JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **121**, 449–464 (1987)

Nonlinear Hyperbolic Systems with Nonlocal Boundary Conditions

EUGENIO SINESTRARI

Dipartimento di Matematica, Università di Roma, P. Aldo Moro 2, Rome, Italy

AND

G. F. WEBB*

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37235

Submitted by Kenneth L. Cooke

Received August 16, 1985

A system of nonlinear hyperbolic equations with boundary conditions of renewal type is studied as a general mathematical model for structured biological populations.

1. INTRODUCTION

The mathematical models of structured population dynamics give rise to first order hyperbolic partial differential equations with nonlocal boundary conditions. In these models one identifies one or more characteristics of individuals which are important in the development of the population as a whole. Such properties as age, size, DNA content, nutritional state, and other physical quantities may be used as structure variables. If the influences of crowding and resource limitation are present, then those models involve nonlinear equations. If the interactions of population subclasses are present, then these models involve systems of equations. It is our purpose here to investigate systems of nonlinear first order hyperbolic partial differential equations with nonlocal boundary conditions applicable to such models. In these models one seeks a density function for the population, and it is therefore natural to recast the partial differential equations that

* G. F. Webb gratefully acknowledges the support of CNR of Italy for a Visiting Professorship at the University of Rome in the Spring of 1982.

0022-247X/87 $3.00

model the population dynamics as integral equations: the integral problem is obtained by integration along the characteristic curves of the differential problem. Sufficiently regular solutions of the differential problem will always be solutions of the integral problem. In [11, 12, 16] general models of populations structured by a single internal variable have been studied from the viewpoint of nonlinear integral equations. In our treatment here we will formulate a general model of a population structured by several internal variables as a nonlinear integral equation and study the existence, uniqueness, and positivity of its solutions.

2. The General Problem

The general problem we study is motivated in the following way. Consider a population of individuals structured by two internal variables a and b. There is a density function $n(a, b, t)$ (which may be vector-valued) whose integral $\int_0^\infty \int_0^\infty n(a, b, t)\, da\, db$ gives the total population at time t. For each $t \geqslant 0$ the density $n(\cdot, \cdot, t)$ lies in a space X of $\mathbb{R}^N$-valued functions on $(0, \infty) \times (0, \infty)$.

The density satisfies the so-called balance law of the population, which accounts for such processes as mortality, migration, or movement into population subclasses. If the density is sufficiently regular, then the balance law may be written in differential form as

$$\begin{aligned} &n_t(a, b, t) + n_a(a, b, t) + n_b(a, b, t) \\ &\quad = H(n(\cdot, \cdot, t))(a, b), \qquad a \geqslant 0, \quad b \geqslant 0, \quad t \geqslant 0 \end{aligned} \tag{2.1}$$

where H is a (nonlinear) operator from X to X. The density also satisfies the so-called birth laws of the population, which account for the input of neonates. If the density is sufficiently regular, then the birth laws may be written as the nonlocal boundary conditions

$$n(a, 0, t) = F(n(\cdot, \cdot, t))(a), \qquad a \geqslant 0, \quad t > 0 \tag{2.2}$$

$$n(0, b, t) = G(n(\cdot, \cdot, t))(b), \qquad b \geqslant 0, \quad t > 0, \tag{2.3}$$

where F and G are (nonlinear) operators from X to a space of $\mathbb{R}^N$-valued functions with independent variables a and b, respectively. Lastly, the density satisfies the initial condition of the population

$$n(a, b, 0) = \phi(a, b), \qquad a \geqslant 0, \quad b \geqslant 0, \tag{2.4}$$

where $\phi \in X$ is prescribed.

For both physical and mathematical reasons it is useful to choose the space $X = L^1((0, \infty) \times (0, \infty); \mathbb{R}^N)$ with norm

$$\|\phi\| = \sum_{i=1}^{N} \int_0^\infty \int_0^\infty |\phi_i(a, b)| \, da \, db, \qquad \phi \in X.$$

In this setting the density $n(a, b, t)$ makes sense only almost everywhere in the variables a and b. Consequently, it is necessary to make a more general formulation of the balance and birth laws. In the formulation we employ, the density is the solution of an integral equation in the three variables a, b, and t. To see the connection of this integral equation to the differential problem (2.1)–(2.4) we will integrate along the characteristic curves of Eq. (2.1).

Let $L^1 = L^1((0, \infty); \mathbb{R}^N)$ with norm

$$\|f\| = \sum_{i=1}^{N} \int_0^\infty |f_i(a)| \, da, \qquad f \in L^1.$$

Let $H\colon X \to X$, let $F, G\colon X \to L^1$, and let $\phi \in X$. We suppose that n, H, F, G, and ϕ are all sufficiently regular so that n satisfies (2.1)–(2.4) and the formal calculations below are valid. Fix a_0, b_0, and t_0 and define

$$w(\xi) = n(a_0 + \xi, b_0 + \xi, t_0 + \xi), \qquad \xi \geqslant 0. \tag{2.5}$$

From (2.1) we obtain

$$w'(\xi) = H(n(\cdot, \cdot, t_0 + \xi))(a_0 + \xi, b_0 + \xi)$$

so that

$$w(\xi) = w(0) + \int_0^\xi H(n(\cdot, \cdot, t_0 + \hat{\xi}))(a_0 + \hat{\xi}, b_0 + \hat{\xi}) \, d\hat{\xi}.$$

By using (2.5) and setting $t_0 + \hat{\xi} = \tau$ inside the integral we obtain

$$n(a_0 + \xi, b_0 + \xi, t_0 + \xi)$$
$$= n(a_0, b_0, t_0) + \int_{t_0}^{t_0 + \xi} H(n(\cdot, \cdot, \tau))(a_0 - t_0 + \tau, b_0 - t_0 + \tau) \, d\tau. \tag{2.6}$$

