(t^*-t_0)}$; when $t^* + s \geqslant t_k$ for all $s \in [-\tau, 0]$, we have, for any $s \in [-\tau, 0]$,

$$\begin{aligned}
V(t^*+s) &= Q(t^*+s) + \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t^*+s-t_0)} \\
&\leqslant \alpha + \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t^*-\tau-t_0)} \\
&\leqslant \bigl(\alpha + \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t^*-t_0)}\bigr)e^{\lambda\tau} \\
&\leqslant V(t^*)e^{\lambda\tau} \leqslant qV(t^*).
\end{aligned}$$

When $t^* + s < t_k$ for some $s \in [-\tau, 0]$, note that $0 \leqslant \psi_k(as) \leqslant a\psi_k(s)$ and $\psi_k(s) \geqslant s$ hold for any $a \geqslant 0$ and $s \geqslant 0$, then we have, for any $s \in [-\tau, 0]$ and $m < k$, $m, k \in N$,

$$\begin{aligned}
&\psi_m\bigl(\psi_{m-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t^*+s-t_0)} \\
&\quad\leqslant \psi_k\bigl(\psi_{k-1}\bigl(\ldots\bigl(\psi_0\bigl(V(t_0)\bigr)\bigr)\ldots\bigr)\bigr)e^{-\lambda(t^*+s-t_0)}.
\end{aligned}$$

So in this case, we can also get $V(t^*+s) \leqslant qV(t^*)$ hold for all $s \in [-\tau, 0]$. Thus by condition (ii), we have $D^+V(t^*) \leqslant -\eta V(t^*)$, and then we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + \lambda\psi_k\big(\psi_{k-1}\big(\dots\big(\psi_0\big(V(t_0)\big)\big)\dots\big)\big)e^{-\lambda(t^*-t_0)} \\ &\leqslant -\eta V(t^*) + \lambda\psi_k\big(\psi_{k-1}\big(\dots\big(\psi_0\big(V(t_0)\big)\big)\dots\big)\big)e^{-\lambda(t^*-t_0)} \\ &\leqslant -\lambda\big(V(t^*) - \psi_k\big(\psi_{k-1}\big(\dots\big(\psi_0\big(V(t_0)\big)\big)\dots\big)\big)e^{-\lambda(t^*-t_0)}\big) \\ &\leqslant -\lambda\alpha < 0. \end{aligned}$$

Again this contradicts the definition of t^*, which implies $Q(t) \leqslant \alpha$ for all $t \in [t_k, t_{k+1})$. Let $\alpha \to 0^+$, we have $Q(t) \leqslant 0$ for all $t \in [t_k, t_{k+1})$. So $Q(t) \leqslant 0$ for all $t \in [t_0, t_{k+1})$ which proves, by the method of induction, $V(t) \leqslant \psi_k(\psi_{k-1}(\dots(\psi_0(V(t_0)))\dots))e^{-\lambda(t-t_0)}$ for $t \in [t_{k-1}, t_k)$, $k \in N$. By condition (iii), we have

$$\begin{aligned} \psi_k\big(\psi_{k-1}\big(\dots\big(\psi_0\big(V(t_0)\big)\big)\dots\big)\big)e^{-\lambda(t-t_0)} &= \psi_k\big(\psi_{k-1}\big(\dots\big(\psi_1\big(V(t_0)\big)\big)\dots\big)\big)e^{-\lambda(t-t_0)} \\ &\leqslant HV(t_0)e^{-\lambda(t-t_0)}, \quad t \geqslant t_0. \end{aligned}$$

Thus by condition (i), we have

$$c_1\|x\|^p \leqslant V(t) \leqslant c_2 H\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \geqslant t_0,$$

i.e.,

$$\|x\| \leqslant \left(\frac{c_2 H}{c_1}\right)^{\frac{1}{p}} \|\phi\|_\tau e^{-\frac{\lambda(t-t_0)}{p}}, \quad t \geqslant t_0,$$

which completes our proof. □

Corollary 3.1. *Assume that hypotheses* (H1)–(H4) *are satisfied and condition* (i), (ii) *of Theorem* 3.2 *hold, condition* (iii) *is replaced by*

