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Representations of Completely Bounded Multilinear Operators

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A definition of a completely bounded multilinear operator from one C^*-algebra into another is introduced. Each completely bounded multilinear operator from a C^*-algebra into the algebra of bounded linear operators on a Hilbert space is shown to be representable in terms of $*$-representations of the C^*-algebra and interlacing operators. This result extends Wittstock's Theorem that decomposes a completely bounded linear operator from a C^*-algebra into an injective C^*-algebra into completely positive linear operators.

INTRODUCTION

Stinespring's Theorem gives a useful representation for a completely positive linear operator from a C^*-algebra into the algebra $\mathrm{BL}(H)$ for continuous linear operators on a Hilbert space H [12, 1]. Using this representation with Wittstock's Theorem that decomposes a completely bounded linear operator as a finite linear combination of completely positive linear operators [14, 10], one obtains a representation of a completely bounded linear operator from a C^*-algebra into $\mathrm{BL}(H)$. Our main result is a representation theorem for completely bounded multilinear operators from a C^*-algebra into $\mathrm{BL}(H)$, which generalizes this representation of completely bounded linear operators. Corollaries give multilinear generalizations of several results known for completely bounded linear operators. We shall briefly describe the type of representations to be

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studied here and state the main theorem before giving the detailed definitions and further discussion.

If $\theta_1, ..., \theta_k$ are *-representations of a C^*-algebra $\mathscr{A}$ on Hilbert spaces $H_1, ..., H_k$ and if $V_j \in \mathrm{BL}(H_{j+1}, H_j)$ for $j = 0, ..., k$, where $H_0 = H = H_{k+1}$, then

$$\Phi(a_1, ..., a_k) = V_0 \theta_1(a_1) V_1 \theta_2(a_2) V_2 \cdots \theta_k(a_k) V_k$$

is clearly an exceedingly nice k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$. Such k-linear operators will be called representable and the infimum of $\|V_0\| \cdot \|V_1\| \cdots \|V_k\|$ over all representations of Φ will be taken as the representation norm $\|\Phi\|_{\mathrm{rep}}$ of Φ. Extending the definition of completely bounded in a suitable way from linear operators to k-linear operators (Definition 1.1) between C^*-algebras leads to the representability of completely bounded k-linear operators and the main result (Theorem 5.2) of this paper.

Let $\mathscr{A}$ be a C^*-algebra, and let H be a Hilbert space. A k-linear operator Φ from $\mathscr{A}^k$ into $\mathrm{BL}(H)$ *is representable if and only if it is completely bounded; when Φ is completely bounded $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{rep}}$, and the representable norm is attained.*

From this representation theorem it follows that a completely bounded k-linear operator from $\mathscr{A}^k$ into an injective C^*-algebra $\mathscr{B}$ is a linear combination of completely bounded completely positive k-linear operators from $\mathscr{A}^k$ into $\mathscr{B}$. This generalizes to completely bounded multilinear operators Wittstock's decomposition theorem that each completely bounded linear operator from a C^*-algebra $\mathscr{A}$ into an injective C^*-algebra $\mathscr{B}$ is a linear combination of completely positive linear operators (see [14, Satz 4.5; 10, Corollary 2.6; 7] for a detailed discussion of this). Because *-representations lift from a C^*-algebra to its enveloping von Neumann algebra, completely bounded multilinear operators from a C^*-algebra into $\mathrm{BL}(H)$ lift to the enveloping von Neumann algebra (Corollary 5.5).

There are essentially three steps in our proof of the representation theorem (5.2). First, a reduction to the case of a symmetric completely bounded k-linear operator by a standard 2×2 matrix corner technique (proof of Theorem 5.2). Second, there is a Grothendieck-type domination of a completely bounded k-linear operator by a completely positive linear operator just on the outer two variables (Theorem 2.8). Third, there is the representation given such domination combined with an inductive reduction argument on k that peels off the outer two variables of a completely bounded symmetric k-linear operator leaving a completely bounded symmetric $(k-2)$-linear operator (Lemma 3.1).

