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Cubic Spline Wavelet Bases of Sobolev Spaces and Multilevel Interpolation

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In this paper, a semi-orthogonal cubic spline wavelet basis of homogeneous Sobolev space $H_0^2(I)$ is constructed, which turns out to be a basis of the continuous space $C_0(I)$. At the same time, the orthogonal projections on the wavelet subspaces in $H_0^2(I)$ are extended to the interpolating operators on the corresponding wavelet subspaces in $C_0(I)$. A fast discrete wavelet transform (FWT) for functions in $C_0(I)$ is also given, which is different from the pyramid algorithm and easy to perform using a parallel algorithm. Finally, it is shown that the singularities of a function can be traced from its wavelet coefficients, which provide an adaptive approximation scheme allowing us to reduce the operation time in computation.

1. INTRODUCTION

The use of polynomial splines is widespread in numerical analysis and many other fields because they have simple structure, good localization properties, and computing stability. (See [7].) Based on this fact, recently splines have often been used to construct wavelet bases. (See [1–3, 6, 11, 12, 14, 17].) In applications, wavelets on a bounded interval are also very useful. Chui and Quak first constructed spline wavelet bases of $L^2(I)$ (see [4]). A general discussion of the construction of wavelets on the interval [0, 1] can be found in Micchelli and Xu [13]. The aim of this paper is to construct a (semiorthogonal) cubic spline wavelet basis for the Sobolev space on a bounded interval and to present its properties.

Let $I = [a, b]$ be a bounded interval. The homogeneous Sobolev space $H_0^2(I)$ can be defined by

$$H_0^2(I) = \{f : f'' \in L^2(I), f(a) = f'(a) = f(b) = f'(b) = 0\}.$$

Equipped with the inner product $\int_I f''g''$, $H_0^2(I)$ is a Hilbert space, in which cubic splines on I with dyadic knots are dense. Hence the cubic B-spline creates a multiresolution analysis for space $H_0^2(I)$. In a way similar to that in [3], we can construct a semiorthogonal cubic spline wavelet and its dual in $H_0^2(I)$. Since $H_0^2(I)$ is dense in the space $C_0(I) := \{f : f \in C(I), f(a) = f(b) = 0\}$ equipped with the uniform norm $\|f\| := \max_{x\in I} |f(x)|$, a basis of $H_0^2(I)$ is also a basis of $C_0(I)$. Furthermore, the interpolation operator in the spline subspace of $C_0(I)$ turns out to be the extension of the orthogonal projection in $H_0^2(I)$. Then the corresponding Fast Wavelet Transform (FWT) can be performed as a multilevel interpolation procedure which computes the wavelet coefficients on the different levels simultaneously. Meanwhile, the singularities of functions in $C_0(I)$ can be characterized by their wavelet coefficients with respect to this basis. Based on this fact, an adaptive approximation scheme is designed to reduce the operation time.

The outline of this paper is as follows. In Section 2, we construct the semiorthogonal cubic spline wavelet basis of $H_0^2(I)$. In Section 3, we describe the singularities of functions by their wavelet coefficients. Finally, an FWT algorithm and an adaptive approximation scheme are presented in Section 4.

2. CUBIC SPLINE WAVELET BASIS OF H_0^2

Let $I = [0, L]$, where L is a positive integer (for the sake of simplicity, we assume that $L \geqslant 4$). For any $j \in Z^+$, we define the cardinal spline space S_j by

$$S_j = \{s(x); s \in C^2(I) \text{ and } s(x) \in \pi_3, x \in [x_k^j, x_{k+1}^j] \quad \text{for } k = 0, \ldots, n_j - 1\},$$

where π_k is the set of all polynomials of degree no greater than k, $x_k^j = k/2^j$, and $n_j = 2^j L$. Later the knot set $\{x_k^j\}_{k=1}^{n_j-1}$ will be denoted by Δ_j.

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Let $S_j^0 := S_j \bigcap H_0^2(I). \{S_j^0\}_{j=0}^{\infty}$ forms a multiresolution approximation of $H_0^2(I)$ in the following sense:

$$S_0^0 \subset S_1^0 \subset S_2^0 \subset \cdots \subset S_j^0 \subset \cdots$$

and

$$\overline{\bigcup_{j=0}^{\infty} S_j^0} = H_0^2(I).$$

We define

$$\phi(x) = \frac{1}{6}\sum_{j=0}^{4}\binom{4}{j}(-1)^j(x-j)_+^3$$

$$\phi_b(x) = \tfrac{3}{2}x_+^2 - \tfrac{11}{12}x_+^3 + \tfrac{3}{2}(x-1)_+^3 - \tfrac{3}{4}(x-2)_+^3,$$

where

$$x_+^n = \begin{cases} x^n & \text{if } x \geqslant 0 \\ 0 & \text{otherwise,}\end{cases}$$

and denote, for any $j, k \in Z$,

$$\phi_{j,k}(x) = \phi(2^j x - k), \quad \phi_{b,j}(x) = \phi_b(2^j x),$$
$$\phi_{r,j}(x) = \phi_{b,j}(L - x). \quad (2.3)$$

The set $\{\phi_{j,k}, \phi_{b,j}, \phi_{r,j}\}_{k=0}^{n_j-4}$ is a basis of the space S_j^0 (see [7]).

In order to construct a wavelet basis, we introduce the following lemma.

LEMMA 2.1. *A function $g \in H_0^2(I)$ is orthogonal to the space S_j^0 if and only if*

$$g(x_k^j) = 0, \quad k = 1, \cdots, n_j - 1. \quad (2.4)$$

Proof. This is derived from the fact that for any $g \in H_0^2(I), s \in S_j^0$,

$$\begin{aligned}\langle g, s\rangle &= \int_I g''(x)s''(x)\,dx \\ &= \sum_{k=1}^{n_j-1} g(x_k^j)[s^{(3)}(x_k^j+) - s^{(3)}(x_k^j-)]. \quad \blacksquare\end{aligned}$$

Let W_j be the orthogonal compliment of S_j^0 in S_{j+1}^0. From Lemma 2.1, we know that any function in W_j satisfies (2.4). Now we define two functions $\psi(x), \psi_b(x) \in W_0$,

$$\psi(x) = -\tfrac{3}{7}\phi(2x) + \tfrac{12}{7}\phi(2x-1) - \tfrac{3}{7}\phi(2x-2) \quad (2.5)$$

$$\psi_b(x) = \tfrac{24}{13}\phi_b(2x) - \tfrac{6}{13}\phi(2x). \quad (2.6)$$

We set

$$\psi_{j,k}(x) = \psi(2^j x - k), \quad \psi_{b,j}(x) = \psi_b(2^j x),$$
$$\psi_{r,j}(x) = \psi_{b,j}(L - x). \quad (2.7)$$

It is easy to verify that $\{\psi_{j,k}\}_{k=0}^{n_j-3} \bigcup \{\psi_{b,j}, \psi_{r,j}\} \subset W_j$ forms a basis of W_j. For convenience, we set $\psi_{-1,k} = \phi_{0,k}, \psi_{-1,-1} = \phi_{b,0}, \psi_{-1,L-3} = \phi_{r,0}, n_{-1} = L - 1$, and $W_{-1} = S_0^0$. Similarly, for $j \geqslant 0$, we set $\psi_{b,j} = \omega_{j,-1}$ and $\psi_{r,j} = \psi_{j,n_j-2}$. By these notations, we have

$$H_0^2(I) = \bigoplus_{j \geqslant -1} W_j$$

and that $\{\psi_{j,k}; -1 \leqslant k \leqslant n_j - 2, j \geqslant 0\}$ forms a basis of $H_0^2(I)$. The following theorem confirms that it becomes an unconditional basis when they are stretched.

