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A Greedy Partition Lemma for directed domination

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ABSTRACT

A directed dominating set in a directed graph D is a set S of vertices of V such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex v in S with v directed to u. The directed domination number of D, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in D. The directed domination number of a graph G, denoted $\Gamma_d(G)$, is the maximum directed domination number $\gamma(D)$ over all orientations D of G. The directed domination number of a complete graph was first studied by Erdős [P. Erdős On a problem in graph theory, Math. Gaz. 47 (1963) 220–222], albeit in a disguised form. In this paper we prove a Greedy Partition Lemma for directed domination in oriented graphs. Applying this lemma, we obtain bounds on the directed domination number. In particular, if α denotes the independence number of a graph G, we show that $\alpha \le \Gamma_d(G) \le \alpha(1 + 2\ln(n/\alpha))$.

1. Introduction

An *asymmetric digraph* or *oriented graph D* is a digraph that can be obtained from a graph G by assigning a direction to (that is, orienting) each edge of G. The resulting digraph D is called an *orientation* of G. Thus if D is an oriented graph, then for every pair u and v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D. A *directed dominating set*, abbreviated DDS, in a directed graph $D = (V, A)$ is a set S of vertices of V such that every vertex in $V \setminus S$ is dominated by some vertex of S; that is, every vertex $u \in V \setminus S$ has an adjacent vertex v in S with v directed to u. Every digraph has a DDS since the entire vertex set of the digraph is such a set.

The *directed domination number* of a *directed graph D*, denoted by $\gamma(D)$, is the minimum cardinality of a DDS in D. A DDS of D of cardinality $\gamma(D)$ is called a $\gamma(D)$-set. Directed domination in digraphs is well studied (cf. [1–10]).

The *directed domination number* of a *graph G*, denoted $\Gamma_d(G)$, is defined in [11] as the maximum directed domination number $\gamma(D)$ over all orientations D of G; that is,

$$\Gamma_d(G) = \max\{\gamma(D)\}$$

where the maximum is taken over all orientations D of G. The directed domination number of a complete graph was first studied by Erdős [12] albeit in a disguised form. In 1962, Schütte [12] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set S of k vertices, there is a vertex u which dominates all vertices in S. Erdős [12] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer k. The proof of the following bounds on the directed domination number of a complete graph is along identical lines to that presented by Erdős [12]. This result can also be found in [10]. Throughout this paper, log is to the base 2 while ln denotes the logarithm in the natural base e.

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Theorem 1 (*Erdős [12]*). *For $n \geq 2$, $\log n - 2\log(\log n) \leq \Gamma_d(K_n) \leq \log(n+1)$.*

In [11] this notion of directed domination in a complete graph is extended to directed domination of all graphs. In this paper we prove a Greedy Partition Lemma for directed domination in oriented graphs. Applying this lemma, we obtain bounds on the directed domination number. In particular, if α denotes the independence number of a graph G, we show that $\alpha \leq \Gamma_d(G) \leq \alpha(1 + 2\ln(n/\alpha))$.

1.1. Notation

For notation and graph theory terminologies we in general follow [13]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. If A and B are subsets of $V(G)$, we let $[A, B]$ denote the set of all edges between A and B in G.

We denote the *degree* of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The average degree in G is denoted by $d_{\text{av}}(G)$. The minimum degree among the vertices of G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. The parameter $\gamma(G)$ denotes the *domination number* of G. The parameters $\alpha(G)$ and $\alpha'(G)$ denote the (vertex) *independence number* and the *matching number*, respectively, of G, while the parameters $\chi(G)$ and $\chi'(G)$ denote the *chromatic number* and *edge chromatic number*, respectively, of G. The *covering number* of G, denoted by $\beta(G)$, is the minimum number of vertices that covers all the edges of G.

A vertex v in a digraph D *out-dominates*, or simply *dominates*, itself as well as all vertices u such that (v, u) is an arc of D. The *out-neighborhood* of v, denoted $N^+(v)$, is the set of all vertices u adjacent from v in D; that is, $N^+(v) = \{u \mid (v, u) \in A(D)\}$. The *out-degree* of v is given by $d^+(v) = |N^+(v)|$, and the maximum out-degree among the vertices of D is denoted by $\Delta^+(D)$. The *in-neighborhood* of v, denoted $N^-(v)$, is the set of all vertices u adjacent to v in D; that is, $N^-(v) = \{u \mid (u, v) \in A(D)\}$. The *in-degree* of v is given by $d^-(v) = |N^-(v)|$. The *closed in-neighborhood* of v is the set $N^-[v] = N^-(v) \cup \{v\}$. The maximum in-degree among the vertices of D is denoted by $\Delta^-(D)$.