To obtain a representation formula for $n(a, b, t)$ we shall consider separately the three possible situations: (1) $0 \leqslant t$, $t \leqslant a$, $t \leqslant b$; (2) $0 \leqslant b$, $b \leqslant a$, $b < t$; and (3) $0 \leqslant a$, $a < b$, $a < t$. We substitute into (2.6) successively (1) $a_0 = a - t$, $b_0 = b - t$, $t_0 = 0$, $\xi = t$; (2) $a_0 = a - b$, $b_0 = 0$, $t_0 = t - b$, $\xi = b$; and (3) $a_0 = 0$, $b_0 = b - a$, $t_0 = t - a$, $\xi = a$, and then use (2.4), (2.2), and (2.3), respectively, to obtain that n satisfies the integral equation

$$
n(a,b,t)=\begin{cases}
\phi(a-t,b-t)+\int_0^t H(n(\cdot,\cdot,\tau))(a-t+\tau,b-t+\tau)\,d\tau, \\
\qquad\qquad t\geqslant 0,\quad a\geqslant t,\quad b\geqslant t \\
F(n(\cdot,\cdot,t-b))(a-b)+\int_{t-b}^t H(n(\cdot,\cdot,\tau))(a-t+\tau,b-t+\tau)\,d\tau, \\
\qquad\qquad t>0,\quad 0\leqslant b<t,\quad a\geqslant b \\
G(n(\cdot,\cdot,t-a))(b-a)+\int_{t-a}^t H(n(\cdot,\cdot,\tau))(a-t+\tau,b-t+\tau)\,d\tau, \\
\qquad\qquad t>0,\quad 0\leqslant a<t,\quad b>a.
\end{cases}
\tag{2.7}
$$

A solution of (2.7) on the interval $[0, T]$ is a function $t\to n(\cdot,\cdot,t)$ which is continuous from $[0, T]$ to X and which satisfies the conditions in (2.7) almost everywhere. Under the assumptions that we make on H in Section 4 the integrals in (2.7) exist.

The integral equation (2.7) may be viewed as the general formulation of a mathematical model of population growth structured by two internal variables subject to renewal conditions. In Section 4 we will prove that under simple continuity assumptions on F, G, and H, this equation has a unique positive solution for every positive initial function $\phi\in X$.

3. Examples

Before presenting the general theory we provide some examples of populations structured by internal variables. The first example is a single species structured population model due to Sinko and Streifer [14]. The density function $p(a, m, t)$ depends on time t, age a, and a variable m corresponding to mass or some other physical attribute of individuals. The differential problem for this model is

$$p_t+p_a+(g(m)\,p)_m=-\mu(a,m)\,p,\qquad a\geqslant 0,\quad m\geqslant 0,\quad t\geqslant 0 \tag{3.1}$$

$$p(0,m,t)=\int_0^\infty\int_0^\infty \beta(\hat{a},\hat{m},m)\,p(\hat{a},\hat{m},t)\,d\hat{m}\,d\hat{a},\qquad m\geqslant 0,\quad t>0 \tag{3.2}$$

$$p(a,0,t)=0,\qquad a>0,\quad t>0 \tag{3.3}$$

$$p(a,m,0)=\psi(a,m),\qquad a\geqslant 0,\quad m\geqslant 0. \tag{3.4}$$

The functions g, μ, and β are the growth, mortality, and birth moduli, respectively. The presence of the growth modulus in the balance law is necessary when m does not increase at the same rate as a. The time required for an individual to grow from m_1 to m_2 is $\int_{m_1}^{m_2} g(\hat{m})^{-1}\, d\hat{m}$.

Under appropriate assumptions on g, μ, and β, a change of variable converts the balance law (2.1) to (3.1). That is, let $z(m)=\int_0^m g(\hat{m})^{-1}\, d\hat{m}$, $m \geqslant 0$ and let

$$H(\phi)(a,b) = -[\mu(a, z^{-1}(b)) + g'(z^{-1}(b))]\, \phi(a,b), \qquad \phi \in X, \quad a \geqslant 0, \quad b \geqslant 0$$

$$F(\phi)=0, \qquad \phi \in X$$

$$G(\phi)(b) = \int_0^\infty \int_0^\infty \beta(\hat{a}, z^{-1}(\hat{b}), z^{-1}(b))\, g(z^{-1}(\hat{b}))\, \phi(\hat{a}, \hat{b})\, d\hat{a}\, d\hat{b}, \qquad \phi \in X, \quad b \geqslant 0$$

$$\phi(a,b) = \psi(a, z^{-1}(b)), \qquad a \geqslant 0, \quad b \geqslant 0.$$

Suppose that a solution $n(a, b, t)$ of (2.1)–(2.4) is obtained for these H, F, G, and ϕ. A simple calculation shows that if n is sufficiently regular, then $p(a, m, t) \equiv n(a, z(m), t)$ is a solution of (3.1)–(3.4).

The second example is a model of cell population growth due to Bell and Anderson [1]. The density $p(a, m, t)$ depends on time t, age a, and a structure variable m corresponding to mass, volume, or some other physically conserved quantity. The differential problem for this model is

$$p_t + p_a + (g(m)\,p)_m = -[\mu(a,m) + \beta(a,m)]\, p, \qquad a \geqslant 0, \quad m \geqslant 0, \quad t \geqslant 0, \tag{3.5}$$

$$p(0, m, t) = 4 \int_0^\infty \beta(a, 2m)\, p(a, 2m, t)\, da, \qquad m \geqslant 0, \quad t > 0 \tag{3.6}$$

$$p(a, 0, t) = 0, \qquad a > 0, \quad t > 0 \tag{3.7}$$

$$p(a, m, 0) = \psi(a, m), \qquad a \geqslant 0, \quad m \geqslant 0. \tag{3.8}$$

