INTERSECTION THEORY ON ALGEBRAIC STACKS AND Q-VARIETIES

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0. Introduction

In his article 1161, Mumford constructed an intersection product on the Chow groups (with rational coefficients) of the moduli space \( \mathcal{M}_g \) of stable curves of genus \( g \) over a field \( k \) of characteristic zero. Having such a product is important in studying the enumerative geometry of curves, and it is reasonable to conjecture its existence even though \( \mathcal{M}_g \) is singular, because \( \mathcal{M}_g \) is locally (in the étale topology) the quotient of a smooth variety by a finite group (a 'Q-variety' in the terminology of op. cit.; see Section 9 for the full definition, which includes compatibility between local charts). Mumford's construction does not apply to general Q-varieties however; for it uses both the assumption that \( \text{char}(k) = 0 \) and the fact that \( \mathcal{M}_g \) is globally the quotient of a Cohen-Macaulay variety by a finite group. In this article we shall see, how, using higher algebraic K-theory, one can remove these restrictions and construct intersection products for general Q-varieties and stacks.

There are really two moduli spaces of curves of genus \( g \); the fine moduli stack \( \mathcal{M}_g \) in the sense of Deligne and Mumford [2] which keeps track of automorphisms of curves and is smooth over \( k \), and the coarse moduli scheme \( \bar{\mathcal{M}}_g \) which para-
meterizes isomorphism classes of curves and is singular. We start by studying intersection theory on $\bar{M}_g$ and on algebraic stacks in general, starting with the Fulton style (cf. [3]) Chow homology groups:

**0.1. Definition.** Let $F$ be an algebraic stack of finite type over a field $k$ (the existence of $k$ is not strictly necessary.) The codimension $i$ Chow group $CH^i(F)$ of $F$ is the quotient of the free abelian group $\mathbb{Z}^i(F)$ generated by the integral substacks of $F$ of codimension $i$ (the 'codimension $i$ cycles on $F$') by the subgroup generated by divisors of rational functions on integral substacks of codimension $i - 1$. (If we grade the cycles by dimension instead of codimension, we get groups $CH_i(F)$.) For the definition of a rational function on an integral stack, and its associated divisor, see Section 2.

These Chow groups behave just like the Chow homology groups of schemes, at least for representable morphisms:

**0.2. Proposition.** Let $f: S \rightarrow T$ be a morphism of algebraic stacks of finite type over $k$.

(i) If $f$ is flat, there is a natural map $f^*: CH^*(T) \rightarrow CH^*(S)$.

(ii) If $f$ is proper and representable, there is a natural direct image map

$$f_*: CH_*(S) \rightarrow CH_*(T).$$

If $X$ is a regular scheme over $k$ recall that we can use the isomorphism

$$H^p_{\text{et}}(X, K^p(\mathcal{O}_X)) \cong CH^p(X)$$

to define the intersection product on the Chow groups using the ring structure on the higher $K$-theory sheaves $\bigoplus_{p \geq 0} K^p(\mathcal{O}_X)$ (see [8], [11], [17] for details). We would like to exploit this result to do intersection theory on stacks; however the Zariski topology is too coarse an invariant of a stack; while $K$-theory is in general not well behaved in the étale topology. However if we are willing to neglect torsion, this difficulty disappears.

**0.3. Theorem.** Let $F$ be a regular algebraic stack of finite type over a field. Then

$$H^p_{\text{et}}(F, K^p(\mathcal{O}_F)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong CH^p(F) \otimes_{\mathbb{Z}} \mathbb{Q};$$

hence $CH^p(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ has a natural ring structure.

In order to link the Chow groups of stacks and $\mathbb{Q}$-varieties we need:

**0.4. Theorem.** The Chow homology functor (0.2) $F \rightarrow CH_*(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ extends naturally to a covariant functor on the category of all proper morphisms between algebraic stacks.
0.5. **Theorem.** Let $F$ be an algebraic stack with coarse moduli space $X$ (see Section 3 for the definition). Then the map $p : F \to X$ is proper, and induces an isomorphism

$$CH^*(F) \otimes_{\mathbb{Q}} \mathbb{Q} = CH^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}.$$ 

If $X$ is a $\mathbb{Q}$-variety, then we shall see (Section 9) that $X$ is the coarse moduli space of a regular algebraic stack, so that the existence of a product on $CH^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}$ follows from:

0.6. **Corollary.** If $X$ is the coarse moduli space of a regular algebraic stack $F$, there is a natural isomorphism

$$CH^p(X) \otimes_{\mathbb{Q}} \mathbb{Q} \sim H^p_{\mathbb{Q}}(F, K_p(\mathcal{O}_F)) \otimes_{\mathbb{Q}} \mathbb{Q}.$$ 

We define the intersection product on $CH^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}$ via this isomorphism. While it might seem that the product depends on the stack $F$, this is not in fact the case. The proofs of the integral versions of most of these theorems for schemes use the Quillen spectral sequence (see Section 1):

$$E_1^{p,q}(X) = \bigoplus_{x \in X'^n} K_{-p-q}(k(x)) \Rightarrow K'_{-p-q}(X),$$

$$E_2^{p,-p} = CH^p(X).$$

(0.7)

(0.8)

For general stacks this spectral sequence is not as well behaved; in particular $E_1^{p,-p}$ will be a direct sum of representation rings of inertia groups and (0.8) will be false. However for algebraic spaces the spectral sequence (0.7) still exists and the isomorphism (0.8) holds. Furthermore the Riemann–Roch theorem for proper morphisms between schemes [4] extends to algebraic spaces:

0.9. **Riemann–Roch Theorem.** There is a natural isomorphism of covariant functors on the category of proper morphisms between algebraic spaces of finite type over $k$:

$$\tau : K'_0(\_\_\_\_\_\_\_\_)_{\mathbb{Q}} \cong CH_*(\_\_\_\_\_\_\_\_)_{\mathbb{Q}}.$$ 

The paper is organized as follows. Section 1 is a review of the $K$-theory and intersection theory needed in the rest of the paper and need only be referred to in order to keep track of notation. Section 2 is a discussion of the basic properties of algebraic groupoids, including proofs of the analogues of the main results in the theory of groups acting on fields. In Sections 3 and 4 we recall the main properties of algebraic stacks and then introduce their Chow groups. Section 5 contains the main technical result on the rational $K$-theory of groupoids which makes everything else work; other versions of these results are well known to the experts (see [18] in particular). The main results of the paper are proved in Section 6, while Sections 7, 8 and 9 compare the intersection theory of this paper with that of [16].
I would like to thank David Mumford for several helpful conversations, and the referee for his constructive criticisms of my treatment of products.

1. Preliminaries

1.1. We will assume throughout the paper, unless stated otherwise, that all schemes are of finite type over a field, even though many of the results do apply more generally, since Bloch's formula (1.3) is known only for such schemes.

1.2. K-theory. Recall that if $X$ is a scheme there are categories:

- $\mathcal{P}(X)$ = category of locally free (finite rank) $\mathcal{O}_X$-modules,
- $\mathcal{M}(X)$ = category of coherent $\mathcal{O}_X$-modules.

Both categories are exact, and we can take their $K$-theory [17, §7]:

$$K_*(X) = K_*(\mathcal{P}(X)), \quad K^*_*(X) = K^*_*(\mathcal{M}(X)).$$

$K_*$ is a covariant ring valued functor on the category of all schemes [17], [7] while $K^*_*$ is covariant with respect to projective [17] or even proper [8], [9] morphisms and contravariant with respect to flat morphisms. If

$$\begin{array}{ccc}
W & \xrightarrow{q} & X \\
\downarrow{p} & & \downarrow{f} \\
Z & \xrightarrow{g} & Y
\end{array} \quad (1.2.1)$$

is a Cartesian diagram with $f$ projective and $g$ flat, then [17, §7]:

$$g^*f_* = p_*q^*: K_*^*(X) \rightarrow K_*^*(Y).$$

1.3. If $X$ is a scheme, let $Z(X)$ be the free abelian group on the set of integral subschemes of $X$ (= group of cycles) which can be graded by dimension ($Z_*(X)$) or codimension ($Z^*(X)$). Let $\mathcal{U}(X) = \bigoplus_{Z \subset X} k(Z)^*$ where the direct sum runs over integral subschemes. $\mathcal{U}(X)$ can also be graded by dimension ($\mathcal{U}_*(X)$) or codimension ($\mathcal{U}^*(X)$) by setting $\mathcal{U}^*(X) = \bigoplus_{Z \subset X} k(Z)^*$ where the direct sum runs over $Z$ of codimension $q - 1$ and where the sum runs over $Z$ of dimension $q + 1$ for $\mathcal{U}_q(X)$. Define the Chow groups of cycles modulo rational equivalence by

$$\text{CH}^q(X) = \text{coker} \left( \mathcal{U}^q(X) \xrightarrow{\partial} Z^q(X) \right),$$

$$\text{CH}_q(X) = \text{coker} \left( \mathcal{U}_q(X) \xrightarrow{\partial} Z_q(X) \right)$$
where $\partial(\sum f_j) = \sum \text{div}(f_j) \in Z(X)$. See [3] for details. The Chow groups $CH^*$ are contravariant for flat maps while the $CH_\ast$ are covariant for proper maps. If $X$ is biequidimensional, then $\text{CH}^*(X) = \text{CH}_{\text{dim} X - \ast}(X)$. If diagram (1.2.1) is Cartesian and $X, Y, Z, W$ are biequidimensional, then

$$p_*q^* = g^*f^*: \text{CH}_\ast(X) \to \text{CH}_\ast(Z)$$

where $q^*$ and $g^*$ are defined via $\text{CH}^*(S) = \text{CH}_{\text{dim}(S) - \ast}(S)$ for $S = Y, Z$ or $W$.

1.4. For $X$ a scheme there is a spectral sequence [17, §7]:

$$E_1^{p,q}(X) = \bigoplus_{x \in X^{(p)}} K_{-p-q}(x) \Rightarrow K_{-p-q}(X) \quad (1.4.1)$$

where $X^{(p)}$ is the set of points of codimension $p$ in $X$, and $E_2^{p,-p} = \text{CH}^p(X)$. If $X$ is regular,

$$E_2^{p,q} = H^p(X, K_q(\mathcal{X}))$$

where $K_{\mathcal{X}}(\mathcal{X})$ is the sheaf in the Zariski topology associated to the presheaf $U \mapsto K_{\mathcal{X}}(U)$, and so

$$\text{CH}^p(X) = H^p(X, K_q(\mathcal{X})).$$

Since $K_{\mathcal{X}}$ has a ring structure, this defines a product on $\text{CH}^*(X)$ which can be seen to coincide with the classical product [11], [8]. This isomorphism is known as Bloch's formula.

In general for $X$ a scheme, define

$$\text{CH}_{r,s}(X) = E_2^{r,-s}(X), \quad \text{CH}^{r,q}(X) = E_2^{p,-q}(X)$$

where $E_r(\cdot)(X)$ is the spectral sequence (1.4.1) and $E'_r, \cdot$, is the spectral sequence [7, §7] with

$$E_1^{r,s} = \bigoplus_{x \in X^{(r)}} K_{r+s}(x) \Rightarrow K_{r+s}(X)$$

($X_l(\cdot)$ = set of points of dimension $r$ in $X$), and write

$$\text{HK}^{p,q}(X) = H^p_{\text{zar}}(X, K_q(\mathcal{X})).$$

Then $\text{CH}^p(X) - \text{CH}^{p,p}(X), \text{CH}_p(X) = \text{CH}_{p,p}(X)$ and if we write $\text{HK}^p(X)$ for $\text{HK}^p(X)$ we have that $\text{HK}^*(X)$ is a contravariant ring valued functor on the category of all schemes. For all $X, \text{CH}_\ast(X)$ is a graded $\text{HK}^*(X)$-module and if $p: X \to Y$ is proper,

$$p_*: \text{CH}_\ast(X) \to \text{CH}_\ast(Y)$$

is a map of $\text{HK}^*(Y)$-modules. (This is the projection formula.)
2. Algebraic groupoids

2.1. Definition. (a) A *groupoid* is a category in which all morphisms are isomorphisms.

(b) If \( C \) is a category with fibre products, a *groupoid object* \( G \) in \( C \) consists of data \((M, X, s, t, \mu, e, i)\); \( M \) and \( X \) which are objects in \( C \), are the 'spaces' of morphisms and objects respectively of \( G \), while \( s : M \to X \), \( t : M \to X \) are the 'source' and 'target' morphisms, \( e : X \to M \) is the 'unit' morphism, \( \mu : M \times_s M \to M \) is 'multiplication' and \( i : M \to M \) is 'inversion'. These maps satisfy the following axioms:

(i) \( se = te = 1_X \).

(ii) The following diagram commutes:

(iii) The diagram
\[
\begin{array}{ccc}
M \times_s M & \xrightarrow{\mu} & M \\
\downarrow{\mu \times 1} & & \downarrow{\mu} \\
M \times M & \xrightarrow{1 \times \mu} & M \\
\end{array}
\]

commutes. (This is the associative law.)

(iv) (Unit)
\[
\begin{array}{ccc}
M & \xrightarrow{e \cdot s \times 1_M} & M \times_s M \\
\downarrow{1_M \times e} & & \downarrow{\mu} \\
M \times M & \xrightarrow{\mu} & M \\
\end{array}
\]

(v) (Inverses)
(a) \( i^2 = 1_M \).
(b) \( i s = t \) (and therefore \( i t = s \)).
(c) The following diagram commutes.
2.2. Definition. (i) An algebraic groupoid $G$ is a groupoid object in the category of algebraic spaces. We shall say that $G$ is finite (respectively étale) if the map $s$ is finite (étale). Note that if $s$ is finite or étale, $t$, $\mu$ and $e$ must also be finite or étale respectively.

(ii) A sheaf of groupoids $G$ on a site $X$ is a groupoid object in the category of sheaves of sets on $X$. Note that if $G$ is a groupoid object in a category $C$ and if $X$ is an arbitrary object in $C$, then $\text{Hom}(X, C)$ is a groupoid object in the category of sets.

2.3. Definition. If $C$ is a category with fibre products and $G = (M \cong X)$ is a groupoid object in $C$, then we can form a simplicial object $B_k G$ in $C$ in a standard fashion [6] with

$$B_k G = M_s \times_s M_s \times_s M_s \times_s \cdots \times_s M$$

(k factors).

Note that the formation of $B_k G$ commutes with products and coproducts of groupoids.

2.4. Definition. Let $G = (M \cong X)$ be as in 2.3. Then if $f: Y \to X$ is an arbitrary morphism, $\text{cosk}^G_0(Y)$ is the groupoid with object ‘space’ $Y$, morphism ‘space’ $Y_f \times_s M_f \times_f Y$ and source and target maps the two projections to $Y$. The rest of the structure of $\text{cosk}^G_0(Y)$ can be deduced from the isomorphism:

$$B_0 \text{cosk}^G_0(Y) = \text{cosk}^B_0^G(Y).$$

If $G = (M, X)$ is a groupoid in the category of sets, then the image of the natural map $(s, t): M \to X \times X$ is an equivalence relation on $X$, and we denote the quotient of $X$ by this relation as $\pi_0(G)$. Note that $x, y \in X$ belong to the same class (or ‘connected component’) in $\pi_0(G)$ if and only if there is a morphism from $x$ to $y$ in $G$. Given an object $x \in X$, $\text{Hom}_G(x, x)$ is a group, $\pi_1(G, x)$. If $x$ and $y$ are in the same connected component of $X$, then the composition maps

$$\text{Hom}_G(x, x) \times \text{Hom}_G(x, y) \to \text{Hom}_G(x, y),$$

$$\text{Hom}_G(x, y) \times \text{Hom}_G(y, y) \to \text{Hom}_G(x, y)$$
make \( \text{Hom}_G(x, y) \) simultaneously a left \( \pi_1(G, y) \)-torsor and a right \( \pi_1(G, x) \)-torsor, hence for each \( f \in \text{Hom}_G(x, y) \) there is an isomorphism \( \pi_1(G, x) = \pi_1(G, y) \), given by \( \theta \mapsto f \theta f^{-1} \).