1.2. Known results

We shall need the following inequality chain established in [11].

Theorem 2 (*[11]*). *For every graph G on n vertices, $\gamma(G) \leq \alpha(G) \leq \Gamma_d(G) \leq n - \alpha'(G)$.*

2. The Greedy Partition Lemma and its applications

In this section we present our key lemma, which we call the Greedy Partition Lemma, and apply it to obtain several upper bounds on the directed domination number of a graph. In particular, using the Greedy Partition Lemma we present an upper bound on the directed domination number of a graph in terms of its independence number and we establish an upper bound on the directed domination number of a graph whose complement is d-degenerate. The Greedy Partition Lemma is a generalization of earlier results by Caro [14,15], Caro and Tuza [16], and Jensen and Toft [17].

First we introduce some additional terminologies. Let G be a hypergraph and let P be a hypergraph property. Let $P(G) = \max\{|V(H)| : H \text{ is an induced subhypergraph of } G \text{ that satisfies property } P\}$. Let $\chi(G, P)$ be the minimum number q such that there exists a partition $V(G) = (V_1, V_2, \ldots, V_q)$ such that V_i induces a subhypergraph having property P for all $i = 1, 2, \ldots, q$. For example, if P is the property of independence, then $P(G) = \alpha(G)$, while $\chi(G, P) = \chi(G)$. If P is the property of edge independence, $P(G) = \alpha'(G)$, while $\chi(G, P) = \chi'(G)$. If P is the property of being d-degenerate (recall that a d-degenerate graph is a graph G in which every induced subgraph of G has a vertex with degree at most d), then $P(G)$ is the maximum cardinality of a d-degenerate subgraph and $\chi(G, P)$ is the minimum partition of $V(G)$ into induced d-degenerate graphs. For a subhypergraph H of a hypergraph G, we let $G - H$ be the subhypergraph of G with vertex set $V(G) \setminus V(H)$. We are now in a position to state the Greedy Partition Lemma.

Lemma 3 (*Greedy Partition Lemma*). *Let $\mathcal{H}$ be a class of hypergraphs closed under induced subhypergraphs. Let $t \geq 2$ be an integer and let $f : [t, \infty) \to [1, \infty)$ be a positive nondecreasing continuous function. Let P be a hypergraph property such that for every hypergraph $G \in \mathcal{H}$ the following holds.*

(a) *If $|V(G)| \leq t$, then $\chi(G, P) \leq |V(G)|$.*
(b) *If $|V(G)| \geq t$, then $P(G) \geq f(|V(G)|)$.*

Then for every hypergraph $G \in \mathcal{H}$ of order n,

$$\chi(G, P) \leq t + \int_t^{\max(n,t)} \frac{1}{f(x)}\, dx.$$

Proof. We proceed by induction on n. We first observe that the value of the given integral is always non-negative. If $n \leq t$, then by condition (a), $\chi(G, P) \leq n \leq t$, and the inequality holds trivially. This establishes the base case. For the inductive hypothesis, assume that the inequality holds for every hypergraph in $\mathcal{H}$ with less than n vertices and let $G \in \mathcal{H}$ be of order n.

As observed earlier, if $n \le t$, then the inequality holds trivially. Hence we may assume that $n > t$. Let $P(G) = z = |V(H)|$ be the cardinality of the largest induced subhypergraph H of G that has property P. By condition (b), $z \ge f(n)$. If $z \ge n - t + 1$, then $n - z = |V(G) \setminus V(H)| \le t - 1$, and so by condition (a), $\chi(G - H, P) \le t - 1$. Hence, $\chi(G, P) \le \chi(G - H, P) + 1 \le t$ and the inequality holds trivially. Therefore we may assume that $z \le n - t$, and so $|V(G) \setminus V(H)| \ge t$. Thus applying the inductive hypothesis to the induced subhypergraph $G - H \in \mathcal{H}$, and using condition (b), we have that