The remainder of this Introduction will be devoted to detailed definitions, notation, motivation, and a discussion of the relationship with previous results.

If $\mathscr{A}$ is a C^*-algebra, $M_n(\mathscr{A})$ denotes the C^*-algebra of $n \times n$ matrices over $\mathscr{A}$, which will be naturally identified with $\mathscr{A} \otimes M_n(\mathbb{C})$. Elements of $M_n(\mathscr{A})$ will often be written $A = (a_{ij})$ or $A_\ell = (a_{\ell ij})$ with $a_{ij}, a_{\ell ij} \in \mathscr{A}$. Recall that if $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras and if $\varphi: \mathscr{A} \to \mathscr{B}$ is a linear operator, then $\varphi_n: M_n(\mathscr{A}) \to M_n(\mathscr{B})$ is defined by $\varphi_n(a_{ij}) = (\varphi(a_{ij}))$. The linear operator $\varphi: \mathscr{A} \to \mathscr{B}$ is called *completely bounded* if $\|\varphi\|_{\mathrm{cb}} = \sup\{\|\varphi_n\|: n \in \mathbb{N}\}$ is finite, and $\|\cdot\|_{\mathrm{cb}}$ is the completely bounded norm. If $\varphi_n \geqslant 0$ for all n, then φ is *completely positive.* Intuitively the completely bounded (positive) linear operators look like generalizations of continuous (positive) linear functionals. This analogy is reasonable when the image algebra is $\mathrm{BL}(H)$ or an *injective* C^*-algebra (see [14, 10, 7]). For our purposes the most convenient definition of an *injective* C^*-algebra $\mathscr{B}$ is that for each C^*-algebra $\mathscr{A}$ containing $\mathscr{B}$ there is a completely positive projection from $\mathscr{A}$ onto $\mathscr{B}$ (see [11, p. 393]).

Multilinear definitions of completely bounded and completely positive will now be introduced. The definitions and theorems are given only in the case where all the domain algebras are the same. This is the most interesting situation, the definitions may be trivially modified to cover the general case, and the general case theorems follow from the particular ones by the following well known technique. If $\mathscr{A}_1, \ldots, \mathscr{A}_k$, $\mathscr{B}$ are C^*-algebras, and if Φ is a k-linear operator from $\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_k$ into $\mathscr{B}$, let $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \cdots \oplus \mathscr{A}_k$ with π_j the projection from $\mathscr{A}$ onto $\mathscr{A}_j$ annihilating $\mathscr{A}_i$ $(i \neq j)$ and let $\Psi = \Phi \circ (\pi_1 \otimes \cdots \otimes \pi_k)$. Then Ψ is a k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$ that is essentially Φ.

1.1. *Definitions*

Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras, and let Φ be a k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$. The k-linear operator Φ_n from $M_n(\mathscr{A})^k$ into $M_n(\mathscr{B})$ is defined by

$$\Phi_n(A_1, \ldots, A_k) = \left(\sum_{r,s,\ldots,t} \Phi(a_{1ir}, a_{2rs}, \ldots, a_{ktj}) \right)$$

for all $A_\ell = (a_{\ell ij}) \in M_n(\mathscr{A})$ $(1 \leqslant \ell \leqslant k)$ and all $n \in \mathbb{N}$. The k-linear operator Φ is said to be *completely bounded* with *completely bounded norm* $\|\Phi\|_{\mathrm{cb}}$ if

$$\|\Phi\|_{\mathrm{cb}} = \sup\{\|\Phi_n\|: n \in \mathbb{N}\}$$

is finite. Note that the definition of Φ_n is intimately related to the definition of matrix multiplication. This observation is intuitively and technically crucial in motivating the definition and explaining why the results follow. A completely bounded k-linear operator Φ from $\mathscr{A}^k$ into $\mathscr{B}$ lifts naturally to a bounded k-linear operator Φ_∞ from $(\mathscr{A} \otimes_* \mathscr{C})^k$ into $\mathscr{B} \otimes_* \mathscr{C}$, where $\mathscr{C}$ is the C^*-algebra of all compact operators on a separable infinite dimensional

