J. Math. Anal. Appl. 377 (2011) 464–470

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a generalized Mazur–Ulam question: Extension of isometries between unit spheres of Banach spaces ☆

Lixin Cheng, Yunbai Dong *

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

ARTICLE INFO

Article history:
Received 27 June 2010
Available online 13 November 2010
Submitted by G. Corach

Keywords:
Isometric extension
Lipschitz mapping
Support point
Unit sphere
Somewhere-flat space
Banach space

ABSTRACT

We call a Banach space X admitting the Mazur–Ulam property (MUP) provided that for any Banach space Y, if f is an onto isometry between the two unit spheres of X and Y, then it is the restriction of a linear isometry between the two spaces. A generalized Mazur–Ulam question is whether every Banach space admits the MUP. In this paper, we show first that the question has an affirmative answer for a general class of Banach spaces, namely, somewhere-flat spaces. As their immediate consequences, we obtain on the one hand that the question has an approximately positive answer: Given $\varepsilon > 0$, every Banach space X admits a $(1+\varepsilon)$-equivalent norm such that X has the MUP; on the other hand, polyhedral spaces, CL-spaces admitting a smooth point (in particular, separable CL-spaces) have the MUP.

1. Introduction

The study of affine extension of isometries between specific sets of Banach spaces has continued on and off for over for over 70 years since Mazur–Ulam's theorem, which was established in the joint work of S. Mazur and S. Ulam [15] in 1932: Every onto isometry between two Banach spaces is necessarily affine. In 1972, P. Mankiewicz [14] extended Mazur–Ulam's theorem in the following: Suppose X and Y are two normed spaces and $A \subset X$, $B \subset Y$. If both A and B are either two connected open sets or two closed convex bodies, then every onto isometry from A to B is the restriction of an affine onto isometry between the two spaces. D. Tingley [18] further proposed a generalized Mazur–Ulam's question in 1987: Whether every onto isometry between the spheres of two Banach spaces must be the restriction of a linear isometry between the two spaces? While the study of the generalized Mazur–Ulam's question has become an active area only recently, it has appeared in literature in a variety of concrete classical Banach spaces since the beginning of this century. It is doubtless that it is an interesting but difficult research area. Though we have known many concrete classical Banach spaces such as $C(\Omega)$, $c_0(\Gamma)$ and (for $p = 1, \infty$) $\ell_p(\Gamma)$ and $L_p(\mu)$ admitting the MUP, many elementary questions have left unknown in general theory. For example, we do not know whether every finite dimensional space has the MUP; see Ding's and Li–Liu's survey papers [5] and [11].

Our purpose in this paper is to give a general approach for the generalized Mazur–Ulam's question. Because existence of non-proper support points of maximal convex sets in unit spheres plays an important role in proof of the main results, we begin with discussion regarding non-proper support points of closed convex sets of Banach spaces. After showing a natural extension property of an onto isometry between two unit spheres, we proceed to prove the main lemma that "every onto

☆ The first author was supported by the Natural Science Foundation of China, grants 10771175 and 11071201. The second author was supported by the Natural Science Foundation of China, grant 11001231.
* Corresponding author.
E-mail addresses: lxcheng@xmu.edu.cn (L. Cheng), baiyunmu301@126.com (Y. Dong).

0022-247X/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2010.11.025

isometry between two unit spheres maps each maximal convex set into a maximal convex set"; and making use of these and the Mankiewicz–Mazur–Ulam theorem, we show then the main result of this paper, which entails that: Somewhere-flat spaces admit the MUP. (A Banach space is somewhere-flat provided that there is a maximal convex set in its unit sphere admitting non-empty relative interior with respect to some closed hyperplane.) As their applications, on the one hand we give the generalized Mazur–Ulam's question an affirmative answer in an approximation sense: For every $\varepsilon > 0$ every Banach space X has a $(1+\varepsilon)$-equivalent norm such that X admits the MUP with respect to the new norm. On the other hand, we show that many classes of Banach spaces, including that of polyhedral spaces, CL-spaces admitting a smooth point (in particular, separable CL-spaces) and of almost CL-spaces with the RNP, belong to the class of somewhere-flat spaces. Combining results of this paper and known results of other mathematicians' work in this topic, some further results about some classical Banach spaces are presented in the last section.