In general if \( G = (M \rightarrow X) \) is a groupoid object within a category with products and fibre products we can construct the equalizer \( I(G) \) of the diagram \( M \rightarrow X \).

This will be a group object (the ‘inertia group object’) in the category \( C/X \).

If \( G \) is a groupoid in the category of sets, then

\[
I(G) = \prod_{x \in X} \pi_1(G, x) \quad \text{(which is a ‘group over } X)\).
\]

2.5. Definition. Let \( S \) be a site with a final object \( F \). If \( G = (M \rightarrow X) \) is a sheaf of groupoids on \( S \) and \( \# \) is a crible (= sieve) on \( T \), then a \( G \)-torsor \( (T, \theta) \) locally trivial on \( \# \) consists of:

(i) for each \( U \in \# \) an object \( T_U \in X(U) \),

(ii) for each \( U \in \# \) and open \( V \subset U \), an isomorphism (i.e. element of \( M(V) \)):

\[
\theta_{UV} : T_U \rightarrow T_U|_V
\]

such that for \( W \subset V \subset U \):

\[
(\theta_{UV|W}) \cdot \theta_{VW} = \theta_{UV}.
\]

If \( (T^1, \theta^1) \) and \( (T^2, \theta^2) \) are \( G \)-torsors locally trivial on \( \# \), a morphism \( \psi : (T^1, \theta^1) \rightarrow (T^2, \theta^2) \) consists of, for every \( U \in \# \) an isomorphism

\[
\psi_U : T^1_U \rightarrow T^2_U
\]

such that

\[
\psi_U \theta^1_U = \theta^2_U \psi_U.
\]

The category of \( G \)-torsors locally trivial on \( \# \) is obviously a groupoid, which we shall denote \( T(G, \#) \). If a crible \( \# \) is a refinement of a crible \( \# \) there is an obvious functor

\[
\varphi : T(G, \#) \rightarrow T(G, \#)
\]

such that if \( \# \subset \# \subset \# \).

(Note that equality holds rather than natural isomorphism because of the concrete nature of \( T(G, \#) \)). We define the category of \( G \)-torsors on \( F \) as \( \lim_{\#} T(G, \#) \).

If \( G = (M \rightarrow X) \) is an algebraic groupoid, we shall often talk of \( G \) acting on \( X \).

If \( G \) is an algebraic groupoid with \( X = \text{Spec}(A) \) affine, we shall talk of \( G \) acting on \( A \). This is by analogy with the action of a group (or group scheme) \( H \) on a set (or scheme) \( X \):
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\( m : H \times X \to X \)

from which we deduce a groupoid:

\[ [X : H] = (H \times X \cong X) \]

in which the maps \( H \times X \to X \) are multiplication and projection onto the second factor, while \( X \cdot H \times X \) sends \( x \) to \( (e, x) \).

In this case, \( \mathcal{I}(X : H) \) is the group (or group scheme) over \( X \) with fibre over \( x \in X \) the subgroup of \( G \) fixing \( x \) (and if \( X \) is a scheme, acting trivially on the scheme \( x \)).

From now on we will be dealing almost exclusively with algebraic groupoids.

2.6. Definition. If \( G = (M \cong X) \) is an algebraic groupoid, we define \( H^0(G, \mathcal{C}_G) \) by the sequence

\[
0 \to H^0(G, \mathcal{C}_G) \to H^0(X, \mathcal{C}_X) \xrightarrow{s^* - t^*} H^0(M, \mathcal{C}_M)
\]

(i.e. \( H^0(G, \mathcal{C}_G) = H^0(B \cdot G, \mathcal{C}_{B \cdot G}) \)). If \( X = \text{Spec}(A) \) is affine we will often write \( A^G \) for \( H^0(G, \mathcal{C}_G) \).

2.7. Definition. Let \( G \) be an algebraic groupoid. \( G \) will be said to be connected if it is not the disjoint union of two proper non-empty subgroupoids. Obviously any \( G \) is a finite disjoint union \( \bigsqcup_i H_i \) of its connected components.

2.8. Lemma. Let \( G \) be a connected (finite) étale groupoid acting on a product of fields \( A = \prod_{i=1}^n F_i \). Then \( H^0(G, \mathcal{C}_G) \) is a field, and for any \( S_i = \text{Spec}(F_i) \), if \( G_i = \text{cosk}_0^G(S_i) \),

\[
H^0(G, \mathcal{C}_G) \Rightarrow H^0(G_i, \mathcal{C}_{G_i}).
\]

Proof. If \( G = (M \cong X) \), so that \( X = \prod_{i=1}^n S_i \), then if we write \( H(S_i, S_j) = \text{Spec}(A_{ij}) \) for the fibre product, \( S_i \times_X M \times_X S_j \), \( M = \prod_{1 \leq i, j \leq n} H(S_i, S_j) \) with \( H(S_i, S_j) \) finite and étale over \( S_i \) and \( S_j \). There are projection maps:

\[
\begin{array}{ccc}
H(S_i, S_j) & \xrightarrow{s} & H(S_i) \\
\downarrow s & & \downarrow t \\
S_i & & S_j
\end{array}
\]

(2.8.1)

a composition map

\[
\mu : H(S_i, S_j) \times_{S_k} H(S_j, S_k) \to H(S_i, S_k)
\]

and an inversion map \( t : H(S_i, S_j) \to H(S_j, S_i) \) satisfying the obvious compatibilities. Note that since \( G \) is connected, each \( H(S_i, S_j) \) is non-empty. An element of \( H^0(G, \mathcal{C}_G) \) is a family \( f_i \in F_i \) such that \( s^* f_i = t^* f_j \in A_{ij} \) for all \( i, j \). Let \( E_i \in F_i \) be the subring:
Obviously $E_i$ is a subfield and I claim that for each $i$, there is a canonical isomorphism $H^0(G, \mathcal{O}_G) = E_i$. Clearly the natural map from $H^0(G, \mathcal{O}_G)$ to $E_i$ is injective since the projection maps in diagram (2.8.1) are surjective, so it is enough to show that the map to $E_i$ is onto which is equivalent to showing that for all $i, j$:

$$s^*(E_i) = t^*(E_j) \subseteq A_{ij}.$$

Consider diagram (2.8.1). If $f \in E_i$ we must first show that $s^*f$ descends to $S_j$. Since $t$ is étale surjective, it is enough to show that

$$\pi_1^*s^*f = \pi_2^*s^*f \in A_{ij} \otimes E_i A_{ij}$$

where $\pi_i$ are the projections:

$$
\begin{array}{ccc}
H(S_i, S_j) \times S_j & \xrightarrow{\pi_1} & H(S_i, S_j) \\
\downarrow{\pi_2} & & \downarrow{t} \\
H(S_i, S_j) & \xrightarrow{t} & S_j \\
\end{array}
$$

We have a commutative diagram:

$$
\begin{array}{ccc}
H(S_i, S_j) \times H(S_i, S_j) & \xrightarrow{1 \times \Delta} & H(S_i, S_j) \times H(S_i, S_j) \\
\downarrow{1 \times t \times 1} & & \downarrow{1 \times t \times 1} \\
H(S_i, S_j) \times H(S_i, S_j) & \xrightarrow{\mu \times 1} & H(S_i, S_j) \\
\downarrow{\mu \times 1} & & \downarrow{\mu \times 1} \\
H(S_i, S_j) & \xrightarrow{g_1} & H(S_i, S_j) \\
\end{array}
$$

where $g_1$ is composition and $g_2$ is projection onto the second factor of the product. To prove $\pi_1^*s^*f = \pi_2^*s^*f$ it is therefore enough to show that $g_1^*s^*f = g_2^*s^*f$. But $g_0$ is projection onto the first factor of the product $H(S_i, S_j) \times H(S_i, S_j)$. $s_0 = sg_0$ and $s_0^* = t_0$, hence $g_1^*f = g_2^*s^*f$ for all $f \in E_i$. By descent $s^*f = t^*g$ where $g \in F_j$; a similar diagram chase shows that $g$ is in fact in $E_j$. Hence $\{f_i\} \subseteq \prod F_i$ lies in $H^0(G, \mathcal{O}_G)$ if and only if $f_i \in E_i$ for all $i$ and $s^*f_i = t^*f_j$ for $i \neq j$.

Note that it follows from the proof of the lemma that

$$H^0(G, \mathcal{O}_G) = H^0(\text{cosk}^G(S_i), \mathcal{O}_{\text{cosk}^G(S_i)}) \quad \text{for all } i.$$
2.9. **Corollary.** Let $G$ be a finite étale groupoid acting on a product of fields. Then $H^0(G, \mathcal{O}_G)$ is a product of fields $\prod E_j$ with $E_j = H^0(G_j, \mathcal{O}_{G_j})$ as $G_j$ runs through the connected components of $G$.

2.10. **Theorem.** Let $G = (M \xrightarrow{s} X)$ be a finite étale groupoid acting on a product of fields $\prod F_i$, i.e. $X = \prod \text{Spec}(F_i)$. Then $\prod F_i$ and hence each $F_i$ is a finite étale $H^0(G, \mathcal{O}_G)$-algebra.

**Proof.** By 2.8 and 2.9, it is sufficient to prove that if $X = \text{Spec}(F)$ ($F$ a field) and $E = H^0(G, \mathcal{O}_G) = F^G$, then $F/E$ is a finite separable field extension. If $G_1 = \text{Spec}(A)$, let us write $G(t)$ for the groupoid with object space $\text{Spec}(F(t))$ and morphisms $\text{Spec}(A(t))$. Obviously $H^0(G(t), \mathcal{O}_G(t)) = E(t)$.

Since $s$ and $t$ are both finite and étale there are norm maps

$$s_*, t_* : A(t)^* \to F(t)^*.$$

2.11. **Lemma.** Let $G = (M \xrightarrow{s} X)$ be a finite étale groupoid acting on a field $F$. Then if $\alpha \in F^*$, $s^*t^*\alpha \in F^G$.

**Proof of 2.11.** If $S = \text{Spec}(F)$, and $M = \text{Spec}(A)$, we want to show that $s^*s^*t^*\alpha = t^*s^*t^*\alpha \in A$.

We have two cartesian diagrams:

$$
\begin{array}{ccc}
M \times_M M & \longrightarrow & M \\
p_1 \downarrow & & \downarrow s \\
M & \longrightarrow & S \\
& \quad \downarrow t & \\
& S
\end{array}
$$

($p_i$ the natural projections) (2.11.1)

$$
\begin{array}{ccc}
M \times_M M & \longrightarrow & M \\
p_1 \downarrow & & \downarrow s \\
M & \longrightarrow & S \\
& \quad \downarrow s & \\
& S
\end{array}
$$

(the fact that this diagram is Cartesian is equivalent to the existence of right inverses).

By a standard property of the norm, if $\beta \in A^*$, then from diagram (2.11.1):

$$t^*s^* \beta = p_1^*p_2^* \beta.$$
While from diagram (2.11.2)
\[ s^*s_\beta = p_1^*\mu^*\beta. \]

If \( \beta = t^*\alpha \), then because \( t \cdot \mu = t \cdot p_2 \), we see that
\[ t^*s_\alpha = s^*s_\eta^*\alpha, \]
concluding the proof of the lemma.

**Proof of 2.10 (contd.).** Returning to the theorem, let \( \alpha \in F \). Then \((t-\alpha) \in F(t)^*\) and so \( f(t) = s_\eta^*(t-\alpha) \in E(t)^* \subset F(t)^* \). Note that \( f(t) \) actually lies in \( F[t] \). I claim that \((t-\alpha)|f(t) \in F[t] \). The map \( s : M \to S \) is finite étale and has a section \( e : S \to M \); hence \( M = \bigsqcup s_j \text{Spec}(L_j) = M_j \) with each \( L_j \) a field, and we may suppose \( L_1 = F \) via \( e \). So if \( t-\alpha \in F(t)^* \),
\[ s_\eta^*(t-\alpha) = \prod s_j^*t^*(t-\alpha) \]
where \( s_j, t_j \) are the restrictions of \( s, t \) to \( M_j \). But \( s_1 \) and \( t_1 : M_0 \to S \) are the identity, so \( s_\eta^*(t-\alpha) = ((t-\alpha) \prod f_j(t)) \in F(t) \). Hence for each \( \alpha \in F, \alpha \) satisfies an equation of degree \( \leq \text{deg}(s) \) over \( E \). Hence by a standard result (see e.g. [21, Ch. VIII §1 Lemma 1]) \( F' \) is a finite extension of \( E \) with:
\[ [F' : E] \leq [A : F] \]
where \( F' \subset F \) is the separable closure of \( E \) in \( F \). It remains to show that \( F/E \) is separable, i.e. that \( F' = F \). First we need a definition:

**2.12. Definition.** (i) An algebraic groupoid of the form \( S \times \Gamma \) where \( \Gamma \) is a groupoid in the category of sets will be said to be split.

(ii) If \( \Gamma = (\Gamma_1 \rightrightarrows \Gamma_0) \) is a groupoid in the category of sets, then a functor \( \phi : \Gamma \rightharpoonup \{\text{schemes}\} \) determines an algebraic groupoid \( \{\phi\} = \{M \rightrightarrows \text{schemes} \} \) as follows:
\[ X = \bigsqcup_{x \in \Gamma_0} \phi(x), \quad M = \bigsqcup_{\alpha \in \Gamma_0} \phi(s(\alpha)). \]

The 'source' and 'target' maps of \( \{\phi\} \) are:
\[ s = \bigsqcup_{\alpha} \text{id}_{\phi(s(\alpha))}, \quad t = \bigsqcup_{\alpha} \phi(\alpha) \]
\[ : M = \bigsqcup_{\alpha \in \Gamma_0} \phi(s(\alpha)) \to X = \bigsqcup_{x \in \Gamma_0} \phi(x) \]
(note \( \phi(\alpha) : \phi(s(\alpha)) \to \phi(t(\alpha)) \)). A groupoid of the form \( \{\phi\} \) will be said to be quasi-split. Note that every split groupoid \( S \times \Gamma \) is quasi-split by the functor sending every morphism in \( \Gamma \) to the identity on \( S \).