The functions g, μ, and β are the rates at which cells grow, die, and divide, respectively. The birth law (3.6) accounts for the fission of a mother cell into two daughter cells of equal mass. As before the change of variable $b \to z(m)$ transforms the problem (2.1)–(2.4) to the problem (3.5)–(3.8), where F and ϕ are as above and

$$H(\phi)(a,b) = -[\mu(a, z^{-1}(b)) + \beta(a, z^{-1}(b)) + g'(z^{-1}(b))]\, \phi(a,b), \qquad \phi \in X, \quad a \geqslant 0, \quad b \geqslant 0$$

$$G(\phi)(b) = 4 \int_0^\infty \beta(\hat{a}, 2z^{-1}(b))\, \phi(\hat{a}, z(2z^{-1}(b)))\, d\hat{a}, \qquad \phi \in X, \quad a \geqslant 0.$$

The third example is a model of an epidemic population incorporating both chronological and disease age. This model, which is similar to models considered by Hoppensteadt [7, 8] and Waltman [15], is of $S \to I \to R$ type. The population is divided into subclasses of susceptibles, infectives, and recovered individuals having density $S(a, t)$, $I(a, b, t)$, and $R(a, t)$, respectively. Here a represents chronological age and b represents disease age of infectives. The total population of infectives at time t is $\int_0^\infty \int_0^a I(a, b, t)\, db\, da$ ($I(a, b, t) = 0$ for $b > a$, since disease age cannot exceed chronological age).

The differential problem for this model is

$$S_t(a, t) + S_a(a, t) = -\mu_1(a)\, S(a, t) - \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, I(a', b', t)\, db'\, da' \right) S(a, t), \qquad a \geqslant 0, \quad t \geqslant 0 \tag{3.9}$$

$$I_t(a, b, t) + I_a(a, b, t) + I_b(a, b, t) = -[\mu_2(a, b) + \mu_3(a, b)]\, I(a, b, t), \qquad a \geqslant 0, \quad b \geqslant 0, \quad t \geqslant 0 \tag{3.10}$$

$$R_t(a, t) + R_a(a, t) = -\mu_4(a)\, R(a, t) + \int_0^a \mu_2(a, b)\, I(a, b, t)\, db, \qquad a \geqslant 0, \quad t \geqslant 0 \tag{3.11}$$

$$S(0, t) = \int_0^\infty [\beta_1(a)\, R(a, t) + \beta_2(a)\, S(a, t)]\, da, \qquad t > 0 \tag{3.12}$$

$$I(a, 0, t) = \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, I(a', b', t)\, db'\, da' \right) S(a, t), \qquad a \geqslant 0, \quad t \geqslant 0 \tag{3.13}$$

$$I(0, b, t) = 0, \qquad b > 0, \quad t > 0 \tag{3.14}$$

$$R(0, t) = 0, \qquad t > 0 \tag{3.15}$$

$$S(a, 0) = S_0(a), \qquad I(a, b, 0) = I_0(a, b), \qquad R(a, 0) = R_0(a), \qquad a \geqslant 0, \quad b \geqslant 0. \tag{3.16}$$

The functions μ_1, μ_3, and μ_4 are the age-specific mortality rates of the susceptible, infective, and recovered classes, respectively, μ_2 is the age-specific rate at which infectives recover, β_1 and β_2 are the age-specific fertility rates of the recovered and susceptible classes, respectively, ρ is the age-specific infection rate, and the birth rates of the infective and recovered classes are 0.

The problem (3.9)–(3.16) may be written as a nonlinear vector system in the form (2.1)–(2.4). Let $X = L^1((0, \infty) \times (0, \infty); \mathbb{R}^3)$ and define for $[\phi_1, \phi_2, \phi_3] \in X$ and $a, b \geqslant 0$:

$$H([\phi_1, \phi_2, \phi_3])(a, b)$$

$$= \Big[-\mu_1(a)\, \phi_1(a, b) - \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, \phi_2(a', b')\, db'\, da' \right) \phi_1(a, b),$$

$$-(\mu_2(a, b) + \mu_3(a, b))\, \phi_2(a, b),\ -\mu_4(a)\, \phi_3(a, b)$$

$$+ e^{-b} \int_0^a \mu_2(a, b')\, \phi_2(a, b')\, db' \Big] \tag{3.17}$$

$$F([\phi_1, \phi_2, \phi_3])(a)$$

$$= \left[0, \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, \phi_2(a', b')\, db'\, da' \right) \int_0^\infty \phi_1(a, b)\, db, 0 \right] \tag{3.18}$$

$$G([\phi_1, \phi_2, \phi_3])(b)$$

$$= \left[e^{-b} \int_0^\infty \left(\beta_1(a) \int_0^\infty \phi_3(a, b')\, db' + \beta_2(a) \int_0^\infty \phi_1(a, b')\, db' \right) da, 0, 0 \right] \tag{3.19}$$

$$\phi(a, b) = [e^{-b} S_0(a), I_0(a, b), e^{-b} R_0(a)]. \tag{3.20}$$

Suppose that $n(a, b, t)$ is a solution of (2.1)–(2.4) for this H, F, G and ϕ and let $n(\cdot, \cdot, t) = [s(\cdot, \cdot, t), \ I(\cdot, \cdot, t), \ r(\cdot, \cdot, t)]$. Define $S(a, t) = \int_0^\infty s(a, b, t)\, db$ and $R(a, t) = \int_0^\infty r(a, b, t)\, db$. We will show that $[S, I, R]$ satisfies (3.9)–(3.16) provided that n is of sufficient regularity so that the calculations below are valid. From (2.1) and (3.17) we obtain

$$S_t(a, t) + S_a(a, t) = \int_0^\infty (s_t(a, b, t) + s_a(a, b, t))\, db$$

$$= -\int_0^\infty \Big[s_b(a, b, t) + \mu_1(a)\, s(a, b, t)$$

$$+ \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, I(a', b', t)\, db'\, da' \right) s(a, b, t) \Big]\, db$$

$$= s(a, 0, t) - \lim_{b \to \infty} s(a, b, t) - \mu_1(a)\, S(a, t)$$

$$- \left(\int_0^\infty \int_0^{a'} \rho(a, a', b')\, I(a', b', t)\, db'\, da' \right) S(a, t).$$

