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On coreflexive coalgebras and comodules over commutative rings

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Abstract

In this paper we study dual coalgebras of algebras over arbitrary (Noetherian) commutative rings. We present and study a generalized notion of coreflexive comodules and use the results obtained for them to characterize the so called coreflexive coalgebras. Our approach in this note is an algebraically topological one.

MSC: 16D90; 16W30; 16Exx

0. Introduction

The concept of *coreflexive coalgebras* was studied, in the case of commutative base fields, by several authors. An algebraic approach was presented by Taft ([27,28]), while a topological one was presented mainly by Radford [12,23] and studied by several authors (e.g. [20,32]). In this paper we present and study a generalized concept of *coreflexive comodules* and use it to characterize coreflexive coalgebras over commutative (Noetherian) rings. In particular we generalize results in the papers mentioned above from the case of base fields to the case of arbitrary (Noetherian) commutative ground rings.

Throughout this paper R denotes a commutative ring with $1_R \neq 0_R$. We consider R as a left and a right linear topological ring with the *discrete topology*. The category of R-(bi)modules will be denoted by $\mathcal{M}_R$. The unadorned $-\otimes-$ and Hom

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mean $-\otimes_R-$ and Hom_R respectively. For R-modules M,N and a submodule K of M, the image of the canonical R-linear mapping $\imath_K\otimes id_N : K\otimes_R N\to M\otimes_R N$ is denoted by $\mathrm{Im}(K\otimes_R N)$. The R-submodule K is called *N-pure*, if $\imath_K\otimes id_N$ is injective (in this case $\mathrm{Im}(K\otimes_R N)=K\otimes_R N$). We call $K\subset M$ *pure* (in the sense of Cohn), if it is N-pure for every R-module N. For every R-module L, we denote with L^* the *algebraic dual* R-module of all R-linear maps from L to R. For two topologies $\mathfrak{T}$ and $\mathfrak{T}'$, we write $\mathfrak{T}\preceq\mathfrak{T}'$ to mean that $\mathfrak{T}$ is *coarser* than $\mathfrak{T}'$.

Let S be a ring. We consider every left (respectively right) S-module K as a right (respectively a left) module over $\mathrm{End}({}_SK)^{\mathrm{op}}$ (respectively $\mathrm{End}(K_S)$) and as a left (respectively a right) module over $\mathrm{Biend}({}_SK):=\mathrm{End}(K_{\mathrm{End}({}_SK)^{\mathrm{op}}})$ (respectively $\mathrm{Biend}(K_S):=\mathrm{End}({}_{\mathrm{End}(K_S)}K)^{\mathrm{op}}$), the *ring of biendomorphisms* of K (e.g. [31, 6.4]).

Let A be an R-algebra and M be an A-module. An A-submodule $N\subseteq M$ will be called *R-cofinite*, if M/N is finitely generated in $\mathscr{M}_R$. The class of all R-cofinite A-submodules of M is denoted with $\mathscr{K}_M$. We call M *cofinitely R-cogenerated*, if M/N is R-cogenerated for every R-cofinite A-submodule N of M. With $\mathscr{K}_A$ we denote the class of *all* R-cofinite A-ideals and define

$$A^\circ := \{f\in A^* \mid f(I)=0 \text{ for some } R\text{-cofinite ideal } I\lhd A\}.$$

If $\mathscr{K}_A$ is a filter (e.g. R is a Noetherian ring) then $A^\circ\subseteq A^*$ is an R-submodule with equality if and only if ${}_RA$ is finitely generated projective.

We assume the reader is familiar with the theory of Hopf Algebras. For any needed definitions or results the reader may refer to any of the classical books on the subject (e.g. [1,25,31]). For an R-coalgebra $(C,\Delta_C,\varepsilon_C)$ and an R-algebra (A,μ_A,η_A) we consider $\mathrm{Hom}_R(C,A)$ as an R-algebra with multiplication the *convolution product* $(f\star g)(c):=\sum f(c_1)g(c_2)$ and unity $\eta_A\circ\varepsilon_C$.

1. Preliminaries

In this section we present some definitions and lemmas.

Definition 1.1. Let $(C,\Delta_C,\varepsilon_C)$ be an R-coalgebra. We call an R-submodule $K\subseteq C$:
an *R-subcoalgebra* if and only if $K\subseteq C$ is pure and $\Delta_C(K)\subseteq K\otimes_R K$;
a *C-coideal* if and only if $K\subseteq\mathrm{Ker}(\varepsilon_C)$ and

$$\Delta_C(K)\subseteq\mathrm{Im}(\imath_K\otimes id_C)+\mathrm{Im}(id_C\otimes\imath_K);$$

a *right C-coideal* (respectively a *left C-coideal*, a *C-bicoideal*), if $K\subseteq C$ is C-pure and $\Delta_C(K)\subseteq K\otimes_R C$ (respectively $\Delta_C(K)\subseteq C\otimes_R K$, $\Delta_C(K)\subseteq(K\otimes_R C)\cap(C\otimes_R K)$).

1.2. Subgenerators. Let A be an R-algebra and K be a left A-module. We say a left A-module N is *K-subgenerated*, if N is isomorphic to a submodule of a K-generated left A-module (equivalently, if N is kernel of a morphism between K-generated left A-modules). The *full* subcategory of ${}_A\mathscr{M}$, whose objects are the K-subgenerated left A-modules is denoted by $\sigma[{}_AK]$. In fact $\sigma[{}_AK]\subseteq{}_A\mathscr{M}$ is the *smallest* Grothendieck full subcategory that contains K. If M is a left A-module then

$$\mathrm{Sp}(\sigma[_AK], M) := \sum f(N) : f \in \mathrm{Hom}_{A-}(N, M), \quad N \in \sigma[_AK]$$

is the biggest A-submodule of M that belongs to $\sigma[_AK]$. The reader is referred to [30,31] for the well-developed theory of categories of this type.

The linear weak topology

1.3. R-**pairings**. An *R-pairing* $P=(V,W)$ consists of R-modules V, W with an R-bilinear form

$$\alpha : V \times W \to R, (v, w) \mapsto \langle v, w \rangle.$$

If the induced R-linear mapping $\kappa_P : V \to W^*$ (respectively $\chi_P : W \to V^*$) is injective then we call P *left non-degenerate* (respectively *right non-degenerate*). If both κ_P and χ_P are injective then we call P *non-degenerate*.

For R-pairings (V,W) and (V',W') a morphism

$$(\xi, \theta) : (V', W') \to (V, W)$$

consists of R-linear mappings $\xi : V \to V'$ and $\theta : W' \to W$, such that

$$\langle \xi(v), w' \rangle = \langle v, \theta(w') \rangle \quad \text{for all } v \in V \quad \text{and} \quad w' \in W'.$$

The R-pairings with the morphisms described above (and the usual composition of pairings) build a category which we denote with $\mathscr{P}$. If $P=(V,W)$ is an R-pairing, $V' \subseteq V$ is an R-submodule and $W' \subseteq W$ is a (pure) R-submodule with $\langle V', W' \rangle = 0$ then $Q := (V/V', W')$ is an R-pairing, $(\pi, \imath_K) : (V/V', W') \to (V, W)$ is a morphism in $\mathscr{P}$ and we call $Q \subseteq P$ a (*pure*) *R-subpairing*.

Notation. Let $P=(V,W)$ be an R-pairing. For $X \subseteq V$ and $K \subseteq W$ set

$$X^{\perp} := \{w \in W | \langle X, w \rangle = 0\} \text{ respectively } K^{\perp} := \{v \in V | \langle v, K \rangle = 0\}.$$

We say $X \subseteq V$ (respectively $K \subseteq W$) is *orthogonally closed* with respect to P, if $X = X^{\perp\perp}$ (respectively $K = K^{\perp\perp}$). In case $V = W^*$, we set for every subset $X \subseteq W^*$ (respectively $K \subseteq W$) $\mathrm{Ke}(X) := \{w \in W | f(w) = 0 \text{ for every } f \in X\}$ (respectively $\mathrm{An}(K) =: \{f \in W^* | f(w) = 0 \text{ for every } w \in K\}$).

1.4. Let $P=(V,W)$ be an R-pairing. Then the class of R-submodules of V:

$$\mathscr{F}(0_V) := \{K^{\perp} | K \subseteq W \text{ is a finitely generated } R\text{-submodule}\}$$

is a filter basis consisting of R-submodule of V and induces on V a topology, the so called *linear weak topology* $V[\mathfrak{T}_{ls}(W)]$, such that $(V, V[\mathfrak{T}_{ls}(W)])$ is a linear topological R-module and $\mathscr{F}(0_V)$ is a neighborhood basis of 0_V. In particular we call $W^*[\mathfrak{T}_{ls}(W)]$ the finite topology. The properties of this topology were studied by several authors in the case of commutative base fields (e.g. [13,14,23]). We refer mainly to the recent work of the author [2] for the case of arbitrary ground rings.

1.1. The α-condition

In a joint work with Gómez-Torrecillas and Lobillo [4] on the category of comodules of coalgebras over arbitrary commutative base rings, we presented the so called *α-condition*. That condition has shown to be a natural assumption in the author's study of *duality theorems* for Hopf algebras [3]. We refer mainly to [2] for the properties of such pairings over arbitrary ground rings.

1.5. α-pairings. We say an R-pairing $P=(V,W)$ satisfies the *α-condition* (or P is an *α-pairing*), if for every R-module M the following map is injective

$$\alpha_M^P : M\otimes_R W \to \mathrm{Hom}_R(V,M), \quad \sum m_i\otimes w_i \mapsto \left[v\mapsto \sum m_i\langle v,w_i\rangle\right]. \tag{1}$$

With $\mathscr{P}^\alpha \subseteq \mathscr{P}$ we denote the *full* subcategory of R-pairings satisfying the α-condition. We call an R-pairing $P=(V,W)$ *dense*, if $\kappa_P(V)\subseteq W^*$ is dense (considering W^* with the finite topology). It's easy to see that $\mathscr{P}^\alpha\subseteq\mathscr{P}$ is closed under pure R-subpairings.

We say an R-module W *satisfies the α-condition*, if the R-pairing (W^*,W) satisfies the α-condition, i.e. for every R-module M the canonical R-linear morphism $\alpha_M^W : M\otimes_R W\to \mathrm{Hom}_R(W^*,M)$ in injective (equivalently, if ${}_RW$ is *locally projective* in the sense of Zimmermann-Huisgen [33]).

Remark 1.6. [2, Remark 2.2] Let $P=(V,W)\in\mathscr{P}^\alpha$. Then ${}_RW$ is R-cogenerated and flat. If R is perfect then ${}_RW$ turns to be projective.

Notation. Let W,W' be R-modules and consider for any R-submodules $X\subseteq W^*$ and $X'\subseteq W'^*$ the canonical R-linear mapping

$$\delta : X\otimes_R X' \to (W\otimes_R W')^*.$$

For $f\in X$ and $g\in X'$ set $f\underline{\otimes}g=\delta(f\otimes g)$, i.e.

$$(f\underline{\otimes}g)\left(\sum w_i\otimes w_i'\right) := \sum f(w_i)g(w_i') \text{ for every } \sum w_i\otimes w_i'\in W\otimes_R W'.$$

2. Measuring R-pairings

2.1. For an R-coalgebra C and an R-algebra A we call an R-pairing $P=(A,C)$ a *measuring R-pairing*, if the induced mapping $\kappa_P : A\to C^*$ is an R-algebra morphism. In this case C is an A-bimodule through the left and the right A-actions

$$a\rightharpoonup c := \sum c_1\langle a,c_2\rangle \quad\text{and}\quad c\leftharpoonup a := \sum\langle a,c_1\rangle c_2 \quad \text{for all } a\in A, c\in C. \tag{2}$$

Let (A,C) and (B,D) be measuring R-pairings. We say a morphism of R-pairings $(\xi,\theta):(B,D)\to(A,C)$ is a *morphism of measuring R-pairings*, if $\xi : A\to B$ is an R-algebra morphism and $\theta : D\to C$ is an R-coalgebra morphism. The category of measuring R-pairings and morphisms described above will be denoted by $\mathscr{P}_m$. With $\mathscr{P}_m^\alpha\subseteq\mathscr{P}_m$ we denote the *full* subcategory of measuring R-pairings satisfying the α-condition (we call these *measuring α-pairings*). If $P=(A,C)$ is a measuring R-pairing,

$D \subseteq C$ is an R-subcoalgebra and $I \triangleleft A$ is an ideal with $\langle I, D\rangle = 0$ then $Q := (A/I, C)$ is a measuring R-pairing, $(\pi_I, \imath_D) : (A/I, D) \to (A, C)$ is a morphism in $\mathscr{P}_m$ and we call $Q \subseteq P$ a *measuring R-subpairing*. Since by convention an R-subcoalgebra is a pure R-submodule, it is easy to see that $\mathscr{P}_m^{\alpha} \subseteq \mathscr{P}_m$ is closed under measuring R-subpairings.

Lemma 2.2. *Let $P=(A,C)$, $Q=(B,D) \in \mathscr{P}_m$ and $(\xi,\theta):(B,D) \to (A,C)$ be a morphism of R-pairings.*

1. *Assume that $P \otimes P := (A \otimes_R A, C \otimes_R C)$ is right non-degenerate (i.e. $\chi := \chi_{P\otimes P} : C \otimes_R C \hookrightarrow (A \otimes_R A)^*$ is an embedding). If ξ is an R-algebra morphism then θ is an R-coalgebra morphism. If A is commutative then C is cocommutative.*
2. *If Q is left non-degenerate (i.e. $B \overset{\kappa_Q}{\hookrightarrow} D^*$ is an embedding) and θ is an R-coalgebra morphism then ξ is an R-algebra morphism. If C is cocommutative and P is left non-degenerate (i.e. $A \subseteq C^*$) then A is commutative.*

Proof. 1. If ξ is an R-algebra morphism then we have for arbitrary $d \in D$, $a, \widetilde{a} \in A$:

$$\begin{aligned}
\chi\left(\sum \theta(d)_1 \otimes \theta(d)_2\right)(a \otimes \widetilde{a}) &= \sum \langle a, \theta(d)_1\rangle\langle \widetilde{a}, \theta(d)_2\rangle \\
&= \langle a\widetilde{a}, \theta(d)\rangle \\
&= \langle \xi(a\widetilde{a}), d\rangle \\
&= \langle \xi(a)\xi(\widetilde{a}), d\rangle \\
&= \sum \langle \xi(a), d_1\rangle\langle \xi(\widetilde{a}), d_2\rangle \\
&= \sum \langle a, \theta(d_1)\rangle\langle \widetilde{a}, \theta(d_2)\rangle \\
&= \chi\left(\sum \theta(d_1) \otimes \theta(d_2)\right)(a \otimes \widetilde{a}).
\end{aligned}$$

By assumption χ is injective and so $\sum \theta(d)_1 \otimes \theta(d)_2 = \sum \theta(d_1) \otimes \theta(d_2)$ for every $d \in D$, i.e. θ is an R-coalgebra morphism.

If A is commutative then we have for all $c \in C$ and $a, \widetilde{a} \in A$:

$$\begin{aligned}
\chi\left(\sum c_1 \otimes c_2\right)(a \otimes \widetilde{a}) &= \sum \langle a, c_1\rangle\langle \widetilde{a}, c_2\rangle \\
&= \langle a\widetilde{a}, c\rangle \\
&= \langle \widetilde{a}a, c\rangle \\
&= \sum \langle a, c_2\rangle\langle \widetilde{a}, c_1\rangle \\
&= \chi\left(\sum c_2 \otimes c_1\right)(a \otimes \widetilde{a}).
\end{aligned}$$

By assumption χ is injective and so $\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1$ for every $c \in C$, i.e. C is cocommutative.

2. The proof is analogous to that of (1). □

Notation. Let A be an R-algebra and N be a left A-module (respectively a right A-module). For subsets $X, Y \subseteq N$ we set

$$(Y : X) := \{a \in A | aX \subseteq Y\} \text{ (respectively } (Y : X) := \{a \in A | Xa \subseteq Y\}).$$

If $Y = \{0_N\}$ then we set also $\mathrm{Ann}_A(X) := (0_N : X)$. If N is an A-bimodule then we set for every $X \subseteq N$:

$$\mathrm{Ann}^l_A(X) := \{a \in A | aX = 0_N\} \text{ and } \mathrm{Ann}^r_A(X) := \{a \in A | Xa = 0_N\}.$$

2.3. The C-adic topology. Let $(A, C) \in \mathrm{P}_m$ and consider C as a left A-module with the left A-action "$\rightharpoonup$" in (2). Then the class of left A-ideals

$$\mathscr{B}_{C-}(0_A) := \{\mathrm{Ann}^l_A(W) = (0_C : W) | W = \{c_1, \ldots, c_k\} \subset C \text{ a finite subset}\}$$

is a neighborhood basis of 0_A and induces on A a topology, the so called *left C-adic-topology* $\mathscr{T}_{C-}(A)$, so that $(A, \mathscr{T}_{C-}(A))$ is a left linear topological R-algebra (see [6,7]). A left A-ideal $I \lhd_l A$ is open with respect to $\mathscr{T}_{C-}(A)$ if and only if A/I is C-subgenerated. If $\mathfrak{T}$ is a left linear topology on A then the category of discrete left $(A, \mathfrak{T})$-modules is equal to the category of C-subgenerated left A-modules $\sigma[_AC]$ if and only if $\mathfrak{T} = \mathscr{T}_{C-}(A)$. In particular we have for every left A-module N:

$$\mathrm{Sp}(\sigma[_AC], N) = \{n \in N | \exists F = \{c_1, \ldots, c_k\} \subset C \quad \text{with } \mathrm{Ann}^l_A(F) \subseteq (0_N : n)\}.$$

By [3, Lemma 2.2.4] the C-adic topology $\mathscr{T}_{C-}(A)$ and the linear weak topology $A[\mathfrak{T}_{ls}(C)]$ coincide. Hence A, with the linear weak topology $A[\mathfrak{T}_{ls}(C)]$, is a left linear topological R-algebra.