THEOREM 2.2. Let $\mathbf{B}_j = \{2^{-3j/2}\psi_{j,k}\}_{k=-1}^{n_j-2}, -1 \leqslant j < \infty$. Then $\mathbf{B} := \bigcup_{j=-1}^{\infty} \mathbf{B}_j$ is an unconditional basis of $H_0^2(I)$.

Proof. Recall that $W_j \perp W_k$, if $k \neq j$. The theorem will be proved if we can verify that, for any sequence $\{d_{j,k}\}_{k=1}^{n_j}, 0 \leqslant j < \infty$,

$$\begin{aligned} c_1\left(\sum_{k=1}^{n_j} d_{j,k}^2\right)^{1/2} &\leqslant \|\sum_{k=1}^{n_j} d_{j,k}2^{-3j/2}\psi_{j,k-2}\|_{H_0^2} \\ &\leqslant c_2\left(\sum_{k=1}^{n_j} d_{j,k}^2\right)^{1/2}, \end{aligned} \quad (2.8)$$

where c_1 and c_2 are two constants independent of j. Note that, for $0 \leqslant j < \infty$,

$$\|\sum_{k=1}^{n_j} d_{j,k}2^{-3j/2}\psi_{j,k-2}\|_{H_0^2} = \sum_{k,l=1}^{n_j} d_{j,k}d_{j,l}\alpha_{k,l}^{(j)}, \quad (2.9)$$

where

$$\alpha_{k,l}^{(j)} = 2^{-3j}\int_I \psi_{j,k-2}''(x)\psi_{j,l-2}''(x)\,dx, \quad 1 \leqslant k, l \leqslant n_j. \quad (2.10)$$

Now we write

$$\mathscr{A}^{(j)} = (\alpha_{k,l}^{(j)})_{n_j \times n_j}. \quad (2.11)$$

By (2.10), $\alpha_{k,l}^{(j)}$ can be represented as

$$\alpha_{k,l}^{(j)} = \int_{2^j I} \psi_{0,k-2}^{(4)}(x)\psi_{0,l-2}(x)\,dx, \quad 1 \leqslant k, l \leqslant n_j,$$

from which it follows that

$$\mathscr{A}^{(j)} = \tfrac{3}{7} \times 2^7\Gamma_j, \quad (2.12)$$

where

$$\Gamma_j = \mathscr{N}_j\mathscr{M}_j \quad (2.13)$$

with

$$\mathcal{N}_j = \begin{pmatrix} \frac{70}{13} & \frac{14}{13} & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & \frac{14}{13} & \frac{70}{13} \end{pmatrix} \tag{2.14}$$

and

$$\mathcal{M}_j = \begin{pmatrix} 1 & -\frac{1}{14} & & & & & \\ -\frac{1}{13} & 1 & -\frac{1}{14} & & & & \\ & -\frac{1}{14} & 1 & -\frac{1}{14} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{14} & 1 & -\frac{1}{14} & \\ & & & & -\frac{1}{14} & 1 & -\frac{1}{13} \\ & & & & & -\frac{1}{14} & 1 \end{pmatrix}. \tag{2.15}$$

Since both $\mathcal{N}_j$ and $\mathcal{M}_j$ are diagonally dominant, (2.8) is true. ■

An unconditional basis of $H_0^2(I)$ provides a one-to-one mapping from $H_0^2(I)$ to l_2. Therefore we have the following.

COROLLARY 2.1. *Let* $f = \sum_{j=-1}^{\infty} \sum_{k=1}^{n_j} d_{j,k}\psi_{j,k-2}$. *Then* $f \in H_0^2(I)$ *if and only if*

$$\sum_{j=-1}^{\infty} \sum_{k=1}^{n_j} 2^{3j} d_{j,k}^2 < +\infty.$$

We remark that an advantage of expanding a function to a series of the basis **B** is that the coefficients in the expansion can be obtained by interpolation. In order to illustrate this fact, we derive representations of the orthogonal projections $P_j : H_0^2(I) \to S_j^0$ and $Q_j : H_0^2(I) \to W_j$, respectively. Let $\bar{\Delta}_j = \Delta_{j+1} \setminus \Delta_j$; that is, $\bar{\Delta}_j = \{\bar{x}_k^j\}_{k=1}^{n_j}$, where $\bar{x}_k^j = (2k - 1)/2^{j+1})$. For any function f, we set

$$\begin{aligned} \mathbf{f}_j &= f(\Delta_j) = \{f(x_k^j)\}_{k=1}^{n_j-1}, \\ \bar{\mathbf{f}}_j &= f(\bar{\Delta}_j) = \{f(\bar{x}_k^j)\}_{k=1}^{n_j}. \end{aligned} \tag{2.16}$$

By the Schoenberg–Whitney Theorem [16], there is a unique (interpolatory) spline $s_f(\bar{w}_f)$ in $S_j^0(W_j^0)$ such that

$$\mathbf{s}_f := s_f(\Delta_j) = \mathbf{f}_j \quad (\bar{\mathbf{w}}_f := w_f(\bar{\Delta}_j) = \bar{\mathbf{f}}_j).$$

Now we define the interpolatory operators $I_j^s : H_0^2(I) \to S_j^0$ by $I_j^s(f) = s_f$ and $I_j^w : H_0^2(I) \to W_j$ by $I_j^w(f) = w_f$. Let I be the identity operator on $H_0^2(I)$.

THEOREM 2.3. $P_j = I_j^s, Q_j = I_j^w(I - I_j^s)$. *Therefore,* $P_{j+1} = P_j + Q_j, I_{j+1}^s = I_j^s \bigoplus I_j^w$, *where* $I_j^s \bigoplus I_j^w$ *is the tensor sum of the operators* I_j^s *and* I_j^w, *defined by* $I_j^s \bigoplus I_j^w = I_j^s + I_j^w - I_j^w I_j^s$.