From (2.2) and (3.18) we see that $s(a, 0, t) = F(n(\cdot, \cdot, t))_1(a) = 0$. Since s is assumed sufficiently regular, $\lim_{b \to \infty} s(a, b, t) = 0$. Thus, (3.9) is satisfied, and similar calculations show that (3.10) and (3.11) are satisfied. From (2.3) and (3.19) we see that

$$S(0, t) = \int_0^\infty s(0, b, t)\, db = \int_0^\infty G(n(\cdot, \cdot, t))_1(b)\, db$$

$$= \int_0^\infty e^{-b} \int_0^\infty \left[\beta_1(a) \int_0^\infty r(a, b', t)\, db' + \beta_2(a) \int_0^\infty s(a, b', t)\, db' \right] da\, db$$

$$= \int_0^\infty [\beta_1(a)\, R(a, t) + \beta_2(a)\, S(a, t)]\, da$$

so that (3.12) is satisfied. From (3.18) we obtain (3.13), from (3.19) we obtain (3.14) and (3.15), and from (3.20) we obtain (3.16).

The general formulation of an epidemic population structured by chronological and disease age is given by the integral equation (2.7) with H, F, G, and ϕ defined as in (3.17), (3.18), (3.19), and (3.20), respectively. Notice that if $I_0(a, b) = 0$ for $b > a$, then $I(a, b, t) = 0$ for $b > a$ and all $t > 0$. To see this claim set

$$i(\xi) = I(a_0 + \xi, b_0 + \xi, t_0 + \xi)$$

and $\mu = \mu_2 + \mu_3$. From (3.10) we obtain

$$i(\xi) = i(0) \exp\left[-\int_0^\xi \mu(a_0 + \xi', b_0 + \xi')\, d\xi' \right].$$

If $t > a$ and $b > a$, set $a_0 = 0$, $b_0 = b - a$, $t_0 = t - a$, and $\xi = a$ to obtain

$$I(a, b, t) = I(0, b - a, t - a) \exp\left[-\int_0^a \mu(\xi', b - a + \xi')\, d\xi' \right].$$

Now use (3.14) to conclude that $I(a, b, t) = 0$ in this case. If $t \leqslant a < b$, set $a_0 = a - t$, $b_0 = b - t$, $t_0 = 0$, and $\xi = t$ to obtain

$$I(a, b, t) = I(a - t, b - t, 0) \exp\left[-\int_0^t \mu(a - t + \xi', b - t + \xi')\, d\xi' \right].$$

The assumption that $I_0(a, b) = 0$ for $a < b$ implies that $I(a, b, t) = 0$ in this case as well.

The three examples above, in differential form, provide applications of the general theory we will develop for the integral equation (2.7). More extensive investigations of these examples with emphasis upon applications will be given in future works of the authors.

4. Existence, Uniqueness, and Positivity of Solutions

We require the following hypotheses on H, F, and G:

$H: X \to X$, $H(0)=0$, and there exists an increasing function $c_1: [0,\infty) \to [0,\infty)$ such that $\|H(\phi_1)-H(\phi_2)\| \leqslant c_1(r)\,\|\phi_1-\phi_2\|$ for all $\phi_1, \phi_2 \in X$ such that $\|\phi_1\|, \|\phi_2\| \leqslant r$ (4.1)

$F: X \to L^1$, $F(0)=0$, and there exists an increasing function $c_2: [0,\infty) \to [0,\infty)$ such that $\|F(\phi_1)-F(\phi_2)\| \leqslant c_2(r)\,\|\phi_1-\phi_2\|$ for all $\phi_1, \phi_2 \in X$ such that $\|\phi_1\|, \|\phi_2\| \leqslant r$ (4.2)

$G: X \to L^1$, $G(0)=0$, and there exists an increasing function $c_3: [0,\infty) \to [0,\infty)$ such that $\|G(\phi_1)-G(\phi_2)\| \leqslant c_3(r)\,\|\phi_1-\phi_2\|$ for all $\phi_1, \phi_2 \in X$ such that $\|\phi_1\|, \|\phi_2\| \leqslant r$. (4.3)

PROPOSITION 4.1. *Let* (4.1), (4.2), (4.3) *hold and let* $r>0$. *There exists* $T>0$ *such that if* $\phi \in X$ *and* $\|\phi\| \leqslant r$, *then there exists a unique continuous function* $t \to n(\cdot,\cdot,t)$ *from* $[0,T]$ *to* X *such that* n *is a solution of* (2.7) *on* $[0,T]$.

Proof. Choose $T>0$ such that

$$T(c_1(2r)+c_2(2r)+c_3(2r))+\tfrac{1}{2} \leqslant 1. \tag{4.4}$$

Let $\phi \in X$ such that $\|\phi\| \leqslant r$. Define the closed subset M of $C([0,T]);X)$ by

$$M=\{n \in C([0,T];X)\colon n(\cdot,\cdot,0)=\phi \text{ and } \sup_{0 \leqslant t \leqslant T} \|n(\cdot,\cdot,t)\| \leqslant 2r\}.$$

Define a mapping K on M as follows: for $n \in M$, $t \in [0,T]$, $(Kn)(a,b,t)$ is given by the right-hand side of (2.7).

