Maximum bipartite subgraphs of cubic triangle-free planar graphs

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Abstract

Thomassen recently proved, using the Tutte cycle technique, that if \( G \) is a 3-connected cubic triangle-free planar graph then \( G \) contains a bipartite subgraph with at least \( \frac{29|V(G)|}{24} - \frac{7}{6} \) edges, improving the previously known lower bound \( \frac{6|V(G)|}{5} \). We extend Thomassen’s technique and further improve this lower bound to \( \frac{39|V(G)|}{32} - \frac{9}{16} \).

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1. Introduction

Erdős [4,5] and Edwards [3] showed that for any graph \( G \) there is a bipartite subgraph of \( G \) with at least \( \frac{|E(G)|}{2} + \frac{|V(G)| - 1}{4} \) edges. Staton [8] and Locke [7] proved that if \( G \) is a connected cubic graph and \( G \neq K_4 \) then \( G \) has a bipartite subgraph with at least \( \frac{7|E(G)|}{9} \) edges. Hopkins and Staton [6] showed that every cubic triangle-free graph \( G \) contains a bipartite subgraph with at least \( \frac{4|E(G)|}{5} \) edges. Bondy and Locke [1] extended the result of Hopkins and Staton to all subcubic graphs, and proved that the Petersen graph and the dodecahedron are the only cubic triangle-free graphs \( G \) whose maximum bipartite subgraphs have exactly \( \frac{4|E(G)|}{5} \) edges.

The result of Hopkins and Staton may be rephrased as follows: If \( G \) is a cubic triangle-free graph then \( G \) contains a bipartite subgraph with at least \( \frac{6|V(G)|}{5} \) edges. Recently, Thomassen [12] improved the lower bound of Bondy and Locke for planar graphs. More precisely, Thomassen proved that if \( G \) is a 3-connected cubic triangle-free planar graph then \( G \) contains a bipartite subgraph with at least \( \frac{29|V(G)|}{24} - \frac{7}{6} \) edges. It is easy to see that if \( G \) is a plane graph and \( S \subseteq E(G) \) such that \( G - S \) is bipartite, then we obtain an even graph from the dual graph of \( G \) by deleting its edges corresponding to edges in \( S \). (A graph is said to be even if all its vertices have even degree.) With this observation and by using the Tutte cycle technique (to be described in the next section), Thomassen proved the following equivalent result: If \( G \) is a planar triangulation with minimum degree at least 4, then \( G \) has a set of at most \( \frac{7|V(G)|}{12} \) edges whose deletion results in an even graph.

In this paper, we extend Thomassen’s technique by proving stronger results on the Tutte cycles, and we show

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Theorem 1.1. If $G$ is a planar triangulation with minimum degree at least 4, then $G$ has a set of at most $\frac{9|V(G)|}{16} - \frac{9}{16}$ edges whose deletion results in an even graph.

The dual version of Theorem 1.1 is

Theorem 1.2. Let $G$ be a 3-connected cubic triangle-free planar graph. Then $G$ has a bipartite subgraph with at least $\frac{39|V(G)|}{32} - \frac{9}{16}$ edges.

Thomassen [12] provided a class of graphs showing that the coefficient $\frac{9}{16}$ in Theorem 1.1 cannot be improved to less than $\frac{10}{17}$, and hence the coefficient $\frac{39}{32}$ in Theorem 1.2 cannot be improved to more than $\frac{47}{36}$. However, it seems possible to improve the coefficient $\frac{9}{16}$ in Theorem 1.1 to $\frac{5}{9}$.

The main technique in [12] is the Tutte subgraph technique. Let $G$ be a graph and $H$ be a subgraph of $G$. An $H$-bridge of $G$ is a subgraph of $G$ which is either (1) induced by an edge of $E(G) - E(H)$ with both incident vertices on $H$ or (2) induced by the edges contained in a component of $G - V(H)$ and the edges from this component to $H$. An $H$-bridge of $G$ defined by (1) (respectively, (2)) is said to be trivial (respectively, nontrivial). If $B$ is an $H$-bridge of $G$, then the vertices in $V(H \cap B)$ are called the attachments of $B$ (on $H$). We say that $H$ is a Tutte subgraph of $G$ if every $H$-bridge of $G$ has at most three attachments on $H$.

The Tutte subgraph technique has been extensively used to prove the existence of Hamilton cycles in graphs embeddable on surfaces. We also mention that Thomassen [11] used the Tutte subgraph technique to edge-partition a planar graph into an outer planar graph and a bipartite graph. This work is related to a conjecture of Chartrand et al. [2] that every planar graph can be edge-partitioned into two outer planar subgraphs.

This paper is organized as follows. In Section 2, we state a few results about the Tutte subgraphs, including two results from [12] (Lemmas 2.4 and 2.5) concerning the existence of the Tutte cycles of certain lengths. In Section 3, we prove three lemmas which improve Lemma 2.4 and generalize Lemma 2.4 to certain near triangulations. In Section 4, we improve Lemmas 2.4 and 2.5 by increasing the lengths of cycles by 1, which are further improved in Section 5. We complete the proofs of Theorems 1.1 and 1.2 in Section 6.

We conclude this section with some notation and terminology. It is well known that the faces of a 2-connected plane graph are bounded by cycles, called facial cycles. The cycle that bounds the infinite face of a plane graph is called its outer cycle. A planar triangulation is a plane graph in which all facial cycles are triangles. A near triangulation is a plane graph whose faces are bounded by triangles, except possibly its outer cycle.

Let $G$ be a plane graph and $C$ be a cycle in $G$. Then by the Jordan Curve Theorem, $C$ divides the plane into two closed regions $R_1$ and $R_2$ whose intersection is $C$, where $R_1$ is bounded and $R_2$ is not bounded. Let $\text{Int}(C)$ and $\text{Ext}(C)$ denote the subgraphs of $G$ contained in $R_1$ and $R_2$, respectively.

Let $G$ be a graph. For any $S \subseteq V(G) \cup E(G)$, define $G - S$ to be the subgraph of $G$ with vertex set $V(G) - (S \cap V(G))$ and edge set $\{e \in E(G) : e \not\in S \text{ or } e \text{ is not incident with any vertex in } S\}$. Let $H$ be a subgraph of $G$. We define $H + S$ as the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{e \in E(G) : e \in S \text{ and } e \text{ is incident with two vertices in } V(H) \cup (S \cap V(G))\}$. When $S = \{s\}$, we simply write $G - s$ and $H + s$ instead of $G - \{s\}$ and $H + \{s\}$.

We write $A := B$ to rename $B$ as $A$. For any graph $G$ and any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$. For any subgraph $H$ of $G$, we write $G[H] := G[V(H)]$.

2. Tutte subgraphs

Let $G$ be a graph and $H \subseteq G$. For any subgraph $S$ of $G$, we say that $H$ is an $S$-Tutte subgraph of $G$ if $H$ is a Tutte subgraph of $G$, and every $H$-bridge of $G$ containing an edge of $S$ has at most two attachments on $H$.

Note that if $C$ and $C'$ are subgraphs of a graph $G$ such that $V(C) = V(C')$, then any nontrivial $C$-bridge of $G$ is also a nontrivial $C'$-bridge of $G$, and vice versa. Hence, we have the following.

Proposition 2.1. Let $C$ and $C'$ be subgraphs of $G$ such that $V(C) = V(C')$. Then $C$ is a Tutte subgraph of $G$ iff $C'$ is a Tutte subgraph of $G$. Moreover, for any $S \subseteq G$, $C$ is an $S$-Tutte subgraph of $G$ iff $C'$ is an $S$-Tutte subgraph of $G$. 


A Tutte cycle in a graph is a cycle that is a Tutte subgraph. The following result is proved in [9], which generalizes earlier results of Tutte [13] and Thomassen [10]. This result was proved for the purpose of finding Hamilton cycles in graphs embeddable on surfaces, see [9,14]. Thomassen used it in [12] to give a lower bound on max-cuts in 3-connected cubic triangle-free planar graphs. It was also used by Thomassen in [11] to edge-partition a planar graph into an outer planar graph and a bipartite graph.

**Theorem 2.2** (Thomas and Yu [9]). Let G be a 2-connected plane graph with outer cycle S, and let \( e_1, e_2, e_3 \) be three edges of S. Then G has an S-Tutte cycle C such that \( \{e_1, e_2, e_3\} \subseteq E(C) \).

Note that if C is a connected subgraph of a plane graph G, and if B is a C-bridge of G with three attachments, then any two attachments of B are incident with a common face of G. Hence, we have the following observation.

**Proposition 2.3.** Let H be a planar triangulation, and let C be a Tutte subgraph of H. If B is a C-bridge of H, then any two attachments of B on C are adjacent in H.

When proving a lower bound on the max-cuts in 3-connected cubic triangle-free planar graphs, Thomassen [12] proved the following lemma for the purpose of extending the Tutte cycles.

**Lemma 2.4** (Thomassen [12]). Let H be a planar triangulation with outer cycle S := xyzx, and assume that \(|V(H)| \geq 4\) and every vertex of H not in S has degree at least 4. Then H − z has a cycle C such that \( xy \in E(C) \), \(|V(C)| \geq 5\), and \( C \cup S \) is a Tutte subgraph of H. Moreover, if H is not the octahedron (see Fig. 1), then \(|V(C)| \geq 6\).

Thomassen [12] also proved the following lemma, which is used in [12] to find a Tutte subgraph whose blocks are cycles of length at least 7. (This was crucial to get the lower bound in [12].)

**Lemma 2.5** (Thomassen [12]). Let H be a planar triangulation with outer cycle S, and assume that \(|V(H)| \geq 9\) and every vertex of H not in S has degree at least 4. Then H has a cycle C such that \(|V(C \cap S)| = 1\), \(|V(C)| \geq 7\), and \( C \cup S \) is a Tutte subgraph of H.

### 3. Cycles of length at least 6

In this section, we prove three lemmas that generalize Lemma 2.4 to certain near triangulations. These results will be used in the next two sections to improve Lemma 2.5.

**Lemma 3.1.** Let H be a triangulation with outer cycle S := xyzx, and assume that \(|V(H)| \geq 4\) and every vertex of H not in S has degree at least 4. Then H contains a Tutte cycle C such that \( xy \in E(C) \), \( V(S) \subseteq V(C) \), and \(|V(C)| \geq 6\).

**Proof.** Since \(|V(H)| \geq 4\) and every vertex of H not in S has degree at least 4, we have \(|V(H)| \geq 6\). Let \( G := H - \{yz, zx\} \), and we consider two cases.

**Case 1.** \( G \) is 2-connected.
Then $G$ is a near triangulation. Let $D$ denote the outer cycle of $G$. Note that $|V(D)| = 5$. Let $e_1 = xy$ and choose edges $e_2, e_3$ from $D$ so that $z$ is incident with both $e_2$ and $e_3$. By applying Theorem 2.2, we find a $D$- Tutte cycle $C'$ in $G$ such that $\{e_1, e_2, e_3\} \subseteq E(C')$. Clearly, $V(S) \subseteq V(C')$ and $|V(C')| \geq 5$. If $|V(C')| \geq 6$, then $C := C'$ gives the desired cycle.

So we may assume $|V(C')| = 5$. Then $C' = D$. Since $|V(H)| \geq 6$, there is a nontrivial $C'$-bridge in $H$. Let $H'$ be an arbitrary nontrivial $C'$-bridge of $H$. Then, $H' \subseteq \text{Int}(C')$ and $H'$ has three attachments on $C'$. Note that $V(H' \cap C') \neq \{x, y, z\}$; for otherwise, the finite face of $G$ incident with both $x$ and $z$ is not bounded by a triangle, a contradiction.

Therefore, there exist two vertices in $V(H' \cap C')$ that are joined by an edge $f$ of $C' - xy$. Let $u$ denote the vertex in $V(H' \cap C')$ not incident with $f$. We apply Lemma 2.4 to find a cycle $C''$ in $H[H'] - u$ such that $f \in E(C'')$, $|V(C'')| \geq 5$, and $C'' + u$ is a Tutte subgraph of $H[H']$. Now $C := (C' \cup C'') - f$ gives the desired cycle in $H$.

Case 2. $G$ is not 2-connected.

Since $H$ is a triangulation and $|V(H)| \geq 6$, we see that $H - z$ is 2-connected and $z$ has a unique neighbor, say $w$, in $H - \{x, y\}$. Moreover, $H - z$ is a triangulation, $|V(H - z)| \geq 5$, and every vertex of $H - z$ not in $\{w, x, y\}$ has degree at least 4. So $(H - z) - wx$ is a near triangulation whose outer cycle has length 4. By applying Theorem 2.2 to $(H - z) - wx$, we obtain a Tutte cycle $C'$ in $H - z$ such that $xy, yw \in E(C')$. Clearly, $|V(C')| \geq 4$.

If $|V(C')| \geq 5$, then $C := (C' - yw) + \{z, wz, yz\}$ gives the desired cycle. So we may assume $|V(C')| = 4$. Then $C'$ is the outer cycle of $(H - z) - wx$. Since $|V(H)| \geq 6$, there is a nontrivial $C'$-bridge in $(H - z) - wx$.

Let $H'$ denote an arbitrary nontrivial $C'$-bridge of $(H - z) - wx$. Clearly, $H' \subseteq \text{Int}(C')$ and $H'$ has three attachments on $C'$. Note that $V(H' \cap C') \neq \{x, y, w\}$; as otherwise, the finite face of $(H - z) - wx$ incident with both $w$ and $x$ is not bounded by a triangle, a contradiction.

Therefore, there exist two vertices in $V(H' \cap C')$ that are joined by an edge $f$ of $C' - \{xy, yw\}$. Let $u$ denote the vertex in $V(H' \cap C')$ not incident with $f$. By Lemma 2.4 we find a cycle $C''$ in $H[H'] - u$ such that $f \in E(C'')$, $|V(C'')| \geq 5$, and $C'' + u$ is a Tutte subgraph of $H[H']$. Now $C := ((C' \cup C'') - \{f, yw\}) + \{z, wz, yz\}$ gives the desired cycle in $H$. \[\blacksquare\]

The next result deals with near triangulations whose outer cycles have length 4.

**Lemma 3.2.** Let $H$ be a near triangulation with outer cycle $S := wxyzw$, and assume that $|V(H)| \geq 6$ and every vertex of $H$ not in $S$ has degree at least 4. Then $H$ contains a Tutte cycle $C$ such that $xy \in E(C)$, $V(S) \subseteq V(C)$, and $|V(C)| \geq 6$.

