Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 308 (2008) 5312–5316
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

Every 4-connected line graph of a quasi claw-free graph is hamiltonian connected

Hong-Jian Lai[a], Yehong Shao[b], Mingquan Zhan[c]

[a]Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
[b]Department of Mathematics, Ohio University Southern Campus, Ironton, OH 45638, USA
[c]Department of Mathematics, Millersville University, Millersville, PA 17551, USA

Received 23 January 2007; received in revised form 16 September 2007; accepted 18 September 2007
Available online 7 November 2007

Abstract

Let G be a graph. For $u, v \in V(G)$ with $\text{dist}_G(u, v) = 2$, denote $J_G(u, v) = \{w \in N_G(u) \cap N_G(v) | N_G(w) \subseteq N_G(u) \cup N_G(v) \cup \{u, v\}\}$. A graph G is called quasi claw-free if $J_G(u, v) \neq \emptyset$ for any $u, v \in V(G)$ with $\text{dist}_G(u, v) = 2$. In 1986, Thomassen conjectured that every 4-connected line graph is hamiltonian. In this paper we show that every 4-connected line graph of a quasi claw-free graph is hamiltonian connected.

Keywords: Hamiltonian connected graph; Collapsible graph; Quasi claw-free graph; Line graph

1. Introduction

We use [1] for terminology and notations not defined here, and consider loopless finite simple graphs only. Let G be a graph. The *degree* and *neighborhood* of a vertex x of G are, respectively, denoted by $d_G(x)$ and $N_G(x)$. Denote $N_G[x] = N_G(x) \cup \{x\}$. A graph G is *essentially k-edge-connected* if $|E(G)| \geqslant k + 1$ and if for every $E_0 \subseteq E(G)$ with $|E_0| < k$, $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$. For any two distinct vertices x and y in G, denote $\text{dist}_G(x, y)$ the distance in G from x to y. For $u, v \in V(G)$ with $dist_G(u, v) = 2$, denote $J_G(u, v) = \{w \in N_G(u) \cap N_G(v) | N_G(w) \subseteq N_G[u] \cup N_G[v]\}$. A graph G is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. A graph G is called *quasi claw-free* if $J_G(u, v) \neq \emptyset$ for any $u, v \in V(G)$ with $\text{dist}_G(u, v) = 2$. Clearly, every claw-free graph is quasi claw-free.

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and deleting the resulting loops. For convenience, if H is a subgraph of G, we write G/H for $G/E(H)$.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

A graph G is *hamiltonian connected* if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path (a path starting from u and ending at v). In [12], Thomassen conjectured that every 4-connected line graph is hamiltonian.

E-mail address: Mingquan.Zhan@millersville.edu (M. Zhan).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.09.045

By [11], this conjecture is equivalent to the conjecture of Matthews and Sumner stating that every 4-connected claw-free graph is hamiltonian [10].

So far it is known that every 7-connected line graph is hamiltonian connected [14], and that every line graph of a 4-edge-connected graph is hamiltonian connected [13], and that every 4-connected line graph of a claw-free graph is hamiltonian connected [6]. Thomassen's conjecture has also been proved to be true for 4-connected line graphs of planar simple graphs [7]. Here we consider the hamiltonicity of the line graph of a quasi claw-free graph and have the following.

Theorem 1.1. *Every* 4-*connected line graph of a quasi claw-free graph is hamiltonian connected.*

2. Preliminaries

A subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless. Let $v_0, v_k \in V(G)$. A (v_0, v_k)-*trail* of G is a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \dots, e_k, v_k$$

such that all the e_i's are distinct and such that for each $i = 1, 2, \dots, k$, e_i is incident with both v_{i-1} and v_i. With the notation above, this (v_0, v_k)-trial is also called an (e_1, e_k)-*trail*. All the vertices in $v_1, v_2, \dots, v_{k-1}$ are internal vertices of trail. A *dominating* (e_1, e_k)-*trail* T of G is an (e_1, e_k)-trail such that every edge of G is incident with an internal vertex of T. A *spanning* (e_1, e_k)-*trail* of G is a dominating (e_1, e_k)-trail such that $V(T) = V(G)$. There is a close relationship between dominating eulerian subgraphs in graphs and hamiltonian cycles in $L(G)$.

Theorem 2.1 (*Harary and Nash-Williams [5]*). *Let G be a graph with $|E(G)| \geqslant 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

With a similar argument in the proof of Theorem 2.1, one can obtain a theorem for hamiltonian connected line graphs.