**Proof of 2.10 (contd.).** Returning to the proof of the theorem, let \( L \) be a separable
Intersection theory on algebraic stacks and \(\mathcal{Q}\)-varieties

The closure of \(E\) and \(\Sigma\) the finite set of embeddings \(\sigma : F' \to L\) fixing \(E\). For each \(\sigma \in \Sigma\), write \(L^\sigma\) for the purely inseparable extension \(L^\sigma \otimes_{F'} F\) of \(L\). Note that \(L^\sigma\) is separably closed, and that we will be done if we can show that \(L = L^\sigma\) for any and hence all \(\sigma\). Since \(L\) is flat over \(E\), by base change we obtain a finite étale groupoid \(G \otimes_E L\): 

\[
F \otimes_E L \cong A \otimes_E L
\]

with \(H^0(G \otimes_E L, F \otimes_E L) \cong L\). Now \(F \otimes_E L\) is isomorphic to \(\prod_{\sigma \in \Sigma} L^\sigma\), so by Lemma 2.8 in order to compute \(\xi^0(G \otimes_E L, F \otimes_E L)\) we may replace \(G \otimes_E L\) with \(G^\sigma = \text{Cosk}_{G \otimes E L}(\text{Spec}(L^\sigma))\). But \(G^\sigma\) is quasisplit, and is in fact a finite group acting on \(L^\sigma\). To see this observe that since \(L^\sigma\) is separably closed \(A \otimes L^\sigma\) is a product of copies of \(L^\sigma\), 

\[
A \otimes L^\sigma = \prod_{h \in H} L^\sigma
\]

and that the index set \(H\) has a natural group structure. Hence 

\[
L = H^0(G \otimes_E L, F \otimes_E L) = (L^\sigma)^H;
\]

and so \(L^\sigma/L\) is separable; but we know that \(L^\sigma/L\) is also purely inseparable and hence \(L = L^\sigma\).

2.13. Definition. Let \(G = (M \rightrightarrows X)\) be an étale groupoid. The order \(o_\sigma(G)\) of \(G\) at a point \(x \in X\) is the degree of \(s\) at \(x\). (Note that since \(s = t\), degree \(s = \text{degree}\), \(t = \text{degree}\).)

Clearly, if \(G\) is quasi-split by a functor \(\phi : \Gamma \to \{\text{Schemes}\}\) and \(x \in \phi(xt)\), then \(o_\sigma(G)\) equals the number of morphisms in \(\Gamma\) with source \(t\).

2.14. Lemma. If \(G = (M \rightrightarrows X)\) is a connected finite étale groupoid, then \(o_\sigma(G)\) is independent of \(x \in X\), and we can write \(o(G)\) without ambiguity.

Proof. If \(X = \coprod X_i\) is the decomposition of \(X\) into connected components, then since the degree of a finite étale morphism is constant on connected components we know that if \(x, y \in X_i\), then \(o_\sigma(G) = o_\sigma(G)\). So it remains to show that \(o_\sigma(G)\) is independent of \(i\). If \(x \in X_i\), then 

\[
o_\sigma(G) = \sum_j \text{deg}(\text{Hom}(X_i, X_j)/X_i)
\]

so it is sufficient to show that for all \(i, j, k\), 

\[
\text{deg}(\text{Hom}(X_k, X_i)/X_i) = \text{deg}(\text{Hom}(X_k, X_j)/X_j). \tag{2.14.1}
\]

Consider the diagram:

\[
\text{Diagram}
\]
The existence of inverses implies that $\mu \times 1$ is an isomorphism, so the lemma follows since the degree of a finite étale morphism is stable under base change.

2.15. Definition. (i) Let $G = (M \subseteq X)$ be a connected (finite) étale groupoid acting on a product of fields. Then if $E = H^0(G, \mathcal{O}_G)$, we define the inertia index $e(G)$ of $G$ by

$$e(G) = o(G) / \text{degree}(X/\text{Spec}(E)).$$

Note that if $G$ is split, $G = \Gamma \times S$, then $e(G) = \# \pi_1(\Gamma)$, where $\# \pi_1(\Gamma)$ is well defined since $\Gamma$ is connected.

2.16. Lemma. Let $G = (M \subseteq X)$ be a connected finite étale groupoid acting on a product of fields. $I(G)$ the inertia group scheme (over $X$) of $G$. Then $e(G) = \text{degree of } I(G) \text{ over } X$.

Proof. If $E = H^0(G, \mathcal{O}_G)$, and $\bar{E}$ is an algebraic closure of $E$, then $G \otimes_E \bar{E}$ is split and since the degree of an étale morphism is stable under base change,

$$e(G) = e(G \otimes_E \bar{E}) = \# \pi_1(\Gamma)$$

where $G \otimes_E E = \Gamma \times \text{Spec}(E)$; but the formation of $I(G)$ is also compatible with étale base change so

$$\text{degree}(I(G)/X) = \text{degree}(I(G \otimes_E \bar{E})/X \otimes_E \bar{E}) = \text{degree}(I(\Gamma)/\Gamma_0) = \# \pi_1(\Gamma).$$

Replacing $\bar{E}$ by any sufficiently large algebraically closed field ($= \text{universal domain}$) $\Omega$, we see that

$$G(\Omega) = \Gamma \times \text{Hom}(E, \Omega)$$

so that $e(G) = \# \pi_1(G(\Omega))$.

The lemma allows us to define $e(G)$ for a general groupoid.

2.17. Definition. Let $G = (M \subseteq X)$ be an étale algebraic groupoid. If $x \in X$ we define
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\[ \varepsilon_x(G) = \# \pi_1(G(\Omega), x) \]

where $\Omega$ is a universal domain and $x : \text{Spec}(\Omega) \to X$ is a geometric point with image $x$.

3. Algebraic stacks, a review

3.1. Definition. (i) By an algebraic stack $F$ we mean (in the terminology of [2]) a stack in groupoids $F$ over the category of schemes with the étale topology such that products in $F$ are represented by algebraic spaces and such that there is an étale surjective 1-morphism $x : X \to F$ with $X$ a scheme.

Note that if $X$ is a scheme or more generally an algebraic space, the associated algebraic stack (which we also write as $X$) has as fibre over a scheme $S$ the set $X(S)$ of $S$-valued points of $X$, viewed as a groupoid with only identity morphisms. If $F$ is an arbitrary stack, a 1-morphism $x : X \to F$ for $X$ a scheme may be identified with the object $x = x(1_X)$ in the fibre $F(X)$ of $F$ over $x$. If

\[
\begin{array}{ccc}
F & \xrightarrow{f} & G \\
\downarrow & & \downarrow \\
H & \xrightarrow{g} & K
\end{array}
\]

is a diagram of stacks, the pseudo-fibred product $F \times_H G$ is the stack such that for each scheme $S$, $F \times_H G(S)$ has objects $(x, y, \alpha)$ with $x \in \text{Ob}(F(S))$, $y \in \text{Ob}(G(S))$ and $\alpha : f(x) \cong g(y)$ an isomorphism in $H(S)$. All the usual rules for products hold up to natural equivalence, e.g.,

\[ (F \times_H G) \times_K (H \times_K L) = (F \times_H G) \times_K L = F \times_G (H \times_K L). \]

(ii) A 1-morphism (or simply, morphism) of algebraic stacks $f : F \to G$ is said to be representable if for any morphism $x : X \to G$ with $X$ a scheme, the fibre product $X \times_G F$ is represented by an algebraic space.

Note that if $x : X \to F$, $y : Y \to F$ with $X, Y$ schemes and $\pi = x(1_X)$, $\pi = y(1_Y)$ then $X \times_F Y \to f$ corresponds to the product $\pi \times \pi'$ in $F$. Hence the representability of morphisms from schemes to $f$ is equivalent to the existence of products in the category $F$. In particular the moduli topologies of Mumford [15] are stacks. Finally, notice that if $H$ is a scheme, then $F \times_H G = F \times_H G$.

For further details and properties of stacks see [1] and [2]; for information regarding algebraic spaces see [13].

If $F$ is an algebraic stack and $X$ is a scheme, we shall write $F(X)$ for the fibre or category of sections of $F$ over $X$. Note that if each component of $F(X)$ (which is a groupoid) has trivial $\pi_1$, then $F$ is an algebraic space.
An étale surjective morphism $x : X \to F$ from a scheme to a stack $F$ will be called an atlas of $F$. Since the morphism $x$ is étale, the projections $X \times_F X \to X$ are étale too, hence $X \times_F X$, which a priori is an algebraic space, is actually a scheme. Note that $G = (X \times_F X \cong X)$ is an algebraic groupoid which represents a stack equivalent to $F$, i.e. for every scheme $S$ the category $G/S$ of $G$-torsors over $S$ is equivalent to $F(S)$. Finally, all sheaves on a stack will be sheaves on the étale site of $F$ [2].

3.2. Definition. Let $F$ be an algebraic stack. Then the quotient space of $F$, $|F|$ is the topological space with points $t$ corresponding to the integral (i.e. reduced and irreducible) substacks $T \subset F$. An open subset $|U| \subset |F|$ corresponds to an open substack $U \subset F$ and contains all points $z$ such that the integral substack $Z \subset F$ has a non-empty intersection with $U$. In this topology $\{z\} = |Z| \subset |F|$.

Observe that there is 1-1 correspondence $T \to p^{-1}(T) = Y$ between substacks $T \subset F$ and subschemes $Y \subset X$ (where $p : X \to F$ is an atlas) for which $s^{-1}(Y) = r^{-1}(Y) \subset X \times_F X$ (where $s$ and $t$ are the projections $X \times_F X \to X$) such that $T$ is open, closed or reduced if and only if $Y = p^{-1}(T)$ is. If $Y \subset X$ is a subscheme, there is therefore a unique minimal integral substack $p(Y) = T \subset F$ such that $Y \subset p^{-1}(T)$.

3.3. Proposition. Let $F$ be an algebraic stack and $p : X \to F$ be an atlas of $F$ with $G = (X \times_F X \cong X)$ the associated groupoid. Then:

(i) $F$ is irreducible if and only if the groupoid $\mathfrak{g}$ of generic points of $G$ is connected.

(ii) There is a 1-1 correspondence between integral substacks $T \subset F$ and $G$ orbits in $|X|$.

(iii) $|F| = |X|/G$ (where $|X|/G$ is the space of $G$ orbits in $|X|$).

(iv) $|F|$ is a Zariski topological space of the same dimension as $|X|$.

(An orbit of $G$ in $|X|$ is an equivalence class for the relation $\text{Im}(|X \times F| \to |X| 	imes |X|)$.)

Proof. (i) Suppose $F = F_1 \cup F_2$ with $F_1$ and $F_2$ non-empty closed substacks of $F$ such that $F_1 \subset F_2$ and $F_2 \subset F_1$. If $U_i = F - F_i$, then $U_1$ and $U_2$ are disjoint open substacks $F$, so that $p^{-1}(U_1)$ and $p^{-1}(U_2)$ are disjoint $G$-stable open subschemes of $X$. But any $G$-stable subset of $|X|$ is a union of orbits; hence the sets of generic points of $X$ contained in $U$ and $V$ contain disjoint $G$-orbits and $G$ cannot be connected. On the other hand, if $g = g_1 \cup g_2$, $g_i$ non-empty, then $X = X_1 \cup X_2$ with each $X_i$ non-empty, closed in $X$ and $G$-stable, and $X_1 \subset X_2, X_2 \subset X_1$. Since each $X_i$ is $G$-stable, $X_i = p^{-1}(F_i)$ with $F_i \subset F$ a closed substack, and $F = F_1 \cup F_2$ with $F_1 \subset F_2, F_2 \subset F_1$ so $F$ is reducible.

(ii) If $T \subset F$ is an integral stack, the generic points of $p^{-1}(T) \subset X$ are a $G$-orbit by (i), and if $S \subset |X|$ is a $G$-orbit, then giving $\{S\}$ its reduced subscheme structure, we get a subscheme $Y \subset X$ satisfying $s^{-1}(Y) = r^{-1}(Y) \subset X \times_F X$ and hence there is a reduced substack $T \subset F$ such that $Y = p^{-1}(T)$. Since $S$ is a $G$-orbit, the groupoid of
generic points of \( \cosk^2(Y) \) is connected and \( T \) is irreducible.

(iii) follows immediately from (ii) and the definition of \( |T| \).

(iv) \( |T| \) is noetherian since \( |X| \) is noetherian, and is Zariski by construction. To see that \( \dim |T| = \dim X \), by (ii) it is sufficient to check that if \( Y \subset Z \) are integral subschemes of \( X \), then \( p(Y) \neq p(Z) \subset T \). But \( p(Y), p(Z) \) correspond by (ii) to the \( G \)-orbits \( t(s^{-1}(y)), t(s^{-1}(z)) \) where \( y, z \) are the generic points of \( Y \) and \( Z \) respectively, and since \( s \) and \( t \) are \( \eta \)tale they preserve codimension, and so \( t(s^{-1}(y)) \cap t(s^{-1}(z)) = \emptyset \).

3.4. Corollary. If \( f : F \to G \) is a morphism of algebraic stacks, there is an induced map \( |f| : |F| \to |G| \).

Proof. Choose atlases of \( F \) and \( G \), \( p : X \to F, q : Y \to G \) so that there is a diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{p} & F \\
\downarrow g & & \downarrow f \\
Y & \xrightarrow{q} & G
\end{array}
\]

then we get an induced diagram:

\[
\begin{array}{ccc}
|X \times_F X| & \xrightarrow{g \times g} & |X| \\
\downarrow g & & \downarrow g \\
|Y \times_G Y| & \xrightarrow{g \times g} & |Y|
\end{array}
\]

and, on passing to quotients, the desired map \( |F| \to |G| \).

3.5. Definition. (i) A stack \( F \) is punctual if it is reduced and \( |F| \) is a single point; by 3.3(iii), this is equivalent to requiring that for any atlas \( p : X \to F, X = \text{Spec}(E) \) with \( E \) a product of fields, and that \( |X \times_F X| \to |X| \times |X| \) is surjective.

(ii) If \( F \) is an arbitrary integral stack, the generic point of \( F \) is the punctual stack \( \phi = F - \bigcup_{\xi \subset F} G \).

Note that the points of \( |F| \) are in 1-1 correspondence with the punctual substacks of \( F \).

(ii) By 2.10, if \( S \) is a punctual stack, \( H^0(S, \mathcal{O}_S) = k \) is a field and \( E/k \) is a finite separable extension. If \( F \) is an integral stack with generic point \( \phi \), we call \( H^0(\phi, \mathcal{O}_\phi) \) the function field of \( F \), and write it as \( k(F) \).

3.6. Lemma. If \( \xi \) is a punctual stack, then for any atlas \( p : U \to \xi \), the ramification index of the finite \( \xi \)-valued groupoid \( G = (U \times_{\xi} U \equiv U) \) is independent of the atlas \( U \),
and equals \( #(\pi_1(F(\Omega))) \) where \( \Omega \) is any separable closure of \( k(\xi) \).

**Proof.** All \( G \)-torsors over \( \text{Spec}(\Omega) \) are trivial, since \( \Omega \) is a point in the étale topology; hence the groupoid of \( G \)-torsors, which is equivalent to \( F(\Omega) \), is equivalent to \( G(\Omega) \) and so \( #(\pi_1 G(\Omega)) = #(\pi_1 F(\Omega)) \).

### 3.7. Definition

If \( F \) is an algebraic stack, \( x \in [F] \) a point of the quotient space, we define the ramification index \( e(x) \) as the ramification index of the associated punctual substack of \( F \).