In this paper, the letter X will always be a Banach space, and as usual, X^* its dual. $B_X(x,r)$ is the closed ball in X centered at x with radius r, and $S_X(x,r)$ the sphere of $B_X(x,r)$. We simply use S_X and B_X to denote the unit sphere and the closed unit ball of X, respectively. For a set $A \subset X$, L_A, X_A, $aff\, A$ and $co\, A$ stand successively for $span\, A$, the closure of $span\, A$, the affine hull of A and the convex hull of A. $\overline{A}$ represents the closure of A. If $B \subset X$ is another set with $A \subset B$, then $int_B\, A$ denotes the relative interior of A with respect to B.

2. Non-proper support points of maximal convex sets of unit spheres

Because non-proper support points, or in other words, relative non-support points of maximal convex sets in unit spheres of Banach spaces play an important role in showing the main lemma and many other results, in this section, we review briefly some concepts and known results about (relative) non-support points of closed convex sets in Banach spaces. We shall begin with the following definition.

Definition 2.1. Suppose that C is a closed convex subset of a Banach space X, and $x \in C$.

i) The point x is said to be a support point of C provided there is a non-zero functional $x^* \in X^*$ such that $\langle x^*, x\rangle = sup_{y\in C}\langle x^*, y\rangle$; the functional x^* is called a support functional of C and support C at x, or simply, a support functional of C at x;
ii) x is called a non-support point of C if it is not a support point of C;
iii) x is called a proper support point of C if there exists a support functional $x^* \in X^*$ of C, which is not identically a constant on C and supports C at x;
iv) we say that x is a non-proper support point of C if it is not a proper support point of C. The subset of all non-proper support points of C is denoted by $N(C)$.

Remark. For a closed convex set C, we also say that $x \in C$ is a relative support point of C if there is $x^* \in X^*$ such that $\langle x^*, x\rangle = \max_C x^*$ and it is not identically a constant on $aff\, C$. Therefore, a non-proper support point is essentially a relative non-support point.

Proposition 2.2. *If C is a non-empty closed separable convex set, then $N(C) \neq \emptyset$.*

Proof. We choose any $x_0 \in C$, and let $A = C - x_0$ and $X_A = \overline{span}\, A$. Then X_A is a separable Banach space and A cannot be contained in any closed hyperplane of X_A. Applying Exercise 2.18 of R.B. Holmes [9, p. 111] to X_A, the set $N(A) = N(C) - x_0$ of all non-support points of A is not empty. □

More related matters can be found in [3] and [17].

Assume that K is a closed subset of a Banach space X. We write $ext\, K$ for the set of all extreme points of K. We call $s(x,K) = \{y \in K:\ [x,y] \subset K\}$ the star-like subset of K around x for each $x \in K$. Clearly, for every $x \in K$ there is (at least) a maximal convex subset of K containing x, and $s(x,K)$ is just the union of all such maximal convex sets of K. Consider next an important particular case where the closed subset K is the unit sphere of a Banach space X, that is, $K = S_X$. For every $x \in S_X$, let $\sigma(x)$ be the set of norm-one support functionals of B_X at x; i.e., $\sigma(x) = \{x^* \in S_{X^*}:\ \langle x^*, x\rangle = 1\}$. Let us agree that if $x^* \in S_{X^*}$ then C_{x^*} denotes the set $\{u \in S_X:\ \langle x^*, u\rangle = 1\}$; in this situation, we say also that C_{x^*} is determined by x^*.

Lemma 2.3. *Suppose X is a Banach space and $x \in S_X$. Then*

i) $s(x, S_X) = \{y \in S_X:\ \|x+y\| = 2\} = \bigcup_{x^* \in ext\,\sigma(x)} C_{x^*}$;
ii) *if C is maximal convex set of S_X and $x \in N(C)$ then $s(x, S_X) = C$.*

Proof. i) The first equality can be easily followed by definitions of star-like set and maximal convex sets. It is clear that $s(x, S_X)$ is just the union of all maximal convex sets of S_X containing x. To show $s(x, S_X) = \bigcup_{x^* \in ext\,\sigma(x)} C_{x^*}$, let $C \subset s(x, S_X)$ be a maximal convex set. Separation theorem and the Krein–Milman theorem (see, for instance, [4, p. 148], [16]) together assert that there exists an extreme point x^* of $\sigma(x)$ such that $C = C_{x^*}$ since $\sigma(x)$ is non-empty convex and w^*-compact. Therefore, $s(x, S_X) \subset \bigcup_{x^* \in ext\,\sigma(x)} C_{x^*}$. The converse inclusion follows from $C_{x^*} \subset \{y \in S_X\colon \|x+y\| = 2\}$ for every $x^* \in \sigma(x)$.