Proposition 2.2. *Let G be a graph with $|E(G)| \geqslant 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denote $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the length 2 path is called *subdividing e*. For a graph G and edges $e_1, e_2 \in E(G)$, let $G(e_1)$ denote the graph obtained from G by subdividing e_1, and let $G(e_1, e_2)$ denote the graph obtained from G by subdividing both e_1 and e_2. Thus

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

From the definitions, one immediately has the following observation.

Proposition 2.3. *Let G be a graph and $e_1, e_2 \in E(G)$. If $G(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail, then G has a dominating (e_1, e_2)-trail.*

In [3] Catlin defined collapsible graphs. For $R \subseteq V(G)$, a subgraph Γ of G is called an R-*subgraph* if $G - E(\Gamma)$ is connected and R is the set of all vertices of Γ with odd degrees. A graph is *collapsible* if G has an R-subgraph for every even set $R \subseteq V(G)$. A graph G is contracted to a graph G' if G contains pairwise vertex-disjoint connected subgraphs $H_1, H_2, \dots, H_k$ with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting $H_1, H_2, \dots, H_k$.

Theorem 2.4. (i) *Catlin* [3]. *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible.*

(ii) *Catlin* [3]. *Let H_1 and H_2 be subgraphs of H such that $H_1 \cup H_2 = H$ and $V(H_1) \cap V(H_2) \neq \emptyset$. If H_1 and H_2 are collapsible, then so is H.*

(iii) *Lai et al.* [9]. *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

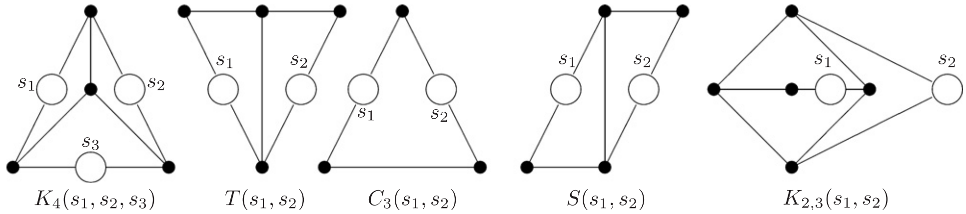

Fig. 1.

We define $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The *edge arboricity* $a_1(G)$ of G is the minimum number of edge-disjoint forests whose union equals G.

Theorem 2.5. *Let G be a graph. Each of the following statements holds*:

(i) *Catlin* [3]. *If* $F(G) \leqslant 1$ *and if G is connected, then G is collapsible if and only if G cannot be contracted to a* K_2.
(ii) *Catlin et al.* [4]. *If* $F(G) \leqslant 2$ *and if G is connected, then G is collapsible if and only if G cannot be contracted to a* K_2 *or a* $K_{2,t}$ *for some integer* $t \geqslant 1$.
(iii) *Catlin* [2]. *If* $a_1(G) \leqslant 2$, *then* $F(G) = 2|V(G)| - |E(G)| - 2$.

Let $s_i \geqslant 1$ $(i = 1, 2, 3)$ be integers. Denote $K_4(s_1, s_2, s_3)$, $T(s_1, s_2)$, $C_3(s_1, s_2)$, $S(s_1, s_2)$ and $K_{2,3}(s_1, s_2)$ to be the graphs depicted in Fig. 1, where the s_i $(i = 1, 2, 3)$ vertices and the two vertices connected by the two lines shown in each of the graphs forms a K_{2,s_i} graph. Denote

$$\mathscr{F}_1 = \{K_4(s_1, s_2, s_3), T(s_1, s_2), C_3(s_1, s_2), S(s_1, s_2), K_{2,3}(s_1, s_2) \mid s_i \geqslant 1\ (i = 1, 2, 3) \text{ is an integer}\}$$

and $\mathscr{F} = \mathscr{F}_1 \cup \{K_{2,t} \mid t \geqslant 2\}$.

Theorem 2.6 (*Lai et al. [8]*). *Let G be a* 2*-edge-connected planar graph. If* $F(G) \leqslant 3$, *then G is collapsible if and only if G cannot be contracted to a graph in* $\mathscr{F}$.

Theorem 2.7 (*Lai et al. [9]*). *Let G be a graph with* $\kappa'(G) \geqslant 3$. *If every* 3*-edge-cut of G has at least one edge in a* 2*-cycle or* 3*-cycle of G, then the graph* $G(e', e'')$ *is collapsible for any* $e', e'' \in E(G)$.