$$\begin{aligned}
\int_t^n \frac{1}{f(x)}\,dx &= \int_t^{n-z} \frac{1}{f(x)}\,dx + \int_{n-z}^n \frac{1}{f(x)}\,dx \\
&\ge \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(x)}\,dx \\
&\ge \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(n)}\,dx \\
&= \chi(G - H, P) - t + z/f(n) \\
&\ge \chi(G, P) - 1 - t + 1 \\
&\ge \chi(G, P) - t,
\end{aligned}$$

which completes the proof of the Greedy Partition Lemma. □

We next discuss several applications of the Greedy Partition Lemma. For this purpose, we recall the Caro–Wei Theorem (see [18,19]).

Theorem 4 (*Caro–Wei Theorem*). *For every graph G, $\alpha(G) \ge \sum_{v \in V(G)} \frac{1}{1+d_G(v)}$.*

We shall also need the following lemma.

Lemma 5. *For $k \ge 1$ an integer, let G be a graph with $k \ge \alpha(G)$ and let D be an orientation of G. Let H be an induced subgraph of G of order $n_H \ge k$ and size m_H, and let D_H be the orientation of H induced by D. Then the following holds.*

(a) $m_H \ge n_H(n_H - k)/2k$.
(b) $\Delta^+(D_H) \ge (n_H - k)/2k$.

Proof. Since H is an induced subgraph of G, every independent set in H is an independent set in G. In particular, $k \ge \alpha(G) \ge \alpha(H)$. Thus applying the Caro–Wei Theorem (see [18,19]), we have

$$k \ge \alpha(H) \ge \sum_{v \in V(H)} \frac{1}{d_H(v) + 1} \ge \frac{n_H}{d_{av}(H) + 1} = \frac{n_H}{(2m_H/n_H) + 1} = \frac{n_H^2}{2m_H + n_H},$$

or, equivalently, $m_H \ge n_H(n_H - k)/2k$. This establishes Part (a). Part (b) follows readily from Part (a) and the observation that

$$n_H \cdot \Delta^+(D_H) \ge \sum_{v \in V(D_H)} d^+_{D_H}(v) = m_H. \quad \square$$

2.1. Independence number

Using the Greedy Partition Lemma we present an upper bound on the directed domination number of a graph in terms of its independence number. First we introduce some additional notation. Let $\alpha \ge 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs G with $\alpha \ge \alpha(G)$. Since every induced subgraph F of $G \in \mathcal{G}_\alpha$ satisfies $\alpha \ge \alpha(G) \ge \alpha(F)$, the class $\mathcal{G}_\alpha$ of graphs is closed under induced subgraphs.

Theorem 6. *For $\alpha \ge 1$ an integer, if $G \in \mathcal{G}_\alpha$ has order $n \ge \alpha$, then*

$$\Gamma_d(G) \le \alpha(1 + 2\ln(n/\alpha)).$$

Proof. If $\alpha = 1$, then $G = K_n$ and by Theorem 1, $\Gamma_d(G) \le \log(n+1) \le 1 + 2\ln n = \alpha(1 + 2\ln(n/\alpha))$. Hence we may assume that $\alpha \ge 2$, for otherwise the desired bound holds. We now apply the Greedy Partition Lemma with $t = \alpha$ and with $f(x)$ the positive nondecreasing continuous function on $[\alpha, \infty)$ defined by $f(x) = (x - \alpha)/2\alpha + 1$ where $x \in [\alpha, \infty)$. Let $P(G) = 1 + \min\{\Delta^+(D)\}$, where the minimum is taken over all orientations D of G. Then, $\Gamma_d(G) \le \chi(G, P)$. To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph $H \in \mathcal{G}_\alpha$, where H has order $|V(H)| = n_H$. If $|V(H)| \le \alpha$, then $\Gamma_d(H) \le \chi(H, P) \le \alpha$ since in this case H may be the empty graph on α vertices. Thus condition (a) of Lemma 3 holds. If $|V(H)| \ge \alpha$ and D is an arbitrary orientation of H, then by Lemma 5, $\Delta^+(D) \ge (n_H - \alpha)/2\alpha$, and so $|V(H)| \geq P(H) \geq (n_H - \alpha)/2\alpha + 1 = f(n_H)$. Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,

$$\begin{aligned}\Gamma_d(G) &\leq \alpha + \int_\alpha^n \frac{1}{(x-\alpha)/2\alpha + 1}\,dx \\ &= \alpha + 2\alpha \int_\alpha^n \frac{1}{x+\alpha}\,dx \\ &= \alpha + 2\alpha \ln((n+\alpha)/2\alpha) \\ &\leq \alpha + 2\alpha \ln(n/\alpha) \\ &= \alpha(1 + 2\ln(n/\alpha)). \quad \square\end{aligned}$$