Proof. For any $f \in H_0^2(I), \langle (I - I_j^s)f, I_j^s f\rangle = 0 \Rightarrow I_j^s = P_j$. Now $\forall f \in S_{j+1}^0, (I - I_j^s)f \in W_j$ and $I_j^w(I - I_j^s)f = (I - I_j^s)f$. It follows that $f = I_j^s f + I_j^w(I - I_j^s)f$. On the other hand, $\forall f \in W_j, I_j^w(I - I_j^s)f = I_j^w f = f$. Hence $I_j^w(I - I_j^s) = Q_j$. ■

The operators I_j^s and I_j^w can be extended on the space $C_0(I)$ in a natural way. Let

$$V_j = S_j \bigcap C_0(I), \quad 0 \leqslant j < \infty,$$

and

$$W_j = \operatorname{span}_{C_0(I)} \{\psi_{j,k}\}_{k=-1}^{n_j-2}.$$

(For simplicity, we use the same notation W_j to denote both the wavelet subspaces in $H_0^2(I)$ and $C_0(I)$.) It is clear that I_j^s and I_j^w are also operators from $C_0(I)$ to V_j and W_j respectively. Furthermore, since $H_0^2(I)$ is dense in $C_0(I)$, we obtain the following corollary from Theorem 2.2.

COROLLARY 2.2. $\forall f \in C_0(I)$,

$$\lim_{j\to\infty} \|(I - I_j^s)f\| = 0$$

and the wavelet basis **B** *of* $H_0^2(I)$ *is also a basis of* $C_0(I)$.

3. DUAL BASIS AND REGULARITY ANALYSIS

From the previous section we know that any function in $C_0(I)$ can be decomposed into the following series

$$f = \sum_{j=-1}^{\infty} \sum_{k=1}^{n_j} d_{j,k}\psi_{j,k-2}. \tag{3.1}$$

In this section, we want to obtain a formula for calculating the coefficients $d_{j,k}$ and then to describe the regularities of the function f by using these coefficients. For this purpose, at first we construct the dual basis of the basis **B** in Theorem 2.2.

Let B_j^* denote the dual basis of $\mathbf{B}_j$ in the subspace $W_j, -1 \leqslant j < \infty$, that is, $B_j^* = \{\psi_{j,k}^*\}_{k=-1}^{n_j-2}$ with $\psi_{j,k}^* \in W_j$ and

$$\langle \psi_{j,k}^*, \psi_{j,k'}\rangle = \delta_{k,k'}, \quad -1 \leqslant k, k' \leqslant n_j. \tag{3.2}$$

Since $W_j, -1 < j < \infty$, are mutually orthogonal, $B^* = \bigcup_{j=-1}^{\infty} B_j^*$ is the dual of **B**. Then the wavelet coefficient $d_{j,k}$ for $f \in H_0^2(I)$ can be computed by the formula

$$d_{j,k} = \langle f, \psi_{j,k-2}^*\rangle. \tag{3.3}$$

Recall that $\psi^*_{j,k-2} \in W_j$. Hence it has an expansion on the basis $\{\psi_{j,k-2}; 1 \leqslant k \leqslant n_j\}$ and the coefficients of its expansion come from the entries of the inverse of the matrix $\mathscr{A}^{(j)}$ in (2.11). By writing $(\mathscr{A}^{(j)})^{-1} := (\beta^{(j)}_{k,l})_{n_j \times n_j}$, we have the following

LEMMA 3.1. *For* $j \geqslant 0$,

$$\psi^*_{j,k-2} = 2^{-3j} \sum_{l=1}^{n_j} \beta^{(j)}_{k,l} \psi_{j,l-2}. \tag{3.4}$$

Therefore, $\forall f \in H^2_0(I), j \geqslant 0$,

$$d_{j,k} = 2^{-3j} \sum_{l=1}^{n_j} \beta^{(j)}_{k,l} \langle f, \psi_{j,l-2} \rangle. \tag{3.5}$$

Now we are ready to estimate $\beta^{(j)}_{k,l}$. First we introduce a general result about the inverse of a tridiagonal matrix.

LEMMA 3.2. *Let* $|a| < 1/2, 0 < |b| < 1/2$, *and* $\mathscr{T}$ *be a tridiagonal matrix in the form*

$$\mathscr{T} = \begin{pmatrix} 1 & b & & & & & \\ a & 1 & b & & & & \\ & b & 1 & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & b & 1 & b & \\ & & & & b & 1 & a \\ & & & & & b & 1 \end{pmatrix}_{n \times n}. \tag{3.6}$$

Then $\mathscr{T}$ *is invertible and the element of* $\mathscr{T}^{-1} := (\tau_{k,l})_{n \times n}$ *satisfies the inequality*

$$|\tau_{l,k}| \leqslant \begin{cases} \left| \dfrac{\alpha \beta^{|k-l|}}{\alpha + a} \right|, & l = 1, n, \\[2ex] \left| \dfrac{\beta^{|k-l|}}{b(\alpha - \beta)} \right|, & 1 < l < n, \end{cases} \tag{3.7}$$

where

$$\alpha = \frac{-1 - \sqrt{1 - 4b^2}}{2b} \quad \text{and} \quad \beta = \frac{-1 + \sqrt{1 - 4b^2}}{2b}.$$

Proof. For any fixed l, the sequence $(\tau_{l,k})^n_{k=1}$ satisfies the difference equation

$$b\tau_{l,k-1} + \tau_{l,k} - b\tau_{l,k+1} = \delta_{l,k}, \quad k = 2, \cdots, n-1 \tag{3.8}$$

with the boundary conditions

$$\begin{cases} \tau_{l,1} + \alpha \tau_{l,2} &= \delta_{l,1} \\ \alpha \tau_{l,n-1} + \tau_{l,n} &= \delta_{l,n}. \end{cases} \tag{3.9}$$

The solution of (3.8) and (3.9) is

$$\begin{cases} \tau_{1,k} = \dfrac{\alpha[(\alpha + a) - (\beta + a)\beta^{2(n-k-1)}]}{(\alpha + a)^2 - (\beta + a)^2 \beta^{2(n-3)}} \beta^{k-1}, \\ \qquad\qquad 1 \leqslant k \leqslant n, \\[2ex] \tau_{l,k} = \dfrac{[(\alpha + a) - (\beta + a)\beta^{2(l-2)}] \times [(\alpha + a) - (\beta + a)\beta^{2(n-k-1)}]}{b(\beta - \alpha)[(\alpha + a)^2 - (\beta + a)^2 \beta^{2(n-3)}]} \beta^{k-l}, \\ \qquad\qquad 2 \leqslant l \leqslant k \leqslant n_j - 1 \\[2ex] \tau_{l,k} = \tau_{n-l+1,n-k+1}, \qquad k \leqslant l. \end{cases} \tag{3.10}$$