**Proof.** Let $G := H - \{wx, yz\}$. Note that $G$ is connected. Suppose that $G$ is 2-connected. Let $D$ denote the outer cycle of $G$. Then $|V(D)| = 6$, and we may pick a matching $\{e_1, e_2, e_3\}$ on $D$ so that $e_1 = xy$, and each of $\{w, z\}$ is incident with $e_2$ or $e_3$. By Theorem 2.2, $G$ has a Tutte cycle $C$ such that $\{e_1, e_2, e_3\} \subseteq E(C)$. Clearly, $V(S) \subseteq V(C)$. Since $\{e_1, e_2, e_3\}$ is a matching in $D$, $|V(C)| \geq 6$. Therefore, we may assume that $G$ is not 2-connected.

Case 1. One of $\{w, x, y, z\}$ is a cut vertex of $G$.

First, assume that $x$ or $y$ is a cut vertex of $G$. By symmetry, assume that $y$ is a cut vertex of $G$. Then $d_H(x) = 2$, and $H - x$ is a triangulation with outer cycle $wyzw$ such that every vertex of $H - x$ not in $\{w, y, z\}$ has degree at least 4. Thus $|V(H - x)| \geq 6$. By Lemma 3.1, there is a Tutte cycle $C'$ in $H - x$ such that $wy \in E(C')$, $\{w, y, z\} \subseteq V(C')$, and $|V(C')| \geq 6$. Now $C := (C' - wy) + \{x, wx, xy\}$ gives the desired cycle in $H$.

So we may assume by symmetry that $z$ is a cut vertex of $G$. Then $d_H(w) = 2$, and $H - w$ is a triangulation with outer cycle $xyzx$ such that every vertex of $H - w$ not in $\{x, y, z\}$ has degree at least 4. Thus $|V(H - w)| \geq 6$. Moreover, $(H - w) - yz$ is a near triangulation whose outer cycle has length 4. So by Theorem 2.2, we can find a Tutte cycle $C'$ in $(H - w) - yz$ such that $\{xy, xz\} \subseteq E(C')$. Clearly, $\{x, y, z\} \subseteq V(C')$ and $|V(C')| \geq 4$.

If $|V(C')| \geq 5$ then $C := (C' - xz) + \{w, wx, wz\}$ gives the desired cycle in $H$. So we may assume $|V(C')| = 4$. Then $C'$ is the outer cycle of $(H - w) - yz$. Since $|V(H - w)| \geq 6$, there is a nontrivial $C'$-bridge in $(H - w) - yz$. Let $H'$ be an arbitrary nontrivial $C'$-bridge of $(H - w) - yz$. Then $H' \subseteq \text{Int}(C')$. Moreover, $H'$ has three attachments on $C'$. Note that $V(H' \cap C') \neq \{x, y, z\}$; for otherwise, the finite face of $(H - w) - yz$ incident with both $y$ and $z$ is not bounded by a triangle, a contradiction. Therefore, there exist two vertices in $V(H' \cap C')$ that are joined by an edge $f$ of $C' - \{xy, xz\}$. Let $u$ denote the vertex in $V(H' \cap C')$ not incident with $f$. We apply Lemma 2.4 to find a cycle $C''$ in $H[H'] - u$ such that $f \in E(C'')$, $|V(C'')| \geq 5$, and $C'' + u$ is a Tutte subgraph of $H[H']$. Now $C := ((C' \cup C'') - \{f, xz\}) + \{w, wx, wz\}$ gives the desired cycle in $H$. \[\blacksquare\]
Lemma 3.1
Then there is a vertex \( v \in V(H) - \{w, x, y, z\} \) such that \( v \) is a cut vertex of \( G \). Since \( H \) is a near triangulation, \( G \) has exactly two \( v \)-bridges, say \( G_1 \) and \( G_2 \), such that \( \{w, z\} \subseteq V(G_1) \) and \( \{x, y\} \subseteq V(G_2) \). Note that \( G_1 \) is a triangulation with outer cycle \( S_1 := uvzv \), and \( G_2 \) is a triangulation with outer cycle \( S_2 := xxyzw \). Since \( |V(H)| \geq 6 \), we know that \( |V(G_i)| \geq 4 \) for some \( i \in \{1, 2\} \). Moreover, if \( |V(G_i)| \geq 4 \) then \( |V(G_i)| \geq 6 \), since every vertex of \( G_i \) not in \( S_i \) must have degree at least 4.

Suppose \( |V(G_1)| \geq 6 \). Then by Lemma 3.1, we find a Tutte cycle \( C_1 \) in \( G_1 \) such that \( uz \in E(C_1) \), \( V(S_1) \subseteq V(C_1) \), and \( |V(C_1)| \geq 6 \). Now \( C := (C_1 - uz) + \{x, y, ux, xy, vz\} \) gives the desired cycle in \( H \).

So we may assume \( |V(G_2)| \geq 6 \). Then \( G_2 - vx \) is a near triangulation whose outer cycle has length 4. So by Theorem 2.2, we find a Tutte cycle \( C_2 \) in \( G_2 - vx \) such that \( xy, vy \subseteq E(C_2) \). Clearly, \( V(S_2) \subseteq V(C_2) \) and \( |V(C_2)| \geq 4 \). Now \( C := (C_2 - yv) + \{w, z, vw, wz, zy\} \) gives the desired cycle in \( H \).  

Our final lemma is about near triangulations whose outer cycles have length 5.

**Lemma 3.3.** Let \( H \) be a near triangulation with outer cycle \( S := uvwxzyv \), and assume that \( |V(H)| \geq 6 \) and every vertex of \( H \) not in \( S \) has degree at least 4. Then one of the following holds:

1. \( d_H(z) = d_H(w) = 2 \), and \( H \) contains a Tutte cycle \( C \) such that \( z \not\in V(C) \), \( xy \in E(C) \), \( V(S - z) \subseteq V(C) \), and \( |V(C)| \geq 6 \); or
2. \( H \) contains a Tutte cycle \( C \) such that \( xy \in E(C) \), \( V(S) \subseteq V(C) \), and \( |V(C)| \geq 6 \).

**Proof.** Suppose \( d_H(z) = d_H(w) = 2 \). Then since \( |V(H)| \geq 6 \), \( H - \{w, z\} \) has a vertex not contained in \( \{v, x, y\} \). Since every vertex of \( H - \{w, z\} \) not in \( \{v, x, y\} \) has degree at least 4, \( |V(H - \{w, z\})| \geq 6 \). By applying Lemma 3.2 to \( H - w \), we see that (i) holds.

So we may assume by symmetry that \( d_H(w) \geq 3 \). Let \( G := H - vw \).

Suppose that \( G \) is 2-connected. Then \( G \) is a near triangulation whose outer cycle, say \( D \), has length 6. We may choose a matching \( \{e_1, e_2, e_3\} \) from \( D \) such that \( xy \in \{e_1, e_2, e_3\} \), and every vertex of \( S \) is incident with some edge in \( \{e_1, e_2, e_3\} \). By applying Theorem 2.2, we find a Tutte cycle \( C \) in \( G \) containing \( \{e_1, e_2, e_3\} \). Clearly, \( V(S) \subseteq V(C) \) and \( |V(C)| \geq 6 \). So \( C \) gives the desired cycle for (ii).

Therefore, we may assume that \( G \) is not 2-connected. Then by planarity, one of \( \{x, y, z\} \) is a cut vertex of \( G \). Since \( d_H(w) \geq 3 \), \( x \) cannot be a cut vertex of \( G \).

**Case 1.** \( y \) is a cut vertex of \( G \).

Since \( H \) is a near triangulation, \( G \) has exactly two \( v \)-bridges, say \( G_1 \) and \( G_2 \), with \( \{u, y, z\} \subseteq V(G_1) \) and \( \{w, x, y\} \subseteq V(G_2) \). Moreover, \( G_1 \) is a triangulation with outer cycle \( S_1 := vzyu \), and \( G_2 \) is a triangulation with outer cycle \( S_2 := wxyzw \). Since \( |V(H)| \geq 6 \), \( |V(G_i)| \geq 4 \) for some \( i \in \{1, 2\} \).

Suppose \( |V(G_1)| \geq 4 \). Then by Lemma 3.1, there is a Tutte cycle \( C_1 \) in \( G_1 \) such that \( uy \in E(C_1) \), \( V(S_1) \subseteq V(C_1) \), and \( |V(C_1)| \geq 6 \). Now \( C := (C_1 - uy) + \{w, x, uw, wx, xy, yz\} \) gives the desired cycle for (ii).

So we may assume \( |V(G_2)| \geq 4 \). Then \( G_2 - wx \) is a near triangulation whose outer cycle has length 4. By applying Theorem 2.2 we find a Tutte cycle \( C_2 \) in \( G_2 - wx \) such that \( xy, vy \subseteq E(C_2) \). Necessarily, \( \{w, x, y\} \subseteq V(C_2) \) and \( |V(C_2)| \geq 4 \). Hence, \( C := (C_2 - wy) + \{v, z, vw, vz, zy\} \) is the desired cycle for (ii).

**Case 2.** \( y \) is not a cut vertex of \( G \).

Then \( d_H(z) \geq 3 \), and \( z \) is a cut vertex of \( G \). Since \( d_H(w) \geq 3 \), \( d_H(z) \geq 3 \), and \( y \) is not a cut vertex of \( G \), \( H - v \) is a near triangulation with outer cycle \( S' := wxyzw \). By Theorem 2.2 we find a Tutte cycle \( C' \) in \( H - v \) such that \( xy, wz \subseteq E(C') \). Clearly, \( \{w, x, y, z\} \subseteq V(C') \) and \( |V(C')| \geq 4 \). If \( |V(C')| \geq 5 \), then \( C := (C' - wz) + \{v, vw, vz\} \) gives the desired cycle for (ii).

So we may assume \( |V(C')| \leq 4 \). Then \( C' \) must be the outer cycle of \( H - v \). Since \( |V(H)| \geq 6 \), there is a nontrivial \( C' \)-bridge in \( H - v \). Let \( H' \) denote an arbitrary nontrivial \( C' \)-bridge of \( H - v \). Clearly, \( H' \subseteq \text{Int}(C') \). Since \( |V(H' \cap C')| = 3 \) and \( |V(C')| = 4 \), there exist two vertices in \( V(H' \cap C') \) that are joined by an edge \( f \) of \( C' = \{xy, wz\} \). Let \( u \) denote the vertex in \( V(H' \cap C') \) not incident with \( f \). We apply Lemma 2.4 to find a cycle \( C'' \) in \( H[H'] - u \) such that \( f \in E(C''), |V(C'')| \geq 5 \), and \( C'' + u \) is a Tutte subgraph of \( H[H'] \). Now \( C := ((C' \cup C'') - \{f, wz\}) + \{v, vw, vz\} \) gives the desired cycle for (ii).
4. Cycles of length at least 8

The aim of this section is to improve Lemmas 2.4 and 2.5, increasing cycle lengths by 1. First, we give the following improvement of Lemma 2.4.

**Lemma 4.1.** Let $H$ be a planar triangulation with outer cycle $S := xyzx$, and assume that $|V(H)| \geq 8$ and every vertex of $H$ not in $S$ has degree at least 4. Then $H - z$ has a cycle $C$ such that $xy \in E(C)$, $|V(C)| \geq 7$, and $C \cup S$ is a Tutte subgraph of $H$.

**Proof.** Since $|V(H)| \geq 8$ and by Lemma 2.4, $H - z$ has a cycle $D$ such that $xy \in E(D)$, $|V(D)| \geq 6$, and $D \cup S$ is a Tutte subgraph of $H$. We may assume that

\[(1) \text{ for any cycle } C' \text{ in } H - z \text{ such that } xy \in E(C') \text{ and } V(C') = V(D), \text{ and for any nontrivial } (C' \cup S)\text{-bridge } H' \text{ of } H, \text{ no two vertices in } V(H' \cap C') \text{ are joined by an edge of } C' - xy.\]

For otherwise, let $C'$ be a cycle in $H - z$ such that $xy \in E(C')$ and $V(C') = V(D)$, let $H'$ be a nontrivial $(C' \cup S)$-bridge of $H$, and assume that $x'y' \in E(C') - \{xy\}$ and $\{x', y'\} \subseteq V(H' \cap C')$. Let $z' \in (V(H') \cap V(C' \cup S)) - \{x', y'\}$ and $S' := x'y'z'x'$. Then $H[H']$ is a triangulation with outer cycle $S'$. By applying Lemma 2.4 to $H[H']$, $S'$, $x'y'$, we find a cycle $C''$ in $H[H'] - z'$ such that $x'y' \in E(C'')$, $|V(C'')| \geq 5$, and $C'' \cup S'$ is a Tutte subgraph of $H[H']$. Now it is easy to see that $C := (C' \cup C'') - x'y'$ is a cycle in $H - z$, $|V(C)| \geq 9$, and $C \cup S$ is a Tutte subgraph of $H$. Clearly, $C$ gives the desired cycle in $H$. This completes the proof of (1).

If $|V(D)| \geq 7$, $C := D$ gives the desired cycle. So we may assume that $|V(D)| = 6$, and let $D = u_1u_2u_3u_4u_5u_6u_1$, with $u_1 = x$ and $u_2 = y$. Since $|V(H)| \geq 8$ and $|V(D)| = 6$, there is a nontrivial $(D \cup S)$-bridge of $H$. Let $H'$ be an arbitrary nontrivial $(D \cup S)$-bridge of $H$. Let $V(H') \cap V(D \cup S) = \{x', y', z'\}$ and $S' := x'y'z'x'$. Note that $S'$ is the outer cycle of the triangulation $H[H']$. We claim that

\[(2) \ E(D) \cap E(S') = \emptyset.\]

Otherwise, it follows from (1) that $xy \in E(S')$. By choosing appropriate notation we may assume $x' = x$ and $y' = y$. Then $H' \subseteq \text{Int}(D)$, since $xy$ is an edge in the outer cycle of $H$. So $z' \in V(D)$. By (1), $z' \in \{u_4, u_5\}$. Therefore, we may further assume by symmetry that $z' = u_4$, and hence, $S' = u_1u_2u_4u_1$.

We claim that $u_3u_5, u_3u_6 \notin E(H)$. For otherwise, let

\[C' := \begin{cases} u_1u_2u_4u_3u_5u_6u_1, & \text{if } u_3u_5 \in E(H), \\ u_1u_2u_3u_5u_4u_1, & \text{if } u_3u_6 \in E(H). \end{cases}\]

Then $C'$ and $H'$ contradict (1).