(iii)* $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \leqslant \psi_k(V(t_k^-, \varphi(0)))$, *where* $\varphi(0^-) = \varphi(0)$ *and* $\psi_k(s) = (1 + \frac{k}{k^3+s^2})s$ *for all* $k \in N$,

then the trivial solution of system (2.1) *is exponentially stable.*

Proof. Notice

$$\psi_k(s) = \left(1 + \frac{k}{k^3+s^2}\right)s \leqslant |s|\left(1 + \frac{1}{k^2}\right), \quad k \in N,$$

then by Theorem 3.2, the result holds. □

4. Examples

To illustrate our results, we consider some examples.

Example 4.1. Consider the impulsive nonlinear delay differential equation

$$\begin{cases} x'(t) = -a(t)x(t) + \frac{b(t)}{1+x^2(t)}x(t-\tau), & t \geqslant t_0 = 0, \\ x(t_k) = (1 + c_k)x(t_k^-), & t_k = k,\ k \in N, \\ x_{t_0} = \phi, \end{cases} \tag{4.1}$$

where constants $\tau, c_k > 0$ with $\sum_{k=1}^{\infty} c_k < \infty$, functions $a \in C(R, R_+)$, $b \in C(R, R)$, $\phi \in PC([-\tau, 0], R^n)$. If $a(t) \geqslant |b(t)|e^{-\lambda\tau} + \lambda$, then the trivial solution of system (4.1) is exponentially stable.

Proof. Set $V(x) = V(t, x) = |x|$, $m(t) = \lambda$ for all $t \geqslant t_0 - \tau$, where $\lambda > 0$ is a constant, then we have

$$\begin{aligned} D^+V\big(t, \varphi(0)\big) &\leqslant \operatorname{sgn}\big(\varphi(0)\big)\left[-a(t)\varphi(0) + \frac{b(t)}{1+\varphi^2(0)}\varphi(-\tau)\right] \\ &\leqslant -a(t)\big|\varphi(0)\big| + \big|b(t)\big| \cdot \big|\varphi(-\tau)\big| \\ &\leqslant -a(t)V\big(\varphi(0)\big) + \big|b(t)\big| \cdot V\big(\varphi(-\tau)\big). \end{aligned} \tag{4.2}$$

For any solution $x(t)$ of Eq. (4.1) such that

$$V\big(t, \psi(0)\big) \geqslant V\big(t+s, \varphi(s)\big)e^{\int_{t-\tau}^{t} m(s)\,ds}, \quad \text{for } s \in [-\tau, 0],$$

we have $V(\varphi(-\tau)) \leqslant e^{-\lambda\tau} V(\varphi(0))$. Therefore,

$$D^+V\big(t, \varphi(0)\big) \leqslant \big[-a(t) + b(t)e^{-\lambda\tau}\big]V\big(\varphi(0)\big).$$

Since $a(t) \geqslant |b(t)|e^{-\lambda\tau} + \lambda$, it follows that

$$D^+V\big(t, \varphi(0)\big) \leqslant -\lambda V\big(\varphi(0)\big) \leqslant -m(t)V\big(\varphi(0)\big),$$

whenever $V(t, \varphi(0)) \geqslant V(t+s, \varphi(s))e^{\int_{t-\tau}^{t} m(s)\,ds}$ for $s \in [-\tau, 0]$, i.e., condition (ii) of Theorem 3.1 holds.