Since

$$\left| \frac{(\alpha + a) - (\beta + a)\beta^{2(n-k-1)}}{(\alpha + a)^2 - (\beta + a)^2 \beta^{2(n-3)}} \right| \leqslant \left| \frac{1}{\alpha + a} \right|$$

and

$$\left| \frac{[(\alpha + a) - (\beta + a)\beta^{2(l-2)}][(\alpha + a) - (\beta + a)\beta^{2(n-k-1)}]}{[(\alpha + a)^2 - (\beta + a)^2 \beta^{2(n-3)}]} \right| \leqslant 1, \quad 2 \leqslant l \leqslant k,$$

we obtain (3.7). ∎

LEMMA 3.3. *Let* $j \geqslant 0$ *and* $\lambda = 2 - \sqrt{3}$. *Then the coefficient* $\beta^{(j)}_{k,l}$ *in* (3.5) *satisfies*

$$|\beta^{(j)}_{k,l}| \leqslant 2^{-7} \lambda^{|l-k|}, \quad 1 \leqslant k, l \leqslant n_j. \tag{3.11}$$

Proof. It is clear that the matrix $\mathscr{M}_j$ in (2.15) is in the form of (3.6) with $a = -1/13$ and $b = -1/14$. Write $\mathscr{M}_j^{-1} = (m^{(j)}_{l,k})_{n_j \times n_j}$. By Lemma 3.2,

$$0 < m^{(j)}_{j,k} < \frac{7\sqrt{3}}{12}(7 - 4\sqrt{3})^{|l-k|}.$$

On the other hand, the matrix $\mathscr{N}_j$ can be factored as $\mathscr{N}_j = \mathscr{D}_j \mathscr{S}_j$, where $\mathscr{D}_j$ is an n_j order diagonal matrix $\mathscr{D}_j = \operatorname{diag}\{\frac{70}{13} 4 \cdots 4 \frac{70}{13}\}$ and $\mathscr{S}_j^{\mathrm{T}}$ is an n_j order tridiagonal matrix in the form of (3.6) with $a = \frac{1}{5}$ and $b = \frac{1}{4}$. Hence, the element of $\mathscr{N}_j^{-1} := (\eta^{(j)}_{k,l})_{n_j \times n_j}$ satisfies

$$0 < (-1)^{l-k} \eta^{(j)}_{l,k} < \frac{\sqrt{3}}{6}(2 - \sqrt{3})^{|l-k|}.$$

Now writing $\Gamma_j^{-1} = (\bar{\gamma}^{(j)}_{k,l})_{n_j \times n_j}$, we have

$$|\bar{\gamma}^{(j)}_{l,k}| = |\sum_{s=1}^{n_j} m^{(j)}_{k,s} \eta^{(j)}_{s,l}| \leqslant \frac{7}{24}\left(1 + \frac{\lambda}{1 + \lambda} + \frac{\lambda^3}{1 - \lambda^3}\right) \lambda^{|l-k|} \leqslant \frac{3}{7} \lambda^{|l-k|}.$$

Recalling that $\mathscr{A}_j^{-1} = \frac{7}{3} \cdot 2^{-7}\Gamma_j^{-1}$, we obtain (3.11). ■

Now we return to expansion (3.1). Before we employ formula (3.5) to calculate the coefficients $d_{j,k}$ for a function $f \in C_0(I)$, we have to extend the inner product $\langle f, \psi_{j,l-2}\rangle$ to it. Such an extension is given in the following theorem.

THEOREM 3.4. *For $f \in C_0(I)$, the inner product $\langle f, \psi_{j,k-2}\rangle$ can be defined by*

$$\langle f, \psi_{j,k-2}\rangle = 2^{3j}u_{j,k}, \quad 1 \leqslant k \leqslant n_j, \tag{3.12}$$

where

$$\begin{cases} u_{j,1} = \frac{3\cdot 2^5}{13}(40f(2^{-(j+1)}) - 24f(2\cdot 2^{-(j+1)}) \\ \qquad\qquad + 8f(3\cdot 2^{-(j+1)}) - f(4\cdot 2^{-(j+1)})) \\ u_{j,l} = -\frac{3\cdot 2^5}{7}(f(2^{-j}(l-2)) - 8f(2^{-j}(l-3/2)) \\ \qquad + 23f(2^{-j}(l-1)) - 32f(2^{-j}(l-1/2)) \\ \qquad + 23f(2^{-j}l) - 8f(2^{-j}(l+1/2)) \\ \qquad + f(2^{-j}(l+1))), 1 < l < n, \\ u_{j,n_j} = \frac{3\cdot 2^5}{13}(40f(L - 2^{-(j+1)}) - 24f(L - 2\cdot 2^{-(j+1)}) \\ \qquad + 8f(L - 3\cdot 2^{-(j+1)}) - f(L - 4\cdot 2^{-(j+1)})). \end{cases} \tag{3.13}$$

Therefore, the coefficient in the decomposition series (3.1) *for $f \in C_0(I)$ is given by*

$$d_{j,k} = \sum_{l=1}^{n_j} \beta_{k,l}^{(j)} u_{j,l}, \quad 1 \leqslant k \leqslant n_j, 0 \leqslant j < \infty. \tag{3.14}$$

Proof. Recall that the functions $\psi_{j,l}$, $j \geqslant 0, -1 \leqslant l \leqslant n_j - 2$, are given by (2.5)–(2.7). Then, $\forall f \in H_0^2(I)$, by direct calculation, we obtain the inner production formula (3.13). Note that $u_{j,k}$ in formula (3.13) is a bounded linear functional on $C_0(I)$. Since $H_0^2(I)$ is dense in $C_0(I)$, we can extend $\langle f, \psi_{j,l-2}\rangle$ to any function $f \in C_0(I)$ by (3.12), and calculate its coefficient by (3.14). ■

Now we consider the regularity of the wavelet series (3.1).

DEFINITION 3.1. *Let α be a positive number and $n = \lfloor \alpha \rfloor$ be the largest integer in the interval $[0, \alpha)$. A function f continuous on I is said to be uniformly* Lipα *on I, if $f^{(n)}$ exists on I and satisfies*

$$|f^{(n)}(x) - f^{(n)}(y)| \leqslant M|x-y|^{\alpha-n}, \quad \forall x, y \in I, \tag{3.15}$$

where $M > 0$ is a constant. The space containing all functions uniformly Lipα *on I is denoted by $\mathscr{C}^\alpha(I)$.*

When α is an integer, the space $C^\alpha(I)$ is a proper subspace of $\mathscr{C}^\alpha(I)$. The subspace $\mathscr{C}_0^\alpha(I)$ of $C_0(I)$ is defined as

$$\mathscr{C}_0^\alpha(I) = \begin{cases} \mathscr{C}^\alpha(I) \bigcap C_0(I), & \text{if } 0 < \alpha < 1, \\ \mathscr{C}^\alpha(I) \bigcap C_0^1(I), & \text{if } 1 \leqslant \alpha, \end{cases}$$

where $C_0^1(I)$ is the space containing all such functions in $C^1(R)$ that are supported on I.