Analogously C_A induces on A a topology, the so called *right C-adic topology* $\mathscr{T}_{-C}(A)$, such that $(A, \mathscr{T}_{-C}(A))$ is a right linear topological R-algebra.

Rational Modules

2.4. Let $P = (A, C)$ be a measuring α-pairing. Let M be a left A-module, $\rho_M : M \to \mathrm{Hom}_R(A, M)$ be the canonical A-linear mapping and $\mathrm{Rat}^C(_AM) := \rho_M^{-1}(M \otimes_R C)$. In case $\mathrm{Rat}^C(_AM) = M$, we call M a *C-rational* left A-module and define

$$\varrho_M := (\alpha^P_M)^{-1} \circ \rho_M : M \to M \otimes_R C.$$

Analogously one defines the *C-rational right A-modules*. With $\mathrm{Rat}^C(_A\mathscr{M}) \subseteq {}_A\mathscr{M}$ (respectively ${}^C\mathrm{Rat}(\mathscr{M}_A) \subseteq \mathscr{M}_A$) we denote the *full* subcategory of C-rational left (respectively right) A-modules.

Lemma 2.5 ([3, Lemma 2.2.7]). *Let $P = (A, C)$ be a measuring α-pairing. For every left A-module M we have*:

1. $\mathrm{Rat}^C(_AM) \subseteq M$ *is an A-submodule*.
2. *For every A-submodule $N \subseteq M$ we have* $\mathrm{Rat}^C(_AN) = N \cap \mathrm{Rat}^C(_AM)$.
3. $\mathrm{Rat}^C(\mathrm{Rat}^C(_AM)) = \mathrm{Rat}^C(_AM)$.
4. *For every $L \in {}_A\mathscr{M}$ and $f \in \mathrm{Hom}_{A-}(M, L)$ we have* $f(\mathrm{Rat}^C(_AM)) \subseteq \mathrm{Rat}^C(_AL)$.

Theorem 2.6 ([3, Lemmas 2.2.8, 2.2.9, Satz 2.2.16]). *Let $P=(A,C)$ be a measuring R-pairing. Then $\mathcal{M}^C \subseteq {}_A\mathcal{M}$ and ${}^C\mathcal{M} \subseteq \mathcal{M}_A$ (not necessarily full subcategories). Moreover the following are equivalent*:

1. *P satisfies the α-condition*;
2. *${}_RC$ is locally projective and $\kappa_P(A) \subseteq C^*$ is dense.*
   If these equivalent conditions are satisfied then $\mathcal{M}^C \subseteq {}_A\mathcal{M}$ and ${}^C\mathcal{M} \subseteq \mathcal{M}_A$ are full *subcategories and we have category isomorphisms*

$$\begin{array}{llllll} \mathcal{M}^C & \simeq & \mathrm{Rat}^C({}_A\mathcal{M}) & = & \sigma[{}_AC] & \\ & \simeq & \mathrm{Rat}^C({}_{C^*}\mathcal{M}) & = & \sigma[{}_{C^*}C] & \text{and} \\ {}^C\mathcal{M} & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_A) & = & \sigma[C_A] & \\ & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_{C^*}) & = & \sigma[C_{C^*}]. & \end{array} \tag{3}$$

Corollary 2.7. *Let $Q=(B,C)\in\mathcal{P}_m$, $\xi : A \to B$ be an R-algebra morphism and consider the induced measuring R-pairing $P := (A,C)$. Then the following statements are equivalent*:

(i) $P \in \mathcal{P}_m^\alpha$;

(ii) *$Q \in \mathcal{P}_m^\alpha$ and $\xi(A) \subseteq B$ is dense (with respect to the left C-adic topology $\mathcal{T}_{C-}(B)$)*;

(iii) *C satisfies the α-condition and $\kappa_P(A) \subseteq C^*$ is dense.*

If these equivalent conditions are satisfied then we get category isomorphisms

$$\begin{array}{lllll} \mathcal{M}^C & \simeq & \mathrm{Rat}^C({}_A\mathcal{M}) & = & \sigma[{}_AC] \\ & \simeq & \mathrm{Rat}^C({}_{C^*}\mathcal{M}) & = & \sigma[{}_{C^*}C] \ \text{and} \\ & \simeq & \mathrm{Rat}^C({}_B\mathcal{M}) & = & \sigma[{}_BC] \\ {}^C\mathcal{M} & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_A) & = & \sigma[C_A] \\ & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_{C^*}) & = & \sigma[C_{C^*}] \\ & \simeq & {}^C\mathrm{Rat}(\mathcal{M}_B) & = & \sigma[C_B]. \end{array} \tag{4}$$

2.8. Let $(C,\Delta_C,\varepsilon_C)$ be an R-coalgebra and denote with $\mathrm{End}^C(C)$ (respectively ${}^C\mathrm{End}(C)$) the ring of all right (respectively left) C-colinear morphisms from C to C with the usual composition. For every right C-comodule M we have an isomorphism of R-modules

$$\Psi : M^* \to \mathrm{Hom}^C(M,C), \quad h \mapsto \left[m \mapsto \sum f(m_{\langle 0\rangle})m_{\langle 1\rangle}\right] \tag{5}$$

with inverse $g \mapsto \varepsilon_C \circ g$. Analogously $N^* \simeq {}^C\mathrm{Hom}(N,C)$ as R-modules for every left C-comodule N. In particular $C^* \simeq \mathrm{End}^C(C)^{\mathrm{op}}$ and $C^* \simeq {}^C\mathrm{End}(C)$ as R-algebras.

If (A,C) is a measuring α-pairing then we have R-algebra isomorphisms

$$\mathrm{Biend}({}_AC) := \mathrm{End}(C_{\mathrm{End}({}_AC)^{\mathrm{op}}}) \simeq \mathrm{End}(C_{\mathrm{End}^C(C)^{\mathrm{op}}}) \simeq \mathrm{End}(C_{C^*}) = {}^C\mathrm{End}(C) \simeq C^*$$

and

$$\mathrm{Biend}(C_A) := \mathrm{End}({}_{\mathrm{End}(C_A)}C)^{\mathrm{op}} \simeq \mathrm{End}({}_{^C\mathrm{End}(C)}C)^{\mathrm{op}} \simeq \mathrm{End}({}_{C^*}C)^{\mathrm{op}}$$

$$= \mathrm{End}^C(C)^{\mathrm{op}} \simeq C^*.$$

In particular, if ${}_RC$ is locally projective then $\mathrm{Biend}({}_{C^*}C) \simeq C^* \simeq \mathrm{Biend}(C_{C^*})$ as R-algebras (i.e. ${}_{C^*}C_{C^*}$ is *faithfully balanced*).

Corollary 2.9. *Let* $P=(A,C)\in\mathscr{P}_m^{\alpha}$ *and consider* A^* *as an* A*-bimodule with the* regular *A-actions*

$$(af)(\widetilde{a}) = f(\widetilde{a}a) \quad \text{and} \quad (fa)(\widetilde{a}) = f(a\widetilde{a}). \tag{6}$$

Then

1. *For every unitary left (respectively right) A-submodule* $D \subseteq A^*$ *we have*
$$\mathrm{Rat}^C({}_AD) = C\cap D \ (respectively\ {}^C\mathrm{Rat}(D_A) = C\cap D).$$
In particular $\mathrm{Rat}^C({}_AA^*) = C = {}^C\mathrm{Rat}(A_A^*)$.
2. *If* $D \subseteq A^*$ *is an A-subbimodule then* ${}_AD$ *is C-rational if and only if* D_A *is C-rational.*
3. *Let R be Noetherian. If* ${}_AA^\circ$ *(equivalently* A_A°*) is C-rational then* $C = A^\circ$.

Proof.

1. Let $D \subseteq A^*$ be a left A-submodule. By Lemma 2.5 (2) $C\cap D$ is a C-rational left A-module, i.e. $C\cap D \subseteq \mathrm{Rat}^C({}_AD)$. On the other hand, if $f\in\mathrm{Rat}^C({}_AD)$ with $\varrho_D(f)=\sum f_i\otimes c_i\in D\otimes_R C$ then we have for every $a\in A$:
$$f(a) = (af)(1_A) = \sum f_i(1_A)\langle a, c_i\rangle,$$
i.e. $f=\sum f_i(1_A)c_i\in C$. Hence $\mathrm{Rat}^C({}_AD) = C\cap D$. The corresponding result for right A-submodules $D\subseteq A^*$ follows by symmetry.
2. Let $D\subseteq A^*$ be an A-subbimodule. Then by (1) $\mathrm{Rat}^C({}_AD) = C\cap D = {}^C\mathrm{Rat}(D_A)$.
3. If R is Noetherian then $A^\circ \subseteq A^*$ is an A-subbimodule under the regular left and right A-actions (6). Obviously $C\overset{\chi_P}{\hookrightarrow}A^\circ$ and it follows by assumption and (1) that $A^\circ = \mathrm{Rat}^C({}_AA^\circ) = C\cap A^\circ = C$. □

An important role by the study of the category of rational representations of measuring α-pairings is played by the

2.10. Finiteness Theorem.

1. Let $P=(A,C)$ be a measuring α-pairing.
If $M\in\mathrm{Rat}^C({}_A\mathscr{M})$ then there exists for every finite set $\{m_1,\ldots,m_k\}\subset M$ some $N\in\mathrm{Rat}^C({}_A\mathscr{M})$, such that ${}_RN$ is finitely generated and $\{m_1,\ldots,m_k\}\subset N$.
If $M\in{}^C\mathrm{Rat}(\mathscr{M}_A)$ then there exists for every finite set $\{m_1,\ldots,m_k\}\subset M$ some $N\in{}^C\mathrm{Rat}(\mathscr{M}_A)$, such that ${}_RN$ is finitely generated and $\{m_1,\ldots,m_k\}\subset N$.

If $M \in {}^C\mathrm{Rat}^C({}_A\mathscr{M}_A)$, then there exists for every finite set $\{m_1, \dots, m_k\} \subset M$ some $N \in {}^C\mathrm{Rat}^C({}_A\mathscr{M}_A)$, such that ${}_RN$ is finitely generated and $\{m_1, \dots, m_k\} \subset N$.

2. Let C be a locally projective R-coalgebra. Then every finite subset of C is contained in a right C-coideal (respectively a left C-coideal, a C-bicoideal), that is finitely generated in $\mathscr{M}_R$.

Proof.

1. Assume that $P = (A, C) \in \mathscr{P}_m^\alpha$. Let $M \in \mathrm{Rat}^C({}_A\mathscr{M})$ and $\{m_1, \dots, m_k\} \subset M$. Then $Am_i \subseteq M$ is an A-submodule, hence a C-subcomodule. Moreover $m_i \in Am_i$ and so there exists a subset $\{(m_{ij}, c_{ij})\}_{j=1}^{n_i} \subset Am_i \times C$, such that $\varrho_M(m_i) = \sum_{j=1}^{n_i} m_{ij} \otimes c_{ij}$ for $i = 1, \dots, k$. Obviously $N := \sum_{i=1}^k Am_i = \sum_{i=1}^k \sum_{j=1}^{n_i} Rm_{ij} \subseteq M$ is a C-subcomodule and contains $\{m_1, \dots, m_k\}$.
   Using analogous arguments one can show the corresponding result for C-rational right A-modules and C-birational A-bimodules.
2. If C is a locally projective R-coalgebra then $(C^*, C) \in \mathscr{P}_m^\alpha$ and the result follows by (1). □

The following result gives topological characterizations of the C-rational left A-modules and generalizes the corresponding result obtained by Radford [23, 2.2] from the case of base fields to the case of arbitrary (Artinian) commutative ground rings (see also [15, Proposition 1.4.4]).

Proposition 2.11. *Let $P = (A, C)$ be a measuring α-pairing and consider A with the left C-adic topology $\mathscr{T}_{C-}(A) = A[\mathfrak{T}_{ls}(C)]$. If M is a unitary left A-module then for every $m \in M$ the following statements are equivalent*:

1. *there exists a finite subset $W = \{c_1, \dots, c_k\} \subset C$, such that $\mathrm{Ann}_A^l(W) \subseteq (0_M : m)$.*
2. *Am is C-subgenerated*;
3. *$m \in \mathrm{Rat}^C({}_AM)$.*
4. *there exists a finitely generated R-submodule $K \subseteq C$, such that $K^\perp \subseteq (0_M : m)$.*
   If R is Artinian then “1–4” *are equivalent to*:
5. *$(0_M : Am)$ contains an R-cofinite closed R-submodule of A*;
6. *$(0_M : Am)$ is an R-cofinite closed A-ideal*;
7. *$(0_M : m)$ contains an R-cofinite closed A-ideal*;
8. *$(0_M : m)$ is an R-cofinite closed left A-ideal.*

Proof. (1) ⇒ (2) By assumption and 2.3 we have $m \in N := \mathrm{Sp}(\sigma[{}_AC], M)$. Since $Am \subseteq N$ is an A-submodule, it is C-subgenerated.

(2) ⇒ (3) By assumption and Theorem 2.6 $m \in Am \subseteq \mathrm{Rat}^C({}_AM)$.

(3) ⇒ (4) Let $\varrho(m) = \sum_{i=1}^k m_i \otimes c_i$ and $K := \sum_{i=1}^k Rc_i \subseteq C$. Then obviously $K^\perp \subseteq (0_M : m)$.

(4) ⇒ (1) For every subset $W \subseteq C$ we have $\mathrm{Ann}_A^l(W) \subseteq W^\perp$.

Let R be Artinian.

(3) ⇒ (5). By Theorem 2.6 $\mathrm{Rat}^C({}_AM)$ is a C-rational left A-module. Assume that $\varrho_M(m) = \sum_{i=1}^k m_i \otimes c_i \in \mathrm{Rat}^C({}_AM) \otimes_R C$, $\varrho_M(m_i) = \sum_{j=1}^{n_i} m_{ij} \otimes c_{ij}$ for $i = 1, \ldots, k$ and set $K := \sum_{i=1}^k \sum_{j=1}^{n_i} Rc_{ij}$. Then we have for every $a \in K^\perp$ and arbitrary $b \in A$:

$$a(bm) = a\left(\sum_{i=1}^k m_i \langle b, c_i\rangle\right) = \sum_{i=1}^k \sum_{j=1}^{n_i} m_{ij}\langle a, c_{ij}\rangle\langle b, c_i\rangle = 0,$$

i.e. $K^\perp \subseteq (0_M : Am)$. The R-module K is finitely generated and it follows from the embedding $A/K^\perp \hookrightarrow K^*$, that $K^\perp \subseteq A$ is an R-cofinite R-submodule. Moreover $K^\perp$ is by [2, Lemma 1.7 (1)] closed.

Under the assumption that R is Artinian, the implications (5) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (4) follow from [2, Lemma 1.7 (4)]. □

Lemma 2.12. *Let C be an R-coalgebra and consider C^* with the finite topology. For every $f \in C^*$ the R-linear mappings*

$$\xi^r_f : C^* \to C^*, g \mapsto g \star f \quad \text{and} \quad \xi^l_f : C^* \to C^*, g \mapsto f \star g$$

are continuous. If R is an injective cogenerator then ξ^r_f and ξ^l_f are linearly closed (i.e. $\xi^r_f(X) \subseteq C^*$ and $\xi^l_f(X) \subseteq C^*$ are closed for every closed R-submodule $X \subseteq C^*$).

Proof. Consider for every $f \in C^*$ the R-linear mappings

$$\theta^l_f : C \to C, c \mapsto f \rightharpoonup c \quad \text{and} \quad \theta^r_f : C \to C, c \mapsto c \leftharpoonup f.$$

Then we have for every $g \in C^*$ and $c \in C$:

$$\xi^r_f(g)(c) = (g \star f)(c) = \sum g(c_1)f(c_2) = g(f \rightharpoonup c) = g(\theta^l_f(c)) = ((\theta^l_f)^*(g))(c).$$

So $\xi^r_f = (\theta^l_f)^*$ and analogously $\xi^l_f = (\theta^r_f)^*$. The result follows then by [2, Proposition 1.10]. □

If $P = (A, C) \in \mathscr{P}^\alpha_m$ then the Grothendieck category $\mathrm{Rat}^C({}_A\mathscr{M}) \simeq \sigma[{}_AC]$ is in general not closed under extensions:

Example 2.13 ([23, p. 520]). Let R be a *base field*, V be an infinite dimensional vector space over R and consider the R-coalgebra $C := R \oplus V$ (with $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\varepsilon(v) = 1_R$). Let $I \subseteq V^*$ be a vector subspace that is not closed, and consider the exact sequence of C^*-modules

$$0 \to V^*/I \to C^*/I \to C^*/V^* \to 0.$$

Then V^*/I and C^*/V^* are C-rational, while C^*/I is not.