If \( G \) is a finite group of order \( n \) acting trivially on a scheme \( X \), then we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & [X : G] \\
\downarrow {1_X} & & \downarrow f \\
X & & \\
\end{array}
\]

where \([X : G]\) is the quotient stack and \( X \) is the quotient scheme. The morphism \( p \) has degree \( n \), for if \( U \to [X : G] \) where \([X : G]\) is the pseudo-quotient stack (this is analogous to the homotopy quotient \( FG \times_G X \) of topology; in particular \( p \) is finite étale with geometric fibres isomorphic to \( G \)) and \( X \) is the quotient scheme. The morphism \( p \) has degree \( n \), for if \( U \to [X : G] \) is any atlas (for any example \( p \) itself) \( X \times_{[X : G]} U \cong U \times G \) has degree \( n \) over \( U \). On the other hand \( 1_X \) has degree 1; so to be consistent we should define the degree of \( f \) to be \((1/n)\).

More generally:

### 3.8. Definition

(i) Let \( f : \xi \to \eta \) be a finite morphism of punctual stacks. Define the degree of \( f \) to be:

\[
[\xi : \eta] = \frac{e(\eta)}{e(\xi)} \cdot [k(\xi) : k(\eta)]
\]

(note that this is a rational number).

(ii) If \( f : F \to G \) is an arbitrary quasi-finite morphism of stacks and \( \eta \subset G \) is a punctual substack, then \( \text{deg}_\eta(f) \) (= degree of the reduced fibre) is

\[
\text{deg}(F(\eta) : \eta) = \sum_{i=1}^{n} [\xi_i : \eta]
\]

where \( F(\eta) \) is the reduced fibre of \( F \) over \( \eta \), and \( \xi_1, \ldots, \xi_n \) are the connected components of \( F(\eta) \).

### 3.9. Note

(i) The degree of the reduced fibre is stable under étale base change,

(ii) If \( f : F \to G, g : G \to H \) are quasi-finite morphisms of stacks, then for any \( \eta \subset H \)
a punctual substack with $\xi_1, \ldots, \xi_n$ the connected components of $G(\eta)$,

$$\deg_\eta(g \cdot f) = \sum_{i=1}^n \deg_{\xi_i}(f)[\xi_i : \eta].$$

3.10. Definition. Let $F$ be an algebraic stack. A coarse moduli space for $F$ is a scheme $X$ together with a proper morphism $p : F \to X$ such that

(i) $|F| \to |X|$ is a homeomorphism,

(ii) $\forall x \in X, k(p^{-1}(x))|k(x)$ is a purely inseparable extension. (We write $i(x) = [k(p^{-1}(x)) : k(x)].$

Conditions (i) and (ii) are equivalent to saying that for any algebraically closed field $\Omega$:

$$\pi_0 F(\Omega) = X(\Omega).$$

Note that if $f : Y \to X$ is an arbitrary morphism, then $Y$ is a coarse moduli space for $F \times_X Y$.

4. Chow groups of stacks

4.1. Definition. Let $T$ be an algebraic stack. A prime cycle $Z$ on $T$ is an integral (i.e. reduced and irreducible) substack $Z \subset T$. By the dimension or codimension of $Z$ we mean the dimension or codimension of $p^{-1}(Z) \subset U$ for $p : U \to T$ any atlas of $T$. A cycle of dimension $p$ (codimension $q$, respectively) on $T$ is an element of the free abelian group $Z_p(T)$ ($Z_q(T)$, respectively) generated by the prime cycles of dimension $p$ (codimension $q$).

This definition is a generalization of the definition of a cycle on a variety. Recall that the cycle groups $Z_q(U)$, $Z_q(U)$ of a scheme $U$ satisfy the following properties:

(i) $U \to Z_q(U)$ is a contravariant functor on the category of flat morphisms between varieties.

(ii) $U \to Z_q(U)$ is a contravariant (respectively covariant) functor on the category of étale (respectively proper) morphisms between varieties.

$Z_q$ and $Z_q$ are therefore presheaves on the étale site of a stack. If $p : U \to V$ is a surjective étale map of schemes, consider the sequence

$$Z'_q(V) \xrightarrow{p^*} Z'_q(U) \xrightarrow{\text{proj}} Z'_q(U \times_V U). \tag{4.1.1}$$

If $z = \sum n_i [Z_i]$ is a cycle on $U$ such that $p_1^* z = p_2^* z$ and $y_1$ and $y_2$ are two points of codimension $q$ on $U$ with $p(y_1) = p(y_2)$ there is a point $w \in U \times_V U$ of codimension $q$ with $p_1(w) = y_i$ for $i = 1, 2$. Since $p_1$ and $p_2$ are étale:

$$v_{y_1}(z) = v_{y_2}(p_1^* z) = v_{y_2}(p_2^* z) = v_{y_2}(z)$$
(where \( v_p(z) \) is the multiplicity of the prime cycle \( \{ y \} \) in \( z \)) and hence we may define a cycle \( z \) on \( V \) by setting \( v_p(y)(z) = v_p(z) \).

By construction, \( p^*(z) = z \) and we conclude that the sequence (4.1.1) is exact. Since dimension, as well as codimension, is stable under étale localization, this conclusion holds if we replace \( Z_q \) by \( Z_q \).

Summarizing:

4.2. Lemma. If \( T \) is an algebraic stack, \( Z_q \) and \( Z_q \) are sheaves on the étale topology of \( T \).

The preceding argument applies equally well if \( V \) is a stack and \( U \) is an atlas, hence:

4.3. Lemma. If \( T \) is an algebraic stack, then \( Z_q(T) = H^0(T, Z_q) \) and \( Z_q(T) = H^0(T, Z_q) \).

If \( X \) is an integral stack with generic point \( \xi \), then its function field \( k(\xi) = H^0(\xi, \mathcal{O}_\xi) \) can also be thought of as \( H^0(X, \mathcal{O}_X) \) where \( \mathcal{O}_X \) is the sheaf of total quotient rings on the étale site of \( X \). Since the formation of Weil divisors commutes with flat pull back, we get a homomorphism of sheaves of abelian groups

\[
\text{div} : \mathcal{O}_X \rightarrow \mathbb{Z}
\]

and an induced map:

\[
\text{div} : k(\xi) \rightarrow \mathbb{Z}(X).
\]

4.4. Definition. If \( T \) is an algebraic stack, we have groups of rational equivalences:

\[
\pi_q(T) = \bigoplus_{x \in |T|_{(q)}} k(x)^*, \quad \pi_q(T) = \bigoplus_{x \in |T|_{(q)}} k(x)^*
\]

(where \( |T|_{(q)} \) and \( |T|_{(q)} \) are the sets of points of \( T \) of codimension \( q \) and dimension \( q \) respectively). Again, by arguments paralleling those leading to 4.2 and 4.3, the presheaves \( U \mapsto \pi_q(U) \) and \( U \mapsto \pi_q(U) \) are actually sheaves on the étale site of \( T \), and

\[
\pi_q(T) = H^0(T, \pi_q), \quad \pi_q(T) = H^0(T, \pi_q).
\]

There are homomorphisms of sheaves

\[
d : \pi_q \rightarrow \mathbb{Z} \quad \text{and} \quad d : \pi_q \rightarrow \mathbb{Z}
\]

defined on sections over \( U \in (T)_c \) by sending each rational equivalence on \( U \) to its associated Weil divisor viewed as a cycle on \( U \) (note that we only take those components of the Weil divisor of the right codimension or dimension on \( U \)). We define the Chow groups of \( T \) as the cokernels of the corresponding maps on global sections:
4.6. **Proposition.** For all \( q \geq 0 \),

(i) \( \text{CH}^q \) is a contravariant functor on the category of flat morphisms between algebraic stacks.

(ii) \( \text{CH}_q \) is a contravariant functor on the category of étale morphisms between algebraic stacks.

(iii) \( \text{CH}_q \) is a covariant functor on the category of proper representable morphisms between algebraic stacks.

**Proof** (i) A morphism \( f: S \to T \) of stacks is flat if and only if there is a diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{p} & S \\
\downarrow g & & \downarrow f \\
V & \xrightarrow{q} & T
\end{array}
\]  

with \( p \) and \( q \) atlases of \( S \) and \( T \), and \( g \) a flat morphism of schemes. There is an induced diagram of flat morphisms, for \( i = 1 \) or \( 2 \):

\[
\begin{array}{ccc}
U \times_S U & \xrightarrow{p_i} & U \\
g \times g & \downarrow & g \\
V \times_T V & \xrightarrow{p_i} & V
\end{array}
\]  

and hence a commutative diagram:

\[
\begin{array}{ccc}
Z^q(U) & \xrightarrow{p_i^*} & Z^q(U \times_S U) \\
g^* & & (g \times g)^* \\
Z^q(V) & \xrightarrow{p_i^*} & Z^q(V \times_T V)
\end{array}
\]  

and hence by 4.3 a map \( f^*: Z^q(T) \to Z^q(S) \). Similar diagrams exist if we replace \( Z^q \) by \( \mathcal{W}^q \); and since the formation of Weil divisors is compatible with flat pull back, we get a commutative diagram:
and an induced map:

\[ f^* : \text{CH}^q(T) \to \text{CH}^q(S) \]

Clearly, if \( f : S \to T \) and \( g : T \to X \) are flat,

\[ f^* g^* = (g \cdot f)^* \]

(ii) \( f : S \to T \) is étale if and only if in diagram (4.6.1) \( g \) is étale and if \( g \) (and hence \( g \times g \)) is étale, \( g^* \) preserves dimension as well as codimension, and the proof of (i) applies, replacing \( Z^q \) by \( Z_q \) and \( \#^q \) by \( \#_q \) to define a map

\[ f^* : \text{CH}^q(T) \to \text{CH}^q(S) \]

(iii) Let \( f : S \to T \) be a proper representable morphism, then if \( p : U \to T \) is an atlas, \( U \times_T S = V \) is an atlas of \( S \), and the induced map \( f_U : V \to U \) is proper. Furthermore in the diagram \((i = 1 \text{ or } 2)\):

\[
\begin{array}{ccc}
V \times_S V & \xrightarrow{p_i} & V \xrightarrow{q} S \\
\downarrow f_{U \times U} & & \downarrow f_U & \downarrow f \\
U \times_T U & \xrightarrow{p_i} & U \xrightarrow{p} T
\end{array}
\]

the left hand square is cartesian. Hence (1.3) we have commutative diagrams:

(4.6.2)

\[
\begin{array}{ccc}
\text{Z}_q(V \times V) & \xleftarrow{p_i^*} & \text{Z}_q(V) \\
\downarrow f_{U \times U} & & \downarrow f_U \\
\text{Z}_q(U \times U) & \xleftarrow{p_i^*} & \text{Z}_q(U)
\end{array}
\]

and

(4.6.3)

\[
\begin{array}{ccc}
\#_q(V \times V) & \xleftarrow{p_i^*} & \#_q(V) \\
\downarrow f_{U \times U} & & \downarrow f_U \\
\#_q(U \times U) & \xleftarrow{p_i^*} & \#_q(U)
\end{array}
\]
and \( d : \mathcal{H}_q \to \mathbb{Z}_q \) induces a map of diagrams (4.6.3)\( \to \)4.6.2, and hence by 4.3 a commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_q(S) & \xrightarrow{d} & \mathbb{Z}_q(S) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{H}_q(T) & \xrightarrow{d} & \mathbb{Z}_q(T)
\end{array}
\]

and hence a map \( f_* : \text{CH}_q(S) \to \text{CH}_q(T) \). If \( V \) is not a scheme, but an algebraic space, then the argument above still applies, using the methods of 7.10.

We would like to generalize the relationship between the Chow groups and \( K \)-theory of a scheme to stacks, but as discussed in the introduction, the analogy is not precise. We first introduce certain variants of the Chow groups.

4.7. Definition. Let \( F \) be an algebraic stack; then for every \( p \in \mathbb{Z} \) we define:

(i) \( R^*(p) \) is the presheaf on the \( \text{étale} \) site of \( F \):

\[ U \mapsto R^*(U, p), \]

(ii) \( R_*(p) \) is the presheaf

\[ U \mapsto R_*(U, p) \]

where \( R^*(U, p) \) is the complex

\[ \bigoplus_{x \in U^{(m)}} K_p k(x) \to \cdots \to \bigoplus_{x \in U^{(1)}} K_p k(x) \to \cdots \to \bigoplus_{x \in U^{(n)}} \mathbb{Z} \]

and \( R_*(U, p) \) is the complex

\[ \cdots \to \bigoplus_{x \in U_{(i)}} K_{p+i} k(x) \to \cdots \to \bigoplus_{x \in U_{(m)}} K_p k(x). \]

Here \( K_* \) are the higher \( K \)-functors of Quillen [17] and \( U^{(i)} \), \( U_{(j)} \) are the sets of points with closure having codimension \( i \) and dimension \( j \) in \( U \) respectively. See [17, §7] for more details on these complexes.

(iii) Viewing \( K_p(\mathcal{O}_F) \) as a presheaf on \((F)_{\text{ét}}\), there are products:

\[ K_p(\mathcal{O}_F) \otimes R^*(q) \to R^*(p + q), \]

\[ K_p(\mathcal{O}_F) \otimes R_*(q) \to R_*(q - p). \]

(iv) We define 'étale' versions of the groups of Section 1:

\[ HK^{p, q}_{\text{ét}}(F) = H^{p}_{\text{ét}}(F, K_q(\mathcal{O}_F)), \]

\[ CH^{i, s}_{\text{ét}}(F) = H^{i}_{\text{ét}}(F, R^*(s)), \]

\[ CH^{r, t}_{\text{ét}}(F) = H^{r}_{\text{ét}}(F, R_*(s)). \]
Note that if $F$ has pure dimension $n$,
\[ \text{CH}_{n,s}^{\text{et}}(F) = \text{CH}_{n}^{\text{et}}(s, n - n)(F). \]

4.8. Proposition. (i) $HK_{\text{et}}^{s,*}(F)$ is a contravariant bigraded ring valued functor of $F$. (ii) $\text{CH}_{\text{et}}^{s,*}$ is contravariant with respect to flat morphisms.

Proof. (i) follows immediately from the fact that $X \mapsto K_{s}(X)$ is a contravariant ring valued functor on the category of all schemes.

(ii) follows from the fact that for each $q \geq 0$, $R^{s}(q)$ is a contravariant functor on the category of flat morphisms between schemes.

4.9. Note. If $F$ is a regular stack, then by Quillen's proof of Gersten's conjecture (1.4), the natural augmentation
\[ K_{q}(\mathcal{O}_{F}) \rightarrow R^{s}(q) \]

is a quasi-isomorphism, and hence
\[ \text{CH}_{\text{et}}^{s,*}(F) = HK_{\text{et}}^{s,*}(F), \]
so that $\text{CH}_{\text{et}}^{s,*}(F)$ is a bigraded ring.

5. Descent theorems for rational $K$-theory

Theorem 5.1. Let $G = (G_{1} \times G_{0})$ be a finite étale groupoid. For $p \geq 0$ consider the cosimplicial abelian group $K_{p}^{r}(B \cdot G)$:
\[ K_{p}(G_{0}) \xrightarrow{s_{*}} K_{p}^{r}(G_{1}) \xrightarrow{t_{*}} K_{p}(G_{1} \times G_{0} \times G_{1}) \]

(recall that $B_{k}G = G_{1} \times G_{0} \times G_{1} \times G_{0} \times G_{1} \times G_{0} \times G_{1}$ is the $k$-fold fibre product of $G_{1}$ over $G_{0}$). Then, for $i > 0$, $H^{i}(K_{p}^{r}(B \cdot G))$ is $n$-torsion where $n$ is the lowest common multiple of the orders of the connected components of $G$.