To see that the integrals in (2.7) exist, let $n \in M$ and let $0 \leqslant t \leqslant T$. Since $\tau \to H(n(\cdot,\cdot,\tau)) \in C([0,t];X)$, Theorem 17, p. 198 in [4] implies there exists a measurable function h from $[0,\infty)\times[0,\infty)\times[0,t]$ to $\mathbb{R}^N$ such that $h(a,b,\tau)=H(n(\cdot,\cdot,\tau))(a,b)$, a.e. $a \geqslant 0$, a.e. $b \geqslant 0$, and a.e. $\tau \in [0,t]$. By the Fubini–Tonelli theorem (see Theorem 13, p. 193 and Corollary 15, p. 194, in [4])

$$\begin{aligned}\int_0^t \|h(\cdot,\cdot,\tau)\|\,d\tau &= \int_0^t \int_0^\infty \int_0^\infty |h(a,b,\tau)|\,da\,db\,d\tau \\ &= \int_0^t \int_{-\tau}^\infty \int_{-\tau}^\infty |h(u+\tau, v+\tau, \tau)|\,du\,dv\,d\tau \\ &= \int_{-t}^\infty \int_{-t}^\infty \int_{\max\{-u,-v,0\}}^t |h(u+\tau, v+\tau, \tau)|\,d\tau\,du\,dv.\end{aligned}$$

Thus, the inner-most integral exists for a.e. $u>-t$, $v>-t$. Now set $u=a-t$ and $v=b-t$ in this integral to obtain the existence of the three integrals in (2.7).

We next show that for $n \in M$, $Kn(\cdot, \cdot, t) \in X$ for each $t \in [0, T]$ and $\sup_{t \in [0,T]} \|Kn(\cdot, \cdot, t)\| \leqslant 2r$. From (4.1), (4.2), and (4.3) we obtain that for each $t \in [0, T]$:

$$\int_0^\infty \int_0^\infty |Kn(a, b, t)|\, db\, da$$

$$= \int_0^t \int_b^\infty \bigg| F(n(\cdot, \cdot, t-b))(a-b)$$

$$+ \int_{t-b}^t H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)\, d\tau \bigg|\, da\, db$$

$$+ \int_0^t \int_a^\infty \bigg| G(n(\cdot, \cdot, t-a))(b-a)$$

$$+ \int_{t-a}^t H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)\, d\tau \bigg|\, db\, da$$

$$+ \int_t^\infty \int_t^\infty \bigg| \phi(a-t, b-t)$$

$$+ \int_0^t H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)\, d\tau \bigg|\, db\, da$$

$$\leqslant \int_0^t \int_0^\infty |F(n(\cdot, \cdot, t-b))(a)|\, da\, db$$

$$+ \int_0^t \int_{t-\tau}^t \int_b^\infty |H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)|\, da\, db\, d\tau$$

$$+ \int_0^t \int_0^\infty |G(n(\cdot, \cdot, t-a))(b)|\, db\, da$$

$$+ \int_0^t \left[\int_{t-\tau}^t \int_{t-\tau}^b + \int_t^\infty \int_{t-\tau}^t \right]$$

$$\times |H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)|\, da\, db\, d\tau$$

$$+ \int_0^\infty \int_0^\infty |\phi(a, b)|\, db\, da$$

$$+ \int_0^t \int_t^\infty \int_t^\infty |H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)|\, da\, db\, d\tau$$

$$\leqslant \int_0^t (c_2(2r) + c_3(2r))\, \|n(\cdot, \cdot, t-\tau)\|\, d\tau + \|\phi\|$$

$$+ \int_0^t \int_{t-\tau}^\infty \int_{t-\tau}^\infty |H(n(\cdot, \cdot, \tau))(a-t+\tau, b-t+\tau)|\, da\, db\, d\tau$$

$$\leqslant (c_2(2r)+c_3(2r)) \int_0^t 2r\, d\tau + \int_0^t c_1(2r)\, \|n(\cdot,\cdot,\tau)\|\, d\tau + r$$

$$\leqslant (T(c_1(2r)+c_2(2r)+c_3(2r)+\tfrac{1}{2}))\, 2r \leqslant 2r. \tag{4.5}$$

Thus, $Kn(\cdot,\cdot,t) \in X$ for each $t \in [0, T]$ and $\sup_{0 \leqslant t \leqslant T} \|Kn(\cdot,\cdot,t)\| \leqslant 2r$.

It must be shown that the function $t \to Kn(\cdot,\cdot,t)$ is continuous from $[0, T]$ to X. The proof of this fact follows from the continuity of F, G, and H, the continuity of the mapping $t \to n(\cdot,\cdot,t)$ from $[0, T]$ to X, and the continuity of translation in X. For example, let $r(t)$ be the function from $[0, T]$ to X defined by

$$r(t)(a,b) = \begin{cases} G(n(\cdot,\cdot,t-a))(b-a) & \text{a.e. } 0 \leqslant a \leqslant t, \text{ a.e. } b > a \\ 0, & a \geqslant t \text{ or } b \leqslant a. \end{cases}$$

For $0 \leqslant t < \hat{t} \leqslant T$

$$\begin{aligned} \|r(t)-r(\hat{t})\| &\leqslant \int_t^{\hat{t}} \int_a^\infty |G(n(\cdot,\cdot,\hat{t}-a))(b-a)|\, db\, da \\ &\quad + \int_0^t \int_a^\infty |G(n(\cdot,\cdot,t-a))(b-a) \\ &\quad - G(n(\cdot,\cdot,\hat{t}-a))(b-a)|\, db\, da \\ &\leqslant \int_0^{\hat{t}-t} \int_0^\infty |G(n(\cdot,\cdot,\hat{a}))(\hat{b})|\, d\hat{b}\, d\hat{a} \\ &\quad + \int_0^t \int_0^\infty |G(n(\cdot,\cdot,\hat{a})) - G(n(\cdot,\cdot,\hat{t}-t+\hat{a}))|(\hat{b})\, d\hat{b}\, d\hat{a} \end{aligned}$$

so that $\lim_{|t-\hat{t}| \to 0} \|r(t)-r(\hat{t})\| = 0$. Also, one can let g be the function from $[0, T]$ to X defined by

$$g(t)(a,b) = \int_{\max\{t-a,\, t-b,\, 0\}}^t H(n(\cdot,\cdot,\tau))(a-t+\tau, b-t+\tau)\, d\tau$$