The index α in the definition reflects the global regularity of a function. The following theorem states that the regularity of a function can be characterized by the decay rate of its wavelet coefficients as the "level index" j tends to infinity.

THEOREM 3.5. *For any positive number $\alpha, 0 < \alpha \leqslant 3$, the wavelet series* (3.1) *is in space $\mathscr{C}_0^\alpha(I)$ if and only if*

$$|d_{j,k}| \leqslant C2^{-\alpha j}, \quad 1 \leqslant k \leqslant n_j, 0 \leqslant j \leqslant \infty, \tag{3.16}$$

where C is a positive constant independent of j.

Proof. Write

$$\Delta_j^n f(x) = \sum_{s=0}^{n} (-1)^s \binom{n}{s} f(2^{-j}(x + s/2)).$$

If $f \in \mathscr{C}_0^\alpha(I)$ for $0 < \alpha \leqslant 3$, by (3.13), for $j \geqslant 0$, we have

$$u_{j,1} = \begin{cases} \frac{3\cdot 2^5}{13}\left(-23\Delta_j f(0) + 17\Delta_j f\left(\frac{1}{2}\right)\right. \\ \qquad \left.-\, 7\Delta_j f(1) + \Delta_j f\left(\frac{3}{2}\right)\right), & 0 < \alpha < 1, \\ \frac{3\cdot 2^5}{13}\left(-\, 12\Delta_j f(0) - 11\Delta_j^2 f(0)\right. \\ \qquad \left.+\, 6\Delta_j^2 f\left(\frac{1}{2}\right) - \Delta_j^2 f(1)\right), & 1 < \alpha < 2, \\ \frac{3\cdot 2^5}{13}\left(-\, 12\Delta_j f(0) - 6\Delta_j^2 f(0)\right. \\ \qquad \left.-\, 5\Delta_j^3 f(0) + \Delta_j^3 f\left(\frac{1}{2}\right)\right), & 2 < \alpha < 3, \end{cases} \tag{3.17}$$

$$u_{j,l} = \begin{cases} \dfrac{-3\cdot 2^5}{7}\Big(\Delta_j f(l-2) - 7\Delta_j f\left(l-\tfrac{3}{2}\right) \\ \quad +16\Delta_j f(l-1) - 16\Delta_j f\left(l-\tfrac{1}{2}\right) \\ \quad + 7\Delta_j f(l) - \Delta_j f\left(l+\tfrac{1}{2}\right)\Big), \quad 0<\alpha<1, \\ \dfrac{-3\cdot 2^5}{7}\Big(\Delta_j^2 f(l-2) - 6\Delta_j^2 f\left(l-\tfrac{3}{2}\right) \\ \quad + 10\Delta_j^2 f(l-1) - 6\Delta_j^2 f\left(l-\tfrac{1}{2}\right) + \Delta_j^2 f(l)\Big), \\ 1<\alpha<2, \\ \dfrac{-3\cdot 2^5}{7}\Big(\Delta_j^3 f(l-2) - 5\Delta_j^3 f\left(l-\tfrac{3}{2}\right) \\ \quad + 5\Delta_j^3 f(l-1) - \Delta_j^3 f\left(l-\tfrac{1}{2}\right)\Big), \quad 2<\alpha<3, \end{cases} \tag{3.18}$$

$$u_{j,n_j} = \begin{cases} \dfrac{3\cdot 2^5}{13}\Big(23\Delta_j f\left(L-\tfrac{1}{2}\right) - 17\Delta_j f(L-1) \\ \quad + 7\Delta_j f\left(L-\tfrac{3}{2}\right) - \Delta_j f(L-2)\Big), \\ \quad 0<\alpha<1, \\ \dfrac{3\cdot 2^5}{13}\Big(12\Delta_j f\left(L-\tfrac{1}{2}\right) - 11\Delta_j^2 f(L-1) \\ \quad + 6\Delta_j^2 f\left(L-\tfrac{3}{2}\right) - \Delta_j^2 f(L-2)\Big), \\ \quad 1<\alpha<2, \\ \dfrac{3\cdot 2^5}{13}\Big(12\Delta_j f\left(L-\tfrac{1}{2}\right) - 6\Delta_j^2 f(L-1) \\ \quad + 5\Delta_j^3 f\left(L-\tfrac{3}{2}\right) - \Delta_j^3 f(L-2)\Big), \\ \quad 2<\alpha<3. \end{cases} \tag{3.19}$$

Since $f \in \mathscr{C}_0^\alpha$, we have

$$|u_{j,k}| \leqslant C2^{-j\alpha}, \quad 1 \leqslant k \leqslant n_j,$$

where C is a constant independent of j. By (3.14),

$$|d_{j,k}| \leqslant \sum_{k=1}^{n_j} |\beta_{k,l}^{(j)}||u_{j,l}| \leqslant C2^{-j\alpha}\sum_{k=1}^{n_j}|\beta_{k,l}(j)| \leqslant C2^{-j\alpha},$$

which proves (3.16).

Now assuming that (3.16) is true, we prove $f \in \mathscr{C}_0^\alpha(I)$. Let $n = \lfloor \alpha \rfloor$. Note that, $\forall x, y \in I$,

$$\begin{aligned} |f^{(n)}(x) - f^{(n)}(y)| &\leqslant \sum_{j=-1}^{\infty}\sum_{k=1}^{n_j} |d_{j,k}||\psi_{j,k}^{(n)}(x) - \psi_{j,k}^{(n)}(y)| \\ &\leqslant C\sum_{j=-1}^{\infty}\sum_{k=1}^{n_j} 2^{-j(\alpha-n)}|\psi^{(n)}(2^j x - k) \\ &\quad - \psi^{(n)}(2^j y - k)|. \end{aligned}$$

If we choose J such that $2^{-J} \leqslant |x-y| \leqslant 2^{-J+1}$, then

$$\begin{aligned} &\sum_{j<J}\sum_{k=1}^{n_j} 2^{j(n-\alpha)}|\psi^{(n)}(2^j x - k) - \psi^{(n)}(2^j y - k)| \\ &\qquad \leqslant C\sum_{j}\sum_{k\in K_{j,x,y}} 2^{j(n+1-\alpha)}M_{n+1}|x-y|, \end{aligned}$$

where $M_n = \max_{x\in I}|\psi^{(n)}(x)|$, $K_{j,x,y} = [(2^j x - 3, 2^j x)\bigcup(2^j y - 3, 2^j y)]\bigcap Z$.