We now show that $u_3u_1, u_3z \in E(H).$ This is true if $u_3$ is not contained in any nontrivial $(D \cup S)$-bridge of $H$ (since $d_H(u_3) \geq 4$). If $u_3$ is contained in some nontrivial $(D \cup S)$-bridge of $H$, then such a $(D \cup S)$-bridge must also contain $\{u_1, z\}$ and $u_3u_1, u_3z \in E(H)$ by (1), planarity and Proposition 2.3. (For convenience, any argument similar to this will simply be referred to as “since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3.”)

By planarity, $u_3u_1, u_3z$ must be contained in $\text{Ext}(D)$. Therefore, since $u_3u_5 \notin E(H)$ and by (1), planarity and Proposition 2.3, the only $(D \cup S)$-bridge of $H$ containing $u_5$ must be the edge $u_5u_1$. Hence $d_H(u_5) \leq 3$, a contradiction.

So we have (2).

We further claim that

\[(3) \ z \in \{x', y', z'\}.\]

For, suppose $z \notin \{x', y', z'\}$. Then $\{x', y', z'\} \subseteq V(D)$. By (2) and without loss of generality, we may assume that $S' = u_2u_4u_6u_2$.

Then $u_3u_1, u_3u_5, u_5u_1 \notin E(H)$. Otherwise, let

\[C' := \begin{cases} u_1u_2u_6u_5u_4u_3u_1, & \text{if } u_3u_1 \in E(H), \\ u_1u_2u_4u_3u_5u_6u_1, & \text{if } u_3u_5 \in E(H), \\ u_1u_2u_3u_4u_6u_5u_1, & \text{if } u_5u_1 \in E(H). \end{cases}\]

Then $C'$ and $H'$ contradict (1).
Suppose $H' \subseteq \text{Int}(D)$. Then $u_3u_6, u_3z \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3). Therefore, because $u_3u_5 \notin E(H)$, $d_H(u_3) = 2$ (by (1), planarity and Proposition 2.3), a contradiction.

So $H' \subseteq \text{Ext}(D)$. But now, since $u_3u_1, u_3u_5 \notin E(H)$, $d_H(u_3) \leq 3$ (by (1), planarity and Proposition 2.3), again a contradiction. This proves (3).

By (3), we may assume by symmetry that $S' \ni \{u_2u_4z, u_2u_5z, u_2u_6z, u_3u_5z, u_3u_6z\}$. So we consider five cases. Note that $H' \subseteq \text{Ext}(D)$.

Case 1. $S' = u_2u_4zu_2$.

Then by (1) and planarity, $u_3$ has no neighbor in $\text{Ext}(D) - V(D)$. Note that $u_3u_5 \notin E(H)$; since otherwise, $C' := u_1u_2u_4u_5u_6u_3u_1$ and $H'$ contradict (1). Therefore, $u_3u_5, u_3u_1 \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3). But now $C' := u_1u_2u_4u_5u_6u_3u_1$ and $H'$ contradict (1).

Case 2. $S' = u_2u_5zu_2$.

Then $u_3u_6 \notin E(H)$; for otherwise, $C' := u_1u_2u_5u_6u_3u_1$ and $H'$ contradict (1). Therefore, $u_3u_5, u_3u_1 \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3). By planarity, $u_3u_1$ must be contained in $\text{Int}(D)$. Then $u_6u_2 \notin E(G)$ by planarity. Hence $u_6u_4, u_6z \in E(H)$ (since $d_H(u_6) \geq 4$ and by (1), planarity and Proposition 2.3).

But then $C' := u_1u_2u_6u_5u_4u_3u_1$ and $H'$ contradict (1).

Case 3. $S' = u_2u_6zu_2$.

Then $u_3u_1 \notin E(H)$; as otherwise, $C' := u_1u_2u_6u_5u_4u_3u_1$ and $H'$ contradict (1). Therefore, $u_3u_5, u_3u_6 \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3).

Suppose that $u_3u_5$ is contained in $\text{Int}(D)$. Then $u_4u_2, u_4u_6 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). By planarity, $u_3u_6$ is contained in $\text{Int}(D)$. But now $d_H(u_5) = 3$ (by (1), planarity and Proposition 2.3), a contradiction.

So $u_3u_5$ is contained in $\text{Ext}(D)$. Suppose $u_4u_6 \in E(H)$. Then $u_5u_2 \in E(H)$ (since $d_H(u_5) \geq 4$ and by (1), planarity and Proposition 2.3). By planarity, $u_3u_6$ is contained in $\text{Int}(D)$. This forces $d_H(u_4) = 3$ (by (1), planarity and Proposition 2.3), a contradiction. So $u_4u_6 \notin E(H)$. Then $u_4u_1, u_4u_2 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). Now $C' := u_1u_2u_6u_5u_4u_3u_1$ and $H'$ contradict (1).

Case 4. $S' = u_3u_6zu_3$.

Then by (1) and planarity, $u_4$ has no neighbor in $\text{Ext}(D) - V(D)$. We claim that $u_4u_2, u_4u_6 \notin E(H)$; otherwise, let

$$C' := \begin{cases} u_1u_2u_4u_3u_5u_6u_1, & \text{if } u_4u_2 \in E(H), \\ u_1u_2u_3u_5u_4u_6u_1, & \text{if } u_4u_6 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). But now $d_H(u_4) \leq 3$ (by (1), planarity and Proposition 2.3), a contradiction.

Case 5. $S' = u_3u_6zu_3$.

Then $u_4u_1, u_5u_2 \notin E(H)$; otherwise, let

$$C' := \begin{cases} u_1u_2u_3u_6u_5u_4u_1, & \text{if } u_4u_1 \in E(H), \\ u_1u_2u_5u_4u_3u_6u_1, & \text{if } u_5u_2 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). Therefore, $u_4u_2, u_4u_6 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). But then, since $u_5u_2 \notin E(H)$, $d_H(u_5) \leq 3$ (by (1), planarity and Proposition 2.3), a contradiction. 

We now prove an improvement of Lemma 2.5.

**Lemma 4.2.** Let $H$ be a planar triangulation with outer cycle $S$, and assume that $|V(H)| \geq 10$ and every vertex of $H$ not in $S$ has degree at least 4. Then $H$ has a cycle $C$ such that $|V(C \cap S)| = 1$, $|V(C)| \geq 8$, and $C \cup S$ is a Tutte subgraph of $H$.

**Proof.** Let $S := xyzx$. By Lemma 2.5, there is a cycle $D$ in $H$ such that $|V(D \cap S)| = 1$, $|V(D)| \geq 7$, and $D \cup S$ is a Tutte subgraph of $H$. If $|V(D)| \geq 8$, then $C := D$ gives the desired cycle. So we may assume $|V(D)| = 7$. Let $D = u_1u_2u_3u_4u_5u_6u_7u_1$ and, without loss of generality, assume that $x \in V(D)$. Let $B$ denote the block of $H - \{y, z\}$ containing $D$, and let $u$ denote the vertex of $B - x$ which is adjacent to both $y$ and $z$ in $H$. Then $|V(B)| \geq 7$. Since $D \cup S$ is a Tutte subgraph of $H$, $u \in V(D)$ and $D$ is a $T$-Tutte subgraph of $B$, where $T$ denotes the outer cycle of $B$.

We may assume that
(1) Int(uyzu) = uyzu.

Otherwise, assume Int(uyzu) \neq uyzu. Suppose xu \in E(G). Since |V(B)| \geq 7, Int(xuyx) \neq xyux or Int(xzux) \neq zxux. By symmetry we may assume Int(xyux) \neq xyux. Since every vertex of H not in S has degree at least 4, we can apply Lemma 2.4 to Int(xyux) and yu, and to Int(uyzu) and yu. Then there exists a cycle C_1 in Int(xyux) − x such that yu \in E(C_1), |V(C_1)| \geq 5, and C_1 ∪ xyux is a Tutte subgraph of Int(xyux); and there exists a cycle C_2 in Int(uyzu) − z such that yu \in E(C_2), |V(C_2)| \geq 5, and C_2 ∪ uzyu is a Tutte subgraph of Int(uyzu). Now C := (C_1 ∪ C_2) − yu is a cycle in H such that |V(C ∩ S)| = 1, |V(C)| \geq 8, and C ∪ S is a Tutte subgraph of H; and the assertion of the lemma holds.

So we may assume xu \notin E(G). By planarity, the blocks of B − x may be labeled as B_1 . . . B_n such that for all 1 \leq i < j \leq n, B_i ∩ B_{i+1} \neq \emptyset and B_i ∩ B_j = \emptyset if j ≥ i + 2. Let u \in V(B_i) for some 1 \leq k \leq n. Since xu \notin E(H), u \notin V(B_i) for all i \neq k. So |V(B_k)| \geq 3. Let K := \bigcup_{i=1}^k B_i and L := \bigcup_{i=k}^n B_i. Note that |V(K)| + |V(L)| ≥ |V(B)| + 2 ≥ 9. So we may assume by symmetry that |V(L)| ≥ 5.

By planarity, L is contained in a block of H − {x, z} containing y, or is contained in a block of H − {x, y} containing z. By symmetry we may assume that L and y are contained in a block of H − {x, z}, say G. Then G also contains Int(uyzu) − z. Clearly, \{y, u\} is a 2-cut of G. Let G' denote the subgraph of G that is the union of the edge yu and the nontrivial \{y, u\}-bridge of G containing L. Then G' is a near triangulation. Since |V(L)| ≥ 5, we must have |V(G')| ≥ 6. Let T' denote the outer cycle of G'.

Next, we find a cycle C' in G. If |V(T')| ≥ 6 then we can pick a matching \{e_1, e_2, e_3\} from T' such that e_1 = yu and the common neighbour of x and z in V(T') − y is incident with e_2 or e_3; and we use Theorem 2.2 to find a T'-Tutte cycle C' in G such that \{e_1, e_2, e_3\} ⊆ E(C') (and hence |V(C')| ≥ 6). If |V(T')| ≤ 4, then we apply Lemmas 3.1 and 3.2 to find a Tutte cycle C' in G such that yu \in E(C'), V(T') ⊆ V(C'), and |V(C')| ≥ 6. Now assume |V(T')| = 5. Note that if a vertex in V(T') − \{y, u\} has degree 2 in G' then it must be adjacent to both x and z (since such a vertex has degree at least 4 in H). Therefore by planarity, at most one vertex in V(T') − \{y, u\} has degree 2 in G'. So by Lemma 3.3(ii), there is a Tutte cycle C' in G' such that yu \in E(C'), V(T') ⊆ V(C'), and |V(C')| ≥ 6.

Let C_2 be the cycle in Int(uyzu) − z found above. Then C := (C_1 ∪ C_2) − yu is a cycle in H such that |V(C ∩ S)| = 1, |V(C)| ≥ 9, and C ∪ S is a Tutte subgraph of H; and the assertion of the lemma holds. So we have (1).

We may also assume that

(2) for any cycle C' in B such that V(C') = V(D), and for any nontrivial (C' ∪ S)-bridge H' of H, no two vertices in V(H' ∩ C') are joined by an edge of C'.

For otherwise, let C' be a cycle in B such that V(C') = V(D), let H' be a nontrivial (C' ∪ S)-bridge of H, and assume that x'y' ∈ E(C') and \{x', y'\} ⊆ V(H' ∩ C'). Let z' ∈ V(H' ∩ V(C') ∪ S) − \{x', y'\} and S' := x'y'z'x'. By applying Lemma 2.4 to H[B'], S', x'y', we find a cycle C'' in H[B'] − z' such that x'y' ∈ E(C''), |V(C'')| ≥ 5, and C'' ∪ S' is a Tutte subgraph of H[B']. Now it is easy to see that C := (C' ∪ C'') − x'y' is a cycle in H − \{y, z\}, x ∈ V(C), |V(C)| ≥ 10, and C ∪ S is a Tutte subgraph of H; and the assertion of the lemma holds. So we have (2).

Since |V(H)| ≥ 10 and |V(D)| = 7, there is a nontrivial (D ∪ S)-bridge of H. Let H' denote an arbitrary nontrivial (D ∪ S)-bridge of H. Let V(H') ∩ V(D ∪ S) = \{x', y', z\} and S' := x'y'z'x', and assume \{x', y'\} ⊆ V(D). We may assume that

(3) z' \notin \{y, z\}.

Otherwise, we may assume by symmetry that z' = z. By applying Lemma 2.4 to H[H'], S', x'y', we find a cycle C'' in H[H'] − z' such that x'y' ∈ E(C''), |V(C'')| ≥ 5, and C'' ∪ S' is a Tutte subgraph of H[H'].

Let D' denote the outer cycle of H[Int(D)]. Clearly, x'y' ∈ E(D'), x, u ∈ V(D'), and H[Int(D)] ⊆ Int(D'). Further, Int(D') is a near triangulation. If Int(D') has a D'-Tutte cycle C' such that x'y' ∈ E(C'), \{x, u\} ⊆ V(C'), and |V(C')| ≥ 6, then C := (C' ∪ C'') − x'y' gives the desired cycle, with V(C ∩ S) = \{x\}. So we may assume that such a cycle C' does not exist.

Then |V(D')| ≤ 5; for otherwise, C' exists by applying Theorem 2.2 to Int(D') (by choosing a matching \{e_1, e_2, e_3\} on D' so that e_1 = x'y', and x, u are incident with edges in \{e_1, e_2, e_3\}). Again by the nonexistence of C', it follows from Lemmas 3.1–3.3(ii), we must have |V(D')| = 5, and the vertices of V(D') can be labeled as v_1, w_1, x_1, y_1, z_1 such that D' = w_1x_1w_1x_1z_1w_1 and w_1 and z_1 have degree 2 in Int(D'), where x_1 = x' and y_1 = y'.

Therefore, D' − x'y' ⊆ D and, since D ⊆ Int(D), D' ∪ S is a Tutte subgraph of H. So for any nontrivial (D' ∪ S)-bridge H'' of H, no two vertices in V(H'' ∩ D') are joined by an edge of D' − x'y'; as otherwise, H'' is also a
nontrivial \((D \cup S)\)-bridge of \(H\) such that two vertices in \(V(H'' \cap D)\) are joined by an edge of \(D\), contradicting (2). Hence, each vertex in \([w_1, z_1] - \{x\}\) must be adjacent to both \(y\) and \(z\) (since it has degree at least 4 in \(H\) and by planarity and Proposition 2.3). So \([w_1, z_1] = \{x, u\}\). By symmetry, we may assume \(x = z_1\).