ii) By i), it is sufficient to prove that there is only a unique maximal convex subset C of S_X containing x. Assume that C_1 is another maximal convex subset of S_X containing x. Then, there exists an extreme point x^* of $\sigma(x)$ such that $C_1 = C_{x^*}$. Since x is a non-proper support point of C, x^* is identically 1 on C. Thus, $C \subset C_1$. This and maximality of C together entail that $C = C_1$. □

Lemma 2.4. *Suppose that X is a Banach space, and that C is a maximal convex set of S_X determined by an extreme point x^* of B_{X^*}. Let $x_0 \in N(C)$, and X_0 be the intersection of X_C and the kernel of x^*. Then*

i) $\bigcup_{n=1}^{\infty} n(C - x_0) = span(C - C)$ *is a dense subspace of* X_0;
ii) *if, in addition,* $0 \in int_{X_C}\, co(C, -C)$, *then* $0 \in int_{X_0}(C - x_0)$.

Proof. i) Since $x_0 \in C$ is a non-proper support point of C, by Proposition 2.2, $span(C - C) = \bigcup_{n=1}^{\infty} n(C - C) = \bigcup_{n=1}^{\infty} n(C - x_0)$. Note $span\, C = \bigcup_{n=1}^{\infty} n\, co(C, -C)$ is dense in X_C and note X_0 is a closed hyperplane of X_C. $span(C - C) = span\, C \cap X_0$ is necessarily dense in X_0.

ii) Since X_C is a closed subspace of X, it is itself a Banach space. $0 \in int_{X_C}\, co(C, -C)$ implies that $0 \in int_{X_0}(co(C, -C) \cap X_0)$. Note $x \in co(C, -C) \cap X_0$ if and only if $x \in \frac{1}{2}(C - C)$. We see that $0 \in int_{X_0}(C - C)$, and which further entails that $span(C - C) = X_0$. Since X_0 is again a Banach space and since $C - x_0$ is a closed convex absorbing set of X_0, by a simple argument of category, $C - x_0$ is necessarily a 0-neighborhood of X_0. □

It should be remarked that Lemma 2.4ii) fails without the assumption that $x_0 \in N(C)$. This disappointment can be illustrated by the following example: For any set Γ and for every maximal convex set C of the unit sphere of $\ell^1(\Gamma)$ we know that $co(C \cup -C) = B_{\ell^1(\Gamma)}$. But if Γ is uncountable, then $N(C) = \emptyset$, and consequently, for every $x_0 \in C$, $int_{X_0}(C - x_0) = \emptyset$.

3. On somewhere-flat spaces, polyhedral spaces and CL-spaces

In this section, we are going to exhibit a few examples of CL-spaces, polyhedral spaces and non-proper support points of maximal convex sets of their spheres. They are also examples of somewhere-flat spaces.

Recall that a Banach space is said to be somewhere-flat provided there is a maximal convex set in its unit sphere such that it has non-empty relative interior with respect to some closed hyperplane. If X is a somewhere-flat space, then we also call the corresponding norm of X somewhere-flat norm. We should emphasize that every equivalent norm on a normed space M is approximated by a somewhere-flat norm. Indeed, given $\varepsilon > 0$ and any $x_0 \in S_M$, let $x^* \in S_{M^*}$ be a support functional of B_M at x_0, and let H_1 be the support hyperplane of B_M at x_0, and further let $C = B(x_0, \varepsilon) \cap H_1$. It is clear that the Minkowski functional generated by $D = \overline{co}(B_M \cup \pm C)$ is a somewhere-flat norm and is $(1+\varepsilon)$-equivalent to the original one. We sum up now the fact as follows.

Fact 3.1. Suppose that $(X, \|\cdot\|)$ is a Banach space. Then for every $\varepsilon > 0$ there is an equivalent somewhere-flat norm $\|\cdot\|_\varepsilon$ on X such that

$$(1+\varepsilon)^{-1}\|x\| \leqslant \|x\|_\varepsilon \leqslant (1+\varepsilon)\|x\|, \quad \forall x \in X.$$

A Banach space is polyhedral [10] if the unit ball of any of its finite dimensional subspaces is a polyhedron. A result of Fonf [7, Theorem A] entails that every separable polyhedral space is somewhere-flat.

Recall that a Banach space is said to be a (an almost) CL-space [8,12] provided for each maximal convex set C of S_X we have $B_X = co(C \cup -C)$ ($B_X = \overline{co}(C \cup -C)$, respectively).

Proposition 3.2. *(See [2].) Suppose that X is an almost CL space and that C is a maximal convex set of S_X. Then a point $x \in C$ is a smooth point if and only if it is a non-proper support point of C.*

The following result follows from Proposition 2.2, Lemma 2.4 and definition of CL-space directly.