Lemma 2.14 ([25, Lemma 6.1.1, Corollary 6.1.2]). *Let $I \lhd A$ be an ideal.*

1. *Let M be a finitely generated left (respectively right) A-module. If ${}_AI$ (respectively I_A) is finitely generated then also $IM \subseteq M$ (respectively $MI \subseteq M$) is a finitely generated A-submodule. If $I \subseteq A$ is R-cofinite then $IM \subseteq M$ (respectively $MI \subseteq M$) is an R-cofinite A-submodule.*

2. *If* ${}_AI$ *(respectively* I_A*) is finitely generated then* ${}_AI^n$ *(respectively* I_A^n*) is finitely generated for every* $n \geqslant 1$*. If moreover* $I \subseteq A$ *is R-cofinite then* $I^n \subseteq A$ *is R-cofinite.*

The following result generalizes [23, 2.5] from the case of base fields to the case of arbitrary commutative QF rings.

Proposition 2.15. *Let R be a QF Ring, C be a projective R-coalgebra and consider an exact sequence of left* C^**-modules*

$$0 \to N \overset{\iota}{\to} M \overset{\pi}{\to} L \to 0.$$

If $N, L \in \mathrm{Rat}^C({}_{C^*}\mathscr{M})$ *and* ${}_{C^*}(0 : l)$ *is finitely generated for every* $l \in L$ *then M is C-rational.*

Proof. Let $m \in M$ and $\{f_1, \ldots, f_k\}$ be a generating system of ${}_{C^*}(0_L : \pi(m))$. By assumption $\pi(m)$ is C-rational and so there exist by Proposition 2.11 R-cofinite closed A-ideals $J_i \subseteq (0_N : f_i m)$ for $i = 1, \ldots, k$. So we have for the closed R-cofinite A-ideal $J := \bigcap_{i=1}^k J_i \lhd C^*$:

$$J(0_L : \pi(m)) \rightharpoonup m = (J \star f_1 + \cdots + J \star f_k) \rightharpoonup m = 0,$$

i.e. $J(0_L : \pi(m)) \subseteq (0_M : m)$. By Lemmas 2.12, 2.14 and [2, Proposition 1.10 (3.d)] $J(0_L : \pi(m)) = \sum_{i=1}^k J \star f_i$ is R-cofinite and closed. It follows then by [2, Lemma 1.7 (4)] that $(0_M : m) \lhd_l C^*$ is R-cofinite and closed, hence $m \in \mathrm{Rat}^C({}_{C^*}M)$ by Proposition 2.11. □

Definition 2.16. An R-algebra A is called *nearly left Noetherian* (respectively *nearly right Noetherian*, *nearly Noetherian*), if every R-cofinite left (respectively right, two-sided) A-ideal is finitely generated in ${}_A\mathscr{M}$ (respectively in $\mathscr{M}_A$, in ${}_A\mathscr{M}_A$).

As a corollary of Theorem 2.6 and Proposition 2.15 we get

Corollary 2.17. *Let R be a QF Ring and C be a projective R-coalgebra. If* C^* *is nearly left Noetherian (respectively nearly right Noetherian) then* $\mathscr{M}^C \simeq \mathrm{Rat}^C({}_{C^*}C) = \sigma[{}_{C^*}C]$ *(respectively* ${}^C\mathscr{M} \simeq {}^C\mathrm{Rat}(\mathscr{M}_{C^*}) = \sigma[C_{C^*}]$*) is closed under extensions.*

Duality relations between substructures

As an application of our results in this section and our observations about the linear weak topology [2] we generalize known results on the duality relations between substructures of a coalgebra and substructures of its dual algebra from the case of base fields (e.g. [25,1] and [10, 1.5.29]) to the case of measuring α-pairings over arbitrary commutative rings.

As a consequence of Theorem 2.6 and [2, Theorem 1.8] we get

Proposition 2.18. *Let* $P = (A, C) \in \mathscr{P}_m$.

1. *Let* $K \subseteq C$ *be an R-submodule.*

If K is a right (respectively a left) C-coideal then $K^{\perp} = \mathrm{Ann}_A^r(K)$ (respectively $K^{\perp} = \mathrm{Ann}_A^l(K)$), hence a right (respectively a left) A-ideal.

If K is a C-bicoideal then $K^{\perp} = \mathrm{Ann}_A^r(K) \cap \mathrm{Ann}_A^l(K)$, hence a two-sided A-ideal.

2. *Let $P \in \mathscr{P}_m^{\alpha}$.*
   (a) *For every R-submodule $I \subseteq A$ we have:*
   If $I \subseteq A$ is a right (respectively a left) ideal then $I^{\perp} \subseteq C$ is a right (respectively a left) coideal;
   If $I \lhd A$ is a two-sided ideal (and $I^{\perp} \subseteq C$ is pure) then $I^{\perp} \subseteq C$ is a bicoideal (an R-subcoalgebra).
   (b) *Let R be an injective cogenerator. For a closed R-submodule $I \subseteq A$ we have:*
   I is a right (respectively a left) ideal if and only if $I^{\perp} \subseteq C$ is a right (respectively a left) coideal.
   I is a two-sided ideal (and $I^{\perp} \subseteq C$ is pure) if and only if $I^{\perp} \subseteq C$ is a bicoideal (an R-subcoalgebra).

Proposition 2.19.

1. *If $P = (A, C)$ is a measuring R-pairing and $K \subseteq C$ is a coideal then $K^{\perp} \subseteq A$ is an R-subalgebra with unity 1_A.*
2. *Let R be a QF Ring, C be a projective R-coalgebra and $A \subseteq C^*$ be an R-subalgebra (with $\varepsilon_C \in A$). If $\mathrm{Ke}(A) \subseteq C$ is pure then $\Delta_C(\mathrm{Ke}(A)) \subseteq \mathrm{Ke}(A) \otimes_R C + C \otimes_R \mathrm{Ke}(A)$ ($\mathrm{Ke}(A) \subseteq C$ is a C-coideal).*

Proof.

1. Obvious.
2. Let $A \subseteq C^*$ be an R-subalgebra and consider the canonical R-linear mappings

$$\kappa : A \otimes_R A \to (C \otimes_R C)^* \quad \text{and} \quad \chi : C \otimes_R C \to (A \otimes_R A)^*.$$

If $\mathrm{Ke}(A) \subseteq C$ is pure then it follows form [2, Proposition 1.10 (3.c), Corollary 2.9] that

$$\begin{aligned} \mathrm{Ke}(A) &= \mathrm{Ke}(\Delta_C^*(\kappa(A \otimes_R A))) \\ &= \Delta_C^{-1}(\mathrm{Ke}(\kappa(A \otimes_R A))) \\ &= \Delta_C^{-1}(\mathrm{Ke}(A) \otimes_R C + C \otimes_R \mathrm{Ke}(A)), \end{aligned} \tag{7}$$

i.e. $\Delta_C(\mathrm{Ke}(A)) \subseteq \mathrm{Ke}(A) \otimes_R C + C \otimes_R \mathrm{Ke}(A)$. If moreover $\varepsilon_C \in A$ then $\varepsilon_C(\mathrm{Ke}(A)) = 0$, i.e. $\mathrm{Ke}(A) \subseteq C$ is a C-coideal. □

As a consequence of Propositions 2.18, 2.19 and [2, Theorem 1.8] we get

Corollary 2.20. *Let R be an injective cogenerator and C a locally projective R-coalgebra. If we denote with $\mathscr{C}$ the class of all R-submodules of C and with $\mathscr{H}$ the class of all R-submodules of C^* then*

$$\mathrm{An}(-) : \mathscr{C} \to \mathscr{H} \quad \text{and} \quad \mathrm{Ke}(-) : \mathscr{H} \to \mathscr{C} \tag{8}$$

induce bijections

$$\begin{aligned}
\{K \subseteq C \text{ a right } C\text{-coideal}\} &\leftrightarrow \{I \lhd_r C^* \text{ a closed right } A\text{-ideal}\},\\
\{K \subseteq C \text{ a left } C\text{-coideal}\} &\leftrightarrow \{I \lhd_l C^* \text{ a closed left } A\text{-ideal}\},\\
\{K \subseteq C \text{ a } C\text{-bicoideal}\} &\leftrightarrow \{I \lhd C^* \text{ a closed two-sided ideal}\}, \qquad (9)\\
\{K \subseteq C \text{ an } R\text{-subcoalgebra}\} &\leftrightarrow \{I \lhd C^* \text{ a closed two-sided ideal},\\
&\qquad \mathrm{Ke}(I) \subseteq C \text{ pure}\}.
\end{aligned}$$

If R is moreover a QF ring then (8) *induces a bijection*

$$\{K \subseteq C \text{ a pure } C\text{-coideal}\} \leftrightarrow \{A \subseteq C^* \text{ a closed } R\text{-subalgebra}, \varepsilon_C \in A, \mathrm{Ke}(A) \subseteq C \text{ pure}\}.$$

3. Dual coalgebras

Every R-coalgebra $(C, \Delta_C, \varepsilon_C)$ has a *dual R-algebra*, namely C^* with multiplication the convolution product

$$\star : C^* \otimes_R C^* \xrightarrow{\delta} (C \otimes_R C)^* \xrightarrow{\Delta_C^*} C^*,$$

where δ is the canonical R-linear mapping, and with unity element ε_C. If (A, μ_A, η_A) is an R-algebra that is finitely generated projective as an R-module then A^* becomes an R-coalgebra with comultiplication given by

$$\mu_A^\circ : A^* \xrightarrow{\mu_A^*} (A \otimes_R A)^* \xrightarrow{\delta^{-1}} A^* \otimes_R A^*,$$

where $\delta : A^* \otimes_R A^* \to (A \otimes_R A)^*$ is the canonical isomorphism, and with counity $\eta_A^* : A^* \to R$. If A is not finitely generated projective then δ in not surjective anymore (and not even injective over arbitrary ground rings), hence μ_A° is not well defined and μ_A includes on A^* no R-coalgebra structure. However, if R is base field and we consider the R-algebra A with the left (respectively the right) cofinite topology $\mathrm{Cf}^l(A)$ (respectively $\mathrm{Cf}^r(A)$), see (3.20), then the *character module* A° of all continuous R-linear mappings from A to R is an R-coalgebra ([25, Proposition 6.0.2]). That result was generalized in [9] to the case of *Dedekind domains* and in [5] to the case of arbitrary *Noetherian* (*hereditary*) commutative rings.

In this section we consider coalgebra structures on the character module of an algebra, considered with a linear topology induced from a filter basis consisting of cofinite ideals over an arbitrary (Noetherian) ring.

3.1. Let A be an R-algebra and $\mathfrak{B}$ be a filter basis consisting of R-cofinite two-sided A-ideals. Then $\mathfrak{B}$ induces on A a left linear topology $\mathfrak{T}^l(\mathfrak{B})$, such that $(A, \mathfrak{T}^l(\mathfrak{B}))$ is a left linear topological R-algebra and $\mathfrak{B}$ is a neighborhood basis of 0_A. With

$$A_{\mathfrak{B}}^\circ := \{f \in A^* \mid \exists I \in \mathfrak{B}, \text{ such that } f(I) = 0\} = \varinjlim_{\mathfrak{B}} (A/I)^* \qquad (10)$$

we denote the *character module* of all continuous R-linear mappings from A to R (where R is considered as usual with the discrete topology). With the *completion* of A with respect to $\mathfrak{B}$ we mean

$$\widehat{A}_{\mathfrak{B}} := \varprojlim \{A/I \mid I \in \mathfrak{B}\}.$$

If $A^{\circ}_{\mathfrak{B}}$ is an R-coalgebra then we call $A^{\circ}_{\mathfrak{B}}$ the *continuous dual R-coalgebra of A with respect to* $\mathfrak{B}$.

Analogously $\mathfrak{B}$ induces on A a right linear topology $\mathfrak{T}^r(\mathfrak{B})$, such that $(A, \mathfrak{T}^r(\mathfrak{B}))$ is a right linear topological R-algebra and $\mathfrak{B}$ is a neighborhood basis of 0_A.

Remark 3.2. Let R be Noetherian and A be an R-algebra. Let I be an R-cofinite left A-ideal, say $A/I = \sum_{i=1}^{k} R(a_i + I)$, and consider the *two-sided* A-ideal

$$J_I := \bigcap_{i=1}^{k} (I : a_i) = (I : A) \subseteq (I : 1_A) = I.$$

Then

$$\varphi_I : A \to \mathrm{End}_R(A/I), \quad a \mapsto [b + I \mapsto ab + I]$$

is an R-algebra morphism with $\mathrm{Ker}(\varphi_I) = J_I$, i.e. J_I is an R-cofinite A-ideal.

Analogously one can show that every R-cofinite right A-ideal contains an R-cofinite *two-sided* A-ideal. □

The following result extends [5, 1.11] and [4, Remark 2.14]:

Theorem 3.3. *Let R be Noetherian and A an R-algebra. If $C \subseteq A^{\circ}$ is an A-*subbimodule *under the regular A-actions in* (6) *and $P := (A, C)$ then the following statements are equivalent*:

1. *${}_RC$ is locally projective and $\kappa_P(A) \subseteq C^*$ is dense*;
2. *${}_RC$ satisfies the α-condition and $\kappa_P(A) \subseteq C^*$ is dense*;
3. *(A, C) is an α-pairing*;
4. *$C \subset R^A$ is pure* (*in the sense of Cohn*);
5. *C is an R-coalgebra and $(A, C) \in \mathscr{P}^m_{\alpha}$*;
   If R is a QF Ring then "1-4" *are equivalent to*
6. *${}_RC$ is projective*.

Proof. The equivalences (1) if and only if (2) and (3) if and only if (4) follow from [2, Lemma 2.12, Proposition 2.5 (3)].

(2) ⇒ (3) follows from [2, Proposition 2.4 (2)].

(4) ⇒ (5) If $C \subset R^A$ is pure then by [5, 1.11] C is an R-coalgebra. It follows moreover for all $f \in C$ and arbitrary $a, \widetilde{a} \in A$ that

$$\kappa_P(a\widetilde{a})(f) = f(a\widetilde{a}) = \sum f_1(a) f_2(\widetilde{a}) = (\kappa_P(a) \underline{\otimes} \kappa_P(\widetilde{a}))(\Delta(f))$$
$$= (\kappa_P(a) \star \kappa_P(\widetilde{a}))(f)$$

and

$$\kappa_P(1_A)(f) = f(1_A) = \varepsilon_C(f) \quad \text{for all } f \in C.$$

So $\kappa_P : A \to C^*$ is an R-algebra morphism, i.e. $P \in \mathscr{P}_m$. By [2, Proposition 2.5] P satisfies the α-condition, hence $P \in \mathscr{P}_\alpha^m$.

(5) $\Rightarrow$ (2) follows from Theorem 2.6.

Let R be a QF ring.

(2) $\Rightarrow$ (6) follows from Remark 1.6.

(6) $\Rightarrow$ (2) If ${}_RC$ is projective then C satisfies the α-condition by [2, Proposition 2.14 (5)]. Consider the R-submodule $\kappa_P(A) \subseteq C^*$. By [2, Theorem 1.8 (1)] we have

$$\overline{\kappa_P(A)} := \mathrm{AnKe}(\kappa_P(A)) = \mathrm{An}(A^{\perp}) = \mathrm{An}(0_C) = C^*,$$

i.e. $\kappa_P(A) \subseteq C^*$ is dense. □

Definition 3.4. An R-algebra A is said to *satisfy the* α*-condition* or to be an α*-algebra*, if the class $\mathscr{K}_A$ of all R-cofinite A-ideals is a filter and the induced R-pairing (A, A°) satisfies the α-condition (in case R is Noetherian this is equivalent to the purity of $A^\circ \subset R^A$). An R-coalgebra C is said to *satisfy the* α*-condition* or to be an α*-coalgebra*, if the R-pairing (C^*, C) satisfies the α-condition (equivalently, if ${}_RC$ is locally projective). With $\mathbf{Alg}_R$ we denote the category of R-algebras and with $\mathbf{Alg}_R^\alpha \subseteq \mathbf{Alg}_R$ the *full* subcategory of α-algebras. Analogously, we denote with $\mathbf{Cog}_R$ the category of R-coalgebras and with $\mathbf{Cog}_R^\alpha \subseteq \mathbf{Cog}_R$ the *full* subcategory of α-coalgebras.

Remark 3.5. Let R be Noetherian and A be an α-algebra. Then there is obviously a 1-1 correspondence

$$\{P = (A, C) | P \in \mathscr{P}_m^\alpha\} \longleftrightarrow \{C | C \subseteq A^\circ \text{ is an } R\text{-subcoalgebra}\}.$$

Lemma 3.6.

1. *If C, D are R-coalgebras and $\theta : D \to C$ is an R-coalgebra morphism then $\theta^* : C^* \to D^*$ is an R-algebra morphism and*

$$(\theta^*, \theta) : (D^*, D) \to (C^*, C)$$

is a morphism in $\mathscr{P}_m$.

2. *Let R be Noetherian, A, B be α-algebras and $\xi : A \to B$ be an R-algebra morphism. Then we have a morphism in $\mathscr{P}_m^\alpha$*

$$(\xi, \xi^\circ) : (B, B^\circ) \to (A, A^\circ).$$

Proof.