Hilbert space and $\otimes_*$ is the (unique for $\mathscr{C}$) C^*-tensor product. Further $\|\Phi_\infty\| = \|\Phi\|_{cb}$. Note that if $a_1,..., a_k \in \mathscr{A}$ and $T_1,..., T_k \in \mathscr{C}$, then

$$\Phi_\infty(a_1 \otimes T_1,..., a_k \otimes T_k) = \Phi(a_1,..., a_k) \otimes (T_1 \cdot \cdots \cdot T_k).$$

Since $\bigcup_1^\infty M_n(\mathbb{C})$ is dense in $\mathscr{C}$, where $M_n(\mathbb{C})$ is regarded as a subalgebra of $\mathscr{C}$ by embedding it in the $n \times n$ top left corner of the matrix representation of $\mathscr{C}$ acting on $\ell_2(\mathbb{N})$, it is enough to show that the equality holds for all $T_1,..., T_k \in M_n(\mathbb{C})$. This follows directly from the definition of Φ_n and the observation that

$$a_\ell \otimes T_\ell = (a_\ell t_{\ell ij}).$$

The form for Φ_∞ again emphasises the product nature of the definition of Φ_n.

The k-linear operator Φ^* from $\mathscr{A}^k$ into $\mathscr{B}$ is defined by

$$\Phi^*(a_1,..., a_k) = \Phi(a_k^*,..., a_2^*, a_1^*)^*$$

for all $a_1,..., a_k \in \mathscr{A}$. The k-linear operator Φ is said to be *symmetric* (when $k = 1$; selfadjoint [7] or real [10]) if $\Phi = \Phi^*$. If Φ is symmetric, then Φ is completely symmetric in that $\Phi_n^* = \Phi_n$ for all n (Lemma 2.1). The technical proofs in Sections 2, 3, 4 are all given for completely bounded symmetric k-linear operators, and the general result is deduced from these by a symmetrization argument in Section 5. Writing $\Phi = (\Phi + \Phi^*)/2 + i(\Phi - \Phi^*)/2i$ gives Φ as a symmetric plus i times a symmetric operator. However, this decomposition does not preserve norms so we use a 2×2 matrix technique in 5.2.

When studying Hermitian forms F on a Hilbert space H a symmetric norm $\|F\|_s = \sup\{|F(x, x)|: x \in H, \|x\| \leqslant 1\}$ occurs as well as the standard norm

$$\|F\| = \sup\{|F(x, y)|: x, y \in H, \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

The two norms eventually turn out to be equal. A similar situation holds for completely bounded symmetric k-linear operators in that there is a symmetric completely bounded norm $\|\cdot\|_{scb}$ useful in calculations but eventually equal to $\|\cdot\|_{cb}$ (Corollary 4.2). When considering symmetric k-linear operators there is a slight difference between the odd and even cases caused by the central variable in the odd case occuring only as a linear term whereas the other variables occur as quadratic terms in the definition of the symmetric norm. Throughout this paper we shall let $m = [(k+1)/2]$ be the greatest integer less than or equal to $(k+1)/2$.

If $\mathbf{A} = (A_1, \dots, A_k) \in M_n(\mathscr{A})^k$, let $\mathbf{A}^* = (A_k^*, \dots, A_1^*)$. Let

$$\|\Phi_n\|_s = \sup\{ \|\Phi_n(A_1, \dots, A_k)\| : A_j \in M_n(\mathscr{A}) \text{ with } \|A_j\| \leqslant 1 \text{ for } 1 \leqslant j \leqslant k \text{ and } \mathbf{A} = (A_1, \dots, A_k) = \mathbf{A}^* \}$$

for all n. Then the *symmetric completely bounded* norm $\|\cdot\|_{\mathrm{scb}}$ is defined by

$$\|\Phi\|_{\mathrm{scb}} = \sup\{ \|\Phi_n\|_s : n \in \mathbb{N} \}.$$

Clearly $\|\Phi\|_{\mathrm{scb}} \leqslant \|\Phi\|_{\mathrm{cb}}$. For symmetric completely bounded k-linear operators equality holds (Corollary 4.2).