Proposition 3.3. *Every CL-space admitting a smooth point (in particular, every separable CL-space) is somewhere-flat.*

We are now going to present several examples of classical CL-spaces and characterizations of non-proper support points of maximal convex sets in their unit spheres.

Example 1. For any set Γ, the Banach space $c_0(\Gamma)$ with the usual sup norm is a CL-space. Let e_j $(j \in \Gamma)$ denote the standard unit vectors of $\ell_1(\Gamma)$, and let $E = \{\pm e_j\colon\ j \in \Gamma\}$. Then (1) each maximal convex set C of $c_0(\Gamma)$ has the following form: $C = \{x \in S_{c_0(\Gamma)}\colon\ \langle x^*, x\rangle = 1\}$ for some $x^* \in E$; (2) a point in C is a non-proper support point if and only if it is a smooth point, which is also a strongly smooth point [19].

Example 2. For any compact Hausdorff space K the Banach space $X = C(K)$ satisfies that every maximal convex set of S_X admits a non-proper support point.

Example 3. Let (Ω, Σ, μ) be a σ-finite measure space. Then $L^1(\Omega, \Sigma, \mu)$ is a separable CL-space. Therefore, each maximal convex set of its unit sphere contains a non-proper support point.

Example 4. With the measure space (Ω, Σ, μ) as above, since $L^\infty(\Omega, \Sigma, \mu)$ is an abstract M space with a strong unit e (the function which is identically 1 on Ω), it is isometric to $C(K)$ for some compact Hausdorff space K [13, Th. 1.b.6]. Thus, it is a CL-space satisfying that every maximal convex set of its unit sphere admits a non-proper support point.

4. Extension of Lipschitz embeddings between unit spheres

In this section we shall show an extension property of Lipschitz embedding between sets of unit spheres. For a general discussion of property of Lipschitz mappings, we refer the reader to Y. Benyamini and J. Lindenstrauss' book [1].

Proposition 4.1. *Suppose that X and Y are Banach spaces, such that $A \subset S_X$, $B \subset S_Y$, and that $f : A \to B$ is a Lipschitz equivalence with*

$$M^{-1}\|x-y\| \leqslant \big\|f(x)-f(y)\big\| \leqslant M\|x-y\|, \quad \forall x, y \in A. \tag{4.1}$$

Then f has a positively homogeneous extension $U : \mathbb{R}^+A \to \mathbb{R}^+B$ such that U is a Lipschitz equivalence from $\mathbb{R}^+A$ to $\mathbb{R}^+B$ with

$$(3M)^{-1}\|x-y\| \leqslant \big\|U(x)-U(y)\big\| \leqslant 3M\|x-y\|, \quad \forall x, y \in \mathbb{R}^+A. \tag{4.2}$$

Proof. Suppose that $f : A \to B$ is a Lipschitz equivalence satisfying (4.1). Now we define an extension f of U for $x \in \mathbb{R}^+$ by

$$f_*(x) = \begin{cases} \|x\| f(\frac{x}{\|x\|}), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{4.3}$$

Clearly, f_* is a positively homogeneous extension of f. For any fixed $x, y \in \mathbb{R}^+A$, we can assume $0 < \|x\| \leqslant \|y\| \leqslant 1$.

$$\begin{aligned} \|f_*y - f_*x\| &= \left\|\left(f_*y - \|x\|f_*\frac{x}{\|x\|}\right)\right\| \\ &\leqslant \|x\|\left\|f_*y - f_*\frac{x}{\|x\|}\right\| + \big(1-\|x\|\big)\|f_*y\| \\ &\leqslant M\big(\big\|(y-x) - \big(1-\|x\|\big)y\big\| + \big(1-\|x\|\big)\big) \\ &\leqslant M\|y-x\| + 2M\big(1-\|x\|\big) \leqslant 3M\|y-x\|. \end{aligned} \tag{4.4}$$

The same argument to f^{-1} and (4.4) yield (4.2). □

Remark 4.2. For the Lipschitz equivalence $f : A \to B$ in Proposition 4.1, the extension f_* defined by (4.5) is called the natural extension of f.