3. Proof of Theorem 1.1

In this section, we assume that G is a quasi claw-free graph with $\kappa(L(G)) \geqslant 4$. Note that then G is essentially 4-edge-connected. The *core* of the graph G, denoted by G_0, is the graph obtained from G by deleting all degree 1 vertices and contracting exactly one edge xy or yz for each path xyz with $d_G(y) = 2$. Denote G_1^* the graph obtained from G by deleting all degree 1 vertices.

Lemma 3.1 (*Lai et al. [9]*). *Let G be a graph and* G_0 *the core of G. If* $G_0(e', e'')$ *has a spanning* $(v(e'), v(e''))$*-trail for any* $e', e'' \in E(G_0)$, *then* $G(e', e'')$ *has a dominating* $(v(e'), v(e''))$*-trail.*

Lemma 3.2. *Let G be a quasi claw-free graph. Then* G_1^* *is also quasi claw-free.*

Proof. Let $u, v \in V(G_1^*)$ with $\mathrm{dist}_{G_1^*}(u, v) = 2$. Then $\mathrm{dist}_G(u, v) = 2$ and $J_G(u, v) \neq \emptyset$. For $x \in J_G(u, v)$, since $N_{G_1^*}(x) \subseteq N_G(x)$, we have $x \in J_{G_1^*}(u, v)$, and hence G_1^* is quasi claw-free. □

Lemma 3.3. *Let* $x \in V(G_0)$ *with* $d_{G_0}(x) = 3$. *Then exactly one of the following holds.*

(i) $G_0[N[x]]$ *contains a triangle.*
(ii) G_0 *contains the graph B, or C, or D in Fig.* 2 *as the induced subgraph.*

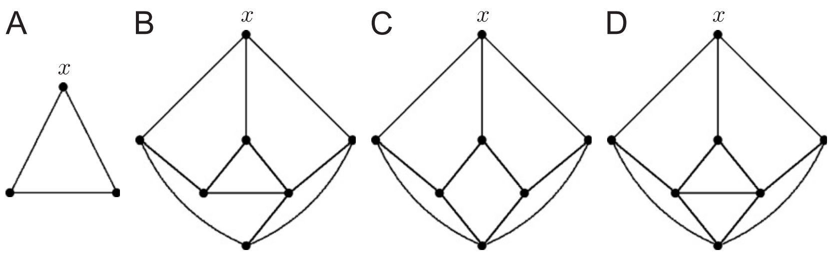

Fig. 2.

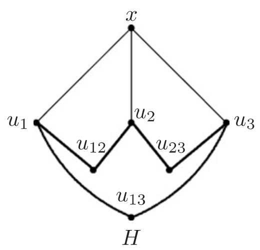

Fig. 3.

Proof. Let $N_{G_0}(x) = \{u_1, u_2, u_3\}$. If $u_i u_j \in E(G_0)$ for some $\{i, j\} \subseteq \{1, 2, 3\}$, then (i) holds. So we may assume that $G_0[\{x, u_1, u_2, u_3\}]$ is a claw in G_0. Since G is essentially 4-edge-connected, $x, u_1, u_2, u_3 \in V(G)$, $d_G(x) = 3$ and $d_G(u_i) \geqslant 3 (i = 1, 2, 3)$. Thus $x, u_1, u_2, u_3 \in V(G_1^*)$. So $G_1^*[\{x, u_1, u_2, u_3\}]$ is a claw in G_1^*. By Lemma 3.2, there exists $u_{ij} \in J_{G_1^*}(u_i, u_j)$ in G_1^*, where $1 \leqslant i < j \leqslant 3$. By the definition of a quasi claw-free graph, $N_{G_1^*}(u_{ij}) \subseteq N_{G_1^*}[u_i] \cup N_{G_1^*}[u_j]$. Since $G_1^*[\{x, u_1, u_2, u_3\}]$ is a claw in G_1^*, the vertices u_{11}, u_{12}, u_{23} are different, $x \notin \{u_{12}, u_{13}, u_{23}\}$ and $u_k \notin N_{G_1^*}(u_{ij})$. Let $H = G_1^*[\{xu_1, xu_2, xu_3, u_{12}u_1, u_{12}u_2, u_{13}u_1, u_{13}u_3, u_{23}u_2, u_{23}u_3\}]$ (see Fig. 3).