Subsequently we shall be concerned with decomposing a completely bounded symmetric k-linear operator as a difference of completely bounded completely positive k-linear operators in a similar way to the Wittstock decomposition of a selfadjoint completely bounded linear operator [14, Satz 4.5]. A k-linear operator Φ from $\mathscr{A}^k$ into $\mathscr{B}$ is said to be *completely positive* if

$$\Phi_n(A_1, \dots, A_k) \geqslant 0$$

for all $(A_1, \dots, A_k) = (A_k^*, \dots, A_1^*) \in M_n(\mathscr{A})^k$ with $A_m \geqslant 0$ if k is odd, where $m = [(k+1)/2]$, and all $n \in \mathbb{N}$.

Each completely positive linear operator φ is completely bounded with $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$. This fails in the case of completely positive bilinear operators (or even forms) as examples exist of such operators which are not completely bounded. Here is the general method.

If K is a Hilbert space, if ψ is a symmetric map ($\psi(x^*) = \psi(x)^*$ for all x) from $\mathscr{A}$ into $\mathrm{BL}(K)$, if $V \in \mathrm{BL}(H, K)$, and if $W = W^* \in \mathrm{BL}(K)$ with $W \geqslant 0$, let $\Psi(a, b) = V^*\psi(a)\, W\psi(b)\, V$ for all $a, b \in \mathscr{A}$. Then Ψ is a completely positive bilinear operator from $\mathscr{A}^2$ into $\mathrm{BL}(H)$ as may be easily seen by adapting the proof of Lemma 5.1. By a suitable choice of ψ this Ψ cannot be representable so is not completely bounded (Theorem 5.2). Similar observations hold for completely positive k-linear operators ($k \geqslant 2$). Using one of the classes of continuous bilinear forms other than RL of Remark 5.3(b) of [9] (see also [4]) shows that there are examples of completely positive bilinear forms on suitably noncommutative C^*-algebras that are not completely bounded.

We now introduce the representability of completely bounded k-linear operators and the representable norm. Let $\mathscr{A}$ be a C^*-algebra and let H be a Hilbert space. If $\theta_1, \dots, \theta_k$ are $*$-representations of $\mathscr{A}$ on Hilbert spaces

$H_1, \dots, H_k$, and if $V_j \in \mathrm{BL}(H_{j+1}, H_j)$ for $j = 0, \dots, k$ where $H_0 = H_{k+1} = H$, then define a k-linear operator Φ from $\mathscr{A}^k$ into $\mathrm{BL}(H)$ by

$$\Phi(a_1, \dots, a_k) = V_0 \theta_1(a_1) V_1 \theta_2(a_2) V_2 \cdots \theta_k(a_k) V_k$$

for all $a_1, \dots, a_k \in \mathscr{A}$. The *representable norm* $\|\cdot\|_{\mathrm{rep}}$ of a representable k-linear operator Φ is defined by

$$\begin{aligned}\|\Phi\|_{\mathrm{rep}} = \inf\{ \|V_0\| \cdot \|V_1\| \cdot \cdots \cdot \|V_k\|: \\ &\Phi \text{ has a representation in terms of} \\ &V_0, V_1, \dots, V_k, \theta_1, \dots, \theta_k\}.\end{aligned}$$