Corollary 4.3. *Suppose that X and Y are two Banach spaces and f is a Lipschitz equivalence between S_X and S_Y. The natural extension of f is a positively homogeneous Lipschitz equivalence between X and Y.*

Lemma 4.4. *Suppose that X and Y are Banach spaces and $C \subset X$, $D \subset Y$ are convex sets* (*linear subspaces*). *Suppose that $f : C \to D$ is a Lipschitz equivalence. Then, for every separable subset* (*linear subspace*) *$A \subset X$ there exist a convex separable subset* (*linear subspace*) *$U \subset C$ and a convex separable subset* (*linear subspace*) *$V \subset D$ such that $f(U) = V$.*

Proof. We can assume that both C and D are closed convex sets. Let $C_0 = A$, and let $C_1 = co\, C_0$ (*span* C_0), $C_{n+1} = co\, f^{-1}(D_n)$ (*span* $f^{-1}(D_n)$) and $D_n = co\, f(C_n)$ (*span* C_n) for every $n \geqslant 1$. Let next $U = \bigcup_{n=1}^\infty C_n$ and $V = \bigcup_{n=1}^\infty D_n$. Then U and V have the desired property. □

5. Invariance of convexity and maximality of maximal convex sets in unit spheres under isometries

In this section we specialize the general notion of Lipschitz equivalence discussed in the proceeding section to that of onto isometry between unit spheres of two Banach spaces. We shall see in the following sections that the next lemma is powerful to enable us to link up the metric property with linearity.

Lemma 5.1 *(Main lemma). Suppose X and Y are Banach spaces and $f : S_X \to S_Y$ is an isometry. Then*

i) *f maps every star-like set of S_X into a star-like set of S_Y, more precisely, for each $x \in S_X$ we have*

$$f\big(s(x, S_X)\big) = -s\big(f(-x), S_Y\big); \tag{5.1}$$

ii) *f maps every maximal convex set of S_X into a maximal convex set of S_Y.*

Proof. i) Let us agree again that f_* denotes the natural extension of f. Note

$$\begin{aligned} z \in s(-x, S_X) \quad &\Leftrightarrow \quad -z \in s(x, S_X) \\ &\Leftrightarrow \quad \|x - z\| = 2 \quad \Leftrightarrow \quad \big\|f(x) - f(z)\big\| = 2 \\ &\Leftrightarrow \quad -f(z) \in s\big(f(x), S_Y\big) \\ &\Leftrightarrow \quad f(z) \in s\big(-f(x), S_Y\big) = -s\big(f(x), S_Y\big). \end{aligned}$$

We have

$$f\big(-s(x, S_X)\big) = f\big(s(-x, S_X)\big) \subset -s\big(f(x), S_Y\big),$$

or equivalently,

$$f\big(s(x, S_X)\big) \subset -s\big(f(-x), S_Y\big).$$

Conversely, note $f^{-1} : S_Y \to S_X$ is again an isometry. Let $u = f(-x)$. Then

$$s(x, S_X) \subset f^{-1}\big(-s(u, S_Y)\big) \subset -s\big(f^{-1}(u), S_X\big) = s(x, S_X).$$

Thus, (5.1) holds.

ii) Suppose that C is a maximal convex set of S_X. We first claim that $f(C)$ is a convex set of S_Y. Otherwise, there are two points $x, y \in C$ such that $[f(x), f(y)] \equiv \{\lambda f(x) + (1-\lambda) f(y)\colon\ \lambda \in [0,1]\} \not\subseteq f(C)$. Applying Lemma 4.4, there are separable subspaces $E \subset X$ with $x, y \in E$ and $F \subset Y$ such that $f_*|_{S_E} = f|_{S_E} : S_E \to S_F$ is an onto isometry. Let $C_E = C \cap E$. It is clear that C_E is a maximal convex set of S_E containing x and y. According to Proposition 2.2, there is a non-proper support point u of C_E. By Lemma 2.3ii) and the equality (5.1) we have just proven, we obtain that

$$f(C_E) = f\big(s(u, S_E)\big) = s\big(-f(-u), S_F\big). \tag{5.2}$$

Let K be a maximal convex subset of S_F containing $-f(-u)$. Then $K \subset s(-f(-u), S_F)$. Therefore, the equality (5.2) entails that $f^{-1}(K) \subset s(u, S_E)$. Since K is separable, by Proposition 2.2 again there is a non-proper support point v of K and which implies that $K = s(v, S_F)$. Note $f^{-1} : S_F \to S_E$ is again an isometry. We observe that

$$\begin{aligned} C_E = s(u, S_E) = f^{-1}\big(s\big(-f(-u), S_F\big)\big) \supset f^{-1}(K) = f^{-1}\big(s(v, S_F)\big) \\ = s\big(-f^{-1}(-v), S_E\big). \end{aligned}$$

Since $-s(f^{-1}(-v), S_E)$ is the union of all maximal convex sets of S_E containing $-f^{-1}(-v)$, there must be a maximal convex set C_1 of S_E containing both x and $-f^{-1}(-v)$ in $s(-f^{-1}(-v), S_E)$. We assert that $C_1 = C_E$. Suppose, to the contrary, that there exists extreme point u^* of B_{E^*} such that

$$C_1 = \big\{u \in S_E\colon \langle u^*, u\rangle = 1\big\} \subsetneqq C_E.$$

It is easy to see that u^* is a support functional of C_E and supporting C_E at u. Since u is a non-proper support point of C_E, we observe that $C_E \subsetneqq C_1$. This contradicts to maximality of C_E. Therefore,

$$f(C_E) = f\big(s\big(f^{-1}(v), S_E\big)\big) = s(u, S_F) = K,$$

and which in turn implies that $f(C)$ is convex.