$$\text{a.e. } a \geqslant 0, \text{ a.e. } b \geqslant 0$$

and show that $\lim_{|t-\hat{t}| \to 0} \|g(t)-g(\hat{t})\| = 0$. Similar assertions yield the claimed continuity of $Kn(\cdot,\cdot,t)$ in t. In conclusion we have shown that K maps M into M. We claim that (4.4) implies that K is a contraction in M. As in (4.5), for $n, \hat{n} \in M$, $t \in [0, T]$, (4.4) yields

$$\int_0^\infty \int_0^\infty |Kn(a,b,t) - K\hat{n}(a,b,t)|\, db\, da$$

$$\leqslant \int_0^t \int_0^\infty |F(n(\cdot,\cdot,t-b))(a) - F(\hat{n}(\cdot,\cdot,t-b))(a)|\, da\, db$$

$$+\int_0^t \int_0^\infty |G(n(\cdot,\cdot,t-a))(b) - G(\hat{n}(\cdot,\cdot,t-\hat{a}))(b)|\, db\, da$$

$$+\int_0^t \int_0^\infty \int_0^\infty |H(n(\cdot,\cdot,\tau))(a,b) - H(\hat{n}(\cdot,\cdot,\tau))(a,b)|\, da\, db\, d\tau$$

$$\leqslant (c_1(2r)+c_2(2r)+c_3(2r))\, T\, \|n-\hat{n}\|_{C([0,T];X)}$$

$$\leqslant \tfrac{1}{2}\, \|n-\hat{n}\|_{C([0,T];X)}.$$

Thus, by the Contraction Mapping Theorem there exists a unique $n \in M$ such that $Kn = n$, and, consequently, n is the unique solution of (2.7) on $[0, T]$. ∎

From the previous proposition one deduces by standard arguments the existence and uniqueness of a maximally defined solution of (2.7) for each $\phi \in X$. For each $\phi \in X$ let $[0, T_\phi)$ be the maximal interval of existence of the solution $n(\cdot,\cdot,t)$ of (2.7) and let $S(t)\,\phi = n(\cdot,\cdot,t)$ for $0 \leqslant t < T_\phi$. The following proposition establishes that the solutions of (2.7) satisfy the semigroup property.

PROPOSITION 4.2. *Let* (4.1), (4.2), (4.3) *hold, let* $\phi \in X$, *and let* $0 < s < T_\phi$. *Then* $T_{S(s)\phi} + s \leqslant T_\phi$ *and* $S(t)\, S(s)\, \phi = S(t+s)\, \phi$ *for* $0 \leqslant t \leqslant T_{S(s)\phi}$.

Proof. Define $g(\cdot,\cdot,t) = S(t)\,\phi$ for $0 \leqslant t \leqslant s$ and $g(\cdot,\cdot,t) = S(t-s)\, S(s)\, \phi$ for $s < t < T_{S(s)\phi} + s$. For $s < t < T_{S(s)\phi} + s$, a.e. $a \geqslant t$, a.e. $b \geqslant t$, $(2.7)_1$ yields

$$\begin{aligned} g(a,b,t) &= S(s)\,\phi(a-t+s, b-t+s) \\ &\quad + \int_0^{t-s} H(S(\tau)\, S(s)\, \phi)(a-t+s+\tau, b-t+s+\tau)\, d\tau \\ &= \phi(a-t, b-t) + \int_0^s H(S(\tau)\,\phi)(a-t+\tau, b-t+\tau)\, d\tau \\ &\quad + \int_s^t H(S(\hat{\tau}-s)\, S(s)\, \phi)(a-t+\hat{\tau}, b-t+\hat{\tau})\, d\hat{\tau} \\ &= \phi(a-t, b-t) + \int_0^t H(g(\cdot,\cdot,\tau))(a-t+\tau, b-t+\tau)\, d\tau. \end{aligned}$$

Thus $g(a,b,t)$ satisfies $(2.7)_1$ for $0 \leqslant t \leqslant T_{S(s)\phi} + s$ and a.e. $a \geqslant t$, a.e. $b \geqslant t$. In a similar fashion one shows that g satisfies $(2.7)_2$ for a.e. $0 \leqslant b < t$, a.e. $a \geqslant b$, and $(2.7)_3$ for a.e. $0 \leqslant a < t$, a.e. $b > a$. By the uniqueness of solutions to (2.7) we have $T_\phi \geqslant T_{S(s)\phi} + s$ and $g(\cdot,\cdot,t) = S(t)\,\phi$ for $0 \leqslant t < T_{S(s)\phi} + s$, that is, $S(t)\, S(s)\, \phi = S(t+s)\, \phi$ for $0 \leqslant t < T_{S(s)\phi}$. ∎

The next proposition establishes that if the solution of (2.7) remains bounded on its maximal interval of existence, then it exists globally.

PROPOSITION 4.3. *Let* (4.1), (4.2), (4.3) *hold and let* $\phi \in X$. *If* $T_\phi < \infty$, *then* $\limsup_{t \to T_\phi^-} \|S(t)\phi\| = \infty$.

Proof. Assume that $T_\phi < \infty$ and there exists $r > 0$ such that $\|S(t)\phi\| \leqslant r$ for $0 \leqslant t \leqslant T_\phi$. By Proposition 4.1 there exists $T \in (0, T_\phi)$ such that if $\hat{\phi} \in X$ and $\|\hat{\phi}\| \leqslant r$, then $T_{\hat{\phi}} \geqslant T$. Let $s = T_\phi - T/2$ and let $\hat{\phi} = S(s)\phi$. By Proposition 4.2, $T \leqslant T_{S(s)\phi} \leqslant T_\phi - s = T/2$, which yields a contradiction. ∎

The next proposition establishes that the solutions of (2.7) depend continuously on the initial values.

PROPOSITION 4.4. *Let* (4.1), (4.2), (4.3) *hold, let* $\phi, \hat{\phi} \in X$, *let* $T < \min\{T_\phi, T_{\hat{\phi}}\}$, *and let* $r = \max_{0 \leqslant t \leqslant T}\{\|S(t)\phi\|, \|S(t)\hat{\phi}\|\}$. *Then, for* $0 \leqslant t \leqslant T$, $\|S(t)\phi - S(t)\hat{\phi}\| \leqslant \|\phi - \hat{\phi}\| \exp[(c_1(r) + c_2(r) + c_3(r))\, t]$.