1. Trivial.
2. If $f \in B^\circ$ then there exists an R-cofinite B-ideal $I \lhd B$, such that $f \in (B/I)^*$. By assumption R is Noetherian and so $\xi^{-1}(I) \subseteq A$ is an R-cofinite A-ideal, i.e. $\xi^\circ(f) \in A^\circ$ and we get a morphism of R-pairings

$$(\xi, \xi^\circ) : (B, B^\circ) \to (A, A^\circ).$$

By assumption ξ is an R-algebra morphism. Moreover the canonical R-linear mapping $A^\circ \otimes_R A^\circ \to (A \otimes_R A)^*$ is by [2, Corollary 2.8 (1)] an embedding, hence $\xi^\circ : B^\circ \to A^\circ$ is an R-coalgebra morphism by Lemma 2.2 (1). □

Lemma 3.7. *Let R be Noetherian, B an α-algebra and consider the α-pairing (B, B°). If $A \subseteq B$ is an α-subalgebra with $1_B \in A$ then $A^\perp := \mathrm{An}(A) \cap B^\circ$ is a B°-coideal.*

Proof. The embedding $\imath_A : A \hookrightarrow B$ is an R-algebra morphism and so $\imath_A^\circ : B^\circ \to A^\circ$ is by Lemma 2.2 (1) an R-coalgebra morphism. Hence $A^\perp := \mathrm{Ker}(\imath_A^\circ) \subseteq B^\circ$ is a B°-coideal. □

The following result follows directly from Propositions 2.18, 2.19 Lemma 3.6 and [2, Theorem 1.8]:

Corollary 3.8. *Let R be a QF Ring, A be an α-algebra, $P := (A, A^\circ)$ and consider A with the linear weak topology $A[\mathfrak{T}_{ls}(A^\circ)]$. Let $I \subseteq A$ be a closed R-submodule and set $I^\perp := \mathrm{An}(I) \cap A^\circ$. Then I is a right (respectively a left) A-ideal if and only if $I^\perp$ is a right (respectively a left) A°-coideal. Moreover $I \subseteq A$ is a two-sided A-ideal (and $I^\perp \subseteq A^\circ$ is pure) if and only if $I^\perp \subseteq A^\circ$ is an A°-bicoideal (an R-subcoalgebra).*

The convolution coalgebra

Dual to the convolution algebra, Radford presented in [23] the so called *convolution coalgebra* in the case of base fields. Over arbitrary Noetherian ground rings the following version of his definition makes sense:

3.9. Let R be Noetherian. If C is an R-coalgebra and A is an α-algebra then we call $A \star C := A^\circ \otimes_R C$ the *convolution coalgebra* of A and C. In the special case $C = R$ we have $A \star R \simeq A^\circ$.

The following result generalizes results of Radford [23] on the convolution coalgebra from the case of base fields to the case of arbitrary Noetherian ground rings:

3.10. Let R be Noetherian, C be a locally projective R-coalgebra and A be an α-algebra. It is easy to see then that $P := (A \otimes_R C^*, A \star C)$ is a measuring R-pairing, which satisfies the α-condition by [2, Lemma 2.8]. By [28, p. 515] the following mappings are R-algebra morphisms:

$$\beta: \ \mathrm{Hom}_R(C, A) \to (A^\circ \otimes_R C)^*, \quad f \mapsto [h \otimes c \mapsto h(f(c))].$$

$$\gamma: \ A \otimes_R C^* \to \mathrm{Hom}_R(C, A), \quad a \otimes g \mapsto [c \mapsto g(c)a].$$

By Corollary 2.7 $(\mathrm{Hom}_R(C, A), A \star C) \in \mathscr{P}_m^\alpha$, $\gamma(A \otimes_R C^*) \subseteq \mathrm{Hom}_R(C, A)$ is dense (with respect to the left C-adic topology) and we get category isomorphisms

$$\begin{aligned}\mathscr{M}^{A\star C} &\simeq \mathrm{Rat}^{A\star C}({}_{A\otimes_R C^*}\mathscr{M}) = \sigma[{}_{A\otimes_R C^*}(A\star C)] \\ &\simeq \mathrm{Rat}^{A\star C}({}_{(A\star C)^*}\mathscr{M}) = \sigma[{}_{(A\star C)^*}(A\star C)] \\ &\simeq \mathrm{Rat}^{A\star C}({}_{\mathrm{Hom}_R(C,A)}\mathscr{M}) = \sigma[{}_{\mathrm{Hom}_R(C,A)}(A\star C)].\end{aligned}$$

Proposition 3.11. *If R is Noetherian then we have bifunctors*

$$-\star- : \mathbf{Alg}_R^{\alpha} \times \mathbf{Cog}_R \to \mathbf{Cog}_R \quad \textit{and} \quad -\star- : \mathbf{Alg}_R^{\alpha} \times \mathbf{Cog}_R^{\alpha} \to \mathbf{Cog}_R^{\alpha}. \tag{11}$$

Proof. Let $A \in \mathbf{Alg}_R^{\alpha}$. Then A° is by Theorem 3.3 a locally projective R-coalgebra (i.e. an α-coalgebra). If C is a (locally projective) R-coalgebra then $A\star C := A^{\circ} \otimes_R C$ is a (locally projective) R-coalgebra by [2, Lemma 2.8]). One can see that (11) describes bifunctors by arguments parallel to those of [22]. □

Continuous dual coalgebras

Definition 3.12. Let A be an R-algebra, $\mathscr{K}_A$ be the class of *all* R-cofinite A-ideals and

$$\mathscr{E}_A := \{I \lhd A | A/I \text{ is finitely generated projective}\}.$$

For every subclass $\mathfrak{F} \subseteq \mathscr{K}_A$ set

$$A^{\circ}_{\mathfrak{F}} := \{f \in A^* | f(I) = 0 \text{ for some } I \in \mathfrak{F}\}.$$

1. We call a filter $\mathfrak{F} = \{I_\lambda\}_\Lambda$ consisting of R-cofinite A-ideals:
   an *α-filter*, if the R-pairing $(A, A^{\circ}_{\mathfrak{F}})$ satisfies the α-condition;
   cofinitary, if $\mathfrak{F} \cap \mathscr{E}_A$ is a filter basis of $\mathfrak{F}$;
   cofinitely R-cogenerated, if A/I is R-cogenerated for every $I \in \mathfrak{F}$.
2. We call A:
   an *α-algebra*, if $\mathscr{K}_A$ is an α-filter;
   cofinitary, if $\mathscr{K}_A$ is a cofinitary filter;
   cofinitely R-cogenerated, if A/I is R-cogenerated for every $I \in \mathscr{K}_A$.

Definition 3.13 ([26]). An R-coalgebra C is called *infinitesimal flat*, if $C = \varinjlim C_\lambda$ for a directed system of *finitely generated projective* R-subcoalgebras $\{C_\lambda\}_\Lambda$.

Proposition 3.14. *Let A be an R-algebra, $\mathfrak{F}$ be a filter consisting of R-cofinite A-ideals, $P := (A, A^{\circ}_{\mathfrak{F}})$ and consider A as a left (respectively a right) linear topological R-algebra with the induced topology $\mathfrak{T}(\mathfrak{F})$.*

1. *Assume $\mathfrak{F}$ to be cofinitely R-cogenerated. Then $\mathfrak{T}(\mathfrak{F})$ is Hausdorff if and only if $\kappa_P : A \to A^*_{\mathfrak{F}^{\circ}}$ is an embedding.*
2. *Assume R to be Noetherian and $\mathfrak{F}$ to be an α-filter. Then $A^{\circ}_{\mathfrak{F}}$ is an α-coalgebra, $(A, A^{\circ}_{\mathfrak{F}}) \in \mathscr{P}^{\alpha}_m$ and $\kappa_P(A) \subseteq A^*_{\mathfrak{F}^{\circ}}$ is dense with respect to the finite topology.*
3. *If A/I is R-reflexive for every $I \in \mathfrak{F}$, e.g. R is an injective cogenerator then $\widehat{A} \simeq A^*_{\mathfrak{F}^{\circ}}$ as left (respectively as right) linear topological R-modules.*

Proof.

1. By assumption A/I is R-cogenerated for every $I \in \mathfrak{F}$, hence

$$\overline{0_A} = \bigcap_{I \in \mathfrak{F}} I = \bigcap_{I \in \mathfrak{F}} \mathrm{KeAn}(I) = \mathrm{Ke}\left(\sum_{I \in \mathfrak{F}} \mathrm{An}(I)\right) = \mathrm{Ke}(A^{\circ}_{\mathfrak{F}}) = \mathrm{Ke}(\kappa_P).$$

2. Every $I \in \mathfrak{F}$ is a two-sided A-ideal and so $A^{\circ}_{\mathfrak{F}} \subseteq A^{\circ}$ is an A-subbimodule. The result follows then from Theorem 3.3.
3. If A/I is R-reflexive for every $I \in \mathfrak{F}$ then we have isomorphisms of topological R-modules

$$\widehat{A} = \underset{\to\ \mathfrak{F}}{\mathit{lim}}\, A/I \simeq \underset{\to\ \mathfrak{F}}{\mathit{lim}}\, (A/I)^{**} \simeq (\underset{\to\ \mathfrak{F}}{\mathit{lim}}\, (A/I)^{*})^{*} =: (A^{\circ}_{\mathfrak{F}})^{*}.$$

   If R is an injective cogenerator then all finitely generated R-modules are R-reflexive (e.g. [31, 48.13]) and we are done. □

The following result extends observations in [16] (respectively [4]) on *cofinitary algebras* over Dedekind domains (respectively Noetherian rings) to the case of *cofinitary filters* for algebras over arbitrary commutative base rings:

Proposition 3.15. *Let A be an R-algebra, $\mathfrak{F}$ be a filter consisting of R-cofinite A-ideals, $P := (A, A^{\circ}_{\mathfrak{F}})$ and consider A as a left (respectively as a right) linear topological R-algebra with the induced left (respectively right) linear topology $\mathfrak{T}(\mathfrak{F})$. If $\mathfrak{F}$ is cofinitary then*

1. *$\mathfrak{T}(\mathfrak{F})$ is Hausdorff if and only if $\kappa_P : A \to A^{*}_{\mathfrak{F}^{\circ}}$ is an embedding.*
2. *$A^{\circ}_{\mathfrak{F}}$ is an infinitesimal flat α-coalgebra, $P \in \mathscr{P}^{\alpha}_{m}$ and $\kappa_P(A) \subseteq A^{*}_{\mathfrak{F}^{\circ}}$ is dense.*
3. *$\widehat{A} \simeq A^{*}_{\mathfrak{F}^{\circ}}$ as left (respectively as right) linear topological R-algebras.*

Proof.

1. For every $I \in \mathscr{E}_A$ the R-module A/I is in particular R-cogenerated and the result follows from Proposition 3.14 (1).
2. For $I, J \in \mathfrak{F} \cap \mathscr{E}_A$ set $I \leqslant J$ if $I \supseteq J$ and consider the canonical R-algebra epimorphism $\pi_{I,J} : A/J \to A/I$. Then

$$\{((A/I)^{*}, \pi^{*}_{I,J}) | I \in \mathfrak{F} \cap \mathscr{E}_A, \pi^{*}_{I,J} : (A/I)^{*} \hookrightarrow (A/J)^{*}\}$$

   is a directed system of finitely generated projective R-coalgebras with R-coalgebra morphisms $\pi^{*}_{I,J} : (A/I)^{*} \to (A/J)^{*}$. Then $A^{\circ}_{\mathfrak{F}} = A^{\circ}_{\mathfrak{F} \cap \mathscr{E}_A} \simeq \underset{\to\ \mathfrak{F} \cap \mathscr{E}_A}{\mathit{lim}}\, (A/I)^{*}$ is an infinitesimal flat R-coalgebra.

   Let M be an arbitrary R-module. If $\sum_{i=1}^{k} m_i \otimes g_i \in \mathrm{Ker}(\alpha^{P}_{M})$ then there exists $I \in \mathfrak{F} \cap \mathscr{E}_A$, such that $\{g_1, \ldots, g_n\} \subset \mathrm{An}(I)$. If $\{(a_l + I, f_l)\}_{l=1}^{k}$ is a dual basis for $(A/I)^{*}$

then

$$\sum_{i=1}^{n} m_i \otimes g_i = \sum_{i=1}^{n} m_i \otimes \left(\sum_{l=1}^{k} g_i(a_l + I) f_l \right)$$
$$= \sum_{i=1}^{n} m_i \otimes \left(\sum_{l=1}^{k} g_i(a_l) f_l \right)$$
$$= \sum_{l=1}^{k} \left(\sum_{i=1}^{n} g_i(a_l) m_i \right) \otimes f_l = 0.$$

Obviously the canonical R-linear mapping $\kappa_P : A \to A^{*}_{\mathfrak{F}^\circ}$ is an R-algebra morphism, i.e. P is a measuring α-pairing. The density of $\kappa_P(A) \subseteq A^{*}_{\mathfrak{F}^\circ}$ follows then by Theorem 3.3.

3. For every $I \in \mathfrak{F} \cap \mathscr{E}_A$ the R-module A/I is finitely generated projective, hence $(A/I)^*$ is an R-coalgebra and $(A/I)^{**} \simeq A/I$ as R-algebras. So we have an isomorphisms of topological R-algebras

$$\widehat{A} = \underset{\leftarrow\ \mathfrak{F}\cap\mathscr{E}_A}{lim} A/I \simeq \underset{\leftarrow\ \mathfrak{F}\cap\mathscr{E}_A}{lim} (A/I)^{**} \simeq (\underset{\rightarrow\ \mathfrak{F}\cap\mathscr{E}_A}{lim} (A/I)^*)^* = (\underset{\overrightarrow{\mathfrak{F}}}{lim}(A/I)^*)^* = (A^{\circ}_{\mathfrak{F}})^*.$$

□

As a consequence of Propositions 3.14, 3.15 and Theorem 2.6 we get

Corollary 3.16. *Let A be an R-algebra and $\mathfrak{F}$ be a filter consisting of R-cofinite A-ideals. If R is Noetherian and $\mathfrak{F}$ is an α-filter, or if $\mathfrak{F}$ is cofinitary, then we have isomorphisms of categories*

$$\begin{aligned} \mathscr{M}^{A^{\circ}_{\mathfrak{F}}} &\simeq \mathrm{Rat}^{A^{\circ}_{\mathfrak{F}}}({}_A\mathscr{M}) = \sigma[{}_A A^{\circ}_{\mathfrak{F}}] \\ &\simeq \mathrm{Rat}^{A^{\circ}_{\mathfrak{F}}}({}_{A^{*}_{\mathfrak{F}^\circ}}\mathscr{M}) = \sigma[{}_{A^{*}_{\mathfrak{F}^\circ}} A^{\circ}_{\mathfrak{F}}] \end{aligned} \quad \textit{and}$$

$$\begin{aligned} {}^{A^{\circ}_{\mathfrak{F}}}\mathscr{M} &\simeq {}^{A^{\circ}_{\mathfrak{F}}}\mathrm{Rat}(\mathscr{M}_A) = \sigma[A^{\circ}_{\mathfrak{F}A}] \\ &\simeq {}^{A^{\circ}_{\mathfrak{F}}}\mathrm{Rat}(\mathscr{M}_{A^{*}_{\mathfrak{F}^\circ}}) = \sigma[A^{\circ}_{\mathfrak{F}A^{*}_{\mathfrak{F}^\circ}}]. \end{aligned}$$

3.17. Let A, B be R-algebras, $\mathfrak{F}_A$, $\mathfrak{F}_B$ be filter bases consisting of R-cofinite A-ideals, B-ideals respectively and

$$\mathfrak{F}_A \times \mathfrak{F}_B := \{\mathrm{Im}(\imath_I \otimes id_B) + \mathrm{Im}(id_A \otimes \imath_J) | I \in \mathfrak{F}_A, J \in \mathfrak{F}_B\}. \quad (12)$$

Obviously $\mathfrak{F}_A \times \mathfrak{F}_B$ is a filter basis consisting of R-cofinite $A \otimes_R B$-ideals and induces so a linear topology $\mathfrak{T}(\mathfrak{F}_A \times \mathfrak{F}_B)$ on $A \otimes_R B$, such that $(A \otimes_R B, \mathfrak{T}(\mathfrak{F}_A \times \mathfrak{F}_B))$ is a linear topological R-algebra and $\mathfrak{F}_A \times \mathfrak{F}_B$ is a neighborhood basis of $0_{A \otimes_R B}$.

One can generalize [4, Proposition 4.9, Theorem 4.10] to obtain

Theorem 3.18. *Let A, B be R-algebras, $\mathfrak{F}_A$, $\mathfrak{F}_B$ be filters consisting of R-cofinite A-deals, B-ideals respectively and consider the canonical R-linear mapping $\delta : A^* \otimes_R B^* \to (A \otimes_R B)^*$.*

1. *If $\mathfrak{F}_A$ and $\mathfrak{F}_B$ are cofinitary then the filter of $A \otimes_R B$-ideals generated by $\mathfrak{F}_A \times \mathfrak{F}_B$ is cofinitary and $(A \otimes_R B)^{\circ}_{\mathfrak{F}_A \times \mathfrak{F}_B}$ is an R-coalgebra. If R is Noetherian then δ induces an R-coalgebra isomorphism*
$$A^{\circ}_{\mathfrak{F}_A} \otimes_R B^{\circ}_{\mathfrak{F}_B} \simeq (A \otimes_R B)^{\circ}_{\mathfrak{F}_A \times \mathfrak{F}_B}.$$
2. *Let R be Noetherian. If $\mathfrak{F}_A$ is an α-filter and $\mathfrak{F}_B$ is cofinitary then the filter generated by $\mathfrak{F}_A \times \mathfrak{F}_B$ is an α-filter, $(A \otimes_R B)^{\circ}_{\mathfrak{F}_A \times \mathfrak{F}_B}$ is an R-coalgebra and δ induces an R-coalgebra isomorphism*
$$A^{\circ}_{\mathfrak{F}_A} \otimes_R B^{\circ}_{\mathfrak{F}_B} \simeq (A \otimes_R B)^{\circ}_{\mathfrak{F}_A \times \mathfrak{F}_B}.$$

Theorem 3.19. *Let R be hereditary and Noetherian.*

1. *All R-algebras satisfy the α-condition, i.e. $\mathbf{Alg}_R^{\alpha} = \mathbf{Alg}_R$.*
2. *There is a* duality *between $\mathbf{Alg}_R$ and $\mathbf{Cog}_R$ through the right-adjoint contravariant functors*
$$(-)^* : \mathbf{Cog}_R \to \mathbf{Alg}_R, (-)^{\circ} : \mathbf{Alg}_R \to \mathbf{Cog}_R.$$

Proof.