Observe that for every graph G of order n, we have $\chi(G) \geq n/\alpha(G)$ and $d_{\mathrm{av}}(G) + 1 \geq n/\alpha(G)$. Hence as an immediate consequence of Theorem 6, we have the following bounds on the directed domination number of a graph.

Corollary 1. *Let G be a graph of order n. Then the following hold.*
(a) $\Gamma_d(G) \leq \alpha(G)(1 + 2\ln(\chi(G)))$.
(b) $\Gamma_d(G) \leq \alpha(G)(1 + 2\ln(d_{\mathrm{av}}(G) + 1))$.

2.2. Degenerate graphs

A d-degenerate graph is a graph G in which every induced subgraph of G has a vertex with degree at most d. The property of being d-degenerate is a hereditary property that is closed under induced subgraphs, as is the property of the complement of a graph being d-degenerate. For $d \geq 1$ an integer, let $\mathcal{F}_d$ be the class of all graphs G whose complement is a d-degenerate graph. Thus the class $\mathcal{F}_d$ of graphs is closed under induced subgraphs. We shall need the following lemma.

Lemma 7. *For $d \geq 1$ an integer, let $G \in \mathcal{F}_d$ and let H be an induced subgraph of G of order n_H. If D is an orientation of G and D_H is the orientation of H induced by D, then $\Delta^+(D_H) > (n_H - 1)/2 - d$.*

Proof. Since $G \in \mathcal{F}_d$, the graph G is the complement of a d-degenerate graph $\overline{G}$. Let G have order n and size m, and let $\overline{G}$ have size $\overline{m}$. It is a well-known fact that we can label the vertices of the d-degenerate graph $\overline{G}$ with vertex labels $1, 2, \ldots, n$ such that each vertex with label i is incident to at most d vertices with label greater than i, implying that $\overline{m} \leq dn - d(d+1)/2$. Therefore, $m \geq n(n-1)/2 - dn + d(d+1)/2$. This is true for every graph G whose complement is a d-degenerate graph. In particular, this is true for the induced subgraph H of G. Therefore if H has size m_H, we have $\sum_{v \in V(H)} d^+_{D_H}(v) = m_H \geq n_H(n_H - 1)/2 - dn_H + d(d+1)/2$. Hence, $\Delta^+(D_H) > (n_H - 1)/2 - d$. $\square$

Theorem 8. *For $d \geq 1$ an integer, if $G \in \mathcal{F}_d$ has order n, then*

$$\Gamma_d(G) \leq 2d + 1 + 2\ln(n - 2d + 1)/2.$$

Proof. We apply the Greedy Partition Lemma with $t = 2d + 1$ and $f(x) = (x-1)/2 - d + 1$ where $x \in [2d+1, \infty)$. Let $P(G) = 1 + \min\{\Delta^+(D)\}$, where the minimum is taken over all orientations D of G. Then, $\Gamma_d(G) \leq \chi(G, P)$. To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph $H \in \mathcal{F}_d$, where H has order $|V(H)| = n_H$. If $|V(H)| \leq 2d + 1$, then $\Gamma_d(H) \leq \chi(H, P) \leq 2d + 1$ since in this case H may be the empty graph on $2d + 1$ vertices. Thus condition (a) of Lemma 3 holds. If $|V(H)| \geq 2d + 1$ and D is an arbitrary orientation of H, then by Lemma 7, $\Delta^+(D) \geq (n_H - 1)/2 - d$, and so $|V(H)| \geq P(H) \geq (n_H - 1)/2 - d + 1 = f(n_H)$. Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,

$$\begin{aligned}\Gamma_d(G) &\leq 2d + 1 + \int_{2d+1}^n \frac{1}{(x-1)/2 - d + 1}\,dx \\ &= 2d + 1 + \int_{2d+1}^n \left(\frac{2}{x - 2d + 1}\right) dx \\ &= 2d + 1 + 2\int_2^{n-2d+1} \frac{1}{x}\,dx \\ &\leq 2d + 1 + 2\ln(n - 2d + 1)/2. \quad \square\end{aligned}$$

2.3. $K_{1,m}$-free graphs

In this section, we establish an upper bound on the directed domination number of a $K_{1,m}$-free graph, where a graph is F-free if it does not contain F as an induced subgraph. We first recall the well-known bound for the usual domination number γ, which was proved independently by Arnautov in 1974 and in 1975 by Lovász and by Payan.