Let D be a maximal convex set of S_Y containing $f(C)$. Then, by the fact we have just proven, $f^{-1}(D)$ is a convex set of S_X containing C. Maximality of C says that $C = f^{-1}(D)$, and further, $f(C) = D$. □

We combine this lemma and Lemma 2.3 to obtain the following consequence.

Corollary 5.2. *Suppose X and Y are Banach spaces and $f : S_X \to S_Y$ is an isometry. Then $f(s(x, S_X)) = s(f(x), S_Y)$ for every $x \in S_X$.*

Remark. We should mention here that Corollary 5.2 has been shown by Fang and Wang [6] in a different way. The proof of the main lemma depends heavily on the hypothesis that f is an onto isometry. We do not know that whether f maps every maximal convex set of S_X into a convex set, or into a set contained in a maximal convex set of S_Y if the mapping f is only an into isometry.

6. Extension of isometries between unit spheres of somewhere-flat spaces

In this section we now establish one of the main results alluded to in the Introduction concerning linear extension of onto isometries between maximal convex sets in unit spheres, which turns out further the main result mentioned in the abstract: Every somewhere-flat space has the MUP.

Theorem 6.1. *Suppose that X, Y are two Banach spaces, and $f : S_X \to S_Y$ is an onto isometry. Suppose that C is a maximal convex set of S_X such that $int_{A_C} C \neq \emptyset$. Then the natural extension f_* of f on $X_C \equiv \overline{span}\, C$ is a linear isometry.*

Proof. According to Lemma 5.1, $f_*(C) = f(C) \equiv D$ is also a maximal convex set of S_Y. Since $int_{A_C} C \neq \emptyset$, given any $x_0 \in int_{A_C} C$, $f(x_0) \in int_{A_D} D$. Therefore, $0 \in int_{X_C} co(C, -C)$, and $0 \in int_{Y_D} co(D, -D)$. Let $x_0^* \in S_{X^*}$ ($y_0^* \in S_{Y^*}$, respectively) be a functional supporting C at x_0 (supporting D at $y_0 = f(x_0)$, respectively). Denote by X_0 (Y_0, respectively) the intersection of $X_C \equiv \overline{span}\, C$ and the kernel of x_0^* ($Y_D \equiv \overline{span}\, D$ and the kernel of y_0^*, respectively). Then by Lemma 2.4ii),

$$X_C = span\, C = \bigcup_{n=1}^{\infty} n(C - x_0) + \mathbb{R}x_0 = X_0 + \mathbb{R}x_0 \quad \text{and}$$

$$Y_D = span\, D = \bigcup_{n=1}^{\infty} n(D - y_0) + \mathbb{R}y_0 = Y_0 + \mathbb{R}y_0, \text{ respectively.}$$

In the following, we claim that $g : C - x_0 \to D - f(x_0)$, defined by $g(u) = f_*(u + x_0) - f_*(x_0)$ for $u \in C - x_0$ is an onto isometry. Indeed, given $u = c_1 - x_0, v = c_2 - x_0 \in C - x_0$, $\|g(u) - g(v)\| = \|f(c_1) - f(c_2)\| = \|c_1 - c_2\| = \|u - v\|$. Since both $C - x_0$ and $D - y_0$ are convex bodies in X_0 and Y_0, respectively, by Mankiewicz' theorem [14], g can be affinely extended to the whole space X_0 to become into an affine onto isometry to Y_0, which is denoted by g_*. Note $g(0) = 0$. Mazur–Ulam's theorem [15] entails that g_* is a linear isometry.