1. Let A be an arbitrary R-algebra. By [5, Proposition 2.11] $A^{\circ} \subset R^A$ is pure and so (A, A°) is an α-pairing by [2, Proposition 2.5].
2. For every R-algebra A the canonical mapping $\lambda_A : A \to A^{\circ *}$ is an R-algebra morphism and for every R-coalgebra C the canonical mapping $\Phi_C : C \to C^{*\circ}$ is an R-coalgebra morphism (compare Lemma 3.6). Moreover for every $A \in \mathbf{Alg}_R$ and every $C \in \mathbf{Cog}_R$
$$\Upsilon_{A,C} : \text{Alg}_R(A, C^*) \to \text{Cog}_R(C, A^{\circ}), \quad \xi \mapsto \xi^{\circ} \circ \Phi_C$$
is an isomorphism with inverse
$$\Psi_{A,C} : \text{Cog}_R(C, A^{\circ}) \to \text{Alg}_R(A, C^*), \quad \theta \mapsto \theta^* \circ \lambda_A.$$
It is easy to see that $\Upsilon_{A,C}$ and $\Psi_{A,C}$ are functorial in A and C. □

Locally finite modules

3.20. The cofinite topology. Let R be Noetherian. For every R-algebra A, the class $\mathcal{K}_A^l$ (resp. $\mathcal{K}_A^r$) of all R-cofinite *left* (respectively *right*) A-ideals is a filter basis. By Remark 3.2 every $I \in \mathcal{K}_A^l$ (resp. $I \in \mathcal{K}_A^r$) contains a *two-sided* A-ideal J_I, such that $J_I \subseteq I$. So $\mathcal{K}_A^l$ (resp. $\mathcal{K}_A^r$) induces on A a *symmetric* left (respectively right) linear topology, the so called *left cofinite topology* $\text{Cf}^l(A)$ (respectively *right cofinite topology* $\text{Cf}^r(A)$), such that $\mathcal{K}_A^l$ (resp. $\mathcal{K}_A^r$) is neighborhood basis of 0_A. If $A^{\circ} := A^{\circ}_{\mathcal{K}_A^l} = A^{\circ}_{\mathcal{K}_A^r}$ is an R-coalgebra then we call it the *continuous dual R-coalgebra* of A.

Consider A with the left cofinite topology $\mathrm{Cf}^l(A)$. Let M be a left A-module and consider the filter $\mathscr{K}_M$ of *all* R-cofinite A-submodules of M. Let $L \subseteq M$ be an R-cofinite A-submodule and consider the R-linear mapping

$$\varphi_L : A \to \mathrm{End}_R(M/L), a \mapsto [m + L \mapsto am + L].$$

Then $A/\mathrm{Ker}(\varphi_L) \hookrightarrow \mathrm{End}_R(M/L)$ and so

$$I_L := \mathrm{Ker}(\varphi_L) = \{a \in A | aM \subseteq L\}$$

is an R-cofinite two-sided A-ideal. If $m \in M$ is arbitrary then $I_L := (L : M) \subseteq (L : m)$, hence $(L : m)$ is open with respect to the left cofinite topology $\mathrm{Cf}^l(A)$. So M becomes a topology, the so called *cofinite topology* $\mathrm{Cf}(M)$, such that $(M, \mathrm{Cf}(M))$ is a linear topological left $(A, \mathrm{Cf}^l(A))$-module and $\mathscr{K}_M$ is a neighborhood basis of 0_M.

Considering A with the right cofinite topology $\mathrm{Cf}^r(A)$ it turns out that for every right A-module M, the filter of R-cofinite right A-submodules of M induces on M a topology, the cofinite topology $\mathrm{Cf}(M)$, such that $(M, \mathrm{Cf}(M))$ is a linear topological right $(A, \mathrm{Cf}^r(A))$-module.

3.21. Let R be Noetherian and A be an R-algebra. A left A-module M is called *locally finite*, if Am is finitely generated for every $m \in M$. For every left A-module M it follows that $\mathrm{Loc}(M) \subseteq M$ is an A-submodule (since the ground ring R is Noetherian) and we get a preradical

$$\mathrm{Loc}(-) : {}_A\mathscr{M} \to {}_A\mathscr{M}, \quad M \mapsto \{m \in M | Am \text{ is finitely generated in } \mathscr{M}_R\}$$

with pretorsion class $\mathrm{Loc}({}_A\mathscr{M}) \subseteq {}_A\mathscr{M}$, the *full* subcategory of locally finite left A-modules.

Analogously one defines the preradical $\mathrm{Loc}(-) : \mathscr{M}_A \to \mathscr{M}_A$ with pretorsion class $\mathrm{Loc}(\mathscr{M}_A) \subseteq \mathscr{M}_A$, the full subcategory of locally finite right A-modules.

Lemma 3.22. *Let R be Noetherian and A be an R-algebra. For every right A-module M we have*

$$\begin{aligned} M^\circ &:= \{f \in M^* \mid f(MI) = 0 \ \textit{for some R-cofinite (right) A-ideal } I \subseteq A\} \\ &= \{f \in M^* \mid Af \ \textit{is finitely generated in } \mathscr{M}_R\} \ (= \mathrm{Loc}({}_AM^*)) \\ &= \{f \in M^* \mid f(L) = 0 \ \textit{for some R-cofinite right A-submodule} \\ &\qquad L \subseteq M\}. \end{aligned} \tag{13}$$

Proof. Let $f \in M^*$ with $f(MI) = 0$ for an R-cofinite right A-ideal I. If $\{a_1 + I, \ldots, a_k + I\}$ is a generating system for A/I over R then $\{a_1 f, \ldots, a_k f\}$ is a generating system for Af over R, i.e. $f \in \mathrm{Loc}({}_AM^*)$.

Let $f \in \mathrm{Loc}({}_AM^*)$ and assume that $Af = \sum_{i=1}^k Rf_i$ with $\{f_1, \ldots, f_k\} \subset M^*$. Then $L := \mathrm{Ke}(Af) = \bigcap_{i=1}^k \mathrm{Ker}(f_i) \subseteq M$ is a right A-submodule and moreover $M/L \hookrightarrow \oplus_{i=1}^k M/\mathrm{Ker}(f_i)$, i.e. $L \subseteq M$ is an R-cofinite A-submodule.

Let $f \in (M/L)^* \simeq \mathrm{An}(L)$ for some R-cofinite A-submodule $L \subseteq M$. Then $I_L := (L : M)$ is an R-cofinite *two-sided* A-ideal (compare 3.20) and moreover $f(MI_L) \subseteq f(L) = 0$, i.e. $f \in M^\circ$. □

It is well known that for an R-algebra A over a base field R, the category of right (respectively left) A°-comodules and the category of locally finite left (respectively right) A-modules coincide, e.g. [1,28]. Over arbitrary commutative rings we have

Proposition 3.23. *Let R be Noetherian and A be an R-algebra.*

1. *Every A°-subgenerated left (respectively right) A-module is locally finite.*
2. *If A is cofinitely R-cogenerated then $\sigma[{}_AA^\circ] = \mathrm{Loc}({}_A\mathscr{M})$ and $\sigma[A^\circ_A] = \mathrm{Loc}(\mathscr{M}_A)$.*
3. *If A is an α-algebra then we have category isomorphisms*

$$\begin{aligned} \mathscr{M}^{A^\circ} &\simeq \mathrm{Rat}^{A^\circ}({}_A\mathscr{M}) = \sigma[{}_AA^\circ] \\ &\simeq \mathrm{Rat}^{A^\circ}({}_{A^{\circ *}}\mathscr{M}) = \sigma[{}_{A^{\circ *}}A^\circ] \quad \textit{and} \\ {}^{A^\circ}\mathscr{M} &\simeq {}^{A^\circ}\mathrm{Rat}(\mathscr{M}_A) = \sigma[A^\circ_A] \\ &\simeq {}^{A^\circ}\mathrm{Rat}(\mathscr{M}_{A^{\circ *}}) = \sigma[A^\circ_{A^{\circ *}}]. \end{aligned} \tag{14}$$

If A is moreover cofinitely R-cogenerated then

$$\mathscr{M}^{A^\circ} \simeq \mathrm{Loc}({}_A\mathscr{M}) \quad \textit{and} \quad {}^{A^\circ}\mathscr{M} \simeq \mathrm{Loc}(\mathscr{M}_A).$$

Proof.

1. Let $M \in \sigma[{}_AA^\circ]$. Then there exists for every $m \in M$ a finite subset $W = \{f_1, \ldots, f_k\} \subset A^\circ$, such that $\mathrm{Ann}^l_A(W) \subseteq (0_M : m)$. Choose for every $i = 1, \ldots, k$ an R-cofinite A-ideal $J_i \subseteq Ke(f_i)$ and consider the R-cofinite A-ideal $J := \bigcap_{i=1}^k J_i$. If $a \in J$ then for every $\widetilde{a} \in A$ and $i = 1, \ldots, k$ we have $(a \rightharpoonup f_i)(\widetilde{a}) = f_i(\widetilde{a}a) = 0$. Consequently $J \subseteq \mathrm{Ann}^l_A(W) \subseteq (0_M : m)$ and so $Am \simeq A/(0_M : m)$ is finitely generated in $\mathscr{M}_R$. Hence ${}_AM$ is locally finite.
2. By (1) $\sigma[{}_AA^\circ] \subseteq \mathrm{Loc}({}_A\mathscr{M})$. Assume now that A is cofinitely R-cogenerated. Let N be a locally finite left A-module. For every $n \in N$ the R-module $A/(0_N : n) \simeq An$ is finitely generated and so there exists by Remark 3.2 an R-cofinite A-ideal $I \subseteq (0_N : n)$. By assumption A/I is R-cogenerated and so $I = \mathrm{KeAn}(I)$ (e.g. [31, 28.1]). If $\mathrm{An}(I) \simeq (A/I)^* = \sum_{i=1}^k Rg_i$ and $W := \{g_1, \ldots, g_k\}$ then it follows for every $a \in \mathrm{Ann}^l_A(W)$ that $g_i(a) = (a \rightharpoonup g_i)(1_A) = 0$. So $\mathrm{Ann}^l_A(W) \subseteq \mathrm{KeAn}(I) = I \subseteq (0_N : n)$, i.e. ${}_AN$ is A°-subgenerated.
3. The category isomorphisms (14) follow from Theorem 2.6. The last statement follows then from (2). □

4. Dual comodules

In this section we discuss for every $(A,C)\in\mathscr{P}_m^{\alpha}$ the *duality* between the category of right (respectively left) A-modules and the category of right (respectively left) C-comodules.

4.1. Let $P=(A,C)\in\mathscr{P}_m$. By Theorem 2.6 $\mathscr{M}^C\subseteq {}_A\mathscr{M}$ is a subcategory and so we have a contravariant functor

$$(-)^*:\mathscr{M}^C\to\mathscr{M}_A,\quad (N,\varrho_N)\mapsto(N^*,\rho_{N^*}), \tag{15}$$

where

$$\rho_{N^*}:N^*\to \mathrm{Hom}_R(A,N^*),\quad f\mapsto\left[a\mapsto\left[n\mapsto\sum f(n_{\langle 0\rangle})\langle a,n_{\langle 1\rangle}\rangle\right]\right]. \tag{16}$$

If moreover P satisfies the α-condition then we get by Theorem 2.6 the contravariant functor

$$(-)^r:\mathscr{M}_A\to\mathscr{M}^C, M\mapsto M^r:=\mathrm{Rat}^C({}_AM^*).$$

If M, $\widetilde{M}$ are right A-modules and $f\in\mathrm{Hom}_{A-}(M,\widetilde{M})$ then $f^*\in\mathrm{Hom}_{-A}(\widetilde{M}^*,M^*)$ and we denote with $f^r\in\mathrm{Hom}^C(\widetilde{M}^r,M^r)$ the restriction of f^* to $\widetilde{M}^r\subseteq\widetilde{M}^*$ (see Lemma 2.5 (4) and Theorem 2.6). For every right A-module M we call M^r the *dual C-comodule* of M with respect to P.

4.2. ([8]) Let ${}_RC$ be a flat R-coalgebra, N be a left C-comodule and consider the R-linear mapping

$$\gamma:N^*\to\mathrm{Hom}_R(N,C),\quad f\mapsto\left[n\mapsto\sum f(n_{\langle -1\rangle})n_{\langle 0\rangle}\right]. \tag{17}$$

If ${}_RN$ is finitely presented then $N^*\otimes_R C\simeq\mathrm{Hom}_R(N,C)$ (e.g. [30, 15.7]) and N^* becomes a structure of a right C-comodule through

$$\varrho_{M^*}:N^*\xrightarrow{\gamma}\mathrm{Hom}_R(N,C)\simeq N^*\otimes_R C. \tag{18}$$

If $N\in\mathscr{M}^C$ and N_R is finitely presented then N^* is analogously a left C-comodule.

Theorem 4.3. *For every $(A,C)\in\mathscr{P}_m^{\alpha}$ there is a* duality *between the category of right C-comodules and the category of right A-modules through the right adjoint contravariant functors*

$$(-)^*:\mathscr{M}^C\to\mathscr{M}_A\quad\textit{and}\quad(-)^r:\mathscr{M}_A\to\mathscr{M}^C.$$

Proof. For every right C-comodule N the canonical mapping $\Phi_N:N\to N^{**}$ is A-linear, hence $\Phi_N(N)\subseteq N^{*r}$ by Lemma 2.5 (4) and it follows by Theorem 2.6 that $\Phi_N:N\to N^{*r}$ is C-colinear. On the other hand, for every right A-module M the canonical mapping $\lambda_M:M\to M^{r*}$ is A-linear. It is easy to see then that we have functorial homomorphisms (in $M \in \mathcal{M}_A$ and $N \in \mathcal{M}^C$)

$$\Upsilon_{N,M}\colon\ \mathrm{Hom}_{-A}(M, N^*) \to \mathrm{Hom}^C(N, M^r), \quad f \mapsto f^r \circ \Phi_N,$$

$$\Psi_{N,M}\colon\ \mathrm{Hom}^C(N, M^r) \to \mathrm{Hom}_{-A}(M, N^*), \quad g \mapsto g^* \circ \lambda_M.$$

Moreover $\Upsilon_{N,M}$ is bijective with inverse $\Psi_{N,M}$. □

Notation. For every R-algebra A denote with $\mathcal{M}_A^f$ (respectively ${}_A\mathcal{M}^f$) the category of *finitely generated* right (respectively left) A-modules.

Lemma 4.4. *Let R be Noetherian. For every $(A, C) \in \mathcal{P}_m^\alpha$ there is a* duality *between* $\mathrm{Rat}^C({}_A\mathcal{M}^f)$ *and* ${}^C\mathrm{Rat}(\mathcal{M}_A^f)$ *through the right-adjoint contravariant functors*

$$(-)^* : {}^C\mathrm{Rat}(\mathcal{M}_A^f) \to \mathrm{Rat}^C({}_A\mathcal{M}^f) \quad \textit{and} \quad (-)^* : \mathrm{Rat}^C({}_A\mathcal{M}^f) \to {}^C\mathrm{Rat}(\mathcal{M}_A^f).$$

Proof. Let $M \in {}^C\mathrm{Rat}(\mathcal{M}_A^f)$ (respectively $M \in \mathrm{Rat}^C({}_A\mathcal{M}^f)$). By [3, Folgerung 2.2.24] every finitely generated C-rational left (respectively right) A-module is finitely generated over R, hence M_R is finitely generated and so ${}_AM^*$ (respectively M_A^*) is finitely generated. By assumption R is Noetherian and so ${}_RM$ is finitely presented. Consequently M^* is by 4.2 a C-rational left (respectively right) A-module. The claimed duality follows then from Theorem 4.3. □

4.5. If C is a locally projective R-coalgebra then we get by Theorem 4.3 right-adjoint contravariant functors

$$(-)^*\colon\ \mathcal{M}^C \to \mathcal{M}_{C^*}, \quad N \mapsto N^*,$$

$$(-)^{\square} : \mathcal{M}_{C^*} \to \mathcal{M}^C, \quad M \mapsto M^{\square} := \mathrm{Rat}^C({}_{C^*}M^*).$$

Lemma 4.6. *Let R be an injective cogenerator and C be a locally projective R-coalgebra. If M a right C^*-module, $L \subseteq M$ is a C^*-submodule and $M^{\square} \subseteq M^*$ is dense then $L^{\square} \subseteq L^*$ is dense.*

Proof. By Lemma 2.5 (4) $\imath_L^*(M^{\square}) \subseteq L^{\square}$ and it follows from [2, Proposition 1.10 (3.b)] that $\overline{\imath_L^*(M^{\square})} = \imath_L^*(\overline{M^{\square}}) = \imath_L^*(M^*) = L^*$. □

Takeuchi [29] studied the category of locally finite modules of a commutative algebra over a base field. In what follows we transfer some results obtained by him to the category $\mathrm{Rat}^C({}_A\mathcal{M})$ corresponding to a measuring α-pairing $P = (A, C) \in \mathcal{P}_m^\alpha$ with A a *commutative* algebra over an arbitrary commutative ground ring.