Moreover,

$$V\big(t_k, \varphi(0) + I_k(t_k, \varphi)\big) = (1 + c_k)V\big(t_k^-, \varphi(0)\big).$$

Thus by Theorem 3.1, the trivial solution of system (4.1) is exponentially stable. The numerical simulation of this example with initial function

$$\phi(t) = \begin{cases} 0, & t \in [-1, 0), \\ 1.7, & t = 0, \end{cases}$$

and $\lambda = \tau = 1$, $b(t) = t^2$, $a(t) = 2 + t^2$, $c_k = \frac{1}{2^k}$ is given in Fig. 1.

It should be noted that when $1 + x^2$ is not there, system (4.1) becomes the well-known linear case which has been studied by several authors, see, for example, [5,20]. □

Example 4.2. Consider the impulsive nonlinear delay differential equations

$$\begin{cases} x'(t) = -y(t)\sin(x(t-1)) - 4x(t) + y(t-1), & t \neq k, \\ y'(t) = x(t)\sin(x(t-1)) - 3y(t), & t \neq k, \\ x(t_k) = \big(1 + \frac{2}{k^2}\big)x(t_k^-), & t = k, \\ y(t_k) = \big(1 - \frac{3}{k^2}\big)y(t_k^-), & t = k,\ k \in N, \\ x_{t_0} = \phi, & t_0 = 0, \end{cases} \tag{4.3}$$

where $\phi \in PC([-\tau, 0], R^n)$.

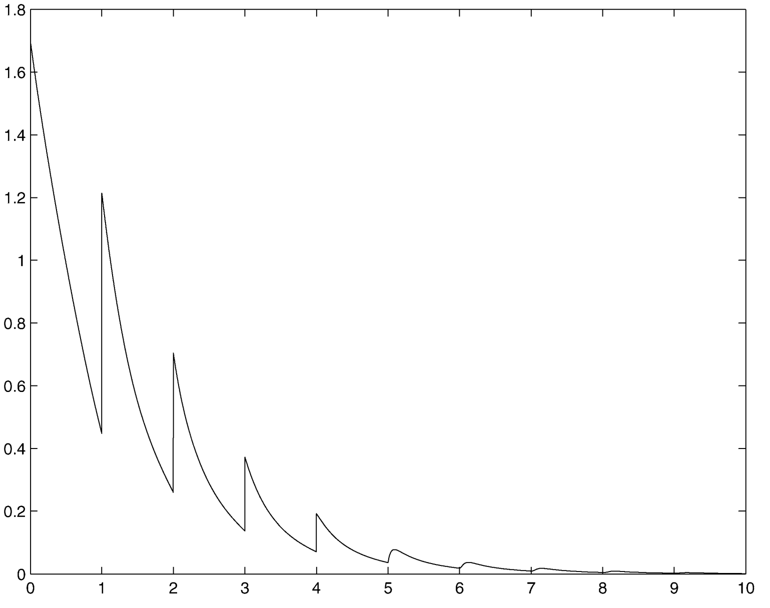

Fig. 1. Numerical simulation of Example 4.1.

Choose $V(x,y) = V(t,x,y) = x^2 + y^2$, then

$$\begin{aligned} D^+V(t,\varphi_1,\varphi_2) &= \varphi_1(0)\big(-\varphi_2(0)\sin\big(\varphi_1(-1)\big) - 4\varphi_1(0) + \varphi_2(-1)\big) \\ &\quad + \varphi_2(0)\big(-\varphi_1(0)\sin\big(\varphi_1(-1)\big) - 3\varphi_2(0)\big) \\ &= -4\varphi_1^2(0) + \varphi_1(0)\varphi_2(-1) - 3\varphi_2^2(0) \\ &\leqslant -4\varphi_1^2(0) + \frac{1}{2}\big(\varphi_1^2(0) + \varphi_2^2(-1)\big) - 3\varphi_2^2(0) \\ &\leqslant -6V\big(\varphi_1(0),\varphi_2(0)\big) + V\big(\varphi_1(-1),\varphi_2(-1)\big), \end{aligned}$$