Noting that $|K_{j,x,y}| \leqslant 6$, we obtain

$$\begin{aligned} \Big|\sum_{j\leqslant J}\sum_{k}\Big| &\leqslant 6M_{n+1}C|x-y|\sum_{j\leqslant J}2^{(n+1-\alpha)j} \\ &\leqslant C|x-y|2^{(n+1-\alpha)J} \leqslant C|x-y|^{\alpha-n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Big|\sum_{j>J}\sum_{k}\Big| &\leqslant 12M_nC\textstyle\sum_{j>J}2^{-j(\alpha-n)} \\ &\leqslant C2^{-J(\alpha-n)} \leqslant C|x-y|^{\alpha-n}. \end{aligned}$$

Combining these two inequalities, we establish (3.15). ■

The decay rate of the wavelet coefficients of a function not only provides the global regularity of the function but also gives its local regularity information. Now we discuss the local regularity of a function in detail.

DEFINITION 3.2. *Let α be a positive number, $n = \lfloor\alpha\rfloor$, and $x_0 \in I$. We say that a function f belongs to $C_{x_0}^\alpha$ if there exists a polynomial P_n of degree n such that*

$$f(x) = P_n(x - x_0) + \circ(|x-x_0|^\alpha). \tag{3.20}$$

Then space $(\mathscr{C}_{x_0}^\alpha)_0$ is defined by

$$(\mathscr{C}_{x_0}^\alpha)_0 = \begin{cases} \mathscr{C}_{x_0}^\alpha \bigcap C_0(I), \ \textit{if } 0<\alpha<1, \\ \mathscr{C}_{x_0}^\alpha \bigcap C_0^1(I), \ \textit{if } 1 \leqslant \alpha. \end{cases}$$

The following theorem gives a wavelet criterion of the local regularity.

THEOREM 3.6. *If $f \in (\mathscr{C}_{x_0}^\alpha)_0$ for $x_0 \in I$ and $0<\alpha<4$, then*

$$|d_{j,k}| \leqslant C2^{-j\alpha}(1 + |2^j x_0 - k|^\alpha). \tag{3.21}$$

Conversely, if (3.21) *holds and $f \in C_0(I)$ is uniformly $\mathscr{C}^\beta$ for a positive number β, then there exists a polynomial of degree $n = \lfloor\alpha\rfloor$ such that*

$$|f(x) - P_n(x-x_0)| \leqslant C|x-x_0|^\alpha \log\frac{2}{|x-x_0|}. \tag{3.22}$$

Proof. From (3.18) we have that for any polynomial $P \in \pi_3$

$$u_{j,l}(P) = 0, \quad 2 \leqslant l \leqslant n_j - 1, \tag{3.23}$$

where $u_{j,l}(f)$ specifies the expression $u_{j,l}$ in (3.14) for a particular function f. By (3.13), we have, for $2 \leqslant l \leqslant n_j - 1$,

$$\begin{aligned}|u_{j,l}(f)| &= |u_{j,l}(f - P(\cdot - x_0))| \\ &\leqslant C \sum_{k=-4}^{2} \left| \frac{l + k/2}{2^j} - x_0 \right|^{\alpha} \\ &\leqslant C 2^{-j\alpha}(1 + |l - 2^j x_0|^{\alpha}).\end{aligned}$$

In order to estimate $u_{j,1}$ and u_{j,n_j}, we write $L^l(x) = P_n(-x_0) - P_n'(-x_0)x, P_n^l(x) = P_n(x - x_0) - L^l(x), L^r(x) = P_n(L - x_0) + P_n'(L - x_0)(L - x)$, and $P_n^r(x) = P_n(x - x_0) - L^r(x)$. Then $u_{j,1}(P_n^l) = u_{j,n_j}(P_n^r) = 0$.

Now if $0 < \alpha \leqslant 1$,

$$\begin{aligned}u_{j,l}(L^l) &= P_n(-x_0)u_{l,1}(1) - P_n^l(-x_0)u_{l,1}(\cdot) \\ &\leqslant \circ(|x_0|^{\alpha}) + \circ(2^{-j}) \\ &\leqslant C2^{-\alpha j}(|1 - 2^j x_0|^{\alpha} + 1).\end{aligned}$$

If $1 < \alpha$, then $f'(0) = 0$ holds. By (3.20), we have $P_n(-x_0) = \circ(|x_0|^{\alpha})$ and $P_n'(-x_0) = \circ(|x_0|^{\alpha-1})$. From (3.17), we obtain $u_{j,1}(\cdot) = \circ(2^{-j})$. It follows that

$$u_{j,1}(L^l) \leqslant \circ(|x_0|^{\alpha}) + \circ(|x_0|^{\alpha-1})2^{-j} \leqslant C2^{-j\alpha}(1 + |2^j x_0 - 1|^{\alpha}).$$

Similarly,

$$u_{j,n_j}(L^r) \leqslant C2^{-j\alpha}(1 + |2^j x_0 - n_j|^{\alpha}), \quad 0 < \alpha \leqslant 4.$$

Finally, we obtain

$$\begin{aligned}|d_{j,k}| &\leqslant \sum_l |\beta_{k,l}^{(j)}||u_{j,l}| \\ &\leqslant C \left[\sum_l \lambda^{|k-l|} 2^{-j\alpha}(|l - 2^j x_0|^{\alpha} + 1) \right] \\ &\leqslant C2^{-j\alpha} \left[\sum_l \lambda^{|l-k|}(|k - 2^j x_0|^{\alpha} + |k - l|^{\alpha}) + 1 \right] \\ &\leqslant C2^{-j\alpha}(|k - 2^j x_0|^{\alpha} + 1).\end{aligned}$$

Assertion (3.21) is proved.

To prove the converse result, we borrow the method used in [10]. (See the proof of Theorem 11 in [10].) Let j_0 and j_1 be two integers determined by $2^{-j_0-1} \leqslant |x - x_0| < 2^{-j_0}$ and $j_1 = (\alpha/\beta)j_0$. Let $f_j(x) = \sum_{k=1}^{n_j} d_{j,k}\psi_{j,k-2}$. Since (3.23) holds, we have

$$\begin{aligned}|f_j(x)| &\leqslant C2^{-\alpha j} \sum_{k=1}^{n_j} (1 + |2^j x_0 - k|^{\alpha})|\psi_{j,k-2}(x)| \\ &\leqslant C2^{-\alpha j}(1 + |2^j(x_0 - x)|^{\alpha})\end{aligned}$$

and for any $l, 0 \leqslant l \leqslant 3$,

$$|f_j^{(l)}(x)| \leqslant C2^{-j(\alpha - l)}(1 + |2^j(x_0 - x)|^{\alpha}).$$

For a function g, let $T_{x_0}(g)$ be the Taylor expansion of g of order $n(= \lfloor \alpha \rfloor)$ at x_0. Then $\forall f \in C_0(I)$,

$$|f(x) - T_{x_0}(f)| \leqslant \sum_{j \leqslant j_0} |f_j(x) - T_{x_0}(f_j)(x)| + \sum_{j \geqslant j_0} |f_j(x)| + \sum_{j \geqslant j_0} |T_{x_0}(f_j)(x)|.$$