Proof. If $G = \coprod_{i=1}^{m} G_{i}$ with the $G_{i}$ connected groupoids, $K_{p}^{r}(B \cdot G) = \oplus_{i=1}^{m} K_{p}^{r}(B \cdot G_{i})$ so it is sufficient to prove the theorem when $G = (G_{1} \times G_{0})$ is connected. Arguing as in 2.11 we know that if $s_{*}, t_{*} : G_{1} \rightarrow G_{0}$ are the 'source' and 'target' maps of $G$, which we may identify with the maps $d_{0}, d_{1} : B_{1}G \rightarrow B_{0}G$, and if $\alpha \in K_{p}(G_{1})$, then $t_{*}s_{*}(\alpha) \in H^{0}(K_{p}^{r}(B \cdot G)) \subset K_{p}(G_{0})$. If $\alpha \in H^{0}(K_{p}^{r}(B \cdot G))$ already, so that $s_{*}t_{*} = \alpha$, then $t_{*}s_{*}(\alpha) = t_{*}t_{*}^{s}(\alpha) = n\alpha$ by the projection formula, so we have a projection operator:
\[ 1/nt_{*}s_{*}^{s} : K_{p}^{r}(G_{0})(1/n) \rightarrow H^{0}(K_{p}^{r}(B \cdot G))(1/n). \]
For $k \geq 1$ we can identify $H^0(K_p'(B,G))$ with a subgroup of $K_p'(B_kG)$ via the map $d_k^*d_{k-1}^* \cdots d_1^* : K_p'(G_0) \rightarrow K_p'(B_kG)$, and the map

$$\pi_k = [d_k^*d_{k-1}^* \cdots d_1^*(d_1^*d_0^*)(d_0^*)^k] / n^{k+1}$$

is a projection operator

$$K_p'(B_kG) \left( \frac{1}{n} \right) \rightarrow H^0(K_p'(B \cdot G)) \left( \frac{1}{n} \right) \subset K_p'(B_kG) \left( \frac{1}{n} \right).$$

Since $d_{i_1}^* \cdots d_{i_{k+1}}^*(\alpha) = d_{k+1}^*d_k^* \cdots d_1^*(\alpha)$ for any $\alpha \in H^0(K_p'(B,G))$ and any $(k+1)$-tuple $(i_1, \ldots, i_{k+1})$ of integers between 1 and $k+1$, we see by the projection formula (remember $d_i : B_kG \rightarrow B_{k-1}G$ is a finite étale map of degree $n$ so $d_*d^* =$ multiplication by $n$) that the $\pi_k$ commute with all coface and codegeneracy maps. Recall (2.11) that from the existence of inverses for $G$, we know that the square

$$G_1 \times_{G_0} G_1 \xrightarrow{d_0} G_1$$

is Cartesian. More generally, for the same reason, know that all squares:

$$B_kG \xrightarrow{d_{i-1}} B_{k-1}G$$

are Cartesian, and hence by (1.2), that

$$d_{i-1}^*d_i^* = d_i^*d_{i-1}^* : K_p'(B_{k-1}G) \rightarrow K_p'(B_{k-1}G). \quad (4.1.1)$$

Applying this equation repeatedly, we see that

$$d_k^*d_{k-1}^* \cdots d_1^*(d_1^*d_0^*)(d_0^*)^k = (d_k^*d_{k-1}^*) \cdots (d_1^*d_0^*).$$

Now for each $k \geq 1$ and $1 \leq i \leq k$, we have maps:

$$h_i : K_p'(B_kG)(1/n) \rightarrow K_p'(B_{k-1}G)(1/n)$$

defined as follows:

$$h_1 = (1/n^k)(d_k^*d_{k-1}^*) \cdots (d_1^*), \quad \ldots$$

$$h_i = (1/n^{k+1-i})(d_k^*d_{k-1}^*) \cdots (d_i^*d_{i-1}^*)d_i^*, \quad \ldots$$

$$h_k = (1/n)d_k^*. $$
By equation (4.1.1) and the projection formula we have the following equalities between maps $K'_p(B_k G) \to K'_p(B_{k-1} G)$

$$h_1 d_0^* = \pi_{k-1},$$
$$h_1 d_i^* = h_{i+1} d_i^*,$$
$$h_k d_k^* = \text{Identity}.$$

Hence [14] the identity is homotopic to $\pi_*$, and so

$$H^i(K'_p(B,G)(1/n)) = H^i(k \to H^0(K'_p(B,G)(1/n)) = 0 \text{ if } i > 0.$$

5.2. Corollary. Let $G = (G_1 \to G_0)$ be a connected finite étale groupoid acting on a product of fields $A$, then for $p \geq 0$

$$H^i(K_p B, G)_{\mathbb{Q}} = \begin{cases} 0, & i > 0 \\ K_i(E)_{\mathbb{Q}}, & i = 0 \end{cases} \text{ where } E = A^G.$$

**Proof.** Since $G_0$ is regular, $K_p B, G = K_p B, G$, so by the theorem $H^i(K_p B, G)$ is $n$-torsion for $i > 0$ where $n$ is the order of $G$. Now by 2.11, $E = A^G$ is a field, and $A$ is a finite étale $E$-algebra. Consider the diagram (where $S = \text{Spec}(E)$):

\[
\begin{array}{ccc}
G_1 & \xrightarrow{d_0} & G_0 \\
\pi \downarrow & & \downarrow p_0 \\
G_0 \times_S G_0 & \xrightarrow{p_1} & G_0 = \text{Spec}(A) \\
\downarrow d_0 & & \downarrow \varepsilon \\
G_0 & \xrightarrow{\varepsilon} & S
\end{array}
\]

Since by definition of $A^G$, the sequence

$$0 \to E \to A \xrightarrow{d_0^* - d_1^*} \Gamma(G_1, \mathcal{O}_{G_1})$$

is exact, the map $\pi$ is finite étale and surjective of degree $n/[A : E] = e(G)$. Hence

$$d_1 d_0^* = p_1 \ast \pi_* \ast p_0^* = e(G) p_1 \ast p_0^* - e(G) \varepsilon \ast \varepsilon_*,$$

but $1/[A : G] \varepsilon \ast \varepsilon_*$ is projection onto $\varepsilon \ast (K_p(E)_{\mathbb{Q}}) \subset K_p(A)_{\mathbb{Q}}$, and so we are done.

5.3. Corollary. Let $F$ be a punctual stack. Then for every $p \geq 0$ the presheaf $K_p(\mathcal{O}_F)_{\mathbb{Q}}$ on the étale site of $F$ is actually a sheaf, and if $E = H^0(F, \mathcal{O}_F)$,

$$H^i_{\text{et}}(F, K_p(\mathcal{O}_F))_{\mathbb{Q}} = \begin{cases} 0, & i > 0 \\ K_i(E)_{\mathbb{Q}}, & i = 0 \end{cases}.$$
**Proof.** Since $K_p(\prod_{i=1}^n S_i) = \prod_{i=1}^n K_p(S_i)$ it is sufficient to observe that for any étale map $x: \text{Spec}(E) \to F$ with $E$ a field and for any finite étale $E$-algebra $A$, we may apply Corollary 4.2 to the groupoid $\text{cosk}^{\text{Spec}(E)}(\text{Spec}(A))$ to see that the sequence

$$0 \to K_p(E) \otimes \to K_p(A) \otimes \to K_p(A \otimes E) \otimes$$

is exact, and hence $K_p(A) \otimes$ is a sheaf. Now, let $x: X = \text{Spec}(A) \to F$ be any étale surjective map; then $G = (X, X \times_X X)$ is a finite étale groupoid acting on the product of fields $A$, so that $A^G = E$. By cohomological descent, since $B.G = \text{cosk}^F(X)$,

$$H^i(B.G, K_p(\mathcal{F}_{B,G} \otimes)) = H^i(F, K_p(\mathcal{F}_B \otimes)).$$

There is the usual spectral sequence

$$H^i(B.G, K_p(\mathcal{F}_{B,G} \otimes)) = H^{i+j}(B.G, K_p(\mathcal{F}_{B,G} \otimes)).$$

Hence, if we prove that

$$H^i(B.G, K_p(\mathcal{F}_{B,G} \otimes)) = 0$$

unless $i = 0$, we will have:

$$H^i(F, K_p(\mathcal{F}_B \otimes)) = H^i(K_p(B.G) \otimes) = \{0, i > 0, K_p(E) \otimes, i = 0\}.$$

Since each $B.G$ is the spectrum of a product of fields, it is sufficient to show that if $L$ is a field and $S = \text{Spec}(L)$, then $H^i(S, K_p(\mathcal{F}_S \otimes)) = 0$ if $i > 0$; but it is well known that Galois cohomology with uniquely divisible coefficients is trivial.

**5.4. Corollary.** Let $F$ be an algebraic stack. Then for all $p, j \in \mathbb{Z}$, $R^j(F) \otimes$ and $R_p(j) \otimes$ are sheaves on the étale site of $F$.

**Proof.** This follows immediately from 4.3 since

$$R^j(F) \otimes \simeq \bigoplus_{\xi \in F^{(p)}} i_\ast K_{j-p}(\mathcal{F}_\xi), \quad R_p(j) \otimes \simeq \bigoplus_{\xi \in F_{i-p}} i_\ast K_{j+p}(\mathcal{F}_\xi)$$

where the sums are over the sets $F^{(p)}(F_{i-p})$, respectively of punctual substacks $\xi$ of $F$ of codimension (dimension) $p$, and $i: \xi \to F$ is the inclusion.

**6. Intersection theory on stacks**

We turn now to the main results of the paper: constructing the product on the Chow groups of a stack $F$, and the comparison with the Chow groups of a coarse moduli space of $F$.

**6.1. Theorem.** Let $F$ be an algebraic stack, $p: U \to F$ an atlas of $F$; $U. = \text{cosk}^U_{F}(U)$ the corresponding simplicial scheme, which has étale face maps. Then there are canonical isomorphisms, for $p \geq 0$:
Proof. Since the proof of (i) is completely parallel to that of (ii), we shall only prove (i). First observe that if \( i : \xi \to F \) is a punctual substack, then for every étale open \( p : U \to F \), \( \xi_U = U \times_F \xi \) is a disjoint union of punctual stacks, so that \( H^*_\text{et}(\xi_U, K_p(\theta_\xi))_\mathbb{Q} = 0 \) if \( n > 0 \) and hence \( R^n i_* K_p(\theta_\xi)_\mathbb{Q} = 0 \) for \( n > 0 \). From the Leray spectral sequence we see that

\[
H^*_\text{et}(F, i_* K_p(\theta_\xi))_\mathbb{Q} \simeq H^*_\text{et}(\xi, K_p(\theta_\xi))_\mathbb{Q}
\]

\[= 0 \quad \text{if } n > 0, \text{ by Lemma 4.3.}\]

Hence, for all \( j, p \geq 0 \):

\[
H^*_\text{et}(F, R^j(p))_\mathbb{Q} = \begin{cases} 
0 & \text{for } n > 0, \\
\bigoplus_{\xi \in |F|'} K_{p-J}(k(\xi)) & n = 0.
\end{cases}
\]

Therefore

\[
H^*_\text{et}(F, R^*(p))_\mathbb{Q} = H^0(\Gamma(F, R^*(p)))_\mathbb{Q}
\]

\[= \text{coker } \Gamma(F, R^{p-1}(p)) \to \Gamma(F, R^p(p))_\mathbb{Q} = \text{CH}^p(F)_\mathbb{Q},
\]

(6.1.1)

and we have the first half of (i). By cohomological descent,

\[
H^*_\text{et}(F, R^*(p))_\mathbb{Q} = H^*_\text{et}(F, R^*(p))_\mathbb{Q}
\]

and there are spectral sequences

\[
H^*_\text{et}(U, R^*(p))_\mathbb{Q} = H^*_\text{et}(U, R^*(p))_\mathbb{Q},
\]

\[
H^*_\text{Zar}(U, R^*(p))_\mathbb{Q} = H^*_\text{Zar}(U, R^*(p))_\mathbb{Q}.
\]

By formula (6.1.1) applied to \( U \),

\[
H^*_\text{et}(U, R^*(p))_\mathbb{Q} = H^r(\Gamma(U, R^*(p)))_\mathbb{Q} = H^r_{\text{Zar}}(U, R^*(p))_\mathbb{Q}
\]

and so via a comparison of spectral sequences,

\[
H^*_\text{et}(U, R^*(p))_\mathbb{Q} \simeq H^r_{\text{Zar}}(U, R^*(p))_\mathbb{Q}
\]

(6.1.2)

and we are done.

Another way of stating the first half of the theorem is:

6.2. Corollary. Let \( F \) be an algebraic stack. Then

\[
\text{CH}^p(F)_\mathbb{Q} = \text{CH}^p(F)_\mathbb{Q}, \quad \text{CH}^p_{\text{et}}(F)_\mathbb{Q} = \text{CH}^p(F)_\mathbb{Q}.
\]
6.3. Corollary. Let $F$ be an algebraic stack; then $\text{CH}^*(F)_q$ and $\text{CH}_*(F)_q$ are graded $\text{HK}^*(F)_q$-modules.

Proof. For $p \geq 0$, $R^*(p)$ and $R_*(p)$ are presheaves of $K_*(\mathcal{F})$-modules and hence $\text{CH}^{*,*}(F)$ and $\text{CH}_,*(F)$ are $\text{HK}^*$-modules (4.7).

6.4. Corollary. Let $f: F \rightarrow G$ be a proper representable morphism of algebraic stacks. Then the direct image map $f_* : \text{CH}_*(F)_q \rightarrow \text{CH}_*(G)_q$ is a map of $\text{HK}^*(G)$-modules.

Proof. Let $p: U \rightarrow G$ be an atlas of $G$. Since $f$ is representable, $V = U \times_G F \rightarrow F$ is an atlas of $F$ and we have a commutative diagram

$$
\begin{array}{c}
\text{cosk}_0^F(V) = V. & \longrightarrow & F \\
\downarrow f & & \downarrow f \\
\text{cosk}_0^G(U) = U. & \longrightarrow & G \\
\end{array}
$$

For each $j \geq 0$ and $0 \leq i \leq j$ the square

$$
\begin{array}{c}
V_j & \longrightarrow & V_{j-1} \\
\downarrow f_j & & \downarrow f_{j-1} \\
U_j & \longrightarrow & U_{j-1} \\
\end{array}
$$

is Cartesian with $f_j$ and $f_{j-1}$ proper, so $f_j \star d_j = a_j \star f_{j-1} : R_*(V_{j-1}, q) \rightarrow R_*(U_j, q)$ and so we have a map of double complexes:

$$
f_* : R_*(V_*, q) \rightarrow R_*(U_*, q)
$$

and a commutative diagram

$$
\begin{array}{c}
\text{CH}_q(F) & \longrightarrow & \text{H}^q(V_*, R_*(q)) \\
\downarrow f_* & & \downarrow f_* \\
\text{CH}_q(G) & \longrightarrow & \text{H}^q(U_*, R_*(q)) \\
\end{array}
$$

in which the horizontal arrows are isomorphisms modulo torsion by 6.1. For each $j \geq 0$ the map

$$
f_j : R_*(V_j, q) \rightarrow R_*(U_j, q)
$$

is a map of $K_*(U_j)$-modules, and more generally (since $R_*(q)_{\overline{\mathbb{Q}}}$ is a sheaf in the étale topology) we have a map
of sheaves of $K_*(\mathcal{O}_U)$-modules and so a map
\[ f_* : \bigoplus_{p \geq 0} H^p_{\text{et}}(V, R_*(p))_\mathbb{Q} \to \bigoplus_{p \geq 0} H^p_{\text{et}}(U, R_*(p))_\mathbb{Q} \]
of $\bigoplus_{p \geq 0} H^p_{\text{et}}(U, K_p(\mathcal{O}_U))_\mathbb{Q}$-modules, hence by 6.1 and cohomological descent for the cover $U \to G$,
\[ f_* : CH_*(F) \to CH_*(G) \]
is a map of $HK^*(G)$-modules.