Proof. As in (4.5)

$$
\begin{aligned}
\|S(t)\phi - S(t)\hat{\phi}\| \leqslant & \int_0^t \int_0^\infty |F(S(t-b)\phi)(a) - F(S(t-b)\hat{\phi})(a)|\, da\, db \\
& + \int_0^t \int_0^\infty |G(S(t-a)\phi)(b) - G(S(t-a)\hat{\phi})(b)|\, db\, da \\
& + \int_0^t \int_0^\infty \int_0^\infty |H(S(\tau)\phi) - H(S(\tau)\hat{\phi})|\, da\, db\, d\tau \\
& + \int_0^\infty \int_0^\infty |\phi(a, b) - \hat{\phi}(a, b)|\, da\, db \\
\leqslant & (c_1(r) + c_2(r) + c_3(r)) \int_0^t \|S(\tau)\phi - S(\tau)\hat{\phi}\|\, d\tau + \|\phi - \hat{\phi}\|.
\end{aligned}
$$

The conclusion then follows by Gronwall's lemma. ∎

We next give sufficient criteria for the solutions of (2.7) to be positive-valued. Let $\mathbb{R}_+^N = \{x = [x_1, ..., x_N] \in \mathbb{R}^N : x_i \geqslant 0,\ i = 1, ..., N\}$. Let $L_+^1 = \{\phi \in L^1 : \phi(a) \in \mathbb{R}_+^N$ for a.e. $a \geqslant 0\}$. Let $X_+ = \{\phi \in X;\ \phi(a, b) \in \mathbb{R}_+^N$ for a.e. $a \geqslant 0$ and a.e. $b \geqslant 0\}$. We will require the following hypotheses:

There exists an increasing function $c_4 : [0, \infty) \to [0, \infty)$ such that if $r > 0$ and $\phi \in X_+$ with $\|\phi\| \leqslant r$, then $c_4(r)\phi + H(\phi) \in X_+$, (4.6)

$$F(X_+) \subset L_+^1 \quad \text{and} \quad G(X_+) \subset L_+^1. \tag{4.7}$$

PROPOSITION 4.5. *Let* (4.1), (4.2), (4.3), (4.6), (4.7) *hold and let* $\phi \in X_+$. *Then* $S(t)\,\phi \in X_+$ *for* $0 \leqslant t < T_\phi$.

Proof. Let $0 < T < T_\phi$ and let $r > 0$ such that $\|S(t)\,\phi\| \leqslant r$ for $0 \leqslant t \leqslant T$. Set $\alpha = c_4(r)$, where c_4 is as in (4.6). Let $0 \leqslant t \leqslant T$. For a.e. $a \geqslant t$ and a.e. $b \geqslant t$, we obtain from $(2.7)_1$ that

$$\begin{aligned}
&\alpha \int_0^t e^{-\alpha(t-\tau)} S(\tau)\,\phi(a-t+\tau, b-t+\tau)\,d\tau \\
&\quad = \alpha \int_0^t e^{-\alpha(t-\tau)} \phi(a-t, b-t)\,d\tau \\
&\qquad + \int_0^t \alpha e^{-\alpha(t-\tau)} \int_0^\tau H(S(\sigma)\,\phi)(a-t+\sigma, b-t+\sigma)\,d\sigma\,d\tau \\
&\quad = (1-e^{-\alpha t})\,\phi(a-t, b-t) \\
&\qquad + \int_0^t (1-e^{\alpha(t-\sigma)})\,H(S(\sigma)\,\phi)(a-t+\sigma, b-t+\sigma)\,d\sigma.
\end{aligned}$$

Again using $(2.7)_1$ we obtain

$$\begin{aligned}
&\int_0^t e^{-\alpha(t-\tau)}(\alpha+H)\,S(\tau)\,\phi(a-t+\tau, b-t+\tau)\,d\tau + e^{-\alpha t}\phi(a-t, b-t) \\
&\quad = \phi(a-t, b-t) + \int_0^t H(S(\tau)\,\phi)(a-t+\tau, b-t+\tau)\,d\tau = S(t)\,\phi(a, b).
\end{aligned}$$

In a similar way one uses $(2.7)_2$ (for a.e. $0 \leqslant b < t$ and a.e. $a \geqslant b$) and $(2.7)_3$ (for a.e. $0 \leqslant a < t$ and a.e. $b > a$) to obtain

$$S(t)\,\phi(a, b) = \begin{cases}
e^{-\alpha t}\phi(a-t, b-t) \\
\quad + \int_0^t e^{-\alpha(t-\tau)}(\alpha+H)\,S(\tau)\,\phi(a-t+\tau, b-t+\tau)\,d\tau, \\
\qquad\qquad t \geqslant 0, \text{ a.e. } a \geqslant t, \text{ a.e. } b \geqslant t \\
a^{-\alpha b}F(S(t-b)\,\phi)(a-b) \\
\quad + \int_{t-b}^t e^{-\alpha(t-\tau)}(\alpha+H)\,S(\tau)\,\phi(a-t+\tau, b-t+\tau)\,d\tau, \\
\qquad\qquad t > 0, \text{ a.e. } 0 \leqslant b < t, \text{ a.e. } a \geqslant b \\
e^{-\alpha a}G(S(t-a)\,\phi)(b-a) \\
\quad + \int_{t-a}^t e^{-\alpha(t-\tau)}(\alpha+H)\,S(\tau)\,\phi(a-t+\tau, b-t+\tau)\,d\tau, \\
\qquad\qquad t > 0, \text{ a.e. } 0 \leqslant a < t, \text{ a.e. } b > a.
\end{cases} \tag{4.8}$$