The first term is bounded by

$$\begin{aligned}&\sum_{j \leqslant j_0} |f_j(x) - T_{x_0}(f_j)(x)| \\ &\leqslant C|x - x_0|^{n+1} \sum_{j \leqslant j_0} \sup_{x \in I} |f_j^{(n+1)}(x)| \\ &\leqslant C|x - x_0|^{n+1} \sum_{j \leqslant j_0} 2^{(n+1-\alpha)j}(1 + 2^{j\alpha}|x - x_0|^{\alpha}) \\ &= C\left[|x - x_0|^{n+1} \sum_{j \leqslant j_0} 2^{(n+1-\alpha)j} + |x - x_0|^{n+1+\alpha} \sum_{j \leqslant j_0} 2^{(n+1)j} \right] \\ &\leqslant C|x - x_0|^{\alpha},\end{aligned}$$

and the second term,

$$\begin{aligned}\sum_{j_0 \leqslant j < j_1} |f_j(x)| &\leqslant C \sum_{j_0 \leqslant J < j_1} (|x - x_0|^{\alpha} + 2^{-\alpha j}) \\ &\leqslant C[(j - j_0)|x - x_0|^{\alpha} + 2^{-\alpha j_0}] \\ &\leqslant C|x - x_0|^{\alpha}[(j_1 - j_0) + 1].\end{aligned}$$

Since $j_1 - j_0 = j_0(\alpha/\beta - 1) \leqslant C\log(2/|x - x_0|)$,

$$\sum_{j_0 \leqslant j < j_1} |f_j(x)| \leqslant C \log \frac{2}{|x - x_0|}(|x - x_0|^{\alpha}).$$

Noting that $f \in C^{\beta}(I) \bigcap C_0(I)$ and $f'(0) = f'(L) = 0$ when $\alpha > 1$, by Theorem 3.4, we have

$$\sum_{j \leqslant j_1} |f_j(x)| \leqslant \sum_{j \leqslant j_1} 2^{-\beta j} \leqslant C|x - x_0|^s.$$

Finally,

$$T_{x_0}(f_j)(x) = \sum_{s=0}^{n} \frac{f_j^{(s)}(x_0)}{s!}(x - x_0)^s$$

and

$$|T_{x_0}^{(s)}(f_j)(x_0)| = |f_j^{(s)}(x_0)| \leqslant C2^{-(\alpha - s)j}.$$

Therefore

$$\sum_{j \geqslant j_0} |T_{x_0}(f_j)(x)| \leqslant \sum_{j \geqslant j_0} \sum_{s=0}^{n} (2^{-(\alpha - s)j}|x - x_0|^{\alpha}) \leqslant C|x - x_0|^{\alpha}.$$

The converse part of the theorem is proved. ■

4. FAST WAVELET TRANSFORM AND ADAPTIVE APPROXIMATION

In this section, we will introduce a Fast Wavelet Transform (FWT) which directly decomposes the sampling data of a function to its wavelet coefficients.

It is observed that for any function $f(x) \in C_0(I)$, there is a sufficiently large J so that $f_J := I_J^s(f) \in V_J$ approximates to f in the designed precision. Hence, in application, we can consider f_J as the initial function to decompose. Note that function $f_J \in V_J$ is uniquely determined by its sampling data $\mathbf{f}_J$. Since in application functions are often represented by their sampling data, considering the sampling data as the initial data in a wavelet transform procedure is practical. As we know, sampling data have hierarchical property: the sampling data of a function in the lower level can be directly obtained from those ones in the higher level; i.e., $\mathbf{f}_k$ and $\bar{\mathbf{f}}_k$ both are subsets of $\mathbf{f}_j$ if $k < j$. Thus, the decomposition algorithm in this case simply contains the step of decomposing $\mathbf{f}_{j+1}$ into $\mathbf{c}_j$ and $\bar{\mathbf{d}}_j$, where $\mathbf{c}_j$ stands for the scaling coefficients of the function $f_j \in V_j$ and $\bar{\mathbf{d}}_j$ for the wavelet coefficients in W_j. Now we present it as the following.

Fast Wavelet Transform (FWT). For $0 \leqslant j < J$,

$$\mathbf{c}_j = \mathscr{B}_j^{-1}\mathbf{f}_j, \tag{4.1}$$

$$\bar{\mathbf{s}}_j = \bar{\mathscr{B}}_j\mathbf{c}_j, \tag{4.2}$$

$$\bar{\mathbf{g}}_j = \bar{\mathbf{f}}_j - \bar{\mathbf{s}}_j, \tag{4.3}$$

$$\bar{\mathbf{d}}_j = \mathscr{M}_j^{-1}\bar{\mathbf{g}}_j, \tag{4.4}$$

where

$$\mathscr{B}_j = \begin{pmatrix} \frac{7}{12} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & \frac{1}{6} & \frac{7}{12} \end{pmatrix},$$

$$\bar{\mathscr{B}}_j = \begin{pmatrix} \frac{25}{96} & \frac{45}{96} & \frac{2}{96} & & & \\ \frac{1}{48} & \frac{23}{48} & \frac{23}{48} & \frac{1}{48} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \frac{1}{48} & \frac{23}{48} & \frac{23}{48} & \frac{1}{48} \\ & & & \frac{2}{96} & \frac{45}{96} & \frac{25}{96} \end{pmatrix},$$

and $\mathscr{M}_j$ is in (2.15). By using formula (3.1), we can completely recover function f_J from the data $\mathbf{c}_0, \mathbf{d}_0, \cdots, \mathbf{d}_{J-1}$.

Now we consider the computing complexity of the full decomposition algorithm. As we assumed that the sampling data is given in J-level, which counts $2^J - 1$ cells. In order to obtain $\mathbf{c}_j$ from $\mathbf{f}_j$, we have to solve the tridiagonal system which involves $7n_j$ operations. To obtain $\bar{\mathbf{s}}_j$ from $\mathbf{c}_j$, we need $5n_j$ operations. We need $8n_j$ operations for getting $\bar{\mathbf{d}}_j$ from $\bar{\mathbf{s}}_j$ and $\bar{\mathbf{f}}_j$. Then the number of operations for obtaining total wavelet coefficients is $\sum_{j=-1}^{J-1} 20n_j = 20L \cdot 2^J$, which is proportional to the quantity of sampling data $\mathbf{f}_j$.

Remark. If we only consider decomposing and recovering the sampling data of functions, step (4.1) can be omitted. It costs only $13L \cdot 2^J$ operations. The corresponding recovering algorithm (to sampling data $\mathbf{f}_j$) is as follows.