Since $d_{G_1^*}(x) = 3$, $xu_{ij} \notin E(G_1^*)$ for all $\{i, j\} \subseteq \{1, 2, 3\}$. Let $t = |E(G_1^*[\{u_{12}, u_{13}, u_{23}\}])|$. If $t \leqslant 1$, without loss of generality, we assume that $u_{12}u_{13}, u_{23}u_{13} \notin E(G_1^*)$. Since $\text{dist}_{G_1^*}(x, u_{13}) = 2$, there exists $a \in J_{G_1^*}(x, u_{13})$. Since $N_{G_1^*}(x) = \{u_1, u_2, u_3\}$ and $u_2u_{13} \notin E(G_1^*)$, we have either $a = u_1$ or u_3. But $u_1u_{12}, u_3u_{23} \in E(G_1^*)$ and $xu_{12}, xu_{23}, u_{13}u_{12}, u_{13}u_{23} \notin E(G_1^*)$, a contradiction. So $t \geqslant 2$. Hence $u_{12}, u_{23}, u_{13} \in V(G_0)$, and H is the subgraph of G_0. So (ii) holds. □

Let $x \in V(G_0)$ with $d_{G_0}(x) = 3$. By Lemma 3.3, denote H_x the induced subgraph in G_0 shown in Fig. 2. Clearly, H_x is a 2-edge-connected planar graph, and $a_1(H_x) \leqslant 2$. By Theorem 2.5(iii),

$$F(H_x) = 2|V(H_x)| - |E(H_x)| - 2 = \begin{cases} 2 \cdot 3 - 3 - 2 = 1 & \text{if } H_x \text{ is a triangle,} \\ 2 \cdot 7 - 11 - 2 = 1 & \text{if } H_x \text{ the graph B or C in Fig. 2,} \\ 2 \cdot 7 - 12 - 2 = 0 & \text{if } H_x \text{ the graph D in Fig. 2.} \end{cases}$$

Denote $\mathscr{H} = \{H_x \mid d_{G_0}(x) = 3\}$, $\mathscr{H}_0 = \{H_x \in \mathscr{H} | H_x \text{is not a triangle}\}$.

Lemma 3.4. *Let $H_x \in \mathscr{H}_0$ and H_x^d be the graph obtained from H_x by subdividing at most two edges of H_x. Then H_x^d is collapsible.*

Proof. Clearly, $H_x \in \mathscr{H}_0$ is 3-edge-connected with $F(H_x) \leqslant 1$. By Theorem 2.5(i), H_x is collapsible.

Let $e' \in E(H_x)$. Then $a_1(H_x(e')) \leqslant 2$. Note that $F(H_x) \leqslant 1$. By Theorem 2.5(iii), $F(H_x(e')) = F(H_x) + 1 \leqslant 2$. Since H_x is 3-edge-connected, and $v(e')$ is the only degree 2 vertex in $H_x(e')$, $H_x(e')$ is collapsible by Theorem 2.5(ii).

Let $e', e'' \in E(H_x)$. Then $a_1(H_x(e', e'')) \leqslant 2$. Note that $F(H_x) \leqslant 1$. By Theorem 2.5(iii), $F(H_x(e', e'')) = F(H_x) + 2 \leqslant 3$. Since H_x is a 3-edge-connected planar graph, and $v(e'), v(e'')$ are only two degree 2 vertices in $H_x(e', e'')$, $H_x(e', e'')$ is collapsible by Theorem 2.6. □

Lemma 3.5. *Let $E_0 = \bigcup_{H_x \in \mathscr{H}_0} E(H_x)$, and $H_0 = G_0[E_0]$. Let H_0^d be the graph obtained from H_0 by subdividing at most two edges of H_0. Then H_0^d is the union of vertex-disjoint collapsible subgraphs.*

Proof. It holds directly by Lemma 3.4 and Theorem 2.4(ii). □

Lemma 3.6. *Let G' be the graph obtained from G_0 by contracting each H_x in $\mathscr{H}_0$. Then every 3-edge-cut of G' has at least one edge in a 2-cycle or 3-cycle of G'.*

Proof. Let $v_1, v_2, \ldots, v_s$ be the new vertices in G' by contracting all $H_x \in \mathscr{H}_0$ in G. Since G is essentially 4-edge-connected, $d_{G'}(v_i) \geqslant 4$ $(i = 1, 2, \ldots, s)$, and every 3-edge-cut of G' must be the 3-edge-cut of G_0. Therefore these three edges are adjacent to some vertex x_0 in G_0 with $d_{G_0}(x_0) = 3$. Let H_{x_0} be the subgraph induced by $N_{G_0}[x_0]$. Since $H_{x_0} \notin \mathscr{H}_0$, $N_{G'}[x_0]$ contains a triangle by Lemma 3.3. Hence every 3-edge-cut of G' has at least one edge in a 2-cycle or 3-cycle of G'. □