We show next that $g_* = f_*$ on X_0. It suffices to show that $h = f_*(\cdot + x_0)$ is affine on X_0. Given $u, v \in X_0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, since $C - x_0$ is a (convex) neighborhood of the origin of X_0, and sine both g_* and f_* are positively homogeneous, we can assume that $0 < |\beta| \leqslant \alpha$ and $\pm u, \pm v, \pm\alpha u, \pm\beta v \in C - x_0$. Therefore

$$g_*(\alpha u + \beta v) = g(\alpha u + \beta v) = f_*(\alpha u + \beta v + x_0) - f(x_0). \tag{6.1}$$

On the other hand,

$$\begin{aligned} g(\alpha u + \beta v) &= \alpha g(u) + \beta g(v) \\ &= \alpha\big(f_*(u + x_0) - f_*(x_0)\big) + \beta\big(f_*(v + x_0) - f_*(x_0)\big). \end{aligned} \tag{6.2}$$

(6.1) and (6.2) together imply

$$\begin{aligned} h(\alpha u + \beta v) &= f_*(\alpha u + \beta v + x_0) \\ &= \alpha f_*(u + x_0) + \beta\big(f_*(v + x_0)\big) \\ &= \alpha h(u) + \beta h(v). \end{aligned}$$

Thus, h is affine on a neighborhood of X_0. Since f_* is positively homogeneous, then h is affine on X_0, and this implies that

$$h(u) = f_*(u + x_0) = f_*(u) + f_*(x_0), \quad \forall u \in X_0. \tag{6.3}$$

Let $H_C^{\pm}$ and $H_D^{\pm}$ denote the upper (lower) half-spaces $\{x \in X_C \colon \langle x_0^*, x\rangle \geqslant (\leqslant) 0\}$ and $\{y \in Y_D \colon \langle y_0^*, y\rangle \geqslant (\leqslant) 0\}$, respectively. Positive homogeneity and (6.3) imply f_* is additive on H_C^+. Indeed, $\forall u, v \in H_C^+$, let $(\alpha, h_1), (\beta, h_2) \in \mathbb{R}^+ \times X_0$ such that $u = \alpha x_0 + h_1$, $v = \beta x_0 + h_2$. Then

$$\begin{aligned} f_*(u + v) &= f_*\big((\alpha + \beta)x_0 + (h_1 + h_2)\big) = (\alpha + \beta) f_*(x_0) + f_*(h_1 + h_2) \\ &= \big(\alpha f_*(x_0) + f_*(h_1)\big) + \big(\beta f_*(x_0) + f_*(h_2)\big) = f_*(u) + f_*(v). \end{aligned}$$

We claim now that $f_* : H_C^+ \to H_D^+$ is an onto isometry. It is trivial that $f_*(H_C^+) = f_*(H_D^+)$, since f is additive and since f_* is a linear isometry from X_0 to Y_0. To show f_* is an isometry on H_C^+, we consider again $u = \alpha x_0 + h_1$, $v = \beta x_0 + h_2$ with $(\alpha, h_1), (\beta, h_2) \in \mathbb{R}^+ \times X_0$ and with $\beta \geqslant \alpha \geqslant 0$. Note

$$\begin{aligned} f_*(v) - f_*(u) &= \big(\beta f_*(x_0) + f_*(h_1)\big) - \big(\alpha f_*(x_0) + f_*(h_2)\big) \\ &= f_*\big((\beta - \alpha)x_0 + (h_2 - h_1)\big) = f_*(v - u). \end{aligned}$$

We have

$$\big\|f_*(v) - f_*(u)\big\| = \big\|f_*(v - u)\big\| = \|v - u\|.$$

We define finally $A : X_C \to Y_D$ by

$$A(x) = \begin{cases} f_*(x), & x \in H_C^+, \\ -f_*(-x), & x \in H_C^-. \end{cases}$$

It is not difficult to observe that A is linear, and this entails further that A is a linear onto isometry.

It remains to prove that f_* is linear on X_C. It is sufficient to show that f_* is an onto isometry. Let us first note that f also maps the maximal convex set $-C$ of S_X into a maximal convex set $f(-C)$ of S_Y. It is easy to observe that $f(-C) = -f(C) = -D$. By a similar argument, we know that there is another linear onto isometry $B : X_C \to X_D$ such that $f_* = B$ on H_C^-. Given $x, y \in X_C$, we want to show $\|f_*(x) - f_*(y)\| = \|x - y\|$. We can assume without loss of generality $x \in H_C^+$ and $y \in H_C^-$. Therefore, there is $z \in X_0$ such that $z = \lambda x + (1 - \lambda)y$, for some $0 \leqslant \lambda \leqslant 1$. Note $f(z) \in Y_0$.

$$\begin{aligned} \|x - y\| &= \|x - z\| + \|z - y\| = \big\|A(x - z)\big\| + \big\|B(z - y)\big\| \\ &= \big\|f_*(x) - f_*(z)\big\| + \big\|f_*(z) - f_*(y)\big\| \geqslant \big\|f_*(x) - f_*(y)\big\|. \end{aligned} \tag{6.4}$$