For $0 \leqslant j < J$,

$$\begin{aligned} \bar{\mathbf{s}}_j &= \bar{\mathscr{B}}_j\mathscr{B}_j^{-1}\mathbf{f}_j, \\ \bar{\mathbf{f}}_{j+1} &= \bar{\mathbf{g}}_j + \bar{\mathbf{s}}_j, \\ \mathbf{f}_j &= \mathbf{f}_{j-1} \bigcup \bar{\mathbf{f}}_{j-1}. \end{aligned}$$

In application, a tolerance is allowable in recovering. Since the wavelet components add the details of the function to its "blur" version, the components with small coefficients can be deleted without causing a big error. Recall that if function f has certain smoothness, for instance $f \in \mathrm{Lip}\alpha$, then, by Theorem 3.4, the wavelet coefficient $d_{j,k} = \circ(2^{-j\alpha})$ which tends to 0 as j tends to ∞. Hence we can reduce a quantity of the wavelet coefficients in order to save the operation time and the memory space. Now we will describe it more precisely.

THEOREM 4.1. *For a given $\epsilon > 0$, if $d_{j,k}^*$ is selected by*

$$d_{j,k}^* = \begin{cases} d_{j,k}, & d_{j,k} > \dfrac{13\epsilon}{15J}, \\ 0, & d_{j,k} \leqslant \dfrac{13\epsilon}{15J}, \end{cases}$$

and $f_J^* = \sum_{j=0}^{J-1}\sum_{k=1}^{n_j} d_{j,k}^*\psi_{j,k-2} + \sum_{k=1}^{L-1}\phi_{0,k-2}$, then

$$\|f_j - f_J^*\| < \epsilon.$$

Proof. We have

$$\|\sum_{k=1}^{n_j}|\psi_{j,k-2}|\| \leqslant \frac{15}{13}.$$

Hence if $|d_{j,k}| \leqslant \epsilon$, for all $k, 1 \leqslant k \leqslant n_j$, then

$$\|\sum_{k=1}^{n_j} d_{j,k}\psi_{j,k-2}\| \leqslant \frac{15\epsilon}{13},$$

from which the theorem follows. ■

Based on this fact, we can compress the set of wavelet coefficients by setting the small coefficients to zero. Recall that the wavelet coefficients are obtained by solving the equation $\bar{\mathbf{g}}_j = \mathcal{M}_j\bar{\mathbf{d}}_j$. The larger j is, the larger the dimension of the matrix $\mathcal{M}_j$ is. On the other hand, the larger j is, the less the nonzero terms in $\bar{\mathbf{d}}_j^*$ are. Hence we can also reduce the operation time of getting $\bar{\mathbf{d}}_j^*$ from $\bar{\mathbf{g}}_j$ by deleting a selected part of the basis $\{\psi_{j,k}\}$ in advance. The following theorem gives such a selection.

THEOREM 4.2. *Assume that, for* $\epsilon > 0$, *the function* $g_j(x) = \sum_{k=1}^{n_j} d_{j,k}\psi_{j,k-2}$ *satisfies*

$$|g_j(\bar{x}_k^j)| \leqslant \epsilon, k_1 \leqslant k \leqslant k_2.$$

Let

$$g_j^*(x) = \sum_{k\in K} d_{j,k}\psi_{j,k-2}(x)$$

where

$$K = \{k \in Z; 1 \leqslant k \leqslant n_j \ \&\ k\bar{\in}[k_1 - l, k_2 + l]\} \tag{4.5}$$

with

$$l = \left\lfloor -\frac{\log\epsilon}{\log(7 + r\sqrt{3})}\right\rfloor.$$

Then

$$\|g_j(x) - g_j^*(x)\| \leqslant \tfrac{7}{5}(1 + \|g_j\|)\epsilon. \tag{4.6}$$

Proof. We have

$$d_{j,k} = \sum_{i=1}^{n_j} m_{k,i}^{(j)} g_j(\bar{x}_k^j), \quad 1 \leqslant k \leqslant n_j.$$

Hence we have, by Lemma 2.3,

$$|d_{j,k}| \leqslant \frac{7}{4\sqrt{3}} \sum_{i=1}^{n_j} \frac{1}{\alpha^{|k-i|}} |g_j(\bar{x}_i^{(j)})|, \tag{4.7}$$

where $\alpha = 7 + 4\sqrt{3}$. Then for $k \in K^c := [k_1 - l, k_2 + l]$, and using (4.7), we obtain

$$|d_{j,k}| \leqslant \frac{7}{4\sqrt{3}}\left\{\epsilon \sum_{|k-i|\leqslant l}\alpha^{-|k-i|} + \|g_j\|\sum_{|k-i|>l}\alpha^{-|k-i|}\right\}$$

$$\leqslant \frac{7}{4\sqrt{3}}\frac{\alpha+1}{\alpha-1}[\epsilon + \alpha^{-(l+1)}\|g_j\|] \leqslant \frac{7}{6}(1 + \|g_j\|)\epsilon.$$

Finally,

$$|g_j(x) - g_j^*(x)|$$

$$= |\sum_{k\in K^c} d_{j,k}\psi_{j,k-2}(x)| \leqslant \frac{7}{6}(1 + \|g_j\|)\epsilon \sum_{k\in K^c}|\psi_{j,k}(x)|$$

$$\leqslant \frac{7}{5}(1 + \|g_j\|)\epsilon. \qquad ■$$

Remark. The norm $\|g_j\|$ in Theorem 4.2 tends to zero as j tents to ∞. For example, if $f \in \mathrm{Lip}_M^\alpha, 0 < \alpha < 4$, then $\|g_j\| \leqslant 2M2^{-j\alpha}$, and if $f \in C^4(I)$, then $\|g_j\| \leqslant \frac{5}{384}\|f^{(4)}\|$. (See [9].)

By Theorem 4.2, the coefficients $d_{j,k}$ of the wavelet expansion $g_j(x)$ can be ignored if the magnitude of function $g_j(x)$ at points $\bar{x}_k^{(j)} \in [x_{k_1+1}^{(j)}, x_{k_2-l}^{(j)}]$ is less than some given error tolerance. This fact provides an adaptive approximation scheme.

Adaptive Approximation Scheme.

Step 1. $\mathbf{c}_0 = \mathcal{B}^{-1}\mathbf{f}_0, \bar{\mathbf{g}}_0 = \bar{\mathbf{f}}_0 - \bar{\mathcal{B}}_0\mathbf{c}_0$.

Step 2. Select the set K_0 stated in Theorem 4.2 for level 0, and choose $\mathbf{B}_0^* = \{\psi_{0,k}; k \in K_0\}$.

Step 3. Obtain $\mathbf{d}_0^*$ by formula $\mathbf{d}_0^* = \mathcal{M}_0^{*-1}\mathbf{g}_0^*$, where $\mathcal{M}_0^*$ is the matrix corresponding to the reduced basis $\mathbf{B}_0^*$ and $\mathbf{g}_0^*$ is compressed from $\bar{\mathbf{g}}_0$.

Step 4. Let $f_1(x) = \sum_{k=1}^{n_0-1} c_{0,k}\phi_{0,k-2} + \sum_{k\in K_0} d_{0,k}^*\psi_{0,k-2}$.

Step 5. Repeat the process above up to J.

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