Then by Lemma 3.3(i), there is a Tutte cycle \(C_1\) in \(\text{Int}(D')\) such that \(z_1 \notin V(C_1)\), \(x'y' \in E(C_1)\), \(\{x_1, y_1, v_1, w_1\} \subseteq V(C_1)\), and \(|V(C_1)| \geq 6\). In \(H[H']\), we apply Lemma 3.1 to find a Tutte cycle \(C_2\) such that \(x'y' \in E(C_2)\), \(\{x', y', z' = z\} \subseteq V(C_2)\), and \(|V(C_2)| \geq 6\). Let \(C'' := (C_1 \cup C_2) - x'y'\). Since \(D' - x'y' \subseteq D\), \(C' \cup S\) is a Tutte subgraph of \(H\). So \(C''\) gives the desired cycle (with \(V(C' \cap S) = \{z\}\)). This proves (3).

By (3), \(\{x', y', z'\} \subseteq V(D)\). By (2) and by symmetry, we may assume \(x' = u_1, y' = u_3, \text{ and } z' = u_5\).

Then \(u_2u_4, u_2u_6, u_2u_7, u_4u_6, u_4u_7 \notin E(H)\). For otherwise, define

\[
C' := \begin{cases}
  u_1u_3u_2u_4u_6u_7u_1, & \text{if } u_2u_4 \in E(H), \\
  u_1u_7u_6u_3u_4u_5u_1, & \text{if } u_2u_6 \in E(H), \\
  u_1u_2u_7u_6u_4u_3u_1, & \text{if } u_2u_7 \in E(H), \\
  u_1u_2u_3u_4u_5u_6u_7u_1, & \text{if } u_4u_6 \in E(H), \\
  u_1u_2u_3u_4u_7u_6u_5u_1, & \text{if } u_4u_7 \in E(H).
\end{cases}
\]

It is then easy to verify that \(C'\) and \(H'\) contradict (2).

We claim that \(u_2y, u_2z \in E(H)\). Suppose this is false. Then since \(x\) and \(u\) are adjacent to both \(y\) and \(z\), we have \(u_2 \notin \{x, u\}\). Also, any nontrivial \((D \cup S)\)-bridge of \(H\) containing \(u_2\) must also contain \(u_5\) and one of \(\{y, z\}\) (by (2), planarity and Proposition 2.3). So \(u_2u_5 \in E(H)\), and \(u_2y \in E(H)\) or \(u_2z \in E(H)\) (since \(d_H(u_2) \geq 4\) and by (2), planarity and Proposition 2.3). By symmetry, we may assume \(u_2y \in E(H)\). So \(u_2u_5\) is contained in \(\text{Ext}(D)\), and \(u_1\) lies in \(\text{Int}(u_2u_3u_4u_5u_2)\); for otherwise, because \(u_4u_2 \notin E(H), d_H(u_4) = 2\) (by (2), planarity and Proposition 2.3). Then by planarity \(H' \subseteq \text{Int}(D)\). Now, since \(u_6u_2 \notin E(H), d_H(u_6) \leq 3\) (by (2), planarity and Proposition 2.3), a contradiction.

By the same argument as above, we also conclude that \(u_4y, u_4z \in E(H)\) (by exchanging the roles of \(u_4\) and \(u_2\) and by exchanging the roles of \(u_1\) and \(u_5\)).

Therefore, \(\{u_2, u_4\} = \{x, u\}\) and \(H' \subseteq \text{Int}(D)\) (by planarity). By symmetry, we may assume \(D \subseteq \text{Ext}(u_2u_3u_4u_5u_2)\). Then \(u_3y \in E(H)\) (since \(u_2u_4 \notin E(H)\) and by (2), planarity and Proposition 2.3). Hence, because \(u_6u_2, u_6u_4 \notin E(H), u_6u_1, u_6z \in E(H)\) (since \(d_H(u_6) \geq 4\) and by (2), planarity and Proposition 2.3). Now, because \(u_7u_2 \notin E(H), d_H(u_7) \leq 3\) (by (2), planarity and Proposition 2.3), a contradiction. 

The graph in Fig. 2 shows that Lemma 4.2 is best possible. We show in the next section that Lemma 4.2 can be further improved if \(H\) is not the graph in Fig. 2.

5. Cycles of length at least 9

In this section we improve Lemmas 4.1 and 4.2, increasing cycle lengths by 1. The approach we take is the same as in the previous section, but the arguments are more complicated. First, we give the improvement of Lemma 4.1.
Lemma 5.1. Let $H$ be a planar triangulation with outer cycle $S := xyzx$, and assume that $|V(H)| \geq 9$ and every vertex of $H$ not in $S$ has degree at least 4. Then $H - z$ has a cycle $C$ such that $xy \in E(C)$, $|V(C)| \geq 8$, and $C \cup S$ is a Tutte subgraph of $H$.

Proof. By Lemma 4.1, $H - z$ has a cycle $D$ such that $xy \in E(D)$, $|V(D)| \geq 7$, and $D \cup S$ is a Tutte subgraph of $H$. If $|V(D)| \geq 8$, then $C := D$ is the desired cycle. So we may assume that $|V(D)| = 7$. Let $D = u_1u_2u_3u_4u_5u_6u_7u_1$, with $u_1 = x$ and $u_2 = y$. By the same argument as for (1) in the proof of Lemma 4.1, we may assume that (1) for any cycle $C'$ in $H - z$ such that $xy \in E(C')$ and $V(C') = V(D)$, and for any nontrivial ($C' \cup S$)-bridge $H'$ of $H$, no two vertices in $V(H' \cap C')$ are joined by an edge of $C' - xy$.

Since $|V(D)| = 7$ and $|V(H)| \geq 9$, there is a nontrivial ($D \cup S$)-bridge of $H$. Let $H'$ be an arbitrary nontrivial ($D \cup S$)-bridge of $H$. Let $V(H') \cap V(D \cup S) = \{x', y', z'\}$ and $S' := x'y'z'x'$, and assume $\{x', y'\} \subseteq V(D)$. We may further assume that (2) $E(D) \cap E(S') = \emptyset$.

Otherwise, suppose $E(D) \cap E(S') \neq \emptyset$. Then by (1) and without loss of generality, we may assume $x'y' = xy$. So $H' \subseteq \text{Int}(D)$, since $xy$ is contained in the outer cycle of $H$. Then by planarity, $z' \in V(D)$. Therefore, we may assume by (1) and by symmetry that $S' = \{u_1u_2u_4u_1, u_1u_2u_5u_1\}$.

Suppose $S' = u_1u_2u_4u_1$. Then $u_3u_5, u_3u_7 \notin E(H)$; otherwise, let

$$C' := \begin{cases} u_1u_2u_4u_3u_6u_7u_1, & \text{if } u_3u_5 \in E(H), \\ u_1u_2u_3u_7u_6u_5u_1, & \text{if } u_3u_7 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). So $u_3u_6 \notin E(H)$; as otherwise, $u_5u_7, u_5u_1 \in E(H)$ (since $d_H(u_5) \geq 4$ and by (1), planarity and Proposition 2.3), and $C' := u_1u_2u_4u_3u_6u_7u_1$ and $H'$ contradict (1). Therefore, $u_3u_1, u_3z \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3). Then, because $u_3u_3, u_7u_3 \notin E(H), u_5u_7, u_5u_1, u_7u_4 \in E(H)$ (since $d_H(u_5) \geq 4 \leq d_H(u_7)$) and by (1), planarity and Proposition 2.3). But now $d_H(u_6) \leq 3$ (by (1), planarity and Proposition 2.3), a contradiction.

Now assume $S' = u_1u_2u_5u_1$. Then $u_3u_6, u_4u_7 \notin E(H)$; otherwise, define

$$C' := \begin{cases} u_1u_2u_5u_4u_6u_7u_1, & \text{if } u_3u_6 \in E(H), \\ u_1u_2u_3u_7u_6u_5u_1, & \text{if } u_4u_7 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). So $u_3u_7 \notin E(H)$; as otherwise, $u_6u_4, u_6u_1 \in E(H)$ (since $d_H(u_6) \geq 4$ and by (1), planarity and Proposition 2.3), and $C' := u_1u_2u_5u_4u_3u_6u_7u_1$ and $H'$ contradict (1). Therefore, $u_3u_1 \notin E(H)$; for otherwise, $d_H(u_7) \leq 3$ (by (1), planarity and Proposition 2.3), a contradiction. Hence $u_3u_5, u_3z \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3). If $u_3u_5$ is contained in $\text{Ext}(D)$, then $d_H(u_4) \leq 3$ (by (1), planarity and Proposition 2.3), a contradiction. So $u_3u_5$ is contained in $\text{Int}(D)$. Now $u_6u_4 \notin E(H)$; otherwise, $C' := u_1u_2u_5u_3u_4u_6u_7u_1$ and $H'$ contradict (1). Hence, because $u_6u_3 \notin E(H), u_6u_1, u_6z \in E(H)$ (since $d_H(u_6) \geq 4$ and by (1), planarity and Proposition 2.3). But now $d_H(u_7) \leq 3$ (by (1), planarity and Proposition 2.3), again a contradiction. This completes the proof of (2).

We now prove (3) $z' \neq z$.

Suppose $z' \neq z$. Then $V(S') \subseteq V(D)$. By (2) and by symmetry between $x$ and $y$, we may assume that $S' \in \{u_2u_4u_6u_2, u_2u_4u_7u_2, u_2u_5u_7u_2, u_3u_4u_7u_3\}$. So we have four cases to consider.

Suppose $S' = u_2u_4u_6u_2$. Then $u_3u_5, u_3u_7, u_5u_7 \notin E(H)$; otherwise, let

$$C' := \begin{cases} u_1u_2u_4u_5u_6u_7u_1, & \text{if } u_3u_5 \in E(H), \\ u_1u_2u_3u_5u_4u_7u_1, & \text{if } u_3u_7 \in E(H), \\ u_1u_2u_3u_4u_5u_7u_1, & \text{if } u_5u_7 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). So $H' \subseteq \text{Int}(D)$; as otherwise, $u_3u_6, u_3u_1 \in E(H)$ (since $d_H(u_3) \geq 4$ and by (1), planarity and Proposition 2.3), and this forces $d_H(u_5) = 2$ (by (1), planarity and Proposition 2.3), a contradiction. If $u_3u_6 \in E(H)$, then because $u_5u_3 \notin E(H)$, we have $d_H(u_5) = 2$ (by (1), planarity and
Proposition 2.3, a contradiction. So \( u_3u_6 \notin E(H) \). Then \( u_3u_1, u_3z \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3). But since \( u_3u_5, u_3u_7 \notin E(H), d_H(u_5) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction.

Now assume \( S' = u_2u_3u_4u_2 \). Then \( u_6u_1, u_6u_3, u_3u_6, u_3u_1 \notin E(H) \); otherwise, define

\[
C' := \begin{cases} 
  u_1u_2u_7u_6u_3u_4u_1, & \text{if } u_3u_1 \in E(H), \\
  u_1u_2u_3u_3u_6u_7u_1, & \text{if } u_3u_5 \in E(H), \\
  u_1u_2u_3u_6u_4u_4u_1, & \text{if } u_3u_6 \in E(H), \\
  u_1u_2u_3u_4u_6u_6u_1, & \text{if } u_3u_1 \in E(H), \\
\end{cases}
\]

and then \( C' \) and \( H' \) contradict (1). Hence, \( u_3u_7, u_3z \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3), and \( H' \subseteq \text{Int}(D) \) (by planarity). But now, since \( u_5u_3 \notin E(H), d_H(u_5) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction.

Suppose \( S' = u_2u_3u_4u_2 \). Then \( u_6u_1, u_6u_3, u_6u_4, u_3u_1 \notin E(H) \); otherwise, define

\[
C' := \begin{cases} 
  u_1u_2u_3u_4u_5u_7u_6u_1, & \text{if } u_6u_1 \in E(H), \\
  u_1u_2u_3u_4u_6u_7u_1, & \text{if } u_6u_3 \in E(H), \\
  u_1u_2u_3u_4u_6u_4u_1, & \text{if } u_6u_4 \in E(H), \\
  u_1u_2u_3u_4u_6u_6u_1, & \text{if } u_3u_1 \in E(H), \\
\end{cases}
\]

and then \( C' \) and \( H' \) contradict (1). So \( u_6u_2, u_6z \in E(H) \) (since \( d_H(u_6) \geq 4 \) and by (1), planarity and Proposition 2.3), and \( H' \subseteq \text{Int}(D) \) (by planarity). Now, since \( u_4u_6 \notin E(H), d_H(u_4) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction. This completes the proof of (3).

By (2) and (3) and by symmetry, we may assume that \( S' \subseteq \{u_2u_4u_2, u_2u_5u_2, u_2u_6u_2, u_2u_7u_2, u_3u_5u_3, u_3u_6u_3, u_3u_7u_3, u_4u_5u_4\} \). Note that \( H' \subseteq \text{Ext}(D) \). We have eight cases to consider.

Case 1. \( S' = u_2u_4u_2 \).

Then \( u_3u_5 \notin E(H) \); otherwise, \( C' := u_1u_2u_4u_3u_5u_7u_1 \) and \( H' \) contradict (1). Moreover, \( u_3u_6 \in E(H) \); otherwise, \( u_3u_7, u_3u_1 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3), and \( C' := u_1u_2u_4u_5u_6u_7u_3u_1 \) and \( H' \) contradict (1).

Now \( u_3u_7 \notin E(H) \); otherwise, \( C' := u_1u_2u_4u_3u_6u_7u_3u_1 \) and \( H' \) contradict (1). Hence, because \( u_5u_3 \notin E(H), u_3u_1, u_5z \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (1), planarity and Proposition 2.3). So \( u_2u_7, u_3u_3 \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (1), planarity and Proposition 2.3). But then \( C' := u_1u_2u_4u_3u_7u_6u_5u_1 \) and \( H' \) contradict (1).

Case 2. \( S' = u_2u_4u_2 \).

Then \( u_3u_6 \notin E(H) \); otherwise, \( C' := u_1u_2u_5u_4u_3u_6u_7u_1 \) and \( H' \) contradict (1).

We claim that \( u_3u_7 \in E(H) \). For, suppose \( u_3u_7 \notin E(H) \). Then \( u_3u_5, u_3u_1 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3). If \( u_3u_5 \) is contained in \( \text{Int}(D) \), then \( d_H(u_5) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction. So \( u_3u_5 \) is contained in \( \text{Ext}(D) \). Now \( u_4u_6 \notin E(H) \); otherwise, \( C' := u_1u_2u_4u_3u_6u_7u_1 \) and \( H' \) contradict (1). So \( u_4u_7, u_4u_1 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3). But now \( C' := u_1u_2u_4u_3u_6u_7u_3u_1 \) and \( H' \) contradict (1).