Proposition 4.7. *Let $P = (A, C) \in \mathcal{P}_m^\alpha$ with A* commutative *and denote with $\mathcal{M}_A^f \subseteq \mathcal{M}_A$ the full subcategory of* finitely generated A-(*bi*)*modules. Then we have an isomorphism of functors*

$$\mathrm{Hom}_A(-, C) \simeq (-)^r : \mathcal{M}_A^f \to \mathcal{M}^C.$$

Proof. *Step* 1: $\mathrm{Hom}_A(-,C) : \mathscr{M}_A^f \to \mathscr{M}^C$ is well-defined.

Let $M \in \mathscr{M}_A^f$ be arbitrary and consider $\mathrm{Hom}_A(M,C)$ with the canonical A-module structure induced by M_A. For arbitrary $f \in \mathrm{Hom}_A(M,C)$ the A-subbimodule $N := f(M) \subseteq C$ is by Theorem 2.6 a C-bicoideal. Moreover N_A is finitely generated and so finitely generated in $\mathscr{M}_R$ (see [3, Folgerung 2.2.24]). Assume that $N = \sum_{i=1}^{l} Rc_i$ with $\Delta(c_i) = \sum_{j=1}^{l_i} c_{ij} \otimes \widetilde{c}_{ij}$ for every $i = 1, \ldots, k$ and set $K := \sum_{i=1}^{l} \sum_{j=1}^{l_i} Rc_{ij}$. Then $K^{\perp} \subseteq (0 : f)$ and so f is by Proposition 2.11 C-rational. By our choice $f \in \mathrm{Hom}_A(M,C)$ is arbitrary, i.e. $\mathrm{Hom}_A(M,C) \in \mathrm{Rat}^C({}_A\mathscr{M})$.

Step 2: $(-)^r \simeq \mathrm{Hom}_A(-,C)$.

Let $N \in \mathscr{M}^C$, $M \in \mathscr{M}_A$ and consider the C-comodule $\mathrm{Hom}_A(M,C)$. The result follows then from the functorial isomorphisms:

$$\begin{aligned}
\mathrm{Hom}^C(N, \mathrm{Hom}_A(M,C)) &= \mathrm{Hom}_A(N, \mathrm{Hom}_A(M,C)) \quad \text{(Theorem 2.6)}\\
&\simeq \mathrm{Hom}_A(M, \mathrm{Hom}_A(N,C))\\
&\simeq \mathrm{Hom}_A(M, \mathrm{Hom}^C(N,C)) \quad \text{(Theorem 2.6)}\\
&\simeq \mathrm{Hom}_A(M, N^*) \quad (5)\\
&\simeq \mathrm{Hom}^C(N, M^r) \quad \text{(Theorem 4.3)}. \quad \square
\end{aligned}$$

As a consequence of Proposition 4.7 we get

Corollary 4.8.

1. *Let R be Noetherian, A be an α-algebra and consider the functor*

   $$(-)^0 = \mathrm{Rat}^{A^\circ}(-) \circ (-)^* : \mathscr{M}_A \to \mathscr{M}^{A^\circ}, \quad M \mapsto M^0 := \mathrm{Rat}^{A^\circ}({}_AM^*)$$

   If A is commutative *then we have a functorial isomorphism*

   $$\mathrm{Hom}_A(-, A^\circ) \simeq (-)^0 : \mathscr{M}_A^f \to \mathscr{M}^{A^\circ}.$$

2. *If C is a* cocommutative *locally projective R-coalgebra then we have a functorial isomorphism*

   $$\mathrm{Hom}_{C^*}(-, C) \simeq (-)^{\square} : \mathscr{M}_{C^*}^f \to \mathscr{M}^C.$$

Corollary 4.9. *Let $P = (A, C) \in \mathscr{P}_m^{\alpha}$, where A is commutative and Noetherian. If $(-)^r : \mathscr{M}_A^f \to \sigma[{}_AC]$ is exact then C is an injective A-module.*

Proof. By *Baer's criteria* (e.g. [31, 16.4]) it is enough to show that C is A-injective. Let I be an A-ideal. Then I_A is finitely generated and by assumption the following set mapping is surjective

$$A^r \xrightarrow{\iota^r} I^r \to 0.$$

By Proposition 4.7 $\mathrm{Hom}_A(-,C) \simeq (-)^r$ and so

$$\mathrm{Hom}_A(A,C) \overset{(\iota,C)}{\to} \mathrm{Hom}_A(I,C) \to 0$$

is a surjective set mapping, i.e. C is A-injective and we are done. □

Continuous dual comodules

In what follows we consider the *dual comodules* of modules of an α-algebra over an arbitrary Noetherian base ring. These were considered in the case of base fields by several authors (e.g. [11,18,19,29]) and in the case of Dedekind domains by R. Larson [16].

4.10. Let R be Noetherian, A be an R-algebra, $\mathfrak{F}$ be a filter consisting of R-cofinite A-ideals and consider A with the induced right linear topology $\mathfrak{T}(\mathfrak{F})$.

1. If $\mathfrak{F}$ is an α-filter then by Proposition 3.14 (2) $A^{\circ}_{\mathfrak{F}}$ is an R-coalgebra and $(A, A^{\circ}_{\mathfrak{F}}) \in \mathscr{P}^{\alpha}_{m}$. By Theorem 4.3 we get right-adjoint contravariant functors

$$(-)^{*}\colon \mathscr{M}^{A^{\circ}_{\mathfrak{F}}} \to \mathscr{M}_A, \quad M \mapsto M^{*},$$

$$(-)^{0}_{\mathfrak{F}}\colon \mathscr{M}_A \to \mathscr{M}^{A^{\circ}_{\mathfrak{F}}}, \quad M \mapsto M^{0}_{\mathfrak{F}} := \mathrm{Rat}^{A^{\circ}_{\mathfrak{F}}}({}_{A}M^{*}).$$

   For every $M \in \mathscr{M}_A$ we call $M^{0}_{\mathfrak{F}}$ the *dual comodule of M with respect to* $\mathfrak{F}$. If A is a α-algebra then we call $M^{0} := \mathrm{Rat}^{A^{\circ}}({}_{A}M^{*})$ the *dual comodule of M*.
2. For every right A-module M we call

$$M^{\circ}_{\mathfrak{F}} := \{f \in M^{*} \mid f(MI) = 0 \text{ for some } I \in \mathfrak{F}\} = \underset{\overrightarrow{\mathfrak{F}}}{lim}(M/MI)^{*}$$

   the *continuous dual module of M with respect to* $\mathfrak{F}$. If $A^{\circ}_{\mathfrak{F}}$ is an R-coalgebra and $M^{\circ}_{\mathfrak{F}}$ is a right $A^{\circ}_{\mathfrak{F}}$-comodule then we call it the *continuous dual comodule* of M with respect to $\mathfrak{F}$. If A° is a R-coalgebra and M° is a right A°-comodule, then we call it the *continuous dual comodule* of M.

Notation. Let R be Noetherian, A be an (α)-algebra and M,N be right A-modules. For every A-linear mapping $\gamma : M \to N$ we denote with $\gamma^{\circ} : N^{\circ} \to M^{\circ}$ ($\gamma^{0} : N^{0} \to M^{0}$) the restriction of γ^{*} on N° (on N^{0}).

The following result generalizes the corresponding one [10, Corollary 2.2.16] stated for the canonical pairing (C^{*}, C) over a base field to an arbitrary measuring α-pairing (A, C) over an arbitrary Noetherian ground ring:

Proposition 4.11. *Let $P = (A,C) \in \mathscr{P}^{\alpha}_{m}$, $N \in {}^{C}\mathrm{Rat}(\mathscr{M}_A)$ and consider for every $f \in N^{*}$ the R-linear mapping*

$$\theta_f : N \to C, \quad \theta_f(n) = \sum n_{\langle -1\rangle} f(n_{\langle 0\rangle}).$$

If R is Noetherian then:

$$\begin{aligned}\mathrm{Rat}^C({}_AN^*) &= \mathrm{Sp}(\sigma[{}_AC], {}_AN^*) := \sum \mathrm{Im}(g)\colon\ g \in \mathrm{Hom}_{A-}(U, N^*),\ \ U \in \sigma[{}_AC]\\ &= \{f \in N^* \mid Af \text{ is finitely generated}\}\ (=\mathrm{Loc}({}_AN^*))\\ &= \{f \in N^* \mid \exists \text{ an } R\text{-cofinite (right) ideal } I \subseteq A \text{ with } f(NI) = 0\}\\ &= \{f \in N^* \mid \exists \text{ an } R\text{-cofinite } A\text{-submodule } L \subseteq N \text{ with } f(L) = 0\}\\ &= \{f \in N^* \mid \exists \text{ an } R\text{-cofinite } C\text{-subcomodule } L \subseteq N \text{ with } f(L) = 0\}\\ &= \{f \in N^* \mid \theta_f(N) \subseteq C \text{ is a finitely generated } R\text{-submodule}\}.\end{aligned}$$

Proof. The equality $\mathrm{Rat}^C({}_AN^*) = \mathrm{Sp}(\sigma[{}_AC],\ {}_AN^*)$ follows from 2.3 and Theorem 2.6. Obviously $\mathrm{Rat}^C({}_AN^*) \subseteq \mathrm{Loc}({}_AN^*)$.

By Theorem 2.6 and Lemma 3.22 $f \in \mathrm{Loc}({}_AN^*)$ if and only if $f(NI) = 0$ for an R-cofinite (right) ideal $I \lhd A$ if and only if $f(L) = 0$ for an R-cofinite right A-submodule $L \subseteq N$ if and only if $f(L) = 0$ for an R-cofinite left C-subcomodule $L \subseteq N$.

Let $f \in N^*$ with $f(L) = 0$ for an R-cofinite left C-subcomodule $L \subseteq N$. Analogous to 5.2 in the next section, $\theta_f : N \to C$ is C-colinear. Notice that $\theta_f(L) = 0$ and so there exists a C-colinear morphism $\overline{\theta_f} : N/L \to C$, such that $\overline{\theta_f} \circ \pi_L = \theta_f$. Consequently $\theta_f(N) = \overline{\theta_f}(N/L)$ is finitely generated in $\mathcal{M}_R$.

To every $f \in N^*$ there corresponds the left C-coideal $\theta_f(N) \subseteq C$. If $\theta_f(N)$ is finitely generated in $\mathcal{M}_R$ then $(\theta_f(N))^*$ is a right C-comodule by 4.2 and we have for every $n \in N$:

$$\varepsilon_C(\theta_f(n)) = \varepsilon_C\left(\sum f(n_{\langle 0\rangle})n_{\langle -1\rangle}\right) = f\left(\sum \varepsilon_C(n_{\langle -1\rangle})n_{\langle 0\rangle}\right) = f(n),$$

i.e. $f \in (\theta_f(N))^* \subseteq \mathrm{Rat}^C({}_AN^*)$. □

As a special case of Proposition 4.11 we get

Corollary 4.12. *Let R be Noetherian. For every locally projective R-coalgebra C we have*

$$\begin{aligned}&\mathrm{Rat}^C({}_{C^*}C^*)\\ &\quad= \mathrm{Sp}(\sigma[{}_{C^*}C], {}_{C^*}C^*) := \sum \mathrm{Im}(g) : g \in \mathrm{Hom}_{C^*-}(U, C^*),\ \ U \in \sigma[{}_{C^*}C]\\ &\quad= \{f \in C^* \mid C^* \star f \text{ is finitely generated in } \mathcal{M}_R\}\\ &\quad= \{f \in C^* \mid \exists \text{ an } R\text{-cofinite (right) ideal } I \lhd C^* \text{ with } f(CI) = 0\}\\ &\quad= \{f \in C^* \mid \exists \text{ an } R\text{-cofinite right } C^*\text{-submodule } K \subseteq C \text{ with } f(K) = 0\}\\ &\quad= \{f \in C^* \mid \exists \text{ an } R\text{-cofinite } left\ C\text{-coideal } K \subseteq C \text{ with } f(K) = 0\}\\ &\quad= \{f \in C^* \mid f \rightharpoonup C \subseteq C \text{ is a finitely generated } R\text{-submodule}\}.\end{aligned}$$

4.13. Cofree comodules. A right C-comodule (M, ϱ_M) is called *cofree*, if there exists an R-module K, such that $(M, \varrho_M) \simeq (K \otimes_R C, id_K \otimes \Delta_C)$ as right C-comodules. Notice that if $K = R^{(\Lambda)}$, a free R-module then $M \simeq R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}$ as right C-comodules (in fact, this is one reason for the terminology *cofree*).

Lemma 4.14. *Let R be Noetherian and A be a cofinitary R-algebra. Let M be an R-module and consider the right A-module $N := M \otimes_R A$. Then $N^\circ \simeq M^* \otimes_R A^\circ$ as A°-comodules (i.e. N° is a* cofree *right A°-comodule).*

Proof. If $N \simeq M \otimes_R A$ as right A-modules then there are isomorphisms in ${}_A\mathscr{M}$:

$$
\begin{aligned}
N^\circ &:= \underset{\rightarrow}{lim}[((M \otimes_R A)/(M \otimes_R A)I)^*] : I \in \mathscr{K}_A \\
&= \underset{\rightarrow}{lim}[((M \otimes_R A)/(M \otimes_R A)\widetilde{I})^*] : \widetilde{I} \in \mathscr{E}_A \quad (A \text{ is cofinitary}) \\
&= \underset{\rightarrow}{lim}[((M \otimes_R A)/(M \otimes_R \widetilde{I}))^*] : \widetilde{I} \in \mathscr{E}_A \\
&\simeq \underset{\rightarrow}{lim}[(M \otimes_R A/\widetilde{I})^*] : \widetilde{I} \in \mathscr{E}_A \\
&\simeq \underset{\rightarrow}{lim}\{[M^* \otimes_R (A/\widetilde{I})^*]\} : \widetilde{I} \in \mathscr{E}_A \quad (A/\widetilde{I} \text{ is f.g. projective in } \mathscr{M}_R); \\
&\simeq M^* \otimes_R \underset{\rightarrow}{lim}[(A/\widetilde{I})^*] : \widetilde{I} \in \mathscr{E}_A \\
&\simeq M^* \otimes_R A^\circ \ (A \text{ is cofinitary}). \quad \square
\end{aligned}
$$

In contrast with [32, Corollary 2] the following example shows that for an arbitrary R-algebra A the preradical $\mathrm{Loc}(-) : {}_A\mathscr{M} \to \mathrm{Loc}({}_A\mathscr{M})$ is in general not a torsion radical:

Counter Example 4.15 (Compare [21, p. 155]). Let R be a field and consider the Hopf R-algebra $H := R[x_1, x_2, \ldots, x_n, \ldots]$, with the usual multiplication in polynomial rings, the usual unity and comultiplication, counity and antipode defined on the generators through

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1, \quad \varepsilon(x_i) = 0, \quad S(x^i) := (-1)^i x^i.$$

If we consider H with the left cofinite topology then $(H, \mathrm{Cf}^l(H))$ is a left linear topological R-algebra with preradical $\mathrm{Loc}(-) : {}_H\mathscr{M} \to {}_H\mathscr{M}$ and pretorsion class $\mathrm{Loc}({}_H\mathscr{M})$ (see 3.20 and Proposition 3.23). If we consider the H-ideal $\omega := Ke(\varepsilon_H)$ then $H/\omega \simeq R$ while $\dim(H/\omega^2) = \infty$, i.e. $\omega^2 \notin \mathscr{K}_H$. So $\mathrm{Cf}^l(H)$ is not a Gabriel-topology and consequently $\mathrm{Loc}({}_H\mathscr{M})$ is not closed under extensions (see [24, Chapter VI, Theorem 5.1, Lemma 5.3]).

5. Coreflexive comodules

In [27,28] Taft developed an algebraic aspect to the study of *coreflexive coalgebras* over base fields (i.e. coalgebras C with $C \simeq C^{*\circ}$). Independently, Heyneman and

Radford [23,12] studied coreflexive coalgebras with the help of the *finite topology* on C^*. In this section we present and study for every $(A,C) \in \mathscr{P}_m^{\alpha}$ over an arbitrary Noetherian ring the notions of *reflexive A-modules* and *coreflexive C-comodules*. We get algebraic as well as topological characterizations for (co)reflexive (co)modules. Our results will be applied then to the study of (co)reflexive (co)algebras, where we generalize also results from the papers mentioned above and from [32].

(A,C)-**Pairings**

In the case of base fields, Radford [23] presented for every measuring R-pairing $P=(A,C)$ the so called right (respectively left) P-pairings. In what follows we consider *duality relations* for such pairings.