The matrix multiplication nature of the definition of Φ_n ensures that

$$\Phi_n(A_1, \dots, A_k) = V_{0,n} \theta_{1,n}(A_1) V_{1,n} \cdots \theta_{k,n}(A_k) V_{k,n},$$

where $V_{\ell,n} = V_\ell \otimes I_n$ is the n-fold amplification of V_ℓ and $\theta_{\ell,n}(A_\ell) = (\theta_\ell(a_{\ell ij}))$ for all $A_\ell = (a_{\ell ij}) \in M_n(\mathscr{A})$ (see Lemma 5.1). Thus Φ_n is also representable and $\|\Phi_n\|_{\mathrm{rep}} = \|\Phi\|_{\mathrm{rep}}$, because $\|V_{l,n}\| = \|V_l\|$. Since $\|\Phi_n\| \leqslant \|\Phi_n\|_{\mathrm{rep}}$, it follows that each representable operator Φ is completely bounded and that $\|\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|_{\mathrm{rep}}$. The main result (Theorem 5.2) of this paper is that each completely bounded k-linear operator Φ from $\mathscr{A}^k$ into $\mathrm{BL}(H)$ is representable, that $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{rep}}$, and that the infimum in the definition of $\|\Phi\|_{\mathrm{rep}}$ is attained. We have been influenced in interpreting our results by Haagerup's discussion of decomposability for completely bounded selfadjoint operators [7, Sect. 1].

There is a definition of a symmetric representable norm for a suitable class of symmetric representable completely bounded k-linear operators. Once again there is a slight difference between the even and odd class; these are written out separately.

k odd. If $\theta_1, \dots, \theta_{m-1}$, ψ_1, ψ_2 are *-representations of $\mathscr{A}$ on Hilbert spaces $H_1, \dots, H_{m-1}$, K_1, K_2, if $V_j \in \mathrm{BL}(H_j, H_{j+1})$ for $j = 0, \dots, m-2$ where $H_0 = H$, and if $W_1 \in \mathrm{BL}(H_{m-1}, K_1)$ and $W_2 \in \mathrm{BL}(H_{m-1}, K_2)$, then let

$$\begin{aligned}\Phi(a_1, \dots, a_k) = {} & V_0^* \theta_1(a_1) V_1^* \theta_2(a_2) \cdots \theta_{m-1}(a_{m-1}) \\ & \times \{ W_1^* \psi_1(a_m) W_1 - W_2^* \psi_2(a_m) W_2 \} \\ & \times \theta_{m-1}(a_{m+1}) \cdots \theta_2(a_{k-1}) V_1 \theta_1(a_k) V_0\end{aligned}$$

for all $a_1, \dots, a_k \in \mathscr{A}$. Then Φ is a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$. The *symmetric representable norm* $\|\cdot\|_{\mathrm{srep}}$ of a symmetrically representable k-linear operator of this type is defined by

$$\|\Phi\|_{\text{srep}} = \inf\{\|V_0\|^2 \cdot \cdots \cdot \|V_{m-2}\|^2 \cdot \|W_1^* W_1 + W_2^* W_2\| : \Phi \text{ has a symmetric representation in terms of } V_0, \dots, V_{m-2}, W_1, W_2, \theta_1, \dots, \theta_{m-1}, \psi_1, \psi_2\}.$$

k even. If $\theta_1, \dots, \theta_m$ are *-representations of $\mathscr{A}$ on Hilbert spaces $H_1, \dots, H_m$, if $V_j \in \mathrm{BL}(H_j, H_{j+1})$ for $j = 0, 1, \dots, m-1$, and if $V_m = V_m^* \in \mathrm{BL}(H_m)$, let

$$\Phi(a_1, \dots, a_k) = V_0^* \theta_1(a_1) V_1^* \cdots V_{m-1}^* \theta_m(a_m) V_m \theta_m(a_{m+1}) V_{m-1} \cdots V_1 \theta_1(a_k) V_0$$

for all $a_1, \dots, a_k \in \mathscr{A}$. Then Φ is a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$. The *symmetric representable norm* $\|\cdot\|_{\text{srep}}$ of a symmetrically representable k-linear operator of this type is defined by

$$\|\Phi\|_{\text{srep}} = \inf\{\|V_0\|^2 \cdot \cdots \cdot \|V_{m-1}\|^2 \cdot \|V_m\| : \Phi \text{ has a symmetric representation in terms of } V_0, \dots, V_m, \theta_1, \dots, \theta_m\}.$$