Theorem 9 (*Arnautov [20], Lovász [21], Payan [22]*). *If G is a graph on n vertices with minimum degree δ, then $\gamma(G) \le n(\log(\delta+1)+1)/(\delta+1)$.*

We show that the above bound on γ is nearly preserved by the directed domination number Γ_d when we restrict our attention to $K_{1,m}$-free graphs. For this purpose, we shall need the following result due to Faudree et al. [23].

Theorem 10 (*[23]*). *If G is a $K_{1,m}$-free graph of order n with $\delta(G)=\delta$ and $\alpha(G)=\alpha$, then $\alpha \le (m-1)n/(\delta+m-1)$.*

We shall prove the following result.

Theorem 11. *For $m \ge 3$, if G is a $K_{1,m}$-free graph of order n with $\delta(G)=\delta$, then*

$$\Gamma_d(G) < (2(m-1)n\ln(\delta+m-1))/(\delta+m-1).$$

Proof. If $\delta < (\sqrt{e}-1)(m-1)$, where e is the base of the natural logarithm, then $\delta < m-1$ and so $(2(m-1)n\ln(\delta+m-1))/(\delta+m-1) > n\ln(\delta+m-1) > n$. Hence we may assume that $\delta \ge (\sqrt{e}-1)(m-1)$, for otherwise the desired upper bound holds trivially. By Theorem 10, $\alpha \le (m-1)n/(\delta+m-1)$. Substituting $\delta \ge (\sqrt{e}-1)(m-1)$ into this inequality, we get $\alpha \le (m-1)n/((\sqrt{e}-1)(m-1)+m-1) = (m-1)n/(\sqrt{e}(m-1)) = n/\sqrt{e}$. Since the function $x(1+2\ln(n/x))$ is monotone increasing in the interval $[1, n/\sqrt{e}\,]$, we get, by Theorem 6, that

$$\begin{aligned}\Gamma_d(G) &\le \alpha(1+2\ln(n/\alpha))\\ &\le ((m-1)n/(\delta+m-1))(1+2\ln(n(\delta+m-1)/(m-1)n))\\ &= ((m-1)n/(\delta+m-1))(1+2\ln((\delta+m-1)/(m-1)))\\ &= 2(m-1)n(1/2+\ln((\delta+m-1)/(m-1)))/(\delta+m-1)\\ &= 2(m-1)n(\ln\sqrt{e}+\ln((\delta+m-1)/(m-1)))/(\delta+m-1)\\ &< (2(m-1)n\ln(\delta+m-1))/(\delta+m-1),\end{aligned}$$

as $\sqrt{e} < m-1$. □

We observe that as a special case of Theorem 11, we have that if G is a claw-free graph of order n with $\delta(G)=\delta$, then $\Gamma_d(G) \le (4n(\log(\delta+2)))/(\delta+2)$.

2.4. Nordhaus–Gaddum-type bounds

In this section we consider Nordhaus–Gaddum-type bounds for the directed domination of a graph. Let $\mathcal{G}_n$ denote the family of all graphs of order n. We define

$$\mathrm{NG}_{\min}(n) = \min\{\Gamma_d(G)+\Gamma_d(\overline{G})\}$$
$$\mathrm{NG}_{\max}(n) = \max\{\Gamma_d(G)+\Gamma_d(\overline{G})\}$$

where the minimum and maximum are taken over all graphs $G \in \mathcal{G}_n$. Chartrand and Schuster [24] established the following Nordhaus–Gaddum inequalities for the matching number: If G is a graph on n vertices, then $\lfloor n/2\rfloor \le \alpha'(G)+\alpha'(\overline{G}) \le 2\lfloor n/2\rfloor$.