Thus, $S(t)\,\phi$ satisfies the integral equation (4.8) for $0 \leqslant t \leqslant T$. Under the hypotheses (4.6) and (4.7) an argument similar to the one used to prove Proposition 4.1 establishes that this integral equation has a unique solution in $C([0, T_1]; X_+)$ for some $T_1 \in (0, T]$. The uniquenes of the solution means that $S(t)\,\phi \in X_+$ for $0 \leqslant t \leqslant T_1$. Assume that there exists $t < T_\phi$ such that $S(t)\,\phi \notin X_+$. Let $s = \inf\{t < T_\phi : S(t)\,\phi \notin X_+\}$. Since X_+ is closed, $S(s)\,\phi \in X_+$. By the argument above there exists $T_1 > 0$ such that $S(t)\,S(s)\,\phi \in X_+$ for $0 \leqslant t \leqslant T_1$. But by Proposition 4.2, $S(t)\,S(s)\,\phi = S(t+s)\,\phi$ for $0 \leqslant t < T_{S(s)\phi}$, which contradicts the definition of s. ▮

Our last proposition provides a sufficient condition for positive solutions of (2.7) to be defined globally.

PROPOSITION 4.6. *Let* (4.1), (4.2), (4.3), (4.6), (4.7) *hold and let there exist* $c \in \mathbb{R}$ *such that for all* $\phi \in X_+$

$$\sum_{i=1}^{N} \left(\int_0^\infty F(\phi)_i(a)\, da + \int_0^\infty G(\phi)_i(b)\, db + \int_0^\infty \int_0^\infty H(\phi)_i(a, b)\, da\, db \right) \leqslant c\, \|\phi\|.$$

Then, $T_\phi = \infty$ *and* $\|S(t)\,\phi\| \leqslant e^{ct}\, \|\phi\|$, $t \geqslant 0$ *for all* $\phi \in X_+$.

Proof. Let $\phi \in X_+$ and define $V(t) = \|S(t)\,\phi\|$ for $0 \leqslant t < T_\phi$. For $0 \leqslant t$, $t + h < T_\phi$ we have, as in (4.5)

$$\lim_{h \to 0} h^{-1}(V(t+h) - V(t))$$

$$= \lim_{h \to 0} h^{-1} \sum_{i=1}^{N} \int_0^\infty \int_0^\infty [(S(t+h)\,\phi)_i(a, b) - (S(t)\,\phi)_i(a, b)]\, da\, db$$

$$= \lim_{h \to 0} h^{-1} \sum_{i=1}^{N} \left[\int_t^{t+h} \int_0^\infty F(S(\tau)\,\phi)_i(a)\, da\, d\tau \right.$$

$$\left. + \int_t^{t+h} \int_0^\infty G(S(\tau)\,\phi)_i(b)\, db\, d\tau + \int_t^{t+h} \int_0^\infty \int_0^\infty H(S(\tau)\,\phi)_i(a, b)\, da\, db \right]$$

$$= \sum_{i=1}^{N} \left[\int_0^\infty F(S(t)\,\phi)_i(a)\, da + \int_0^\infty G(S(t)\,\phi)_i(b)\, db \right.$$

$$\left. + \int_0^\infty \int_0^\infty H(S(t)\,\phi)_i(a, b)\, da\, db \right] \leqslant c V(t).$$

Thus $V'(t) \leqslant c V(t)$ and the conclusion follows by Proposition 4.3. ▮

References

1. G. I. Bell and E. C. Anderson, Cell growth and division. I. A mathematical model with applications to cell volume distributions in mammalian suspension culture, *Biophys. J.* **7** (1967), 329–351.
2. G. Di Blasio, A problem arising in the mathematical theory of epidemics, *in* "Nonlinear Phenomena in Mathematical Sciences," pp. 313–327, Academic Press, Orlando, Fla., 1982.
3. K. Dietz, Transmission and control of arbovirus diseases, *in* "Proc. SIMS Conference on Epidemics," Alta, Utah, 1974.
4. N. Dunford and J. T. Schwartz, "Linear Operators I," Interscience, New York, 1957.
5. R. E. Elderkin, Nonlinear, globally age-dependent population models: Some basic theory, *J. Math. Anal. Appl.* **108** (1985), 546–562.
6. J. C. Frauenthal, "Mathematical Modeling in Epidemiology," Springer-Verlag, Berlin, 1980.
7. F. Hoppensteadt, An age dependent epidemic model, *J. Franklin Inst.* **297** (1974), 325–333.
8. F. Hoppensteadt, "Mathematical Theories of Populations: Demographics, Genetics, and Epidemics," SIAM Regional Conference Series in Applied Mathematics, 1975.
9. V. Lakshmikantham and S. Leela, "Differential and Integral Inequalities I," Academic Press, New York, 1969.
10. L. Lamberti and P. Vernole, An age structured epidemic model: Asymptotic behavior, *Boll. Un. Mat. Ital. Anal. Funzion.* **1** (1982), 147–158.
11. M. Marcus and V. Mizel, Semilinear hereditary hyperbolic systems with nonlocal boundary conditions, A, *J. Math. Anal. Appl.* **76** (1980), 440–475.
12. M. Marcus and V. Mizel, Semilinear hereditary hyperbolic systems with boundary conditions, B, *J. Math. Anal. Appl.* **77** (1980), 1–19.
13. G. Oster, Internal variables in population dynamics, *in* "Some Mathematical Questions in Biology VII," Amer. Math. Soc., Providence, R.I., 1976.
14. J. W. Sinko and W. Streifer, A new model for age-size structure of a population, *Ecology* **48** (1967), 910–918.
15. P. Waltman, "Deterministic Threshold Models in the Theory of Epidemics," Lecture Notes in Biomathematics, Springer-Verlag, Berlin, 1974.
16. G. F. Webb, "Theory of Nonlinear Age-Dependent Population Dynamics," Monographs and Textbooks in Pure and Applied Mathematics Series, Vol. 89, Dekker, New York/Basel, 1985.