For a symmetrically representable operator the definitions yield directly

$$\|\Phi\|_{\text{scb}} \leqslant \|\Phi\|_{\text{cb}} \leqslant \|\Phi\|_{\text{rep}} \leqslant \|\Phi\|_{\text{srep}}$$

except for a minor little twist in the case k odd for the inequality $\|\Phi\|_{\text{rep}} \leqslant \|\Phi\|_{\text{srep}}$. The other two inequalities have already been discussed. When relating the representable and symmetrically representable norms for k odd the central variable

$$\{W_1^* \psi_1(a_m) W_1 - W_2^* \psi_2(a_m) W_2\}$$

in the representation of Φ has to be written $V^* \psi(a_m) W$. Take $K = K_1 \oplus K_2$ the direct sum of Hilbert spaces, $\psi = \psi_1 \oplus \psi_2$ the *-representation of $\mathscr{A}$ on K, $W = W_1 \oplus W_1 \in \mathrm{BL}(H_{m-1}, K)$, and $V = W_1 \oplus (-W_2) \in \mathrm{BL}(H_{m-1}, K)$. The minus sign introduced in V is what makes this representation give Φ when the other $V_0, \dots, V_{m-1}, \theta_0, \dots, \theta_{m-1}$ are introduced in the correct way. Now

$$\|W\|^2 = \|V\|^2 = \|W^* W\| = \|W_1^* W_1 + W_2^* W_2\|$$

so that $\|\Phi\|_{\text{rep}} \leqslant \|\Phi\|_{\text{srep}}$ as required. Theorem 4.1 shows that each symmetric completely bounded k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$ is symmetrically representable, that

$$\|\Phi\|_{\mathrm{scb}} = \|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{rep}} = \|\Phi\|_{\mathrm{srep}},$$

and that the infimum in the definition of $\|\Phi\|_{\mathrm{srep}}$ is attained.

The representable and symmetric representable norms may both be shown to be norms by taking suitable direct sums of the Hilbert spaces, representations, and operators involved. The methods are standard and are omitted because the equalities $\|\cdot\|_{\mathrm{cb}} = \|\cdot\|_{\mathrm{rep}}$ and $\|\cdot\|_{\mathrm{scb}} = \|\cdot\|_{\mathrm{srep}}$ ensure that $\|\cdot\|_{\mathrm{rep}}$ and $\|\cdot\|_{\mathrm{srep}}$ are norms.

Sections 2, 3, and 4 are all concerned with symmetric k-linear operators. Section 2 contains the definition of a matricial sublinear functional from Wittstock's paper [14] and the lemmas required to apply his matricial Hahn-Banach Theorem to obtain Theorem 2.8. In the case $k=2$, Theorem 2.8 is reminiscent of Grothendieck's inequality in that here a completely bounded symmetric bilinear operator is dominated by a completely positive linear operator. However Theorem 2.8 is not a generalization of Grothendieck's inequality, which essentially is that a continuous bilinear form on a commutative C^*-algebra is completely bounded (see Corollary 5.6). Section 3 contains a result for representing a completely bounded symmetric k-linear operator that is suitably dominated by a completely positive linear operator in terms of a completely bounded symmetric $(k-2)$-linear operator. When $k=2$ this is an operator theoretic version of representation theorems for continuous bilinear operators on C^*-algebras (see [9] and [4]). This result (3.1) uses the Stinespring construction in the proof in an analogous way to that in which the Gelfand–Naimark–Segal construction is used in the proof of [9, Remark 5.3(a)] (see also [4]). The technical tools of Sections 2 and 3 are used in Section 4 to prove the representation theorem (4.1) for a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$.

Theorem 5.2 is the main result of this paper and it is deduced from the symmetric version Theorem 4.1. The decomposition and representability of completely bounded k-linear operators in terms of completely bounded $(k-2)$-linear operators are also discussed in Section 5. A bilinear form Φ on a C^*-algebra $\mathscr{A}$ is completely bounded if and only if Φ is continuous in the norm $\|\cdot\|_{\mathrm{h}}$ of Effros and Kishimoto [4, Theorem 2.1] and then $\|\Phi\|_{\mathrm{h}} = \|\Phi\|_{\mathrm{cb}}$ by Theorem 5.2. Finally in Corollary 5.6 there is a characterization of Grothendieck's constant in terms of the completely bounded norm of bilinear forms on commutative C^*-algebras.