Theorem 12. *The following holds.*

(a) $c_1\log n \le \mathrm{NG}_{\min}(n) \le c_2(\log n)^2$ *for some constants c_1 and c_2.*
(b) $n+\log n-2\log(\log n) \le \mathrm{NG}_{\max}(n) \le n+\lceil n/2\rceil$.

Proof. (a) By Ramsey's theory, for all graphs $G \in \mathcal{G}_n$ we have $\max\{\alpha(G),\alpha(\overline{G})\} \ge c\log n$ for some constant c. Hence by Theorem 2(a), $\Gamma_d(G)+\Gamma_d(\overline{G}) \ge \alpha(G)+\alpha(\overline{G}) \ge c_1\log n$ for some constant c_1. Further by Ramsey's theory there exists a graph $G \in \mathcal{G}_n$ such that $\max\{\alpha(G),\alpha(\overline{G})\} \le d\log n$ for some constant d. Hence by Theorem 6, $\Gamma_d(G)+\Gamma_d(\overline{G}) \le 2d\log n(1+2\log(n/d\log n)) \le c_2(\log n)^2$ for some constant c_2. This establishes Part (a).

(b) By Theorem 1, $\Gamma_d(K_n)+\Gamma_d(\overline{K}_n) \le n+\log n-2\log(\log n)$. Hence, $\mathrm{NG}_{\max}(n) \ge n+\log n-2\log(\log n)$. By Theorem 2(b) and by the Nordhaus–Gaddum inequalities for the matching number, we have that $\Gamma_d(G)+\Gamma_d(\overline{G}) \le 2n-(\alpha'(G)+\alpha'(\overline{G})) \le 2n-\lfloor n/2\rfloor = n+\lceil n/2\rceil$. □

3. Two generalizations

In this section, we present two general frameworks of directed domination in graphs.

3.1. Directed multiple domination

For an integer $r \ge 1$, a *directed r-dominating set*, abbreviated DrDS, in a directed graph $D=(V,A)$ is a set S of vertices of V such that for every vertex $u \in V\setminus S$, there are at least r vertices v in S with v directed to u. The *directed r-domination number* of a *directed graph D*, denoted by $\gamma_r(D)$, is the minimum cardinality of a DrDS in D. A DrDS of D of cardinality $\gamma_r(D)$

is called a $\gamma_r(D)$-set. The *directed r-domination number* of a *graph G*, denoted $\Gamma_{d,r}(G)$, is defined as the maximum directed r-domination number $\gamma_r(D)$ over all orientations D of G; that is, $\Gamma_{d,r}(G) = \max\{\gamma_r(D)\}$, where the maximum is taken over all orientations D of G. In particular, we note that $\Gamma_d(G) = \Gamma_{d,1}(G)$.

Theorem 13. *Let $r \geq 1$ be an integer. Let G be a graph of order n with $\alpha(G) = \alpha$. Then the following hold.*

(a) $\Gamma_{d,r}(K_n) \leq r \log(n+1)$.
(b) $\Gamma_{d,r}(G) \leq r\alpha(1 + 2\ln(n/\alpha))$.

Proof. (a) By Theorem 1, $\Gamma_d(K_n) \leq \log(n+1)$. Let D_1 be an orientation of K_n and let S_1 be a $\gamma(D_1)$-set. Then, $|S_1| \leq \log(n+1)$. We now remove the vertices of the DDS S_1 from D_1 to produce an orientation D_2 of K_{n_1} where $n_1 = n - |S|$. Let S_2 be a $\gamma(D_2)$-set. By Theorem 1, $|S_2| \leq \log(n_1+1) < \log(n+1)$. We now remove the vertices of the DDS S_2 from D_2 to produce an orientation D_3 of K_{n_2} where $n_3 = n - |S_1| - |S_2|$ and we let S_3 be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2, \ldots, S_r$ of sets whose union is a DrDS of K_n of cardinality $\sum_{i=1}^{r} |S_i| \leq r\log(n+1)$. This is true for every orientation D of K_n. Hence, $\Gamma_{d,r}(K_n) \leq r\log(n+1)$. This establishes Part (a).