6.5. Corollary. Let $F$ be a regular stack. Then $CH^*(F)_\mathbb{Q}$ has a natural graded ring structure.

Proof. By 6.1 and 4.9, the augmentations for $p > 0$: $K_p(\mathcal{O}_F) \to R^*(p)$ induce isomorphisms
\[ CH^*(F)_\mathbb{Q} \cong HK^*(F)_\mathbb{Q}, \]
and so the Chow groups of $F$ inherit the ring structure of $HK^*(F)_\mathbb{Q}$.

6.6. Theorem. For every $p \in \mathbb{Z}$, the covariant functor
\[ F \mapsto R_*(F, j) \]
on the category of proper representable morphisms between stacks extends to a covariant functor on the category of all proper morphisms between stacks:
\[ F \mapsto R_*(F, j)_\mathbb{Q}. \]

Proof. First recall [2] that a morphism of stacks $p : T \to S$ is representable if there exists a diagram:

\[ \begin{array}{ccc}
X & \xrightarrow{g} & T_U & \xrightarrow{q_T} & T \\
\downarrow{f} & & \downarrow{p_U} & & \downarrow{p} \\
U & \xrightarrow{q} & S
\end{array} \]

with $U$ a scheme, $q$ étale and surjective, $f$ representable (so that $X$ is a scheme) and proper (as a map of schemes) and $g$ surjective (i.e. for all maps $V \to T_U$ with $V$ a scheme, $X \times T_U \to V$ is a surjective map of schemes). Note that since $X$ is a scheme, $g$ is automatically representable.

If $p : T \to S$ is a proper morphism of stacks, define a map of graded abelian groups
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by setting, for $t \in |T|_i$ and $s = p(t) \in |S|$, $p_*: K_{q-i}(k(t))_Q \to K_{q-i}(k(s))_Q$ equal to zero if $s \in |S|_k$ for $k > i$ and equal to $(e(s)/e(t))N_{k(t)/k(s)}$ if $s \in |S|_{(i)}$. Note that in the latter case, by the results of Section 3, $k(k)|k(s)$ is a finite extension. Evidently, this definition of $p^*$ makes $R_*(F, j)_Q$ a covariant functor of $F$. It remains to check that $p_*$ is actually a map of complexes.

6.7. Lemma. Let $q: U \to S$ be an étale map with $U$ a scheme, $P_U: T_U = T \times_S U \to U$ the induced proper morphism. Then we have a commutative diagram

$$
\begin{array}{ccc}
R_*(T, j)_Q & \xrightarrow{q^*} & R_*(T_U, j)_Q \\
p_* & & p_{U*} \\
R_*(S, j)_Q & \xrightarrow{q^*} & R_*(U, j)_Q
\end{array}
$$

Proof of 6.7. We may suppose that $U$, $T$ and $S$ are all punctual. Pick atlases $W \to T$ and $V \to T_U$ (we may suppose $W$ is punctual, too) to obtain a diagram (in which all squares are Cartesian and every stack except for $S$, $T$ and $T_U$ is representable):

$$
\begin{array}{cccc}
W' & \to & W_U & \to & W \\
\downarrow & & \downarrow & & \downarrow \\
V & \to & T_U & \to & T \\
\downarrow & & \downarrow & & \downarrow \\
U & \to & S & & S
\end{array}
$$

To prove the lemma, it will be sufficient to show that:

$$[k(T_U): k(U)] = [k(T): k(S)] \frac{e(S)e(T_U)}{e(T)} .$$

We shall start by computing $[k(W_V): k(S)]$. To do this we use the following sublemma.

6.8. Sublemma. Let $F$ be a punctual stack, $U = \text{Spec}(K) \to F$, $V = \text{Spec}(L) \to F$ finite (not necessarily étale) morphisms with $K$ and $L$ fields. Then $U \times_F V$ is finite of constant degree equal to $e(F)$ over $\text{Spec}(K \otimes_{k(F)} L)$ or equivalently $[k(U \times V): k(V)] = [k(U): k(F)]e(F)$.
Proof of 6.8. Let $F_{\text{iso}}$ be the stack $U \to F(U)^{\text{iso}}$, the groupoid of isomorphisms $\alpha : x \to y$ of objects in $F(U)$. Note that $F_{\text{iso}}$ is equivalent to $F$ and that there is a diagonal functor $\Delta : F_{\text{iso}} \to F \times_k F$ sending $\alpha : x \to y$ to the pair $(x, y)$, where we write $k = k(F)$. We then have a strictly Cartesian diagram of Categories:

$$\begin{array}{ccc}
U \times_F V & \longrightarrow & F_{\text{iso}} \\
p \downarrow & & \downarrow \Delta \\
U \times_k V & \longrightarrow & F \times_k F
\end{array}$$

Let us compute the degree of $p$ on geometric fibres; if $(u, v) \in U \times_k U$ is an $\Omega$-valued point for $\Omega$ a universal domain and $(\xi, \psi)$ the induced point of $F \times_k F$, then

$$p^{-1}(u, v) = \Delta^{-1}(\xi, \psi)$$

= set of ismorphisms $\alpha : \xi \to \psi$ in the groupoid $F(\Omega)$.

Hence $\# p^{-1}(u, v) = e(F)$. Note that if $X \to F$ is an atlas, and $M = X \times_F X = (X \times_k X) \times_F F_{\text{iso}}$, then by 2.10, $M \to X \times_k X$ is étale. By base change the map $U \times_F V \to U \times_k V$ is étale; since we have shown that it has constant degree it is also finite.

From the sublemma we see that in diagram (6.7.1) we have

$$[k(W_U) : k(W)] = [k(V) : k(T_U)] e(T_U)$$

and hence the lemma follows from

$$[k(W_U) : k(V)][k(V) : k(T_U)][k(T_U) : k(U)][k(U) : k(S)]$$

and

$$[k(W_U) : k(S)]$$

$$= [k(W_U) : k(W)][k(W_U) : k(W)][k(W) : k(T)][k(T) : k(S)].$$

Proof of 6.6 (contd.). Returning to the theorem, by definition of properness, there exists a diagram

$$\begin{array}{ccc}
X & \longrightarrow & T_U \\
\downarrow f & & \downarrow p_U \\
U & \longrightarrow & S
\end{array}$$

with $U$ a scheme, $q$ étale and surjective, $f$ representable and proper, and $g$ surjective.
and representable. Since \(q^*\) and \(q_f^*\) are injective, by the lemma, it suffices to prove that \(P_{ij}^*\) is a map of complexes, hence let us assume \(S = U\). Since \(f\) and \(g\) are proper and representable, in the commutative diagram

\[
\begin{array}{ccc}
R_*(X, j) & \xrightarrow{g^*} & R_*(T, j) \\
\downarrow f_* & & \downarrow p_* \\
R_*(S, j) & & \\
\end{array}
\]

\(f_*\) and \(g_*\) are actually maps of complexes. It will be enough therefore, to show that \(g_*\) is surjective. But since \(g\) is surjective, for every field valued point \(\tau : \text{Spec}(E) \to T\), the fibre \(X_\tau\) of \(X\) over \(\text{Spec}(E)\) is a scheme proper over \(\text{Spec}(E)\). Hence there exists a point \(x \in X_\tau\) such that \(k(x)\) finite algebraic. For any point \(t \in T\), there exists a point \(\tau\) as above, with \(E|_{k(t)}\) finite algebraic; hence a point \(x \in X\) such that \([k(x) : k(t)] < \infty\). But then \(Nm_{k(s)|k(t)} : K_f(k(x)_\mathbb{Q}) \to K_f(k(t)_\mathbb{Q})\) is surjective, and we are done.

**Proposition 6.1.** Let \(p : T \to S\) be a proper morphism between algebraic stacks. Then

\[
p_* : \mathcal{H}_*(T)_\mathbb{Q} \to \mathcal{H}_*(S)_\mathbb{Q}
\]

is a map of \(HK^*(S)-\text{modules}\).

**Proof.** It is enough to check for every étale map \(U \to S\) with \(U = \text{Spec}(A)\), that if we write \(T_U = T \times_S U\),

\[
p_{U*} : \bigoplus_{j \in \mathbb{Z}} \mathbb{R}_*(T_U, j)_\mathbb{Q} \to \bigoplus_{j \in \mathbb{Z}} \mathbb{R}_*(U, j)_\mathbb{Q}
\]

is a map of \(K_*(A)\)-modules. But as before, we can assume there is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & T_U \\
\downarrow f & & \downarrow p_U \\
U & & \\
\end{array}
\]

with \(f\) proper and representable and \(g\) surjective. The proposition is true for \(f_*\), and hence by the surjectivity of \(g_*\) for \(p_{U*}\) too.

**6.8. Theorem.** Let \(F\) be an irreducible stack with coarse moduli space \(X\); then

\[
p_* : \mathcal{H}_*(F)_\mathbb{Q} \to \mathcal{H}_*(X)_\mathbb{Q}
\]
is an isomorphism. Hence, if $F$ is regular, $\text{CH}^*(X)_\mathbb{Q}$ has a ring structure induced by the ring structure of $\text{CH}^*(F)$. (Note that if $n=\dim X$, $\text{CH}_i(X)=\text{CH}_n-i(X)$ and $\text{CH}_i(F)=\text{CH}_n-i(F)$.) Furthermore, if $HK^*_\text{Zar}(X)\to \text{CH}^*(X)$ is the natural map, the induced map $HK^*_\text{Zar}(X)\to \text{CH}^*(X)_\mathbb{Q}$ is a ring homomorphism. Finally, the ring structure on $\text{CH}^*(X)_\mathbb{Q}$ is independent of the stack $F$.

**Proof.** To see that $p^*$ is an isomorphism it is enough to observe that

1. $p: |F| \to X$ is a bijection.
2. $\forall y \in |F|, x=p(y) \in X, [k(y):k(x)]$ is purely inseparable and hence $Nm_{k(y)/k(x)}: K_j(k(y))_\mathbb{Q} \to K_j(k(x))_\mathbb{Q}$ is an isomorphism [17, §7].

It follows that for all $j \in \mathbb{Z}$:

$$p^*: R^*(F,j)_\mathbb{Q} \to R^*(X,j)_\mathbb{Q}$$

is an isomorphism, and hence $p^*: \text{CH}^*(F)_\mathbb{Q} \to \text{CH}^*(X)_\mathbb{Q}$ and therefore if $F$ is regular, $\text{CH}^*(X)$ inherits the ring structure of $\text{CH}^*(F)_\mathbb{Q} \cong HK^*(F)_\mathbb{Q}$. However instead of using the product which makes $p^*$ a ring homomorphism, we shall use one which makes $[X] \in \text{CH}^0(X)_\mathbb{Q}$ the unit element. So define

$$p^*: \text{CH}^*(X)_\mathbb{Q} \to \text{CH}^*(F)_\mathbb{Q}, \quad p^* = \frac{i(F/X)}{e(F)} (p^*)^{-1}$$

where $e(F)$ is the ramification index of the generic point $\phi$ of $F$ while $i(F/X)$ is the inseparable degree of $k(\phi)$ over $k(X)$. By the projection formula, since

$$p_*(|F|) = \frac{i(F/X)}{e(F)} [X]$$

we know that the following diagram commutes:

$$\begin{array}{ccc}
HK^*(F)_\mathbb{Q} \to [F] & \text{CH}^*(F)_\mathbb{Q} \\
\uparrow p^* & \quad & \uparrow p^* \\
HK^*_\text{Zar}(X)_\mathbb{Q} \to [X] & \text{CH}^*(X)_\mathbb{Q}
\end{array}$$

and hence if we define the ring structure on $\text{CH}^*(X)_\mathbb{Q}$ via $p^*$, the lower horizontal arrow is a ring homomorphism since the left vertical and top horizontal arrows already preserve products.

Finally let us turn to the uniqueness of the product structure. Suppose that $X$ is the coarse moduli space for two different regular stacks: $p:F \to X$ and $q:G \to X$. If we form the fibre product
then $f$ and $g$ are proper and induce homeomorphisms between $|F \times G|$ and $|F|$ and $|G|$ respectively and $X$ is a coarse moduli space for $F \times G$. Since $F$ and $G$ are both regular, and hence $\text{CH}^*(F)_Q \simeq HK^*(F)_Q$ and $\text{CH}^*(G)_Q \simeq HK^*(G)_Q$, we can define maps

$$a \mapsto f*p*(a) \cap [F \times G] \quad \text{and} \quad a \mapsto g*q*(a) \cap [F \times G]$$

(6.8.1)

from $\text{CH}^*(X)_Q$ to $\text{CH}^*(F \times, G)_Q$. These two maps coincide, and in fact

$$f*p*(a) \cap [F \times G] = \frac{i(F \times G/X)}{e(F \times G)} (h_*)^{-1}(a) = g*q*(a) \cap [F \times G], \quad (6.8.2)$$

for by the projection formula,

$$h_*(f*p*(a) \cap [F \times G]) = p_*(p*(a) \cap f_*[F \times G])$$

$$= p_* \left( p*(a) \cap [F] \cdot \frac{[k(F \times G) : k(F)][e(F)]}{e(F \times G)} \right)$$

$$= a \cdot \frac{[k(F) : k(X)][k(F \times G) : k(F)]}{e(F \times G)}$$

with the same formula holding for $h_*(g*q*(a) \cap [F \times G])$. Since $X$ is a coarse moduli space for $F \times G$, observe that $h_*$ is an isomorphism, and hence the two maps (6.8.1) are isomorphisms. Now suppose that $\zeta$ and $\omega$ are elements of $\text{CH}^*(X)_Q$. Then their products $\beta$ and $\gamma$ relative to $F$ and $G$ are defined by

$$p^*(\zeta)p^*(\omega) = \beta \quad \text{and} \quad q^*(\zeta)q^*(\omega) = q^*(\gamma).$$

But

$$f*p^*(\beta) \cap [F \times G]$$

$$= f*(p^*(\zeta)p^*(\omega)) \cap [F \times G] \quad \text{by definition}$$

$$= f^*(p^*(\zeta)) \cdot f^*(p^*(\omega)) \cap [F \times G] \quad \text{since } f^* \text{ is a ring homomorphism}$$

$$= f*p^*(\zeta)(g*q^*(\omega)) \cap [F \times G] \quad \text{by } (6.8.2)$$

$$= g*q^*(\omega) \cdot f*p^*(\zeta) \cap [F \times G]$$

$$= g*q^*(\omega)g*q^*(\zeta) \cap [F \times G] \quad \text{by } (6.8.2)$$

$$= g*q^*(\gamma) \cap [F \times G],$$
hence \( \beta = \gamma \) since \( h_* \) is an isomorphism, and we are done.