Thus \( u_6u_1 \notin E(H) \); otherwise, \( C' := u_1u_2u_4u_3u_7u_6u_3u_1 \) and \( H' \) contradict (1). Therefore, because \( u_6u_3 \notin E(H), u_6u_4, u_6z \in E(H) \) (since \( d_H(u_6) \geq 4 \) and by (1), planarity and Proposition 2.3). But then \( C' := u_1u_2u_5u_6u_4u_3u_7u_1 \) and \( H' \) contradict (1).

Case 3. \( S' = u_2u_6u_2 \).

Then \( u_3u_7 \notin E(H) \); otherwise, \( C' := u_1u_2u_6u_5u_4u_3u_7u_1 \) and \( H' \) contradict (1).
We claim that \( u_3u_1 \notin E(H) \). For, suppose \( u_3u_1 \in E(H) \). If \( u_7u_5 \in E(H) \), then \( C' := u_1u_2u_6u_7u_5u_4u_3u_1 \) and \( H' \) contradict (1). So \( u_7u_5 \notin E(H) \). Hence, because \( u_7u_3 \notin E(H) \), \( u_7u_4, u_7z \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (1), planarity and Proposition 2.3). Then \( u_3u_2, u_5u_3 \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (1), planarity and Proposition 2.3). But now \( C' := u_1u_2u_6u_7u_4u_5u_1 \) and \( H' \) contradict (1).

Therefore, \( u_3u_5, u_3u_6 \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (1), planarity and Proposition 2.3). Suppose that \( u_3u_5 \) is contained in \( \text{Int}(D) \). Then \( u_4u_2, u_4u_6 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3). By planarity, \( u_3u_6 \) is contained in \( \text{Int}(D) \). But now \( d_H(u_5) = 3 \) (by (1), planarity and Proposition 2.3), a contradiction.

So \( u_3u_5 \) is contained in \( \text{Ext}(D) \). Then \( u_3u_6 \) is contained in \( \text{Ext}(D) \); otherwise, \( d_H(u_4) \leq 3 \) (by (1), planarity and Proposition 2.3). We claim that \( u_7u_4, u_7u_5 \notin E(H) \); otherwise, define

\[
C' := \begin{cases} 
    u_1u_2u_6u_7u_3u_4u_7u_1, & \text{if } u_7u_4 \in E(H), \\
    u_1u_2u_6u_7u_4u_5u_1, & \text{if } u_7u_5 \in E(H),
\end{cases}
\]

and then \( C' \) and \( H' \) contradict (1). Then, because \( u_7u_3 \notin E(H) \), \( u_7u_2, u_7z \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (1), planarity and Proposition 2.3). Moreover, because \( u_5u_7 \notin E(H) \), \( u_5u_2 \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (1), planarity and Proposition 2.3). Hence \( d_H(u_4) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction.

**Case 4.** \( S' = u_2u_7u_2 \).

Then \( u_3u_1 \notin E(H) \); otherwise, \( C' := u_1u_2u_7u_6u_5u_3u_1 \) and \( H' \) contradict (1). So \( u_1 \) is adjacent to at least one vertex in \( \{u_4, u_5, u_6\} \); otherwise, since \( u_2u_7 \) is contained in \( \text{Ext}(D) \), the finite face of \( H \) incident with both \( u_1 \) and \( u_2 \) is not bounded by a triangle, a contradiction.

We claim that \( u_1u_4 \notin E(H) \). For, suppose \( u_1u_4 \in E(H) \). Then \( u_3u_5 \notin E(H) \); otherwise \( C' := u_1u_2u_6u_7u_5u_4u_3u_1 \) and \( H' \) contradict (1). Therefore, because \( u_3u_1 \notin E(H) \), \( u_3u_6, u_3u_7 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3). Hence \( C' := u_1u_2u_7u_3u_6u_5u_4u_1 \) and \( H' \) contradict (1).

We further claim that \( u_1u_5 \notin E(H) \). Otherwise, suppose that \( u_1u_5 \in E(H) \). Note from planarity that \( u_1u_5 \) is contained in \( \text{Int}(D) \). Then \( u_3u_6 \notin E(H) \); as otherwise, \( C' := u_1u_2u_7u_6u_3u_4u_5u_1 \) and \( H' \) contradict (1). So because \( u_3u_1 \notin E(H) \), \( u_3u_5, u_3u_7 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3). Then \( u_3u_5 \) must be contained in \( \text{Int}(D) \); otherwise, since \( u_4u_1 \notin E(H) \), \( d_H(u_4) \leq 3 \) (by (1), planarity and Proposition 2.3), a contradiction. By planarity, \( u_3u_7 \) is contained in \( \text{Ext}(D) \). Now \( u_4u_6, u_4u_7 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3). Hence \( C' := u_1u_2u_7u_3u_6u_5u_4u_1 \) and \( H' \) contradict (1).

Therefore \( u_1u_6 \in E(H) \). Then \( u_3u_7 \notin E(H) \); otherwise, \( C' := u_1u_2u_7u_3u_4u_5u_6u_1 \) and \( H' \) contradict (1). Hence, because \( u_3 u_1 \notin E(H) \), \( u_3u_5, u_3u_6 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (1), planarity and Proposition 2.3). Suppose that \( u_3u_5 \) is contained in \( \text{Ext}(D) \). Then, because \( u_4u_1 \notin E(H) \), \( u_4u_2, u_4u_6 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3), a contradiction. So \( u_3u_5 \) is contained in \( \text{Int}(D) \). Note that \( u_4u_7 \notin E(H) \); otherwise \( C' := u_1u_2u_7u_3u_4u_5u_6u_1 \) and \( H' \) contradict (1). Thus \( u_4u_2, u_4u_6 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3). By planarity, \( u_3u_6 \) is contained in \( \text{Int}(D) \). But again, \( d_H(u_5) = 3 \) (by (1), planarity and Proposition 2.3), a contradiction.

**Case 5.** \( S' = u_3u_5u_3u_3 \).

We claim that \( u_4u_2, u_4u_6 \notin E(H) \). Otherwise, we define

\[
C' := \begin{cases} 
    u_1u_2u_6u_7u_5u_4u_7u_1, & \text{if } u_4u_2 \in E(H), \\
    u_1u_2u_5u_4u_6u_7u_1, & \text{if } u_4u_6 \in E(H).
\end{cases}
\]

Then \( C' \) and \( H' \) contradict (1).

Therefore, \( u_4u_7, u_4u_1 \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (1), planarity and Proposition 2.3). Then \( C' := u_1u_2u_3u_6u_7u_4u_1 \) and \( H' \) contradict (1).

**Case 6.** \( S' = u_3u_6u_3 \).

We first show that \( u_4u_7, u_5u_2 \notin E(H) \). Otherwise, let

\[
C' := \begin{cases} 
    u_1u_2u_3u_6u_7u_5u_1, & \text{if } u_4u_7 \in E(H), \\
    u_1u_2u_5u_4u_6u_7u_1, & \text{if } u_5u_2 \in E(H)
\end{cases}
\]

Then \( C' \) and \( H' \) contradict (1).

We now show \( u_5u_1 \in E(H) \). Suppose \( u_5u_1 \notin E(H) \). Then \( u_5u_3, u_5u_7 \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (1), planarity and Proposition 2.3). By planarity, \( u_5u_7 \) is contained in \( \text{Int}(D) \). If \( u_5u_3 \) is contained in \( \text{Int}(D) \), then
Proposition 2.3, a contradiction. So $u_5u_3$ is contained in Ext($D$). Therefore, because $u_4u_7 \notin E(H)$, we have $u_4u_1, u_4u_2 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). But now $C := u_1u_2u_3u_4u_5u_6u_7u_1$ and $H'$ contradict (1).

So $u_1u_5, u_7 \in E(H)$ (since $d_H(u_7) \geq 4$ and by (1), planarity and Proposition 2.3). If $u_4u_1 \in E(H)$, then $C := u_1u_2u_3u_4u_5u_6u_7u_1$ and $H'$ contradict (1). So $u_4u_1 \notin E(H)$. Therefore, $u_4u_2, u_4u_6 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). But then $C := u_1u_2u_3u_4u_5u_7u_1$ and $H'$ contradict (1).

Case 7. $S' = u_3u_7u_3$.

First, $u_4u_1, u_6u_2 \notin E(H)$. Otherwise, define

$$C' := \begin{cases} u_1u_2u_3u_7u_6u_5u_4u_1, & \text{if } u_4u_1 \in E(H), \\ u_1u_2u_6u_5u_4u_3u_7u_1, & \text{if } u_6u_2 \in E(H). \end{cases}$$

Then $C'$ and $H'$ contradict (1).

We now show $u_4u_6 \in E(H)$. For, suppose $u_4u_6 \notin E(H)$. Then, because $u_4u_1 \notin E(H), u_4u_2, u_4u_7 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). By planarity, $u_4u_2$ is contained in Int($D$). If $u_4u_7$ also lies in Int($D$), then because $u_4u_6 \notin E(H)$ we must have $d_H(u_6) \leq 3$ by (1), planarity and Proposition 2.3), a contradiction. So $u_4u_7$ is contained in Ext($D$). Since $u_6u_2, u_6u_4 \notin E(H), d_H(u_6) \leq 3$ by (1), planarity and Proposition 2.3, a contradiction.

Suppose that $u_4u_6$ is contained in Int($D$). Then $u_5u_3, u_5u_7 \in E(H)$ (since $d_H(u_5) \geq 4$ and by (1), planarity and Proposition 2.3). Further, $u_6u_3 \notin E(H)$; as otherwise, $d_H(u_6) = 3$ by (1), planarity and Proposition 2.3, a contradiction. Therefore, because $u_5u_2 \notin E(H), u_5u_1 \in E(H)$ (since $d_H(u_6) \geq 4$ and by (1), planarity and Proposition 2.3). Now $C := u_1u_2u_3u_7u_5u_4u_6u_1$ and $H'$ contradict (1).

So $u_4u_6$ is contained in Ext($D$). Then $u_5u_1, u_5u_2 \notin E(H)$; for otherwise, we define

$$C' := \begin{cases} u_1u_2u_3u_7u_6u_5u_4u_1, & \text{if } u_5u_1 \in E(H), \\ u_1u_2u_6u_5u_4u_3u_7u_1, & \text{if } u_5u_2 \in E(H), \end{cases}$$

and it is easy to check that $C'$ and $H'$ contradict (1). So $u_5u_3, u_5u_7 \in E(H)$ (since $d_H(u_5) \geq 4$ and by (1), planarity and Proposition 2.3). Then $u_4u_7 \in E(H)$ (since $d_H(u_4) \geq 4$ and by (1), planarity and Proposition 2.3). But now $d_H(u_6) = 3$ by (1), planarity and Proposition 2.3), a contradiction.

Case 8. $S' = u_4u_6u_4$.

Then $u_5u_3, u_5u_7 \notin E(H)$; otherwise, define

$$C' := \begin{cases} u_1u_2u_3u_5u_4u_6u_7u_1, & \text{if } u_5u_3 \in E(H), \\ u_1u_2u_3u_4u_6u_5u_7u_1, & \text{if } u_5u_7 \in E(H), \end{cases}$$

and then $C'$ and $H'$ contradict (1). So $u_5u_3, u_5u_2 \in E(H)$ (since $d_H(u_5) \geq 4$ and by (1), planarity and Proposition 2.3). Then, because $u_3u_5 \notin E(H), d_H(u_3) \leq 3$ by (1), planarity and Proposition 2.3, a contradiction. □

We can now improve Lemma 4.2.

Lemma 5.2. Let $H$ be a planar triangulation with outer cycle $S$, and assume that $|V(H)| \geq 11$ and every vertex of $H$ not in $S$ has degree at least 4. Suppose that $H$ is not the graph in Fig. 2. Then $H$ has a cycle $C$ such that $|V(C \cap S)| = 1, |V(C)| \geq 9$, and $C \cup S$ is a Tutte subgraph of $H$.

Proof. By Lemma 4.2, there is a cycle $D$ in $H$ such that $|V(D \cap S)| = 1, |V(D)| \geq 8$, and $D \cup S$ is a Tutte subgraph of $H$. If $|V(D)| \geq 9$, then $C := D$ gives the desired cycle. So we may assume $|V(D)| = 8$. Let $D = u_1u_2u_3u_4u_5u_6u_7u_8u_1$ and $S := xyzz$, and without loss of generality assume that $x \in V(D)$.

Let $B$ denote the block of $H - [y, z]$ containing $D$, and let $u$ denote the vertex of $B - x$ which is adjacent to both $y$ and $z$ in $H$. Then $|V(B)| \geq 8$. Since $D \cup S$ is a Tutte subgraph of $H$, $u \in V(D)$ and $D$ is a $T$-Tutte subgraph of $B$, where $T$ denotes the outer cycle of $B$. Note that by planarity, $x$ and $u$ are the only vertices of $B$ that are adjacent to both $y$ and $z$. We may assume that

(1) if Int($uyzv$) $\neq uyvz$ then $xv \notin E(G)$. 

Suppose \( \text{Int}(uyzu) \neq uyzu \) and \( xu \in E(G) \). Then \( \text{Int}(xyux) \neq xyux \) or \( \text{Int}(xzux) \neq xzux \), and we may assume by symmetry that \( \text{Int}(xyux) \neq xyux \). Since every vertex of \( H \) not in \( S \) has degree at least 4, we can apply Lemma 2.4 to \( \text{Int}(xyux) \) and \( yu \), and to \( \text{Int}(xyux) \) and \( yu \). Then there exists a cycle \( C_1 \) in \( \text{Int}(xyux) \) such that \( yu \in E(C_1) \), \( |V(C_1)| \geq 5 \), and \( C_1 \cup xyux \) is a Tutte subgraph of \( \text{Int}(xyux) \); and there exists a cycle \( C_2 \) in \( \text{Int}(uyzu) \) such that \( yu \in E(C_2) \), \( |V(C_2)| \geq 5 \), and \( C_2 \cup uyzu \) is a Tutte subgraph of \( \text{Int}(uyzu) \). We choose \( C_1 \) and \( C_2 \) as long as possible. Now \( C' := (C_1 \cup C_2) - yu \) is a cycle in \( H \) such that \( |V(C' \cap S)| = 1 \), \( |V(C')| \geq 8 \), and \( C' \cup S \) is a Tutte subgraph of \( H \).