Conversely, let $u \in H_D^+$, $v \in H_D^-$ such that $x = f_*^-(u)$ and $y = f_*^-(v)$. Let $w = \lambda u + (1 - \lambda)v$, for some $0 \leqslant \lambda \leqslant 1$. Then

$$\begin{aligned} \big\|f_*(x) - f_*(y)\big\| &= \|u - w\| + \|w - v\| \\ &= \big\|A^{-1}(u - w)\big\| + \big\|B^{-1}(w - v)\big\| \\ &= \big\|A^{-1}(u) - A^{-1}(w)\big\| + \big\|B^{-1}(w) - B^{-1}(v)\big\| \\ &= \big\|f_*^-(u) - f_*^-(w)\big\| + \big\|f_*^-(w) - f_*^-(v)\big\| \\ &\geqslant \big\|f_*^-(u) - f_*^-(v)\big\| = \|x - y\|. \end{aligned} \tag{6.5}$$

Hence, f_* is a linear onto isometry from X_C to Y_D. □

Corollary 6.2. *Every polyhedral Banach space admits the MUP.*

Proof. Suppose that X is a polyhedral space and Y is a Banach space. We want to prove that if $f : X \to Y$ is an onto isometry, then it is linear. Lemma 4.4 and heredity of polyhedral spaces allow us to assume that X is separable. The proof is complete by noting that every separable polyhedral space is somewhere-flat. □

The next result is an immediate consequence of Theorem 6.1, Propositions 3.2 and 3.3.

Corollary 6.3. *Every CL-space admitting a smooth point* (*in particular, every separable CL-space*) *has the MUP.*

References

[1] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Amer. Math. Soc. Colloq. Publ., vol. 1, 2000.
[2] L. Cheng, M. Li, Extreme points, exposed points, differentiability points in CL-spaces, Proc. Amer. Math. Soc. 136 (2008) 2445–2451.
[3] L. Cheng, W. Zhang, A note on non-support points, negligible sets, Gâteaux differentiability and Lipschitz embeddings, J. Math. Anal. Appl. 350 (2009) 531–536.
[4] J. Diestel, Sequences and Series in Banach Spaces, Springer-Verlag, 1984.
[5] G. Ding, On isometric extension problem between two unit spheres, Sci. China Ser. A 52 (2009) 2069–2083.
[6] X. Fang, J. Wang, Extension of isometries between the unit spheres of normed space E and $C(\Omega)$, Acta Math. Sin. (Engl. Ser.) 22 (2006) 1819–1824.
[7] V. Fonf, Three characterizations of polyhedral Banach spaces, Ukrainian Math. J. 42 (9) (1990) 1145–1148.
[8] R.E. Fullerton, Geometrical characterization of certain function spaces, in: Proc. Inter. Sympos. Linear Spaces, Jerusalem, 1960, Jerusalem Academic Press/Pergamon, Jerusalem/Oxford, 1961, pp. 227–236.
[9] R.B. Holmes, Geometric Functional Analysis, Springer-Verlag, 1977.
[10] V. Klee, Polyhedral sections of convex bodies, Acta Math. 103 (1960) 243–267.
[11] L. Li, R. Liu, On the isometric extension problem: a survey. Stability of functional equations and applications, Tamsui Oxf. J. Math. Sci. 24 (4) (2008) 339–354.
[12] A. Lima, Intersection properties of balls in spaces of compact operator, Ann. Inst. Fourier (Grenoble) 28 (1978) 35–65.
[13] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Function Spaces, Springer-Verlag, 1979.
[14] P. Mankiewicz, On extension of isometries in normed linear spaces, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 20 (1972) 367–371.
[15] S. Mazur, S. Ulam, Sur less transformations isometriques déspaces vectoriels normés, C. R. Math. Acad. Sci. Paris 194 (1932) 946–948.
[16] R.R. Phelps, Lectures on Choquet's Theorem, Von Nostrand Math. Stud., vol. 7, Von Nostrand, Princeton, 1966.
[17] R.R. Phelps, Some topological properties of support points of convex sets, in: Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces, Jerusalem, 1972, Israel J. Math. 13 (1972) 327–336.
[18] D. Tingley, Isometries of the unit spheres, Geom. Dedicata 22 (1987) 371–378.
[19] C. Wu, L. Cheng, X. Yao, Characterizations of differentiability points of norms on $c_0(\Gamma)$ and $l_\infty(\Gamma)$, Northeast. Math. J. 12 (1996) 153–160.