Note that if \( Y \) and \( Z \) are prime cycles on \( X \) meeting properly, with \( X \) the coarse moduli space of a regular stack \( F \), then in \( \text{CH}^*(X)_Q \)

\[
[Y] \cdot [Z] = \sum_S I(Y, Z; S)[S]
\]

where the sum is over the irreducible components of \( Y \cap Z \) and the intersection multiplicity is:

\[
I(Y, Z; S) = \frac{i(F/X)(\eta) e(\zeta) i(\sigma)}{e(F)i(\eta) i(\zeta) e(\sigma)} I(\eta, \zeta; \sigma) \quad (6.8.3)
\]

where \( \eta = p^{-1}(Y)_{\text{red}}, \zeta = p^{-1}(Z)_{\text{red}}, \sigma = p^{-1}(S)_{\text{red}} \) where \( p: F \to X \) is the given map and \( i(\eta) = [k(\eta): k(Y)] \) etc. The intersection multiplicity on \( F \) may be computed locally in the étale topology i.e. by taking an étale map \( q: U \to F \) the image of which contains the generic point of \( \sigma \); then if \( f = p \cdot q \),

\[
I(\eta, \zeta; \sigma) = I(f^{-1}(Y)_{\text{red}}, f^{-1}(Z)_{\text{red}}, f^{-1}(S)_0)
\]

where \( f^{-1}(S_0) \) is any irreducible component of \( f^{-1}(S) \).

6.9. Theorem. Let \( X \) be a coarse moduli space of a regular stack \( F \) and let \( f: Y \to X \) be an arbitrary morphism of schemes. Then \( \text{CH}^*(Y)_Q \) has a natural \( \text{CH}^*(X)_Q \)-module structure. If \( Y \) is itself the coarse moduli space of a regular stack, this module structure is induced by a ring homomorphism \( f^*: \text{CH}^*(X)_Q \to \text{CH}^*(Y)_Q \). If \( f \) is proper, then \( f_*: \text{CH}_*(Y)_Q \to \text{CH}_*(X)_Q \) is a map of \( \text{CH}^*(X)_Q \)-modules.

Proof. If \( p: F \to X \) is the given map, consider the diagram:

\[
\begin{array}{ccc}
F \times_X Y & \xrightarrow{f} & F \\
p_Y \downarrow & & \downarrow p \\
Y & \xrightarrow{p_Y} & X
\end{array}
\]

Then \( Y \) is a coarse moduli space for \( F \times_X Y \), so

\[
p_Y: \begin{cases}
\text{CH}^*(F \times_X Y)_Q \to \text{CH}^*(Y)_Q \\
\text{CH}_*(F \times_X Y)_Q \to \text{CH}_*(Y)_Q
\end{cases}
\]

is an isomorphism. But since

\[
f^*: \text{HK}^*(F) \to \text{HK}^*(F \times_X Y)
\]

is a ring homomorphism, and \( \text{CH}^*(F \times_X Y) \) is an \( \text{HK}^*(F \times_X Y) \)-module there is a natural \( \text{CH}^*(X)_Q = \text{HK}^*(F)_Q \)-module structure on \( \text{CH}^*(Y)_Q \). Similarly, by reindexing, \( \text{CH}_*(Y)_Q \) is naturally a \( \text{CH}^*(X)_Q \)-module. If \( f \) is proper, then \( f^* \) is proper too, and
is a map of $HK^*(F)\mathbb{Q}$-modules (6.7) and hence $f_* : CH_*(Y)\mathbb{Q} \to CH_*(X)\mathbb{Q}$ is a map of $CH^*(X)\mathbb{Q} = HK^*(F)\mathbb{Q}$-modules. If $q : G \to Y$ is a map making $Y$ the coarse moduli space of a regular stack, then consider the diagram:

$$
\begin{array}{c}
G \times_X Y \\
\downarrow q \\
Y
\end{array}
\quad
\begin{array}{c}
G \\
\downarrow p_Y
\end{array}
\quad
\begin{array}{c}
F \\
\downarrow f
\end{array}
\quad
\begin{array}{c}
F \times_X Y \\
\downarrow p
\end{array}
\quad
\begin{array}{c}
X
\end{array}
$$

Then $CH^*(G \times_X Y)\mathbb{Q} = CH^*(Y)\mathbb{Q}$ and is both a principle torsion free $HK^*(G)\mathbb{Q}$-module and an $HK^*(F)\mathbb{Q}$-module, hence there is a natural homomorphism $HK^*(F)\mathbb{Q} \to HK^*(G)\mathbb{Q} = CH^*(G)\mathbb{Q}$.

7. **K-theory of stacks**

7.1. **Definition.** Let $T$ be an algebraic stack. A sheaf $\mathcal{F}$ (on the étale site of $T$) of $\Theta_T$-modules is said to be quasi-coherent (or coherent, or locally free of rank $r$) if $p^*\mathcal{F}$ is quasi-coherent (or coherent, or locally free of rank $r$) for some (and hence for every) atlas $p : U \to T$. Since we assume that $T$ is noetherian, the category of coherent sheaves on $T$ is a full subcategory, closed under taking subobjects, and quotient objects, of the category of quasi-coherent sheaves. The categories of coherent and quasi-coherent sheaves on $T$ are abelian categories. We shall use the notation:

- $\mathcal{C}(T) =$ category of quasi-coherent sheaves on $T$,
- $\mathcal{M}(T) =$ category of coherent sheaves on $T$,
- $\mathcal{P}(T) =$ category of locally free sheaves of finite rank on $T$.

Note that $\mathcal{P}(T)$ is a sub-exact category of $\mathcal{M}(T)$.

7.2. **Definition.** Just as in the case of schemes, we can define the Quillen $K$-theory of the stack $T$:

$$
K_*^s(T) = K_*^s(\mathcal{M}(T)), \quad K_*^s(T) = K_*^s(\mathcal{P}(T)).
$$

$K_*^s(T)$ is a contravariant ring-valued functor on the full category of algebraic stacks, while $K_*'(T)$ is a contravariant graded abelian group-valued functor on the category of flat maps.
If $T$ and $S$ are algebraic stacks, recall that $\text{Hom}(T, S)$ is actually a groupoid rather than a discrete set. However for a given $f: T \to S$, the homomorphism

$$f^*: K_*(S) \to K_*(T)$$

(or $f^*: K'(S) \to K'(T)$ if $f$ is flat) only depends on the class of $f$ in $\pi_0\text{Hom}(T, S)$. To see this observe that we actually have a functor:

$$(\ )^*: \text{Hom}(T, S) \to \text{Cat}(\mathcal{P}(S), \mathcal{P}(T))$$

such that any morphism $p: f \to g$ in $\text{Hom}(T, S)$ induces a natural isomorphism $p^*: f^* = g^*$ and hence a homotopy between maps $B\mathcal{O}(S) \to B\mathcal{O}(T)$.

The covariance of $K_*$ with respect to proper morphisms is less clear. If $p: T \to S$ is a projective morphism of algebraic stacks (i.e. $p$ is representable and factors through: a closed immersion $j: T \to \mathbb{P}^n_S$), then by the same method as in the case of schemes ([17]). see also [9, §1.3]) we can construct a map $p_*: K_*(T) \to K_*(S)$ making $K_*$ a covariant functor on the category of projective morphisms between algebraic stacks.

We should not expect there to be a straightforward definition of $p_*: K'(T) \to K'(S)$ for a non-representable proper morphism of stacks $p: T \to S$, since even though the sheaves $R^i p_* \mathcal{F}$ are coherent if $\mathcal{F}$ is a coherent sheaf on $T$, they may be non-vanishing for infinitely many $i$, making nonsense of the equation (valid for proper morphisms of schemes):

$$p_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i p_* \mathcal{F}] \in K_0(S).$$

We can still pursue the analogy between the $K$-theory of coherent sheaves on stacks and on schemes in other directions however.

7.3. Lemma. Let $T$ be an algebraic stack; then the closed immersion $j: T_{\text{red}} \to T$ induces an isomorphism $K_*(T_{\text{red}}) = K_*(T)$.

Proof. The immersion $j$ is defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_T$ such that the restriction $\mathcal{I}_U$ of $\mathcal{I}$ to an atlas $p: U \to T$ defines the immersion $j_U: U_{\text{red}} \to U$; $\mathcal{I}_U$ must therefore be nilpotent, say $\mathcal{I}_U^n = 0$ and so $\mathcal{I}$ itself is nilpotent, with $\mathcal{I}^n = 0$. Any coherent $\mathcal{O}_T$-module $\mathcal{M}$ will therefore have a finite filtration $\mathcal{M} \supseteq \mathcal{I} \mathcal{M} \supseteq \cdots \mathcal{I}^n \mathcal{M} = 0$ the quotients of which are $\mathcal{O}_{T_{\text{red}}}$-modules, and we may apply devissage for $K$-theory [17, §5] to complete the proof.

7.4. Lemma. Let $Y \subset X$ be a closed substack, with complement $U = X - Y$. Then there is a long exact localization sequence:

$$\cdots \to K_q'(Y) \to K_q'(X) \to K_q'(U) \to K_{q - 1}'(Y) \to \cdots$$

Proof. We can follow Quillen's proof of the same theorem for schemes [17, §7]; by devissage (7.3 above) if $\mathcal{M}(Y)$ is the Serre subcategory of $\mathcal{M}(X)$ consisting of
those sheaves supported on $Y$, then $K_*(\bar{M}(Y)) = K_*(M(Y))$ and so we need only verify that the abelian category $M(U)$ is equivalent to the quotient category $M(X)/\bar{M}(Y)$. If $i : U \to X$ is the natural inclusion, then the exact functor

$$i^* : M(X) \to M(U)$$

has kernel $\bar{M}(Y) \subseteq M(X)$. If $i^*$ had a right adjoint we would be done, however it does not; but if we replace $M(X), M(U)$ by the corresponding categories of quasi-coherent sheaves $C(X), C(U)$ then we find that $i_* : C(U) \to C(X)$ is right adjoint to $i^*$ and $i_*i_* = \text{Id}$. Hence the natural functor $(\bar{C}(Y) = \text{sheaves in } C(X) \text{ supported on } Y)$

$$C(X)/\bar{C}(Y) \to C(U)$$

is an equivalence of categories (Gabriel [5, p. 374]). Consider the diagram of functors:

$$\begin{array}{ccc}
C(X)/\bar{C}(Y) & \xrightarrow{g} & C(U) \\
a \downarrow & & \downarrow b \\
M(X)/\bar{M}(Y) \xrightarrow{f} M(U)
\end{array}$$

We know $g$ is an equivalence and that $b$ is a fully faithful embedding. To complete the lemma we must show that $a$ is fully faithful and that every object of $M(U)$ is equivalent to an object in the image of $f$. By the definition of quotient categories, the first assertion follows from the fact that $M(X) \subseteq C(X)$ is closed under sub- and quotient-objects, and that $M(X) \cap \bar{C}(Y) = \bar{M}(Y)$. The second assertion follows from the extension lemma:

7.5. Lemma. Let $i : U \to X$ be an open substack of an algebraic stack $X$. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$, and $\mathcal{G}_i \to i^*\mathcal{F}$ a coherent sub-$\mathcal{O}_U$-module of $i^*\mathcal{F}$. Then there is a coherent sheaf $\mathcal{G}$ on $X$ with $\mathcal{G}_i = i^*\mathcal{G}$.

Proof. The proof of this lemma, and the completeness lemma, on which it depends, are identical to the proofs of the same results for algebraic spaces [13, pp. 155–7].

7.6. Proposition. The covariant functor $K_*$ on the category of projective morphisms between algebraic stacks, when restricted to the subcategory of algebraic spaces has a canonical extension to a covariant functor on the category of all proper morphisms between algebraic spaces.

Proof. Every algebraic space $X$ has a dense open subspace $U \subseteq X$ which is a scheme; further if $X$ is of finite type over a field there is a proper birational (i.e. isomorphic on a dense open set) map $p : \tilde{X} \to X$ where $\tilde{X}$ is a quasi-projective scheme. We may therefore apply the machinery of hyperenvelopes [9] to construct direct image maps
for proper morphisms \( f: X \to Y \).

7.7. **Theorem.** Let \( X \) be an algebraic stack. Then there is a spectral sequence:

\[
E_1^{p,q}(X) = \bigoplus_{\xi \in X^{(p)}} K_{-p-q}(\xi) = K_{-p-q}(X) \quad (7.7.1)
\]

where \( X^{(p)} \) is the set of punctual substacks of codimension \( p \) in \( X \). This spectral sequence is contravariant with respect to flat maps, and if \( X \) is pure dimensional, covariant (with a degree shift) with respect to projective maps.

**Proof.** We can filter the category \( \mathcal{M}(X) \) by codimension and proceed given Lemmas 7.3, 7.4 and 7.5 just as in Quillen’s article [17, §7.5]. (For the projective covariance see [7, §7]).

In trying to compute the \( E_2 \) term of this spectral sequence, the analogy with the \( K \)-theory of schemes fails in the following sense. For a punctual substack \( \xi \) of \( X \), \( K_{i}(\xi) \neq K_{i}(\mathbb{k}(\xi)) \) if \( e(\xi) \neq 0 \), since \( \mathbb{P}(\xi) \) can be identified with the category of representations of the inertia group of \( \xi \); for example if \( \xi = [\text{Spec}(\mathbb{C}): G] \) the quotient stack of a finite group acting trivially on Spec(\( \mathbb{C} \)), \( K_{0}(\xi) \) is the classical representation ring of \( G \). However the situation improves if we restrict our attention to algebraic spaces:

7.8. **Proposition.** If \( X \) is an algebraic space, then in the spectral sequence \((7.7.1)\):

\[
E^{p,q}_2 = \bigoplus_{\xi \in X^{(p)}} K_{-p-q}(\xi) = K_{-p-q}(X)
\]

the \( E_2^{p,-p} \) term may be identified with \( \text{CH}^p(X) \).

**Proof.** Since every punctual algebraic space is the spectrum of a field, we know that:

\[
E_1^{p,-p}(X) = \text{Z}^p(X), \quad E_1^{p,-1}(X) = \mathcal{W}^p(X)
\]

(the notation is that of Section 4). By flat contravariance of the spectral sequence \((7.7.1)\) and the fact that \( \text{Z}^p \) and \( \mathcal{W}^p \) are sheaves in the étale topology, because the differential

\[
d_1: E_1^{p,-1,-p}(X) \to E_1^{p,-p}(X)
\]

coincides with the ‘Weil divisor’ map

\[
d: \mathcal{W}^p(X) \to \text{Z}^p(X)
\]

if \( X \) is replaced by an atlas of \( X \) (Section 4), they must be equal on \( X \) itself. Hence \( E_2^{p,-p} = \text{coker}(d: \mathcal{W}^p(X) \to \text{Z}^p(X)) \), i.e.

\[
E_2^{p,-p}(X) = \text{CH}^p(X).
\]
7.9. Note. If \( X \) is a stack, filtering \( \mathcal{M}(X) \) by dimension of support, we would obtain a spectral sequence:

\[
E^1_{p,q} = \bigoplus_{\xi \in X_{\text{in}}} K_{p+q}(\xi) \Rightarrow K_{p+q}(X)
\]

covariant for projective morphisms between stacks or (by the methods of [7], see 7.6) for all proper morphisms between algebraic spaces. If \( X \) is an algebraic space, then

\[
E^2_{p,-p}(X) = \text{CH}_p(X).
\]

7.10. Proposition. The spectral sequence of 7.9 is a covariant functor on the category of proper morphisms between algebraic spaces. In particular, for all \( p \geq 0 \), \( \mathcal{M}(X) \), \( Z_p(X) \) and \( \text{CH}_p(X) \) are all covariant with respect to proper morphisms between algebraic spaces.