If \( |V(C')| \geq 9 \), then \( C := C' \) gives the desired cycle for the lemma. So we may assume \( |V(C')| = 8 \). This implies that \( |C_1| = |C_2| = 5 \). By Lemma 2.4 again, both \( \text{Int}(xyux) \) and \( \text{Int}(uyzu) \) are the octahedron. Since \( H \) is not the graph in Fig. 2 and \( |V(H)| \geq 11 \), \( \text{Int}(xyux) \neq xzux \) and \( \text{Int}(xzux) \) is not the octahedron. Hence, we can apply Lemma 2.4 to \( \text{Int}(xzux) \) and \( xu \), and find a cycle \( D_1 \) in \( \text{Int}(xzux) \) such that \( xu \in E(D_1) \), \( |V(D_1)| \geq 6 \), and \( D_1 \cup xzux \) is a Tutte subgraph of \( \text{Int}(xzux) \). Let \( D_2 \) be a Hamilton cycle of \( \text{Int}(xyux) \) containing \( xu \). Then \( |V(D_2)| = 5 \). Hence, \( C := (D_1 \cup D_2) - xu \) is a cycle in \( H \) such that \( |V(C \cap S)| = 1 \), \( |V(C)| \geq 9 \), and \( C \cup S \) is a Tutte subgraph of \( H \); and the assertion of the lemma holds. This completes the proof of (1).

We may further assume that

(2) \( \text{Int}(uyzu) = uyzu \).

Otherwise, suppose \( \text{Int}(uyzu) \neq uyzu \). Then by (1), \( xu \notin E(H) \).

By planarity, the blocks of \( B - x \) may be labeled as \( B_1 \ldots B_s \) such that for all \( 1 \leq i < j \leq n \), \( B_i \cap B_{i+1} \neq \emptyset \) and \( B_i \cap B_j = \emptyset \) if \( j \geq i + 2 \). Let \( u \in V(B_k) \) for some \( 1 \leq k \leq n \). Since \( xu \notin E(H) \), \( u \notin V(B_k) \) for all \( i \neq k \). So \( |V(B_k)| \geq 3 \). Let \( K := \bigcup_{j=1}^k B_j \) and \( L := \bigcup_{k}^{n} B_i \). Note that \( |V(K)| + |V(L)| \geq |V(B)| + 2 \geq 10 \). So we may assume by symmetry that \( |V(L)| \geq 5 \).

By planarity, \( L \) is contained in a block of \( H - \{x, y\} \) containing \( z \), or is contained in a block of \( H - \{x, y\} \) containing \( z \). By symmetry we may assume that \( L \) and \( y \) are contained in a block of \( H - \{x, z\} \), say \( G \). Then \( G \) also contains \( \text{Int}(uyzu) - y \). Clearly, \( \{y, u\} \) is a 2-cut of \( G \). Let \( G' \) denote the subgraph of \( G \) that is the union of the edge \( yu \) and the nontrivial \( \{y, u\} \)-bridge of \( G \) containing \( L \). Note that \( G' \) is a near triangulation. Since \( |V(L)| \geq 5 \), we must have \( |V(G')| \geq 6 \). Let \( T' \) be the outer cycle of \( G' \).

Next, we find a cycle \( C' \) in \( G' \). If \( |V(T')| \geq 6 \) then we can choose a matching \( \{e_1, e_2, e_3\} \) on \( T' \) such that \( e_1 = yu \) and the common neighbor of \( x \) and \( z \) in \( V(T') - y \) is incident with \( e_2 \) or \( e_3 \); and we use Theorem 2.2 to find a \( T' \)-Tutte cycle \( C' \) in \( G' \) such that \( \{e_1, e_2, e_3\} \subseteq E(C') \) and \( |V(C')| \geq 6 \). If \( |V(T')| \leq 4 \), then we apply Lemmas 3.1 and 3.2 to find a Tutte cycle \( C' \) in \( G' \) such that \( yu \in E(C') \), \( V(T') \subseteq V(C') \), and \( |V(C')| \geq 6 \). Now assume \( |V(T')| = 5 \). Note that every vertex in \( V(T') - \{y, u\} \) with degree 2 in \( G' \) must be adjacent to both \( x \) and \( z \) (since such a vertex has degree at least 4 in \( H \)). So by planarity, at most one member of \( V(T') - \{y, u\} \) has degree 2 in \( G' \). Therefore, by Lemma 3.3(ii), we find a Tutte cycle \( C' \) in \( G' \) such that \( yu \in E(C') \), \( V(T') \subseteq V(C') \), and \( |V(C')| \geq 6 \).

By Lemma 2.4, there is a cycle \( C_2 \) in \( \text{Int}(uyzu) - z \) such that \( yu \in E(C_2) \), \( |V(C_2)| \geq 5 \), and \( C_2 \cup uyzu \) is a Tutte subgraph of \( \text{Int}(uyzu) \). Then \( C := (C_1 \cup C_2) - yu \) is a cycle in \( H \) such that \( |V(C \cap S)| = 1 \), \( |V(C)| \geq 9 \), and \( C \cup S \) is a Tutte subgraph of \( H \); and the assertion of the lemma holds. So we have (2).

We may also assume that

(3) for any cycle \( C' \) in \( B \) such that \( V(C') = V(D) \), and for any nontrivial \( (C' \cup S) \)-bridge \( H' \) of \( H \), no two vertices in \( V(H' \cap C') \) are joined by an edge of \( C' \).

For otherwise, let \( C' \) be a cycle in \( B \) such that \( V(C') = V(D) \), let \( H' \) be a nontrivial \( (C' \cup S) \)-bridge of \( H \), and assume that \( x'y' \in E(C') \) and \( \{x', y'\} \subseteq V(H' \cap C') \). Let \( z' \in (V(H') \cap V(C' \cup S)) - \{x', y'\} \) and \( S' := x'y'z'x' \). By applying Lemma 2.4 to \( H[H'] \), \( S', x'y' \), we find a cycle \( C'' \) in \( H[H'] - z' \) such that \( x'y' \in E(C''), |V(C'')| \geq 5 \), and \( C'' \cup S' \) is a Tutte subgraph of \( H[H'] \). Now it is easy to see that \( C := (C' \cup C'') - x'y' \) is a cycle in \( H - \{y, z\}, x \in V(C), |V(C)| \geq 11 \), and \( C \cup S \) is a Tutte subgraph of \( H \); and the assertion of the lemma holds. This proves (3).

Since \( |V(H)| \geq 11 \) and \( |V(D)| = 8 \), there is a nontrivial \( (D \cup S) \)-bridge of \( H \). So let \( H' \) denote an arbitrary nontrivial \( (D \cup S) \)-bridge of \( H \). Let \( V(H') \cap V(D \cup S) = \{x', y', z'\} \) and \( S' := x'y'z'x' \), and assume \( \{x', y'\} \subseteq V(D) \). Then by the same argument as for (3) in the proof of Lemma 4.2, we may assume

(4) \( z' \notin \{y, z\} \).
Therefore, \( \{ x', y', z' \} \subseteq V(D) \). By (3) and by symmetry, we may assume that \( x' = u_1, y' = u_3, \) and \( z' \in \{ u_5, u_6 \} \).

**Case 1.** \( z' = u_5 \).

We claim that \( u_2u_4, u_2u_6, u_2u_8, u_4u_6, u_4u_8 \notin E(H) \). Otherwise, we define

\[
C' := \begin{cases} 
  u_1u_2u_4u_3u_6u_7u_8u_1, & \text{if } u_2u_4 \in E(H), \\
  u_1u_2u_4u_3u_6u_7u_8u_1, & \text{if } u_2u_6 \in E(H), \\
  u_1u_2u_4u_3u_6u_7u_8u_1, & \text{if } u_2u_8 \in E(H), \\
  u_1u_2u_4u_3u_6u_7u_8u_1, & \text{if } u_4u_6 \in E(H), \\
  u_1u_2u_4u_3u_6u_7u_8u_1, & \text{if } u_4u_8 \in E(H).
\end{cases}
\]

Then \( C' \) and \( H' \) contradict (3).

We also claim that \( u_2u_5 \notin E(H) \). Otherwise, suppose \( u_2u_5 \in E(H) \). Assume that \( u_2u_5 \) is contained in \( \text{Int}(D) \). Then \( H' \subseteq \text{Ext}(D) \) (by planarity). Since \( u_2u_4 \notin E(H) \), the finite face of \( H' \) incident with both \( u_2 \) and \( u_5 \), which is also a face of \( \text{Int}(u_2u_3u_4u_5u_2) \), is not bounded by a triangle, a contradiction. So \( u_2u_5 \) is contained in \( \text{Ext}(D) \). Then by planarity, \( H' \subseteq \text{Int}(D) \). Moreover, \( u_1 \) lies in \( \text{Int}(u_2u_3u_4u_5u_2) \); otherwise, since \( u_4u_2 \notin E(H), d_H(u_4) = 2 \) (by (3), planarity and Proposition 2.3), a contradiction. Now, \( u_6u_8, u_6u_1 \in E(H) \) (since \( d_H(u_6) \geq 4 \) and by (3), planarity and Proposition 2.3). Suppose that \( u_6u_8 \) is contained in \( \text{Ext}(D) \). Then \( u_7u_1, u_7u_5 \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (3), planarity and Proposition 2.3). By planarity, \( u_6u_1 \) is contained in \( \text{Ext}(D) \). But then \( d_H(u_8) = 3 \) (by (3), planarity and Proposition 2.3), a contradiction. So \( u_6u_8 \) is contained in \( \text{Int}(D) \). Furthermore, \( u_6u_1 \) is contained in \( \text{Int}(D) \); otherwise, \( d_H(u_7) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. Then, because \( u_8u_2 \notin E(H), u_8u_5 \in E(H) \) (since \( d_H(u_8) \geq 4 \) and by (3), planarity and Proposition 2.3). But again, \( d_H(u_7) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

We further claim that \( u_2y, u_2z \in E(H) \). Otherwise, \( u_2 \notin \{ x, u \} \). Then \( u_2u_7 \in E(H) \), and \( u_2y, u_2z \in E(H) \) or \( u_2z \in E(H) \) (since \( d_H(u_2) \geq 4 \) and by (3), planarity and Proposition 2.3). By symmetry, assume \( u_2y \in E(H) \). Now \( u_2u_7 \) is contained in \( \text{Ext}(D) \) and \( u_3 \) lies in \( \text{Ext}(u_1u_2u_7u_8u_1) \); otherwise, because \( u_3u_2, u_4u_6 \notin E(H), d_H(u_4) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. Hence \( H' \subseteq \text{Int}(D) \). Then, because \( u_6u_2 \notin E(H), u_8u_5, u_8u_6 \in E(H) \) (since \( d_H(u_8) \geq 4 \) and by (3), planarity and Proposition 2.3). But now \( C' := u_1u_2u_4u_5u_6u_7u_8u_1 \) and \( H' \) contradict (3).

By the same argument as above, we can prove that \( u_4y, u_4z \in E(H) \).

Now \( \{ u_2, u_4 \} = \{ x, u \} \) and \( H' \subseteq \text{Int}(D) \) (by planarity). Without loss of generality, we may assume \( u_4 = x \) and \( D \subseteq \text{Ext}(u_2u_3u_4u_2) \). So \( u_3y \in E(H) \) (since \( u_2u_4 \notin E(H) \) and by (3), planarity and Proposition 2.3). By Lemma 2.4, there is a cycle \( D' \) in \( H[H'] - u_1 \) such that \( u_3u_5 \in E(D'), |V(D')| \geq 5 \), and \( D' \cup S' \) is a Tutte subgraph of \( H[H'] \). Now \( C := ((D - \{ u_2u_3, u_4 \}) \cup (D' - u_3u_5)) \cup \{ y, yu_2, yu_3 \} \) gives the desired cycle for this lemma (with \( V(C \cap S) = \{ y \} \)).

**Case 2.** \( z' = u_6 \).

We claim that \( u_2u_4, u_2u_5, u_2u_7, u_2u_8, u_4u_7, u_5u_8 \notin E(H) \). Otherwise, we define

\[
C' := \begin{cases} 
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_2u_4 \in E(H), \\
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_2u_5 \in E(H), \\
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_2u_7 \in E(H), \\
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_2u_8 \in E(H), \\
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_4u_7 \in E(H), \\
  u_1u_2u_4u_5u_6u_7u_8u_1, & \text{if } u_4u_8 \in E(H).
\end{cases}
\]

Then \( C' \) and \( H' \) contradict (3).

We also claim that \( u_2 \notin \{ x, u \} \). Otherwise, \( u_2 \notin \{ x, u \} \). Then \( u_2u_6 \in E(H) \), and \( u_2y, u_2z \in E(H) \) (since \( d_H(u_2) \geq 4 \) and by (3), planarity and Proposition 2.3). By symmetry, assume \( u_2y \in E(H) \). Then \( u_2u_6 \) is contained in \( \text{Ext}(D) \), and \( u_3 \) lies in \( \text{Ext}(u_2u_4u_7u_8u_1) \); otherwise, because \( u_4u_2 \notin E(H), d_H(u_4) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. Hence \( H' \subseteq \text{Int}(D) \). But now, since \( u_7u_2 \notin E(H), d_H(u_7) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

Therefore \( H' \subseteq \text{Int}(D) \); as otherwise, since \( u_2u_4, u_2u_5, u_2u_7, u_2u_8 \notin E(H) \), the finite face of \( H \) incident with both \( u_2 \) and \( u_3 \), which is also a face of \( \text{Int}(D) \), is not bounded by a triangle, a contradiction.