2. Domination of Completely Bounded Symmetric Multilinear Operators

The main result (Theorem 2.8) of this section may be viewed as a type of Grothendieck inequality in that a suitable completely bounded multilinear operator is dominated by a completely positive operator. Here the completely bounded multilinear operator would correspond to the continuous bilinear form on the commutative C^*-algebra, and the completely positive operator would correspond to the state that is given by Grothendiecks's theorem. However, as is pointed out in the Introduction this result is not a generalization of Grothendiecks's inequality and the parallel must not be carried too far. Theorem 2.8 depends on Wittstock's matricial Hahn–Banach Theorem [14, Kor. 2.2.4]; this result is the crucial tool in our proof together with his splitting of a completely bounded symmetric linear operator into a difference of completely positive linear operators. Before embarking on the proof we shall prove a couple of elementary little technical lemmas and recall some notation from Wittstock's papers [14, 15]. Throughout this section Φ will be a completely bounded k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras with $\mathscr{B}$ injective ($k \geq 2$). Recall that Φ^* is defined by

$$\Phi^*(a_1,..., a_k) = \Phi(a_k^*,..., a_1^*)^*$$

for all $a_1,..., a_k \in \mathscr{A}$.

2.1. Lemma. (i) *For all* $n \in \mathbb{N}$,

$$(\Phi^*)_n = (\Phi_n)^*.$$

(ii) *If* Φ *is symmetric, that is* $\Phi = \Phi^*$, *then* Φ *is completely symmetric in that* $\Phi_n^* = \Phi_n$ *for all* n.

(iii) $\|\Phi^*\|_{\text{cb}} = \|\Phi\|_{\text{cb}}$.

Proof. Parts (ii) and (iii) follow directly from (i); in the case of (iii) using the result that the involution is an isometry on a C^*-algebra.

Part (i) is an elementary direct calculation. Let $A_1,..., A_k \in M_n(\mathscr{A})$. Then the (i, j)th component is

$$\begin{aligned}(\Phi_n)^*(A_1,..., A_k)_{ij} &= (\Phi_n(A_k^*,..., A_1^*)^*)_{ij} \\ &= \Phi_n(A_k^*,..., A_1^*)_{ji}^* \\ &= \sum_{r,s,...,t} \Phi(a_{krj}^*, a_{k-1,sr}^*,..., a_{1it}^*)^* \\ &= \sum_{r,s,...,t} \Phi^*(a_{1it},..., a_{k-1,sr}, a_{krj}) \\ &= (\Phi^*)_n(A_1,..., A_k)_{ij},\end{aligned}$$

which completes the proof.

The next little lemma is well known; it is needed in the proof of Lemma 2.6.

2.2. LEMMA. *Let a, b be in a C^*-algebra $\mathscr{C}$. If $a^*a = b^*b$, then there exists a sequence (z_n) in $\mathscr{C}$ such that $\|z_n\| = 1$ for all n, and*

$$\lim \|z_n a - b\| = 0.$$

Proof. Let $h^2 = a^*a = b^*b$ with $h \geqslant 0$, and let $z_n = b(h^2 + 1/n)^{-1} a^*$ for all n. Either by representing $\mathscr{C}$ as an algebra of operators on a Hilbert space and operator theoretic calculations, or direct exploitation of $\|W^*W\| = \|W\|^2$ gives the result.

To simplify the calculations and notation in the main lemmas we introduce some notation unifying the liftings Φ_n.