Theorem 3.7. *For every pair of edges e', e'' of G_0, $G_0(e', e'')$ is collapsible.*

Proof. Let $E_0 = \bigcup_{H_x \in \mathscr{H}_0} E(H_x)$, and $H_0 = G_0[E_0]$. Then H_0 is the union of some collapsible subgraphs of G_0 by Lemma 3.4.

If $e', e'' \notin E_0$, let $G^{\triangle}$ be the graph obtained from G_0 by contracting H_0. By Lemma 3.6, every 3-edge-cut of $G^{\triangle}$ has at least one edge in a 2-cycle or 3-cycle of $G^{\triangle}$. Thus $G^{\triangle}(e', e'')$ is collapsible by Theorem 2.7. By Lemma 3.5, H_0 is the union of vertex-disjoint collapsible subgraphs. Thus $G_0(e', e'')$ is collapsible by Theorem 2.4(i).

If exactly one of $\{e', e''\}$ is in E_0, without loss of generality, we assume that $e' \in E_0$ and $e'' \notin E_0$. Let $G^{\triangle}$ be the graph obtained from G_0 by contracting $H_0(e')$. Note that $G_0/H_0 = G_0/H_0(e')$. By Lemma 3.6, every 3-edge-cut of $G^{\triangle}$ has at least one edge in a 2-cycle or 3-cycle of $G^{\triangle}$. Thus $G^{\triangle}(e'')$ is collapsible by Theorem 2.7. By Lemma 3.5, $H_0(e')$ is the union of vertex-disjoint collapsible subgraphs. Thus $G_0(e', e'')$ is collapsible by Theorem 2.4(i).

If $e', e'' \in E_0$, let $G^{\triangle}$ be the graph obtained from G_0 by contracting $H_0(e', e'')$. Note that $G_0/H_0 = G_0/H_0(e', e'')$. By Lemma 3.6, every 3-edge-cut of $G^{\triangle}$ has at least one edge in a 2-cycle or 3-cycle of $G^{\triangle}$. Thus $G^{\triangle}$ is collapsible by Theorem 2.7. By Lemma 3.5, $H_0(e', e'')$ is the union of vertex-disjoint collapsible subgraphs. Thus $G_0(e', e'')$ is collapsible by Theorem 2.4(i). □

Proof of Theorem 1.1. Let G be a quasi claw-free graph with $\kappa(L(G)) \geqslant 4$. By Theorem 3.7, $G_0(e', e'')$ is collapsible for any $e', e'' \in E(G_0)$. By Theorem 2.4(iii), $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$-trail . By Lemma 3.1, $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail for any $e', e'' \in E(G)$. By Propositions 2.2 and 2.3, Theorem 1.1 holds. □

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, Elsevier, New York, 1976.
[2] P.A. Catlin, Supereulerian graph, collapsible graphs and 4-cycles, Congr. Numer. 56 (1987) 223–246.
[3] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.
[4] P.A. Catlin, Z. Han, H.-J. Lai, Graphs without spanning closed trails, Discrete Math. 160 (1996) 81–91.
[5] F. Harary, C.St.J.A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–710.
[6] M. Kriesell, All 4-connected line graphs of claw-free graphs are hamiltonian-connected, J. Combin. Theory Ser. B 82 (2001) 306–315.
[7] H.-J. Lai, Every 4-connected line graph of a planar graph is hamiltonian, Graphs Combin. 10 (1994) 249–253.
[8] H.-J. Lai, D. Li, J. Mao, M. Zhan, Supereulerian planar graphs, Ars Combin. LXXV, 2005.
[9] H.-J. Lai, Y. Shao, G. Yu, M. Zhan, Hamiltonian connectedness in 3-connected line graphs, submitted for publication.
[10] M.M. Matthews, D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory 8 (1984) 139–146.
[11] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.
[12] C. Thomassen, Reflections on graph theory, J. Graph Theory 10 (1986) 309–324.
[13] S. Zhan, Hamiltonian connectedness of line graphs, Ars Combin. 22 (1986) 89–95.
[14] S. Zhan, On hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89–95.