Next we show that \( \{ u_1, u_3 \} \cap \{ x, u \} \neq \emptyset \).
First, we show \( u_6 \notin \{ x, u \} \). Otherwise, assume \( u_6 \in \{ x, u \} \). By planarity, \( u_6, y, z \) are contained in \( \text{Ext}(D) \). By symmetry, we may assume \( D \subseteq \text{Ext}(u_2u_3u_4u_5yu_2) \). Hence, because \( u_2u_4 \notin E(H), u_4u_6, u_4y \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (3), planarity and Proposition 2.3). But now \( d_H(u_5) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

Now we show \( u_4, u_7 \notin \{ x, u \} \). For, suppose by symmetry that \( u_5 \in \{ x, u \} \). By planarity, \( u_5y, u_5z \) are contained in \( \text{Ext}(D) \). Then \( D \subseteq \text{Ext}(u_2u_3u_4u_5yu_2) \) or \( D \subseteq \text{Ext}(u_2u_3u_4u_5zu_2) \). Therefore, because \( u_4u_2 \notin E(H), u_4u_6, u_4y \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (3), planarity and Proposition 2.3). Then \( u_7u_5 \notin E(H) \); otherwise, \( C' := u_1u_2u_3u_4u_5u_7u_8u_1 \) and \( H' \) contradict (3). Hence, because \( u_7u_2 \notin E(H), u_7u_1, u_7z \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (3), planarity and Proposition 2.3). But then, because \( u_2u_3 \notin E(H), d_H(u_8) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

It remains to show that \( u_4, u_9 \notin \{ x, u \} \). Otherwise, we may assume by symmetry that \( u_4 \in \{ x, u \} \). By planarity, \( u_4y, u_4z \) are contained in \( \text{Ext}(D) \). By symmetry between \( y \) and \( z \), we may assume \( D \subseteq \text{Ext}(u_2u_3u_4u_5zu_2) \). Now \( u_5u_1 \notin E(H) \); for otherwise, since \( u_5u_2 \notin E(H) \), we would have \( d_H(u_8) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. Suppose \( u_5u_2 \notin E(H) \). Then, because \( u_5u_2, u_5u_8 \notin E(H), u_5u_7, u_5z \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (3), planarity and Proposition 2.3). Since \( H \) is a triangulation and we assume \( u_5u_3 \notin E(H), u_5u_6 \) is contained in \( \text{Int}(D) \) (by (3), planarity and Proposition 2.3). But now \( C := u_1u_2u_3u_4u_5u_7u_8u_1 \) and \( H' \) contradict (3). So \( u_5u_3 \in E(H) \). By planarity, \( u_5u_3 \) is contained in \( \text{Int}(D) \). Then \( u_5u_4 \notin E(H) \); otherwise, \( C' := u_1u_2u_3u_5u_8u_7u_6u_1 \) and \( H' \) contradict (3). Therefore, because \( u_5u_2, u_5u_5 \notin E(H), u_5u_6, u_5z \in E(H) \) (since \( d_H(u_5) \geq 4 \) and by (3), planarity and Proposition 2.3). If \( u_5u_6 \) is contained in \( \text{Ext}(D) \), then \( d_H(u_7) \geq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. So \( u_5u_6 \) is contained in \( \text{Int}(D) \) and, because \( u_7u_4 \notin E(H), u_7u_5, u_7z \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (3), planarity and Proposition 2.3). But in this case, \( C' := u_1u_2u_3u_4u_5u_7u_8u_1 \) and \( H' \) contradict (3).

Therefore, we have shown that either \( e \in \{ x, u \} \) or \( u_1 \in \{ x, u \} \). By symmetry between \( u_1 \) and \( u_3 \) and by symmetry between \( y \) and \( z \), we may assume \( u_3 \in \{ x, u \} \) and \( D \subseteq \text{Ext}(u_2u_3u_4u_5zu_2) \).

Then \( u_5u_1 \notin E(H) \). Otherwise, since \( u_5u_5 \notin E(H), d_H(u_8) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

We claim that \( u_3 \notin E(H) \). Otherwise, suppose \( u_3 \in E(H) \). Then, because \( u_3u_2, u_3u_5 \notin E(H) \), we have \( u_3u_6, u_3z \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (3), planarity and Proposition 2.3). If \( u_3u_6 \) is contained in \( \text{Ext}(D) \), then \( d_H(u_7) \geq 3 \) (by (3), planarity and Proposition 2.3), a contradiction. So \( u_3u_6 \) is contained in \( \text{Int}(D) \). Hence \( u_3u_5, u_3z \in E(H) \) (since \( d_H(u_7) \geq 4 \) and by (3), planarity and Proposition 2.3). But then \( C' := u_1u_2u_3u_4u_5u_7u_8u_1 \) and \( H' \) contradict (3).

Therefore, because \( u_3u_2, u_3u_5 \notin E(H), \) we have \( u_3u_3, u_3u_7 \in E(H) \) (since \( d_H(u_3) \geq 4 \) and by (3), planarity and Proposition 2.3). Moreover, \( u_5u_3 \) is contained in \( \text{Int}(D) \); for otherwise, \( d_H(u_3) \leq 3 \) (by (3), planarity and Proposition 2.3). Then \( u_3u_5 \notin E(H) \); otherwise, \( C' := u_1u_2u_3u_4u_5u_7u_6u_1 \) and \( H' \) contradict (3). Hence, because \( u_3u_2, u_3u_7 \notin E(H), \) we have \( u_3u_4, u_3z \in E(H) \) (since \( d_H(u_4) \geq 4 \) and by (3), planarity and Proposition 2.3). But now, since \( u_3u_4, u_3u_5 \notin E(H), d_H(u_8) \leq 3 \) (by (3), planarity and Proposition 2.3), a contradiction.

6. Maximum cuts

In this section, we prove the main result of this paper. For a vertex \( v \) in a graph \( H \), we use \( \delta_H(v) \) to denote the set of edges of \( H \) incident with \( v \). We show that given a planar triangulation \( H \) with outer cycle \( S \) such that \( 4 \leq |V(H)| \leq 10 \) and every vertex of \( H \) not in \( S \) has degree at least 4, we can delete at most \( \frac{|V(H)|}{2} - |V(S)| \) edges from \( H - E(S) \) so that all vertices not in \( S \) have even degree in the new graph, and the degree of at most one vertex in \( S \) changes parity.

**Lemma 6.1.** Let \( H \) be a planar triangulation with outer cycle \( S \), and assume that \( 4 \leq |V(H)| \leq 10 \) and every vertex of \( H \) not in \( S \) has degree at least 4. Let \( H' := H - V(S) \). Then there exists \( M \subseteq E(H) - E(S) \) such that

(i) \( |M| \leq \frac{|V(H')|}{2} \),
(ii) for any \( v \in V(H') \), \( d_H(v) = |\delta_H(v) \cap M| \pmod 2 \), and
(iii) \( |\{ v \in V(S) : |\delta_H(v) \cap M| = 1 \pmod 2 \}| \leq 1 \).

**Proof.** Since \( |V(H)| \geq 4 \) and every vertex of \( H \) not in \( S \) has degree at least 4, \( |V(H)| \geq 6 \). If \( H' \) has no odd vertex of \( H \), then the assertion of the lemma holds with \( M := \emptyset \). So we may assume that \( H' \) has at least one odd vertex of \( H \). Then \( |V(H)| \geq 7 \). Therefore, since \( |V(H)| \leq 10 \), we have...
(1) $4 \leq |V(H')| \leq 7$.

For convenience, let $S := xyzx$. We may assume that

(2) $H'$ has at least four odd vertices of $H$.

Otherwise, at most three odd vertices of $H$ are contained in $H'$.

First, assume that $H'$ has just one odd vertex of $H$, say $u$. If $w$ is adjacent to some vertex of $S$, say $x$, then $M := \{wx\}$ gives the desired set. So we may assume that $w$ is not adjacent to any vertex of $S$. Then by (1), $d_H(w) = 5$, and at most one vertex of $H'$ is not adjacent to $w$. Since $|V(H)| \geq 7$, we may assume without loss of generality that $d_H(x) \geq 4$. Hence, $x$ is adjacent to some neighbor of $w$, say $y$. Then $M := \{wv, vx\}$ gives the desired set.

Now assume that $H'$ has exactly two odd vertices of $H$, say $w$ and $v$. If $wv \in E(H)$ then $M := \{wv\}$ gives the desired set. So we may assume $wv \notin E(H)$. Since $|V(H)| \leq 10$, $N_H(w) \cap N_H(v) \neq \emptyset$. Let $u \in N_H(w) \cap N_H(v)$. Then $M := \{wu, uv\}$ gives the desired set.

Therefore, we may assume that $H'$ has exactly three odd vertices of $H$; otherwise, (2) holds. Let $u, v, w$ be the odd vertices of $H$ that are contained in $H'$.

We claim that each of $\{u, v, w\}$ is adjacent to some vertex of $S$. Otherwise, assume without loss of generality that $w$ is not adjacent to any vertex of $S$. Then by (1), $d_H(w) = 5$ and $|V(H')| \geq 5$. Moreover, at most one vertex of $H'$ is not adjacent to $w$. By symmetry, we may assume $v \in N_H(w)$. If $u$ is adjacent to some vertex of $S$, say $x$, then $M := \{wu, ux\}$ gives the desired set. So we may assume that $u$ is not adjacent to any vertex of $S$. By (1), $d_H(u) = 5$ and at most one vertex of $H'$ is not adjacent to $u$. Since $|V(H)| \geq 7$, some vertex of $S$, say $x$, has degree at least 4 in $H$. Then $x$ is adjacent to some neighbor of $u$, say $t$. Now, since $|V(H')| \geq 6$, $M := \{wu, ut, tx\}$ gives the desired set.

Since $|V(H)| \leq 10$, either $vw \in E(H)$ or $N_H(u) \cap N_H(w) \neq \emptyset$. Therefore, let $P$ be a path in $H$ of length at most two that is from $v$ to $w$. Note that if no two vertices of $\{u, v, w\}$ are adjacent, then $|V(H')| \geq 6$. So we may further assume that $|E(P)| = 1$ if $|V(H')| \leq 5$. Without loss of generality, let $ux \in E(H)$. Then $M := E(P) \cup \{ux\}$ gives the desired set. This completes the proof of (2).

We may also assume that

(3) if there exists $w \in V(H')$ such that $wx, wy \in E(H)$, then $Int(wxwy) = wxwy$ or $Int(wxzyw) = wxzyw$.

Suppose there exists $w \in V(H')$ such that $wx, wy \in E(H)$, and assume $K := Int(wxwy) \neq wxwy$ and $L := Int(wxzyw) \neq wxzyw$. Since every vertex of $K$ not in $\{w, x, y\}$ has degree at least 4, $|V(K)| \geq 6$.

Suppose that one or two vertices in $V(L) - \{w, x, z, y\}$ have odd degree in $H$. Then $|V(L)| \geq 7$. By (2), at least one vertex in $V(K) - \{w, x, y\}$ has odd degree in $H$. So $|V(K)| \geq 7$, and $|V(H)| \geq 11$, a contradiction.

If at least three vertices in $V(L) - \{w, x, z, y\}$ have odd degree in $H$, then $|V(L)| \geq 8$, and hence, $|V(H)| \geq 11$, a contradiction.

Therefore, no vertex in $V(L) - \{w, x, z, y\}$ has odd degree in $H$. Let $S' := wxwy$ and $K' := K - V(S')$. Note that $K$ is a triangulation with outer cycle $S'$, $4 \leq |V(K)| < |V(H)|$, and every vertex of $K$ not in $S'$ has degree at least 4. So by applying induction to $K$, we find a subset $M' \subseteq E(K) - E(S')$ such that

(i) $|M'| \leq \frac{|V(K')|}{2}$,

(ii) for any $v \in V(K')$, $d_K(v) = |\delta_K(v) \cap M'|$ (mod 2), and

(iii) $|\{v \in V(S') : \delta_K(v) \cap M' \equiv 1 \pmod{2}\}| \leq 1$.

Since we assume $L \neq wxzyw$, $|V(K')| \leq |V(H')| - 2$. Let $M := M' \cup \{wu\}$, where $u \in \{x, y\}$ and, if $\{x, y\} \cap \{v \in V(S') : |\delta_K(v) \cap M' \equiv 1 \pmod{2}\} \neq \emptyset$, then $|\delta_K(u) \cap M'| \equiv 1 \pmod{2}$. Then $M$ is the desired set. So we have (3).

By the same argument, we may assume that

(4) if there exists $w \in V(H')$ such that $wy, wz \in E(H)$, then $Int(wyzw) = wyzw$ or $Int(wxyzw) = wxyzw$, and

(5) if there exists $w \in V(H')$ such that $wz, wx \in E(H)$, then $Int(wzxw) = wzxw$ or $Int(wzyxw) = wzxyw$.

Note that $H'$ must be connected. Indeed,

(6) $H'$ is 2-connected.
Otherwise, let \( w \) be a cut vertex of \( H' \). Then \( w \) is adjacent to at least two vertices of \( S \). We may assume \( wx, wy \in E(H) \); the cases when \( wy, wz \in E(H) \) or \( wz, wx \in E(H) \) can be treated in the same way by using (4) or (5) instead of (3). By (3), \( \text{Int}(wxyw) = wxyw \) or \( \text{Int}(wzyw) = wzyw \). If \( \text{Int}(wxyw) = wzyw \), then \( H' \subseteq \text{Int}(wxyw) \), which is impossible, since \( w \) is a cut vertex of \( H' \). So \( \text{Int}(wxyw) = wxyw \). Then \( H' \subseteq \text{Int}(wzyw) \).

Since \( w \) is a cut vertex of \( H' \), \( wz \in E(H) \), \( \text{Int}(wzyw) \neq wzyw \), and \( \text{Int}(wzyw) \neq wzyw \). Since every vertex of \( H \) not in \( S \) has degree at least 4 and by (1), we see that both \( \text{Int}(wzyw) \) and \( \text{Int}(wzyw) \) are the octahedron. But now, \( w \) is the unique odd vertex of \( H \) contained in \( H' \), contradicting (2).

Let \( C \) denote the outer cycle of \( H' \). Since \( H \) is a triangulation, \( H' \) is a near triangulation and each vertex on \( C \) must be adjacent to at least one vertex of \( S \). We may assume that

\[(7) \quad H' - V(C) \text{ has an odd vertex of } H.\]

Otherwise, suppose that \( H' - V(C) \) has no odd vertex of \( H \). If \( V(C) = V(H') \) then at least two vertices of \( H' \) have degree 2 in \( H' \), and at least one of these vertices must have degree 4 in \( H \); hence \( H' \) has at most 6 odd vertices of \( H \), and all lie on \( C \). If \( V(C) \neq V(H') \) then, since we assume that \( H' - V(C) \) has no odd vertex of \( H \), \( H' \) has at most 6 odd vertices of \( H \), and all lie on \( C \).

Suppose that \( H' \) has exactly four odd vertices of \( H \). If there is a matching \( M \) of two edges in \( C \) which saturates these four odd vertices of \( H \), then \( M \) gives the desired set. So we may assume that such a matching does not exist. Then \(|V(C)| = 6\) or \(|V(C)| = 7\). Now these four odd vertices can be labeled as \( t, u, v, w \) such that \( vw \in E(C) \) and \( C - vw \) has a subpath \( tsu \), where \( s \) is a vertex on \( C \). Then \( M := \{vw, ts, su\} \) gives the desired set.