Proof. As in 7.6 we can apply the machinery of [9].

7.11. Remark. Recent work of R. Thomason indicates that more of the results of [17] apply to the \( K \)-theory of stacks, in particular the calculation of the \( K \)-theory of projective bundles; see [20].

8. Chern classes and the Riemann–Roch theorem for algebraic spaces

If \( T \) is an algebraic stack and \( V \) is a locally free sheaf on \( T \) of constant rank we can define Chern classes:

\[
C_i(V) \in \begin{cases} 
H^i_{\text{et}}(T, K_i(\theta_T)) \overset{\text{def}}{=} HK^i(T), \\
H^{2i}_{\text{et}}(T, \mathbb{Z}/i!n(i)) \quad \text{if } 1/i \in \mathbb{Q}
\end{cases}
\]

since any such \( V \) can be trivialized over some atlas \( p : Z \to T \) and hence can be classified by a map \( x : \cosk_0(Z) \to B.\mathbb{G}_m/k \), so that we can define \( C_i(V) = \chi^*C_i \)

where

\[
C_i \in \begin{cases} 
H^i(B.\mathbb{G}_m/k, K_i(\ell_B.\mathbb{G}_m)), \\
H^{2i}(B.\mathbb{G}_m/k, \mathbb{Z}/i!n(i))
\end{cases}
\]

are the universal classes constructed in [7] and [12].

By standard methods these classes extend to maps

\[
C_i : K_0(T) \to \begin{cases} 
HK^i(T) = H^i_{\text{et}}(T, K; (\ell_T)), \\
H^{2i}_{\text{et}}(T, \mathbb{Z}/i!n(i))
\end{cases}
\]

which satisfy the usual formulae with respect to sums and products; in particular we can define ring homomorphisms:

\[
\text{ch} : K_0(T) \to H^*(T) \tag{8.0}
\]
where $H^k(T) = \text{either } HK^k(T)_0 \text{ or } \bigoplus_{i>0} H^{2i}(T, \mathcal{O}_T(i))$ which are natural transformations of ring-valued functors.

Let $M^f(T)$ be the full exact subcategory of $M(T)$ consisting of those coherent $\mathcal{O}_T$-modules which locally have finite free resolutions. Then if $T$ is a scheme, the inclusion $P(T) \subseteq M^f(T)$ induces an isomorphism,

$$K_0(T) = K_0(M^f(T)) \overset{\text{def}}{=} K_0^f(T).$$

However if $T$ is not a quasi-projective scheme, we do not know that we can path together the local resolutions to construct global resolutions, so that the natural ring homomorphism

$$\alpha : K_0(T) \to K_0^f(T)$$

may not be an isomorphism. (Note that the product on $K_0(T)$ is given by

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i \geq 0} (-1)^i [\text{Tor}^i_T(\mathcal{F}, \mathcal{G})]$$

and that if $f : T \to S$ is a morphism of stacks $f^* : K^f(S) \to K^f(T)$ is defined by

$$f^*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [\text{Tor}^i_T(\mathcal{F})].$$

If $\alpha$ were an isomorphism, we could immediately deduce the existence of Chern classes for coherent sheaves locally of finite projective dimension. In general we can still construct such classes, but since the construction is somewhat technical I shall only state their main properties, leaving the proofs till another paper [10].

8.1. Theorem. (i) There are natural transformations of ring-valued contravariant functors

$$\text{ch} : K_0^f(\quad) \to H^*(\quad)$$

extending the Chern character for $K_0$ in the sense that $\text{ch} \cdot \alpha$ coincides with the Chern character maps (8.0).

(ii) There are also Chern classes and a Chern character with supports:

$$C^2(\mathcal{F}) \quad \text{ch}^2(\mathcal{F}) \in H^*_Z(F)$$

if $\mathcal{F}$ is a coherent sheaf locally of finite projective dimension, supported on a closed substack $Z \subseteq F$, where $H^*_Z(F)$ is either $H^*_Z(F, K_*(\mathcal{O}_F))$ or $H^*_Z(F, \mathcal{O}_*(\quad))$, which is compatible with products and pullbacks.

8.2. Corollary. If $f : X \to T$ is a morphism from a quasi-projective variety to a regular stack, there is a commutative diagram of ring homomorphisms
If \( F \) is a smooth algebraic stack, then it has a well defined tangent bundle \( T_F \), and so we can define the Todd class \( Td(F) \in H^*(F) \) by the usual formula \([19]\); and hence obtain a map:

\[
\text{ch}(\ ) \cdot Td(F) : K_0(F) \to H^*(F).
\] (8.2.1)

Given such a homomorphism it is natural to ask whether one can prove a Riemann–Roch theorem for the \( K \)-theory of algebraic stacks. The only situation in which such a theorem is obviously true is for proper morphisms between algebraic spaces, and even there it is not immediately obvious that the ‘Riemann–Roch’ transformation is given by the formula (8.2.1) for smooth algebraic spaces.

8.3. **Theorem.** There is a natural transformation

\[ \tau : K_0'(\ ) \to H^*_a(\ ) \]

of covariant functors on the category of proper morphisms between algebraic spaces of finite type over \( k \), where \( H^*_a(X) \) is either Chow homology (Section 5) with rational coefficients or étale homology \( \bigoplus_{i \geq 0} H^{2i}_\text{ét}(\ , \Omega^i(i)) \) satisfying the following properties:

1. If \( \alpha \in K_0'(X) \) and \( \beta \in K_0(X) \), then
   \[
   \tau(\alpha \cdot \beta) = \tau(\alpha) \text{ch}(\beta).
   \]

2. If \( X \) is a smooth variety, then
   \[
   \tau([F]) = Td(X) \cdot \text{ch}([F]).
   \]

3. \( \tau : K_0'(\ ) \to \text{CH}^*_a(\ )_{\mathbb{Q}} \) is an isomorphism.

**Proof.** The proof is identical to that of \([4, \S 1]\). The existence of Chow envelopes comes from Chow’s lemma for algebraic spaces \([13, \text{Ch. 4.3}]\).

9. **Comparison with Mumford’s product**

9.1. **Definition.** A \( Q \)-variety is an algebraic variety (or more generally an algebraic space) \( X \) over a field, together with a surjective finite family of morphisms \( \{q_a : Z_a \to X\}_{a \in A} \) such that:

1. Each \( Z_a \) is quasi-projective and smooth over \( k \) and quasifinite over \( X \).
(ii) For each \((a, \beta) \in A \times A\), the normalization \(Z_{a\beta}\) of \(Z_a \times_X Z_\beta\) is smooth over \(k\) (and hence étale over \(Z_a\) and \(Z_\beta\), since smooth and quasifinite implies étale).

(iii) For each \(\alpha \in A\), there is a finite group \(G_\alpha\) acting faithfully on \(Z_\alpha\) such that \(U_\alpha = Z_\alpha / G_\alpha\) is étale over \(X\). (Note that \(\{U_\alpha\}_\alpha \in A\) is then an étale cover of \(X\).

Since each \(Z_\alpha\) is smooth, the diagonal map \(\Delta: Z_\alpha \to Z_\alpha \times X Z_\alpha\) factors through \(Z_{aa}\) and hence we have a diagram in the category of quasi-projective varieties:

\[
\coprod_{a, \beta} Z_{a\beta} \xrightarrow{\sim} \coprod_a Z_a
\]

which naturally has the structure of an étale algebraic groupoid. (The multiplication map,

\[
\coprod_{a, \beta, \gamma} (Z_{a\beta\gamma} = Z_{a\beta} |_{Z_\beta} Z_{\beta\gamma}) \to \coprod_{a, \gamma} Z_{a\gamma}
\]

is induced by the natural projections \(Z_{a\beta\gamma} \to Z_a \times_X Z_\beta\) which, since the \(Z_{a\beta\gamma}\) are smooth, must factor through the \(Z_{a\gamma}\).) If we write

\[Z = \left( \coprod_{a, \beta} Z_{a\beta} \xrightarrow{\sim} \coprod_a Z_a \right),\]

then we have:

**9.2. Proposition.** \(X\) is a coarse moduli space for the stack \(B_Z\) of \(Z\)-torsors.

**Proof.** Since the map

\[
\coprod_{a, \beta} Z_{a\beta} \to \coprod_{a, \beta} Z_a \times_X Z_\beta
\]

is surjective, for any algebraically closed field \(\Omega\) we have an exact sequence:

\[
\coprod_{a, \beta} Z_{a\beta}(\Omega) \xrightarrow{\sim} \coprod_a Z_a(\Omega) \to X(\Omega).
\]

Hence it only remains to show that the map \(B_Z \to X\) is proper. We can check this locally on the étale topology of \(X\), so it is enough to show that for each \(\alpha\), the map

\[B_Z \times_X U_\alpha \to U_\alpha\]

is proper. Consider the diagram:

\[
\begin{array}{ccc}
Z_\alpha & \xrightarrow{f_\alpha} & B_Z \times_X U_\alpha & \longrightarrow & R_Z \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
Since $\coprod_{\beta} Z_{\beta}$ is an atlas of $B_Z$, $\coprod_{\beta} Z_{\beta} \times_X U_a$ is an atlas of $B_Z \times_X U_a$. For each $\beta$ we have a Cartesian diagram

$$
\begin{array}{ccc}
Z_{\alpha} & \overset{p_{\alpha}}{\longrightarrow} & B_Z \\
\downarrow & & \downarrow \\
U_a & \longrightarrow & X
\end{array}
$$

in which the top horizontal arrow is surjective; hence since $Z_{\alpha} \rightarrow U_a$ is proper, the original map $B_Z \rightarrow X$ is proper [2, 4.11].

In [16], Mumford proves that if char($k$) = 0 and $X$ has a normal, Cohen-Macauley global cover $\tilde{X}$ (dominating the $Z_{\alpha}$) with group $G$ (i.e. $X = \tilde{X}/G$), then there is an isomorphism

$$
y : \text{CH}^*(X)_0 \rightarrow (\text{OpCH}^*(\tilde{X})_0)^G,
$$

and therefore a ring structure on $\text{CH}^*(X)_0$.

**9.3. Theorem.** Let $X$ be a $Q$-variety. Then if char($k$) = 0 and $X$ has a normal, Cohen-Macauley global cover, the product structure of Section 6 coincides with the product structure defined via the map $y$.

**Proof.** Let $p : \tilde{X} \rightarrow \tilde{X}/G = X$ be the Cohen-Macauley global cover of $X$. Then $p$ factors locally:

$$
\begin{array}{ccc}
\tilde{X} & \supset \tilde{X}_a \\
\downarrow & & \downarrow \\
p & \tilde{X}_a/H_a = Z_{\alpha} & q_{\alpha} \\
\downarrow & & \downarrow \\
X
\end{array}
$$

where $\tilde{X}_a$ is open in $\tilde{X}$ and is stabilized by $H_a \subset G$ and $H_a$ is a normal subgroup of
$H'_a$ such that $G_a = H'_a/H_a$ acts on $Z_a$ with quotient $U_a$ which is étale over $X$, where the $Z_a$ are the charts of the atlas described in 9.1 (see [16, I §2] for details). Then the $\{X_a\}$ are a Zariski open cover of $\mathcal{X}$. Since $X$ is normal there is a canonical map from the nerve $\mathcal{X}$ of the cover $\{X_a\}_{a \in A}$ to $Z$ (the nerve of the cover $\{Z_a\}$ of $B_Z$) which in degree $n$ is the disjoint union of maps:

$$f_{a_0 \cdots a_n} : X_{a_0} \cap \cdots \cap X_{a_n} \to Z_{a_0} \times_X \cdots \times_X Z_{a_n}$$

and which since $\mathcal{X}_{a_0} \cap \cdots \cap \mathcal{X}_{a_n}$ is normal must factor through the normalization $Z_{a_0 \cdots a_n}$ of $Z_{a_0} \times_X \cdots \times_X Z_{a_n}$.

Since we have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{X}_{a_0} \cap \cdots \cap \mathcal{X}_{a_k} & \to & Z_{a_0} \cdots a_k \\
\downarrow & & \downarrow d_1 d_2 \cdots d_k \\
\mathcal{X}_{a_0} & \overset{f_{a_0}}{\rightarrow} & Z_{a_0}
\end{array}$$

with $f_{a_0}$ flat and $d_1 \cdots d_k$ étale, $f_{a_0 \cdots a_k}$ must be flat, hence $f : \mathcal{X} \to Z$ is a flat map of simplicial schemes. If $\mathcal{F}$ is a coherent sheaf on $Z$, i.e. a $Q$-sheaf in the terminology of [16], then $f^* \mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, and hence by descent on $\mathcal{X}$ itself. Now let $Y$ be a prime cycle on $X$; then (note that $i(Y) = i(X) = e(X) = 0$ since $G$ acts faithfully on $X$ and $\text{char } k = 0$)

$$q^{-1}(Y) = e(Y)[p^{-1}(Y)_{\text{red}}]$$

where $q^{-1}(Y)_{\text{red}}$ is the inverse image of $Y$ with its reduced structure. Now $\mathcal{O}_{q^{-1}(Y)}$ is a coherent sheaf on $Z$, which is smooth, and so by 8.1 we have Chern classes

$$C_i(\mathcal{O}_{q^{-1}(Y)}_{\text{red}}) \in H^{i-1}(Y)(Z, K_i(\mathcal{O}_Z))_{\mathbb{Q}}.$$
Intersection theory on algebraic stacks and \(Q\)-varieties

\[
q^*[Y] = \frac{(-1)^d e(Y)}{(d-1)!} C_d(\ell_q^{-1}(Y)_{\text{red}})
\]

\[
= e(Y) \text{ch}_d(\ell_p^{-1}(Y)_{\text{red}})
\]
since the \(C_i\) vanish for \(i < d\).

By the universal properties of the operational Chow groups, there is a commutative diagram of ring homomorphisms:

\[
\begin{array}{ccc}
K_0(\mathcal{X}) & \xrightarrow{\text{ch}} & \text{OpCH}^*(\mathcal{X})_\mathbb{Q} \\
\downarrow & & \downarrow \phi \\
HK^*(\mathbb{Z})_\mathbb{Q} & \xrightarrow{f^*} & HK^*(\mathcal{X})_\mathbb{Q} \\
\end{array}
\]

In order to prove the theorem, it is sufficient to show that if \(Y\) is a prime cycle on \(X\), then in the group \(E\) we have:

\[
\phi \alpha[Y] = \psi f^* q^*[Y].
\]

However, by definition, \(\alpha[Y] = e(Y) \text{ch}^\text{op}_d(f^* \ell_q^{-1}(Y)_{\text{red}})\), while by (9.1.4)

\[
f^* q^*[Y] = f^* \left( \frac{(-1)^{d-1} e(Y)}{(d-1)!} C_d(\ell_q^{-1}(Y)_{\text{red}}) \right)
\]

\[
= \frac{(-1)^{d-1} e(Y)}{(d-1)!} C_d(f^* \ell_q^{-1}(Y)_{\text{red}}) \quad \text{by Theorem 8.1}
\]

\[
= e(Y) \text{ch}_d(f^* \ell_q^{-1}(Y)_{\text{red}}) \quad \text{by (9.1.4)}
\]

and we are done, by the commutativity of (9.1.6).

References

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203-289.