let $q = 2$, $\eta = 4$, whenever $qV(\varphi_1(0),\varphi_2(0)) \geqslant V(\varphi_1(0),\varphi_2(0))$ for $s \in [-1,0]$, we have

$$\begin{aligned} D^+V(t,\varphi_1,\varphi_2) &\leqslant -6V\big(\varphi_1(0),\varphi_2(0)\big) + V\big(\varphi_1(-1),\varphi_2(-1)\big) \\ &\leqslant -6V\big(\varphi_1(0),\varphi_2(0)\big) + 2V\big(\varphi_1(0),\varphi_2(0)\big) \\ &\leqslant -4V\big(\varphi_1(0),\varphi_2(0)\big), \end{aligned}$$

i.e., condition (ii) of Theorem 3.2 holds.

At last, to check condition (iii), let $\psi_k(s) = (1 + \frac{5}{k^2})(s)$, $k \in N$, $s \in R$, then for any $k \in N$,

$$\begin{aligned} &V\left(\varphi_1(0) + \frac{2}{k^2}\varphi_1(0), \varphi_2(0) - \frac{3}{k^2}\varphi_2(0)\right) \\ &\quad = \frac{1}{2}\left(\left(1 + \frac{2}{k^2}\right)^2\varphi_1^2(0) + \left(1 - \frac{3}{k^2}\right)^2\varphi_2^2(0)\right) \leqslant \psi_k\big(V\big(\varphi_1(0),\varphi_2(0)\big)\big), \end{aligned}$$

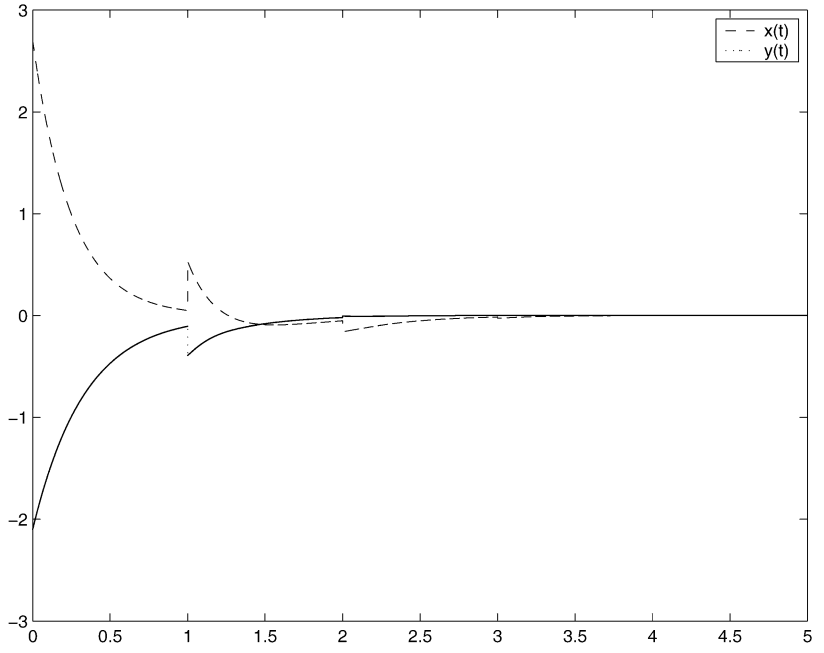

Fig. 2. Numerical simulation of Example 4.2.

i.e., condition (iii) is satisfied. Thus by Theorem 3.2, the trivial solution of system (4.3) is exponentially stable. The numerical simulation of this example with initial function

$$\phi_1(t) = \begin{cases} 0, & t \in [-1, 0), \\ 2.7, & t = 0, \end{cases} \qquad \phi_2(t) = \begin{cases} 0, & t \in [-1, 0), \\ -2.1, & t = 0, \end{cases}$$

is given in Fig. 2.

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