Now assume that \( H' \) has exactly five odd vertices of \( H \). Note that each odd vertex of \( H' \), being on \( C \), must be adjacent to some vertex of \( S \). So these five odd vertices can be labeled as \( s, t, u, v, w \) such that \( st, uv \in E(C) \), and \( w \) is adjacent to some vertex of \( S \), say \( x \). Now \( M := \{st, uv, wx\} \) gives the desired set.

Finally, assume that \( H' \) has 6 odd vertices of \( H \). Then we see that these vertices can be labeled as \( r, s, t, u, v, w \) such that \( rs, tu, vw \in E(C) \). Now \( M := \{rs, tu, vw\} \) gives the desired set. This proves (7).

By (1) and (7), \(|V(C)| \leq 6\). We now consider four cases according to the length of \( C \). Note that in arguments below, some cases might not occur at all; but it is more convenient to treat them than to prove their nonexistence.

Case 1. \(|V(C)| = 6\).

Then by (1) and (7), let \( w \) be the unique vertex in \( H' - V(C) \) which is an odd vertex of \( H \). Let \( C = w_1w_2w_3w_4w_5w_6w_1 \), and assume that \( w_1 \) is the unique vertex of \( C \) not adjacent to \( w \). Then \( w_2w_6 \in E(H) \). Since \( d_{H}(w_1) \geq 4 \), \( w_1 \) is adjacent to at least two vertices of \( S \).

Suppose that \( w_1 \) is adjacent to all vertices of \( S \). Then \( w_1 \) is an odd vertex of \( H \). Without loss of generality, assume \( C \subseteq \text{Int}(w_1xw_1) \). Now each \( w_i, 2 \leq i \leq 6 \), is adjacent to some vertex of \( \{x, y\} \), and exactly one of \( \{w_2, \ldots, w_6\} \) is adjacent to both \( x \) and \( y \). If \( w_2 \) or \( w_4 \) is adjacent to both \( x \) and \( y \), say \( w_2 \), then \( w, w_1, w_6 \) are the only odd vertices of \( H \) contained in \( H' \), contradicting (2). So some \( w_i, i \in \{3, 4, 5\} \), is adjacent to both \( x \) and \( y \). Then \( w, w_1, w_2, w_6, w_i \) are the odd vertices of \( H \) that are contained in \( H' \). Now \( M := \{ww_1, w_2w_6, w_1x\} \) is the desired set.

So we may assume that \( w_1x, w_1y \in E(H) \) and \( w_1z \notin E(H) \). Then \( w_1 \) is an even vertex of \( H \). Clearly, \( C \subseteq \text{Int}(w_1xw_1) \). By planarity, at least one of \( \{w_3, w_4, w_5\} \) has degree exactly 4 in \( H \). So \( H' \) contains at most five odd vertices of \( H \).

Suppose that \( H' \) has four odd vertices of \( H \). Then three of them are on \( C - w_1 \), say \( t, u, v \), and two of them must be adjacent, say \( t \) and \( u \). Now \( M := \{tu, uv\} \) gives the desired set.

Now assume that \( H' \) has five odd vertices of \( H \). Then four of these vertices are on \( C - w_1 \), say \( s, t, u, v \). Note that each vertex of \( \{s, t, u, v\} \) is adjacent to some vertex of \( S \). So we may assume that \( st \in E(C) \) and \( u \) is adjacent to some \( u' \in V(S) \). Then \( M := \{st, uu', vv\} \) gives the desired set.

Case 2. \(|V(C)| = 5\).

Let \( C = w_1w_2w_3w_4w_5w_1 \). Let \( w \in V(H') - V(C) \) such that \( w \) is an odd vertex of \( H \). By (1), \( w \) is adjacent to at least four vertices of \( C \).

Suppose \(|V(H')| = 6 \). Then \( ww_1 \in E(H) \) for \( 1 \leq i \leq 5 \). By planarity, at least two vertices of \( C \) have degree 4 in \( H \). So by (2), \( H' \) has exactly four odd vertices of \( H \). Then three of these odd vertices are on \( C, \) say \( t, u, v \), and two of them must be adjacent, say \( t \) and \( u \). Now \( M := \{tu, uv\} \) gives the desired set.

So by (1), \(|V(H')| = 7 \). Let \( v \in V(H') \) such that \( v \neq w \). Since \( d_H(v) \geq 4 \), \( w \) is not adjacent to all vertices of \( C \). Therefore, we may assume without loss of generality that \( wu_1 \in E(H) \) for \( 1 \leq i \leq 4 \), and \( uv, wu_j \in E(H) \)
for \( j \in \{1, 4, 5\} \). Then \( v \) is an even vertex of \( H \). By (2), \( H' \) contains 4 or 5 or 6 odd vertices of \( H \), and all of which, with the exception of \( w \), lie on \( C \).

Suppose that \( H' \) has exactly four odd vertices of \( H \). If \( w_5 \) is an even vertex of \( H \), then the odd vertices of \( H' \) on \( C \) can be labeled as \( s, t, u \) such that \( st \in E(C) \), and \( M := \{st, wu\} \) gives the desired set. So \( w_5 \) is an odd vertex of \( H \).

If \( w_1 \) or \( w_4 \), say \( w_1 \), is an odd vertex of \( H \), then \( M := \{w_1 w_5, wu\} \) is the desired set, where \( w_1, i \in \{2, 3, 4\} \), is the other odd vertex of \( H \). So \( w_2, w_3 \) are odd vertices of \( H \). Then \( M := \{w_2 w_3, wv, wvw\} \) gives the desired set.

Now assume that \( H' \) has exactly five odd vertices of \( H \). Note that each vertex on \( C \) is adjacent to some vertex of \( S \). If \( w_5 \) is an even vertex of \( H \), then \( M := \{wvw_1, wvw_3, wvw_4\} \) gives the desired set, for some \( w' \in V(S) \). So \( w_5 \) is an odd vertex of \( H \). By symmetry between \( w_1 \) and \( w_4 \), we may assume that \( w_1 \) is also an odd vertex of \( H \). Then the other two odd vertices of \( H \) on \( C \) are \( w_j, w_j \) with \( i \neq j \) and \( \{i, j\} \subseteq \{2, 3, 4\} \). Then \( M := \{w_1 w_5, w_1, w_j w'\} \), for some \( w' \in V(S) \), gives the desired set.

Finally, assume that \( H' \) has 6 odd vertices of \( H \). Then \( M := \{w_1 w_2, w_2 w_3, w_4 w_5\} \) gives the desired set.

Case 3. \( |V(C)| = 4 \).

Let \( C = w_1 w_2 w_3 w_4 \). By (7), let \( w \in V(H') - V(C) \) such that \( w \) is an odd vertex of \( H \). Then by (1), \( w \) is adjacent to at least three vertices of \( C \). Without loss of generality, assume that \( w \) is adjacent to \( u \) and \( v \) and \( w \) is not in \( E(C) \) and \( i \neq w_4 \). Now \( M := \{rs, ut\} \) gives the desired set.

Case 4. \( |V(C)| = 3 \).

Let \( C = w_1 w_2 w_3 \). By (1), (2) and (7), \( |V(H')| = 7 \).

Suppose there is a vertex \( w \in V(H') - V(C) \) such that \( w \) is adjacent to all vertices of \( C \). Then since \( |V(H')| = 7 \) and every vertex of \( H \) not in \( S \) has degree at least 4, \( \text{Int}(w_1 w_2 w_3) \neq w_1 w_2 w_3 w_4 \). Then \( \text{Int}(w_1 w_2 w_3) \) is the octahedron. By (2), \( H' \) has 4 or 5 odd vertices of \( H \). Note that each vertex on \( C \) is adjacent to some vertex of \( S \).

If \( H' \) has five odd vertices of \( H \), then \( M := \{w_1 w_2, w_2 w_3, w_4 w_5\} \) gives the desired set, for some \( t \in V(S) \). So we may assume that \( H' \) has four odd vertices of \( H \). Then three of them are on \( C \), which may be labeled as \( r, s, t \) so that \( rs \in E(C) \) and \( i \neq w \). Now \( M := \{rs, ut\} \) gives the desired set.

The following result immediately implies Theorem 1.1.

**Theorem 6.2.** Let \( H \) be a planar triangulation with minimum degree at least 4. Let \( F \) be a smallest set of edges such that \( H - F \) is an even graph. Then \( |F| \leq \frac{9|V(H)|}{16} - \frac{9}{16} \).

**Proof.** If \( H \) is the graph in Fig. 2, then the assertion of the theorem is obvious. If \( |V(H)| \leq 10 \), then the assertion of the theorem follows from Lemma 6.1. So we may assume that \( |V(H)| \geq 11 \) and \( H \) is not the graph in Fig. 2. Let \( xyzx \) denote the outer cycle of \( H \).

By Lemma 5.1, \( H - z \) has a cycle \( C \) such that \( xy \in E(C) \), \( |V(C)| \geq 8 \), and \( C \cup xyzx \) is a Tutte subgraph of \( H \). Now \( T := (C - xy) + [z, xz, yz] \) is a Tutte cycle in \( H \) and \( |V(T)| \geq 9 \). Note that \( T \) may be viewed as a connected Tutte subgraph of \( H \) such that every block of \( T \) is a cycle of length at least 9. So we may choose a connected Tutte subgraph \( T \) of \( H \) such that

1. every block of \( T \) is a cycle of length at least 9, and subject to this, \( |V(T)| \) is maximum.

We claim that
for any $T$-bridge $H'$ of $H$, either $|V(H')| \leq 10$ or $H[H']$ is the graph in Fig. 2.

For, suppose $H'$ is a $T$-bridge of $H$ such that $|V(H')| \geq 11$ and $H[H']$ is not the graph in Fig. 2. Let $V(H' \cap T) = \{x', y', z', \}$ and let $S' := x'y'z'x'$. Then $H[H']$ is a triangulation with outer cycle $S'$. By Lemma 5.2, there is a cycle $C'$ in $H[H']$ such that $|V(C' \cap S')| = 1$, $|V(C')| \geq 9$, and $C' \cup S'$ is a Tutte subgraph of $H[H']$. Clearly, $T \cup C'$ is a connected Tutte subgraph of $H$ such that every block of $T \cup C'$ is a cycle of length at least 9. But this contradicts the choice of $T$ with (1).

Let $r$ denote the number of blocks in $T$. Note that $|E(T)| = |V(T)| + r - 1$ and $|E(T)| \geq 9r$. Therefore, we have

$$|E(T)| \leq \frac{9|V(T)| - 9}{8}.$$

Let $H_1, \ldots, H_k$ denote the nontrivial $T$-bridges of $H$. For each $1 \leq i \leq k$, let $S'_i$ denote the outer cycle of $H[H_i]$, and let $H'_i := H[H_i] - V(S'_i)$. By applying Lemma 6.1 to each $H[H_i]$, $1 \leq i \leq k$, we obtain a set of edges $M_i \subseteq E(H_i)$ such that

(i) $|M_i| \leq \frac{|V(H'_i)|}{2},$

(ii) for any $v \in V(H'_i)$, $d_{H_i}(v) = |\delta_{H_i}(v) \cap M_i| \pmod{2}$, and

(iii) $|\{v \in V(S'_i) : |\delta_{H_i}(v) \cap M_i| = 1 \pmod{2}\}| \leq 1.$

Note that for all $v \in V(H'_i)$ ($1 \leq i \leq k$), $d(H_i)(v) = d(H_i)(v)$ and $\delta_{H_i}(v) = \delta_{H_i}(v)$. Let $M := \bigcup_{i=1}^{k} M_i$ and $G := H - M$. Then all odd vertices of $G$ are contained in $T$ and

$$|M| \leq \sum_{i=1}^{k} |V(H'_i)|.$$

Since $T$ is connected and every block of $T$ is a cycle, $T$ is an Eulerian graph. Let $v_1v_2 \ldots v_kv_1$ denote an Eulerian circuit in $T$. If $G$ has no odd vertex, then let $M' := \emptyset$ (that is we need not delete more edges to make all vertices of $H$ even). Suppose that $G$ does have odd vertices. Then the odd vertices of $G$ may be labeled as $v_1, v_2, \ldots, v_l$ such that if we start from $v_1$ and proceed along the Eulerian circuit $v_1v_2 \ldots v_lv_1$, $v_1$ is the first odd vertex of $G$ we encounter, and for any $2 \leq j \leq 2l$, $v_j$ is the first odd vertex of $G$ we encounter that is not in $\{v_1, \ldots, v_{j-1}\}$.

Let $P_j$ (defined in the order $j = 1, 2, \ldots, t$) denote the subsequence of $v_1v_2 \ldots v_lv_1$ between $v_{2j-1}$ and $v_{2j}$. Let $Q_j$ (defined in the order $j = 1, 2, \ldots, t$) denote the subsequence of $v_1v_2 \ldots v_lv_1$ between $v_{2j}$ and $v_{2j+1}$, where $v_{2t+1} := v_1$. Then $P_j$ and $Q_j$ are edge-disjoint trails in $T$ whose union is $T$.

Clearly, $G - E(\bigcup_{j=1}^{t} P_j)$ and $G - E(\bigcup_{j=1}^{t} Q_j)$ have no odd vertex. Let $M'$ denote the smaller (in size) of $E(\bigcup_{j=1}^{t} P_j)$ and $E(\bigcup_{j=1}^{t} Q_j)$. Then

$$|M'| \leq \frac{|E(T)|}{2}.$$

By (3) and (5), we see that $|M'| \leq \frac{9|V(H)| - 9}{16}$. Now let $F := M \cup M'$. Then $|F| \leq \sum_{i=1}^{k} |V(H'_i)| + \frac{9|V(H)| - 9}{16} \leq \frac{9|V(H)| - 9}{16} + \frac{9|V(T)| - 9}{16} = \frac{9|V(H)| - 9}{16}$. Clearly, no vertex of $H - F$ has odd degree. \hfill \blacksquare

**Proof of Theorem 1.2.** Let $H$ be the dual graph of $G$. Then $H$ is a planar triangulation with minimum degree at least 4. By Euler’s formula, $|V(H)| = \frac{V(G)}{2} + 2$. By Theorem 6.2, $H$ has a set $F$ of edges such that $H - F$ is an even graph and $|F| \leq \frac{9|V(H)| - 9}{16}$. Therefore, the dual edges of $H - F$ form a bipartite subgraph in $G$. Clearly, the number of edges in this bipartite graph is at least $\frac{3|V(G)}{2} - \frac{9|V(H)| - 9}{16} = \frac{9|V(G)}{32} - \frac{9}{16}$. \hfill \blacksquare

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