5.1. Let $P=(A,C) \in \mathscr{P}_m$. A pairing of R-modules $Q=(M,N)$ is called a *right* (respectively *a left*) *P-pairing*, if M is a right (respectively a left) A-module, N is a right (respectively a left) C-comodule and the induced mapping $\kappa_Q : M \to N^*$ is right A-linear. By $\mathscr{Q}_P^r \subseteq \mathscr{P}$ (respectively $\mathscr{Q}_P^l \subseteq \mathscr{P}$) we denote the subcategory of right (respectively left) P-pairings with morphisms described as follows: for right (respectively left) P-pairings (M,N), (M',N'), a morphism of R-pairings

$$(\xi,\theta) : (M',N') \to (M,N)$$

is a morphism in $\mathscr{Q}_P^r$ (respectively in $\mathscr{Q}_P^l$) if $\xi : M \to M'$ is A-Linear and $\theta : N' \to N$ is C-colinear.

A *P-bi-pairing* is an R-pairing (M,N), where M is an A-bimodule, N is a C-bicomodule and $\kappa_Q : M \to N^*$ is A-bilinear. With $\mathscr{Q}_P$ we denote the category of P-bi-pairings with morphisms described as follows: for P-bi-pairings (M,N), (M',N'), a morphism of R-pairings

$$(\xi,\theta) : (M',N') \to (M,N)$$

is a morphism in $\mathscr{Q}_P$ if $\xi : M \to M'$ is A-bilinear and $\theta : N' \to N$ is C-bicolinear. In particular every measuring R-pairing P is itself a P-bi-pairing.

5.2. Let $P=(A,C) \in \mathscr{P}_m$, $Q=(M,N) \in \mathscr{Q}_P^r$ and define for every $m \in M$:

$$\xi_m \colon A \to M,\ a \mapsto ma \qquad \text{for all } a \in A,$$
$$\theta_m \colon N \to C,\ n \mapsto \sum \langle m, n_{\langle 0 \rangle} \rangle n_{\langle 1 \rangle} \qquad \text{for all } n \in N.$$

Then we have for all $a \in A$ and $n \in N$:

$$\langle \xi_m(a), n \rangle = \langle ma, n \rangle = \langle m, an \rangle = \sum \langle m, n_{\langle 0 \rangle} \rangle \langle a, n_{\langle 1 \rangle} \rangle = \langle a, \theta_m(n) \rangle.$$

Obviously $\xi_m : A \to M$ is A-linear. Moreover, it follows for all $n \in N$ and $a \in A$ that

$$\begin{aligned}\alpha_N^P\left(\sum \theta_m(n)_1 \otimes \theta_m(n)_2\right)(a) &= \sum \theta_m(n)_1 \langle a, \theta_m(n)_2\rangle \\ &= a \rightharpoonup \theta_m(n) \\ &= \sum \langle m, n_{\langle 0\rangle}\rangle (a \rightharpoonup n_{\langle 1\rangle}) \\ &= \sum \langle m, n_{\langle 0\rangle}\rangle n_{\langle 1\rangle 1} \langle a, n_{\langle 1\rangle 2}\rangle \\ &= \sum \langle m, n_{\langle 0\rangle\langle 0\rangle}\rangle n_{\langle 0\rangle\langle 1\rangle} \langle a, n_{\langle 1\rangle}\rangle \\ &= \alpha_N^P\left(\sum \theta_m(n_{\langle 0\rangle}) \otimes n_{\langle 1\rangle}\right)(a).\end{aligned}$$

If $\alpha_N^P : N \otimes_R C \to \mathrm{Hom}_R(A, N)$ is injective then

$$\sum \theta_m(n)_1 \otimes \theta_m(n)_2 = \sum \theta_m(n_{\langle 0\rangle}) \otimes n_{\langle 1\rangle} \quad \text{for every } n \in N,$$

i.e. $\theta_m : N \to C$ is C-colinear and

$$(\xi_m, \theta_m) : (M, N) \to (A, C)$$

is a morphism in $\mathscr{Q}_P^r$.

Notation. Let $P = (A, C) \in \mathscr{P}_m$ and $Q = (M, N) \in \mathscr{Q}_P^r$. For R-submodules $L \subseteq M$, $K \subseteq N$ we set

$$K^\perp := \{m \in M \mid \langle m, K\rangle = 0\}, \quad \mathrm{Ann}_M(K) := \{m \in M \mid \theta_m(k) = 0 \ \forall k \in K\},$$
$$L^\perp := \{n \in N \mid \langle L, n\rangle = 0\}, \qquad \mathrm{Ann}_N(L) := \{n \in N \mid \theta_m(n) = 0 \ \forall m \in L\}.$$

As a consequence of Theorem 2.6 one can easily derive the following result:

Lemma 5.3. *Let $P = (A, C) \in \mathscr{P}_m$ and $Q = (M, N) \in \mathscr{Q}_P^r$ (respectively $Q \in \mathscr{Q}_P^l$, $Q \in \mathscr{Q}_P$).*

1. *Every right C-subcomodule (respectively left C-subcomodule, C-subbicomodule) $K \subseteq N$ is a left A-submodule (respectively a right A-submodule, an A-subbimodule) and $K^\perp \subseteq M$ is a right A-submodule (respectively a left A-submodule, an A-subbimodule).*
2. *Let $(A, C) \in \mathscr{P}_m^\alpha$. If $L \subseteq M$ a right A-submodule (respectively a left A-submodule, an A-subbimodule) then $L^\perp \subseteq N$ is a right C-subcomodule (respectively a left C-subcomodule, a C-subbicomodule).*

The topology $\mathfrak{T}_N^r(M)$

Let $P = (A, C)$ be a measuring R-pairing and consider A as a right linear topological R-algebra with the right C-adic topology $\mathscr{T}_{-C}(A)$. For every $Q = (M, N) \in \mathscr{Q}_P^r$ we present on M a topology $\mathfrak{T}_N^r(M)$, such that $(M, \mathfrak{T}_N^r(M))$ is a linear topological $(A, \mathscr{T}_{-C}(A))$-module.

5.4. Let $P=(A,C)\in\mathscr{P}_m$, $Q=(M,N)\in\mathscr{Q}_P^r$ and consider C with the canonical right A-module structure and A as a right linear topological R-algebra with the right C-adic topology $\mathscr{T}_{-C}(A)$ (compare 2.3). If $K\subseteq N$ is an R-submodule and $m\in \mathrm{Ann}_M(K)$ then we have for arbitrary $n\in K$ and $a\in A$:

$$\begin{aligned}\theta_{ma}(n) &:= \sum \langle ma, n_{\langle 0\rangle}\rangle n_{\langle 1\rangle}\\ &= \sum \langle m, an_{\langle 0\rangle}\rangle n_{\langle 1\rangle}\\ &= \sum \langle m, n_{\langle 0\rangle\langle 0\rangle}\rangle\langle a, n_{\langle 0\rangle\langle 1\rangle}\rangle n_{\langle 1\rangle}\\ &= \sum \langle m, n_{\langle 0\rangle}\rangle\langle a, n_{\langle 1\rangle 1}\rangle n_{\langle 1\rangle 2}\\ &= \left[(\alpha_C^P\circ\Delta_C^{cop})\left(\sum \langle m, n_{\langle 0\rangle}\rangle n_{\langle 1\rangle}\right)\right](a)\\ &= [(\alpha_C^P\circ\Delta_C^{cop})(\theta_m(n))](a)=0,\end{aligned}$$

i.e. $\mathrm{Ann}_M(K)\subseteq M$ is an A-submodule. Let $K=\sum_{i=1}^{l} Rn_i\subseteq N$ be an arbitrary finitely generated R-submodule with $\varrho_N(n_i)=\sum_{j=1}^{l_i} n_{ij}\otimes c_{ij}$ for $i=1,\ldots,l$ and set $W:=\sum_{i=1}^{l}\sum_{j=1}^{l_i} Rc_{ij}$. Let $m\in M$ be arbitrary. If $a\in\mathrm{Ann}_A^r(W)$ then for $i=1,\ldots,l$:

$$\begin{aligned}\theta_{ma}(n_i) &= \sum_{i=1}^{l}\sum_{j=1}^{l_i}\langle ma, n_{ij}\rangle c_{ij}\\ &= \sum_{i=1}^{l}\sum_{j=1}^{l_i}\langle m, an_{ij}\rangle c_{ij}\\ &= \sum_{i=1}^{l}\sum_{j=1}^{l_i}\sum_{n_{ij}}\langle m, n_{ij\langle 0\rangle}\rangle\langle a, n_{ij\langle 1\rangle}\rangle c_{ij}\\ &= \sum_{i=1}^{l}\sum_{j=1}^{l_i}\sum_{c_{ij}}\langle m, n_{ij}\rangle\langle a, c_{ij1}\rangle c_{ij2}\\ &= \sum_{i=1}^{l}\sum_{j=1}^{l_i}\langle m, n_{ij}\rangle(c_{ij}\leftharpoonup a)=0,\end{aligned}$$

i.e. $(\mathrm{Ann}_M(K):m)\supseteq \mathrm{Ann}_A^r(W)$ and so it is open with respect to the right C-adic topology $\mathscr{T}_{-C}(A)$. So

$$\mathscr{B}(0_M):=\{\mathrm{Ann}_M(K)\,|\,K\subseteq N \text{ is a finitely generated } R\text{-submodule}\}$$

is neighborhood basis of 0_M consisting of A-submodules of M and M becomes a topology $\mathfrak{T}_N^r(M)$, such that $(M,\mathfrak{T}_N^r(M))$ is a linear topological right $(A,\mathscr{T}_{-C}(A))$-module.

Remark 5.5. Let $P=(A,C)$ be a measuring R-pairing. Considering P itself as a right P-pairing, it turns out that the right linear topology $\mathfrak{T}_C^r(A)$ and the right C-adic topology $\mathcal{T}_{C-}(A)$ coincide. In fact our definition of $\mathfrak{T}_N^r(M)$ was motivated by that of $\mathcal{T}_{C-}(A)$.

5.6. Let $P=(A,C)\in\mathcal{P}_m$ and $Q=(M,N)\in\mathcal{Q}_P^r$. Then

$$\mathcal{F}(0_M):=\{K^{\perp}|K\subseteq N \text{ is a finitely generated } R\text{-submodule}\}$$

is a filter basis consisting of R-submodules of M and induces on M the *linear weak topology* $M[\mathfrak{T}_{ls}(N)]$, such that $(M,M[\mathfrak{T}_{ls}(N)])$ is a linear topological R-module and $\mathcal{F}(0_M)$ is a neighborhood basis of 0_M.

5.7. Let R be Noetherian, $P=(A,C)\in\mathcal{P}_m$ and consider C^* with the *right cofinite topology* $\mathrm{Cf}^r(C^*)$ (see 3.20). The R-algebra morphism $\kappa_P:A\to C^*$ induces on A a right linear topology κ_P-$\mathrm{Cf}^r(A)$ with neighborhood basis of 0_A:

$$\mathcal{B}_{\kappa_P}(0_A):=\{\kappa_P^{-1}(J)|J\lhd C^* \text{ is an } R\text{-cofinite right ideal}\}.$$

By definition κ_P-$\mathrm{Cf}^r(A)$ the *finest* linear topology $\mathfrak{T}$ on A, such that $(A,\mathfrak{T})$ is a right linear topological R-algebra and $\kappa_P:(A,\mathfrak{T})\to(C^*,\mathrm{Cf}^r(C^*))$ is continuous.

Let $Q=(M,N)\in\mathcal{Q}_P^r$ and consider N_A^* with the cofinite topology $\mathrm{Cf}^r(N^*)$. The A-linear mapping $\kappa_Q:M\to N^*$ induces on M a topology κ_Q-$\mathrm{Cf}(M)$ with neighborhood basis of 0_M

$$\mathcal{B}_{\kappa_Q}(0_M):=\{\kappa_Q^{-1}(L)|L\subseteq N^* \text{ is an } R\text{-cofinite } A\text{-submodule}\}.$$

Clearly κ_Q-$\mathrm{Cf}(M)$ is a linear topological right $\mathrm{Cf}^r(A)$-module and is the *finest* topology $\mathfrak{T}$ on M, such that $(M,\mathfrak{T})$ is a linear topological right $(A,\mathrm{Cf}^r(A))$-module and $\kappa_Q:(M,\mathfrak{T})\to(N^*,\mathrm{Cf}(N^*))$ is continuous.

Lemma 5.8. *Let $P=(A,C)\in\mathcal{P}_m$ and $Q=(M,N)\in\mathcal{Q}_P^r$.*

1. *The linear weak topology $M[\mathfrak{T}_{ls}(N)]$ and the topology $\mathfrak{T}_N^r(M)$ coincide. So M, considered with the linear weak topology, is a linear topological right $(A,\mathcal{T}_{-C}(A))$-module.*
2. *If R is Noetherian and P satisfies the α-condition then*

$$M[\mathfrak{T}_{ls}(N)]\preceq\kappa_Q\text{-}\mathrm{Cf}(M)\preceq\mathrm{Cf}(M). \tag{19}$$

Proof.

1. Let $U\subseteq M$ be a neighborhood of 0_M with respect to $M[\mathfrak{T}_{ls}(N)]$. Then there exists a finitely generated R-submodule $K\subseteq N$, such that $K^{\perp}\subseteq U$. If $m\in\mathrm{Ann}_M(K)$ then we have for arbitrary $n\in K$:

$$\langle m,n\rangle=\left\langle m,\sum n_{\langle 0\rangle}\varepsilon_C(n_{\langle 1\rangle})\right\rangle=\varepsilon_C\left(\sum\langle m,n_{\langle 0\rangle}\rangle n_{\langle 1\rangle}\right)=\varepsilon_C(\theta_m(n))=0.$$

So $\mathrm{Ann}_M(K)\subseteq K^{\perp}\subseteq U$, i.e. U is a neighborhood of 0_M with respect to $\mathfrak{T}_N^r(M)$.

On the other hand, let $U \subseteq M$ be a neighborhood of 0_M with respect to $\mathfrak{T}_N^r(M)$. Then there exists a finitely generated R-submodule $K = \sum_{i=1}^{l} Rn_i \subseteq N$, such that $\mathrm{Ann}_M(K) \subseteq U$. Assume now that $\varrho_N(n_i) = \sum_{j=1}^{l_i} n_{ij} \otimes c_{ij}$ and set $W := \sum_{i=1}^{l} \sum_{j=1}^{l_i} Rn_{ij}$. Then $W^\perp \subseteq \mathrm{Ann}_M(K) \subseteq U$, i.e. U is a neighborhood of 0_M with respect to $M[\mathfrak{T}_{ls}(N)]$. Consequently $M[\mathfrak{T}_{ls}(N)] = \mathfrak{T}_N^r(M)$.

2. Let R be Noetherian and $P \in \mathscr{P}_m^\alpha$. Let $U \subseteq M$ be a neighborhood of 0_M with respect to $M[\mathfrak{T}_{ls}(N)]$, i.e. there exists a finitely generated R-submodule $K \subseteq N$ such that $K^\perp \subseteq U$. By assumption $P \in \mathscr{P}_m^\alpha$ and so there exists by the *Finiteness Theorem* 2.10 a left A-submodule $\widetilde{K} \subseteq N$, such that $K \subseteq \widetilde{K}$ and $\widetilde{K}_R$ is finitely generated. Moreover $N^*/\mathrm{An}(\widetilde{K}) \hookrightarrow \widetilde{K}^*$, i.e. $\mathrm{An}(\widetilde{K}) \subseteq N^*$ is an R-cofinite right A-submodule. It follows then that $\kappa_Q^{-1}(\mathrm{An}(\widetilde{K})) := \widetilde{K}^\perp \subseteq K^\perp \subseteq U$, i.e. U is a neighborhood of 0_M with respect to κ_Q-$\mathrm{Cf}(M)$.
   On the otherhand, let $U \subseteq M$ be a neighborhood of 0_M with respect to κ_Q-$\mathrm{Cf}(M)$, i.e. there exists an R-cofinite A-submodule $L \subseteq N^*$ such that $\kappa_Q^{-1}(L) \subseteq U$. Then $M/\kappa_Q^{-1}(L) \hookrightarrow N^*/L$, and so $\kappa_Q^{-1}(L) \subseteq M$ is an R-cofinite A-submodule. Consequently U is a neighborhood of 0_M with respect to $\mathrm{Cf}(M)$. □

Definition 5.9. Let $P = (A, C) \in \mathscr{P}_m$ and $Q = (M, N) \in \mathscr{Q}_P^r$.

1. If $P \in \mathscr{P}_m^\alpha$ then we call Q *weakly coreflexive*, if $N = M^r$.
2. If R is Noetherian then we call Q *coreflexive*, if $M[\mathfrak{T}_{ls}(N)] = \mathrm{Cf}(M)$.
3. We call Q *proper* (respectively *weakly reflexive*, *reflexive*), if $\kappa_Q : M \to N^*$ is injective (respectively surjective, bijective).

Definition 5.10.

1. Let C be an R-coalgebra and N be a right C-comodule.
   (a) If ${}_RC$ is locally projective then we call N *weakly coreflexive*, if $N = N^{*\square}$.
   (b) If R is Noetherian then we call N *coreflexive*, if $N^*[\mathfrak{T}_{ls}(N)] = \mathrm{Cf}^r(N^*)$.
2. Let R be Noetherian and A be an R-algebra. We call a right A-module M *proper* (respectively *weakly reflexive*, *reflexive*), if the canonical A-linear mapping $\lambda_M : M \to M^{\circ *}$ is injective (respectively surjective, bijective).

Remark 5.11.