(b) By Theorem 6, $\Gamma_d(G) \leq \alpha(1 + 2\ln(n/\alpha))$. We first consider the case when $\alpha \geq n/\sqrt{e}$. Then, $r\alpha(1 + 2\ln(n/\alpha)) > n$ for $r = 2$. However the function $x(1 + 2\ln(n/x))$ is monotone increasing in the interval $[1, n/\sqrt{e}\,]$ and we may therefore assume that $\alpha \leq n/\sqrt{e}$, for otherwise the desired result holds trivially.

Let D_1 be an arbitrary orientation of G and let S_1 be a DDS of G. We now remove the vertices of S_1 from D_1 to produce an orientation D_2 of the graph $G_1 = G - S_1$ where G_1 has order $n_1 = n - |S|$. Let $\alpha(G_1) = \alpha_1$. Since G_1 is an induced subgraph of G, we have $\alpha_1 \leq \alpha$. By Theorem 6, $\Gamma_d(G_1) \leq \alpha_1(1 + 2\ln(n_1/\alpha_1)) < \alpha_1(1 + 2\ln(n/\alpha_1))$. Since $\alpha_1 \leq \alpha \leq n/\sqrt{e}$, the monotonicity of the function $x(1+2\ln(n/x))$ in the interval $[1, n/\sqrt{e}\,]$ implies that $\alpha_1(1+2\ln(n/\alpha_1)) \leq \alpha(1+2\ln(n/\alpha))$. Hence, $\Gamma_d(G_1) < \alpha(1 + 2\ln(n/\alpha))$.

Let S_2 be a $\gamma(D_2)$-set, and so $|S_2| < \alpha(1 + 2\ln(n/\alpha))$. We now remove the vertices of the DDS S_2 from D_2 to produce an orientation D_3 of $G_2 = G_1 - S_2$ where $n_2 = n - |S_1| - |S_2|$ and we let S_3 be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2, \ldots, S_r$ of sets whose union is a DrDS of G of cardinality $\sum_{i=1}^{r} |S_i| \leq r\alpha(1 + 2\ln(n/\alpha))$. This is true for every orientation D of G. Hence, $\Gamma_{d,r}(G) \leq r\alpha(1 + 2\ln(n/\alpha))$. This establishes Part (b). □

3.2. Directed distance domination

Let $D = (V, A)$ be a directed graph. The distance $d_D(u, v)$ from a vertex u to a vertex v in D is the number of edges on a shortest directed path from u to v. For an integer $d \geq 1$, a *directed d-distance dominating set*, abbreviated DdDDS, in D is a set U of vertices of V such that for every vertex $v \in V \setminus U$, there is a vertex $u \in U$ with $d_D(u, v) \leq d$. The *directed d-distance domination number* of a *directed graph D*, denoted by $\gamma(D, d)$, is the minimum cardinality of a DdDDS in D. The *directed d-distance domination number* of a *graph G*, denoted $\Gamma_d(G, d)$, is defined as the maximum directed d-distance domination number $\gamma_d(D, d)$ over all orientations D of G; that is, $\Gamma_d(G, d) = \max\{\gamma(D, d)\}$, where the maximum is taken over all orientations D of G. In particular, we note that $\Gamma_d(G) = \Gamma_d(G, 1)$.

An independent set U of vertices in D is called a *semi-kernel* of D if for every vertex $v \in V(D) \setminus U$, there is a vertex $u \in U$ such that $d_D(u, v) \leq 2$. For the proof of our next result we will use the following theorem due to Chvátal and Lovász [25].

Theorem 14 (*Chvátal, Lovász [25]*)**.** *Every directed graph contains a semi-kernel.*

Theorem 15. *For every integer $d \geq 2$, $\gamma_d(G, d) = \alpha(G)$.*

Proof. Let S be a maximum independent set in G and let D be an orientation obtained from G by directing all edges in $[S, V \setminus S]$ from S to $V \setminus S$ and directing all other edges arbitrarily. Every directed d-distance dominating set must contain S since no vertex of S is reachable in D from any other vertex of $V(D)$. Hence, $\Gamma_d(G, d) \geq |S| = \alpha(G)$. However if D^* is an arbitrary orientation of the graph G, then by Theorem 14 the oriented graph D^* has a semi-kernel S^*. Thus, $\gamma(D, d) \leq |S^*| \leq \alpha(G)$. Since this is true for every orientation of G, we have that $\Gamma_d(G, d) \leq \alpha(G)$. Consequently, $\gamma_d(G, d) = \alpha(G)$. □

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