1. Consider the ground ring R as a trivial R-bialgebra. Then $R^* \simeq R$, $\mathscr{M}_R \simeq \mathscr{M}^R$ and for every R-(co-)module N we have $N^{**} = \mathrm{Rat}^R(N^{**}) = \mathrm{Loc}({}_{R^*}N^{**})$. So N is (co)reflexive if and only if N is reflexive in the usual sense, i.e. if the canonical R-linear mapping $\Phi_N : N \to N^{**}$ is bijective.
2. For every $P = (A, C) \in \mathscr{P}_m^\alpha$ we have $C = A^r$ (by Corollary 2.9 (1)) and so $P \in \mathscr{Q}_P^r$ is weakly coreflexive.

Proposition 5.12. *Let R be Noetherian, A be an R-algebra and denote with* $\mathrm{Cf}(A)$ *the left (or the right) cofinite topology.*

1. *If A is proper, i.e. the canonical mapping* $\lambda_A : A \to A^{\circ *}$ *is injective, then* $\mathrm{Cf}(A)$ *is Hausdorff.*
2. *Let A be cofinitely R-cogenerated. Then A is proper if and only if* $\mathrm{Cf}(A)$ *is Hausdorff.*
3. *Assume R to be a QF ring. Then A is proper if and only if* $\mathrm{Cf}(A)$ *is Hausdorff if and only if* $A^{\circ} \subseteq A^{*}$ *is dense.*

Proof.

1. Obviously $\overline{0_A} := \bigcap_{\mathscr{K}_A} I \subseteq \mathrm{Ker}(\lambda_A)$ and the result follows.
2. Assume $\mathrm{Cf}(A)$ to be Hausdorff. If A is not proper then there exists some $0 \neq \widetilde{a} \in A$, such that $f(\widetilde{a}) = 0$ for every $f \in A^{\circ}$. If $I \lhd A$ is an arbitrary R-cofinite two-sided A-ideal then $\widetilde{a} \in \mathrm{KeAn}(I) = I$ (compare [31, 28.1]) and so $\bigcap_{\mathscr{K}_A} I \neq 0$ (*contradiction*).
3. By [2, Theorem 1.8 (1)] we have

$$\overline{A^{\circ}} = \mathrm{AnKe}(A^{\circ}) = \mathrm{An}\left(\mathrm{Ke}\left(\sum_{I \in \mathscr{K}_A} \mathrm{An}(I)\right)\right)$$
$$= \mathrm{An}\left(\bigcap_{I \in \mathscr{K}_A} \mathrm{KeAn}(I)\right) = \mathrm{An}\left(\bigcap_{I \in \mathscr{K}_A}\right).$$

So $A^{\circ} \subseteq A^{*}$ is dense if and only if $\bigcap_{\mathscr{K}_A} I = 0$. □

Lemma 5.13 (Krull's Theorem). *Let A be a* commutative Noetherian *ring. For every* finitely generated *A-module M and every A-ideal* $I \lhd A$ *we have*

$$\bigcap_{k=0}^{\infty} MI^{k+1} = \{m \in M \mid \exists b \in I, \text{ such that } m(1_A - b) = 0\}.$$

The following result was obtained in [25, 6.1.3] for *commutative affine algebras* over base fields:

Lemma 5.14. *Let R be a QF ring and A be a* commutative Noetherian *R-algebra. If every maximal A-ideal is R-cofinite then* $A^{\circ} \subseteq A^{*}$ *is dense.*

Proof. Let $0 \neq a \in A$ be arbitrary and consider the A-ideal $J := (0 : a)$. Let $\mathfrak{m} \lhd A$ be a maximal A-ideal, such that $J \subseteq \mathfrak{m}$. Sine A is Noetherian, $\mathfrak{m}_A$ is finitely generated. If $a \in \bigcap_{k=0}^{\infty} \mathfrak{m}^{k+1}$ then there exists by Krull's Theorem some $b \in \mathfrak{m}$, such that $a(1_A - b) = 0$ and so $1_A \in \mathfrak{m}$ (*contradiction*). So there exists $k \geqslant 0$, such that $a \notin \mathfrak{m}^{k+1}$. By assumption $\mathfrak{m} \subseteq A$ is R-cofinite and it follows then from Lemma 2.14 that $\mathfrak{m}^{k+1} \subseteq A$ is R-cofinite, i.e. $a \notin \bigcap_{\mathscr{K}_A} I$. Since $0 \neq a \in A$ is arbitrary by our choice, it follows that $\bigcap_{\mathscr{K}(A)} I = 0$, i.e. A is proper and consequently $A^{\circ} \subseteq A^{*}$ is dense by Proposition 5.12. □

Analogous to the proof of Proposition 5.12 we get

Proposition 5.15. *Let R be Noetherian, A be an R-algebra and M be a right A-module.*

1. *If M is proper then* $\mathrm{Cf}(M)$ *is Hausdorff.*
2. *Let M be cofinitely R-cogenerated. Then* $\mathrm{Cf}(M)$ *is Hausdorff if and only if M is proper.*
3. *Assume R to be a QF ring. Then M is proper if and only if* $\mathrm{Cf}(M)$ *is Hausdorff if and only if $M^\circ \subseteq M^*$ is dense.*

Theorem 5.16. *Let R be Noetherian, $P = (A, C) \in \mathscr{P}_m^\alpha$ and $Q = (M, N) \in \mathscr{Q}_P^r$.*

1. *If Q is coreflexive then $M^r = M^\circ$.*
2. *Let M be cofinitely R-cogenerated.*
   (a) *If $N \overset{\chi_Q}{\simeq} M^\circ$ then Q is coreflexive.*
   (b) *Let Q be weakly coreflexive. Then Q is coreflexive if and only if $N \overset{\chi_Q}{\simeq} M^\circ$.*

Proof.

1. Assume Q to be coreflexive and consider A and M with the linear weak topology $A[\mathfrak{T}_{ls}(C)]$, $M[\mathfrak{T}_{ls}[N])$ respectively. Let $f \in M^*$ with $f(L) = 0$ for an R-cofinite A-submodule $L \subseteq M$, say $M/L = \sum_{i=1}^k R(m_i + L)$. By assumption $M[\mathfrak{T}_{ls}(N)] = \mathrm{Cf}(M)$ and so L is open with respect to $M[\mathfrak{T}_{ls}(N)]$. By [2, Corollary 1.9] $\xi_{m_i} : A \to M$ is continuous and so there exist finitely generated R-submodules $Z_1, \ldots, Z_k \subseteq C$, such that $Z_i^\perp \subseteq \xi_{m_i}^{-1}(L)$. Consequently $(\sum_{i=1}^k Z_i)^\perp = \bigcap_{i=1}^k Z_i^\perp \subseteq (0_{M^*} : f)$, i.e. $f \in M^r$ (by Proposition 2.11). Obviously $M^r \subseteq M^\circ$ and the result follows.
2. Let M be cofinitely R-cogenerated.
   (a) Assume that $N \overset{\chi_Q}{\simeq} M^\circ$. Let $L \subseteq M$ be an R-cofinite A-submodule with $\{f_1, \ldots, f_k\}$ a generating system of $\mathrm{An}(L) \simeq (M/L)^*$. Then there exists by assumption $\{n_1, \ldots, n_k\} \subset N$, such that $\chi_Q(n_i) = f_i$. By [31, 28.1] we have then
   $$\left(\sum_{i=1}^k R n_i\right)^\perp = \bigcap_{i=1}^k \mathrm{Ker}(f_i) = \mathrm{Ke}\left(\sum_{i=1}^k R f_i\right) = \mathrm{KeAn}(L) = L,$$
   i.e. L is open with respect to $M[\mathfrak{T}_{ls}(N)]$. Consequently $\mathrm{Cf}(M) \preceq M[\mathfrak{T}_{ls}(N)]$. By Lemma 5.8 (2) $M[\mathfrak{T}_{ls}(N)] \preceq \mathrm{Cf}(M)$ and so $M[\mathfrak{T}_{ls}(N)] = \mathrm{Cf}(M)$, i.e. Q is coreflexive.
   (b) The result follows from (1) and (a). □

Corollary 5.17. *Let R be Noetherian and C be a locally projective R-coalgebra.*

1. *If N is coreflexive then $N^{*\square} = N^{*\circ}$.*
2. *Let N^* be cofinitely R-cogenerated.*
   (a) *If $N \simeq N^{*\circ}$ then N is coreflexive.*
   (b) *Let N be weakly coreflexive. Then N is coreflexive if and only if $N \simeq N^{*\circ}$.*

Theorem 5.18. *Let R be Noetherian and $P=(A,C)\in\mathscr{P}_m^{\alpha}$.*

1. *If P is coreflexive then $C=A^{\circ}$.*
2. *Assume R to be Artinian. Then P is coreflexive if and only if all R-cofinite A-ideals are closed with respect to $A[\mathfrak{T}_{ls}(C)]=\mathscr{T}_{-C}(A)$.*
3. *If A is cofinitely R-cogenerated then the following statements are equivalent:*
   (i) *P is coreflexive;*
   (ii) $C=A^{\circ}$.
   (iii) *every locally finite left A-module is C-rational, i.e.* $\mathrm{Loc}({}_A\mathscr{M})=\sigma[{}_AC]$.

Proof.

1. By Corollary 2.9 (1) $C=A^r$ and so the result follows from Theorem 5.16 (1).
2. Let R be Artinian. By [2, Lemma 1.7 (4)] every R-cofinite closed A-ideal is open and the result follows.
3. (i) if and only if (ii) follows from Theorem 5.16 (3).
   (ii) $\Rightarrow$ (iii) By assumption and Proposition 3.23 (2) $\mathrm{Loc}({}_A\mathscr{M})=\sigma[{}_AA^{\circ}]=\sigma[{}_AC]$.
   (iii) $\Rightarrow$ (ii) Assume all locally finite left A-modules to be C-rational. Then in particular ${}_AA^{\circ}$ is C-rational and it follows from Corollary 2.9 (2) that $C=A^{\circ}$. □

Corollary 5.19. *Let R be Noetherian and C be a locally projective R-coalgebra.*

1. *If C is coreflexive then the canonical R-linear mapping $\phi_C:C\to C^{**}$ induces an isomorphism $C\overset{\Phi_C}{\simeq}C^{*\circ}$.*
2. *Let R be Artinian. Then C is coreflexive if and only if all R-cofinite C^*-ideals are closed with respect to the finite topology.*
3. *If C^* is cofinitely R-cogenerated then the following statements are equivalent:*
   (i) *C is coreflexive;*
   (ii) $C\simeq C^{*\circ}$;
   (iii) *every locally finite left C^*-module is C-rational.*

As a consequence of Lemma 3.22 and Theorem 5.18 (3) get we

Proposition 5.20. *Let R be Noetherian. If A is a cofinitely R-cogenerated α-algebra and M is a right A-module with structure map $\phi_M:M\otimes_R A\to M$ then for every $f\in M^*$ the following statements are equivalent:*

1. $f\in M^{\circ}$,
2. $\phi_M^*(f)\in M^{\circ}\otimes_R A^{\circ}$,
3. $\phi_M^*(f)\in M^{\circ}\otimes_R A^*$,
4. $\phi_M^*(f)\in M^*\otimes_R A^*$,
5. *Af is finitely generated in $\mathscr{M}_R$,*
6. *$f(MI)=0$ for an R-cofinite (right) A-ideal,*
7. *$f(L)=0$ for an R-cofinite right A-submodule $L\subseteq M$.*

Analogous to [28] we get

Corollary 5.21. *Let R be a QF ring.*

1. *A projective R-coalgebra C is coreflexive if and only if C^* is a reflexive R-algebra.*
2. *Let A be an α-algebra. If A is weakly reflexive then A° is a coreflexive R-coalgebra.*

Example 5.22 ([17, Example 5]). Let R be a field and consider the Hopf R-algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ with countable basis $\{h_0, h_1, h_2, \ldots\}$ and

$$\mu(h_n \otimes h_k) := \binom{n+k}{n} h_{n+k}, \quad \Delta(h_n) := \sum_{i+j=n} h_i \otimes h_j, \quad S(h_n) := (-1)^n h_n.$$

$$\eta(1_R) := h_0, \quad \varepsilon(h_n) := \delta_{0,n}.$$

1. $H^* \simeq R[[x]]$ is a principal ideal domain and

   $$\mathscr{M}^H \simeq \mathrm{Rat}^H({}_{H^*}\mathscr{M}) = \{M \in {}_{H^*}\mathscr{M} \mid M \text{ is a } \textit{torsion module}\}.$$

   So $\mathrm{Rat}^H(-)$ is a radical and $\mathrm{Rat}^H({}_{H^*}\mathscr{M})$ is closed under extensions.
2. $H^{\square} := \mathrm{Rat}^H({}_H H^*) = 0$.
3. There exists no finite dimensional nonzero projective right H-comodules.
4. $H \simeq H^{*\circ}$, i.e. H is a coreflexive R-coalgebra.

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References

[1] E. Abe, Hopf algebras, in: Cambridge Tracts in Mathematics, Vol. 74, Cambridge University Press, Cambridge, 1980.

[2] J.Y. Abuhlail, On the linear weak topology and dual pairings over rings, preprint math.RA/0306356.

[3] J.Y. Abuhlail, Dualitätstheoreme for Hopf-Algebren über Ringen, Ph.D. Dissertation, Heinrich-Heine Universität, Düsseldorf-Germany, 2001. http://www.ulb.uni-duesseldorf.de/diss/mathnat/2001/abuhlail.html

[4] J.Y. Abuhlail, J. Gómez-Torrecillas, F. Lobillo, Duality and rational modules in Hopf algebras over commutative rings, J. Algebra 240 (2001) 165–184.

[5] J.Y. Abuhlail, J. Gómez-Torrecillas, R. Wisbauer, Dual coalgebras of algebras over commutative rings, J. Pure Appl. Algebra 153 (2) (2000) 107–120.
[6] T. Albu, R. Wisbauer, M-density, M-adic completion and M-subgeneration, Rend. Sem. Mat. Univ. Padova 98 (1997) 141–159.
[7] J. Berning, Beziehungen zwischen links-linearen Toplogien und Modulkategorien, Dissertation, Universität Düsseldorf, 1994.
[8] T. Brzeziński and R. Wisbauer, Corings and Comodules, Lond. Math. Soc. Lec. Not. Ser. 309, Cambridge University Press (2003).
[9] C. Chen, W. Nichols, A duality theorem for Hopf module algebras over Dedekind rings, Comm. Algebra 18 (1990) 3209–3221.
[10] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, Hopf algebras: an introduction, in: Pure and Applied Mathematics, Vol. 235, Marcel Dekker, New York, 2001.
[11] L. Grünenfelder, T. Košir, Koszul cohomology for finite families of comodule maps and applications, Comm. Algebra 25 (1997) 459–479.
[12] R. Heyneman, D. Radford, Reflexivity and coalgebras of finite type, J. Algebra 28 (1974) 215–246.
[13] J. Kelley, I. Namioka, Linear topological spaces, Reprint of the Van Nostrand ed., in: Graduate Texts in Mathematics, Vol. 36, Springer, Berlin, 1963.
[14] G. Köthe, Topologische lineare Räume I, in: Die Grundlehren der mathematischen Wissenschaften, Vol. 107, Springer, Berlin, 1966.
[15] L. Lambe, D. Radford, Introduction to the quantum Yang-Baxter equation and quantum groups: an algebraic approach, in: Mathematics and its Applications, Vol. 423, Kluwer Academic Publishers, Dordrecht, 1997.
[16] R. Larson, Topological Hopf algebras and braided monoidal categories, Appl. Categ. Struct. 6 (1998) 139–150.
[17] B. Lin, Semiperfect coalgebras, J. Algebra 49 (1977) 357–373.
[18] G. Liu, The duality between modules and comodules, Acta Math. Sin. 37 (1994) 150–154.
[19] D. Lu, Monoidal functors on the category of representations of a triangular Hopf algebra, Sci. China (Ser. A) 412 (1998) 139–146.
[20] H. Miyamoto, A note on coreflexive coalgebras, Hiroshima Math. J. 5 (1975) 17–22.
[21] S. Montogomery, Hopf Algebras and their Actions on Rings, in: Reg. Conf. Series in Mathematics, Vol. 82, American Mathematical Society, Providence, RI, 1993.
[22] B. Pareigis, Endliche Hopf-Algebren, Algebra Berichte, 1973.
[23] D. Radford, Coreflexive coalgebras, J. Algebra 26 (1973) 512–535.
[24] B. Stenström, Rings of Quotients: An Introduction to Methods of Ring Theory, in: Die Grundlagen der mathematischen Wissenschaften, Vol. 217, Springer, Berlin, 1975.
[25] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[26] M. Takeuchi, Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra 9 (1981) 841–882.
[27] E. Taft, Reflexivity of algebras and coalgebras II, Comm. Algebra 5 (1977) 1549–1560.
[28] E. Taft, Reflexivity of algebras and coalgebras, Am. J. Math. 94 (1972) 1111–1130.
[29] M. Takeuchi, Tangent coalgebras and hyperalgebras I, Jap. J. Math. 42 (1974) 1–144.
[30] R. Wisbauer, Modules and Algebras: Bimodule Structure and Group Action on Algebras, in: Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 81, Addison Wesely, Longman Limited, Reading, MA/New York, 1996.
[31] R. Wisbauer, Grundlagen der Modul- und Ringtheorie, München, Verlag Reinhard Fischer, 1988; Foundations of Module and Ring Theory, Gordon and Breach, London, 1991.
[32] L. Witkowski, On coalgebras and linearly topological rings, Colloq. Math. 40 (1979) 207–218.
[33] B. Zimmermann-Huisgen, Pure submodules of direct products of free modules, Math. Ann. 224 (1976) 233–245.