2.3. DEFINITION. Let $\mathbb{F}(\mathscr{A})$ denote the algebra of all infinite matrices with entries from $\mathscr{A}$ indexed by $\mathbb{N} \times \mathbb{N}$ for which only a finite number of entries are nonzero. Thus $\mathbb{F}(\mathscr{A})$ is the algebraic tensor of $\mathscr{A}$ and the algebra of finite rank continuous linear operators on an infinite dimensional separable Hilbert space H. We shall regard $\mathbb{F}(\mathscr{A})$ as a dense subalgebra of the C^*-tensor product $\mathscr{A} \otimes_* \mathrm{CL}(H)$ of $\mathscr{A}$ and $\mathrm{CL}(H)$, the algebra of compact operators on H. Let Φ_∞ be the k-linear operator from $\mathbb{F}(\mathscr{A})^k$ into $\mathbb{F}(\mathscr{B})$ defined by

$$\Phi_\infty(A_1, \ldots, A_k)_{ij} = \sum_{r,s,\ldots,t} \Phi(a_{1ir}, a_{2rs}, \ldots, a_{ktj})$$

for all $i, j \in \mathbb{N}$, $A_1, \ldots, A_k \in \mathbb{F}(\mathscr{A})$.

Note that the sum defining Φ_∞ is finite because each A_ℓ has finite support.

2.4. LEMMA. *If $A_1, \ldots, A_k$ are in $\mathbb{F}(\mathscr{A})$ and γ is in $\mathbb{F}(\mathbb{C})$, then*

$$\Phi_\infty(A_1, \ldots, A_k)\gamma = \Phi_\infty(A_1, \ldots, A_{k-1}, A_k\gamma),$$
$$\gamma\Phi_\infty(A_1, \ldots, A_k) = \Phi_\infty(\gamma A_1, A_2, \ldots, A_k).$$

Proof. The (i, j)th component is

$$\begin{aligned}(\Phi_\infty(A_1, \ldots, A_k)\gamma)_{ij} &= \sum_{r,s,\ldots,t,u} \Phi(a_{1ir}, a_{2rs}, \ldots, a_{ktu})\gamma_{uj} \\ &= \sum_{r,s,\ldots,t} \Phi\left(a_{1ir}, a_{2rs}, \ldots, \sum_u a_{ktu}\gamma_{uj}\right) \\ &= \Phi_\infty(A_1, \ldots, A_{k-1}, A_k\gamma)_{ij}\end{aligned}$$

as required.

The following definition is [14, 2.1.2] and is used in applying Wittstock's matricial Hahn–Banach Theorem [14, 2.2.4].

2.5. DEFINITION. (i) If H and K are nonempty subsets of the selfadjoint part $\mathscr{A}_h$ of a C^*-algebra $\mathscr{A}$, then write $H \prec K$ if and only if for each $a \in K$ there is a $b \in H$ such that $b \leqslant a$.

(ii) A sequence (θ_n) of set valued mappings of $M_n(\mathscr{A})_h$ into $M_n(\mathscr{B})_h$ is said to be *matricial sublinear* if

(a) $\theta_n(X) \neq \varnothing$ for all $n \in \mathbb{N}$ and $X \in M_n(\mathscr{A})_h$,

(b) $\theta_n(X_1 + X_2) \prec \theta_n(X_1) + \theta_n(X_2)$ for all $n \in \mathbb{N}$ and $X_1, X_2 \in M_n(\mathscr{A})_h$,

(c) $0 \prec \theta_n(0)$ for all $n \in \mathbb{N}$, and

(d) $\theta_m(\gamma^* X \gamma) \prec \gamma^* \theta_n(X) \gamma$ for all $m, n \in \mathbb{N}$, $\gamma \in M_{n,m}(\mathbb{C})$, and $X \in M_n(\mathscr{A})_h$.

The following lemma contains the definition of the matricial sublinear functional used in the proof of Theorem 2.8; it also contains the proof that the (θ_n) defined are matricial sublinear.

2.6. LEMMA. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with $\mathscr{B}$ injective, and let Φ be a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$ with $\|\Phi\|_{\text{scb}} = 1$. For each A in $M_n(\mathscr{A})_h$, let*

$$\begin{aligned} \Gamma(A) = \{(X, Y, \gamma; \mathbf{B}) \colon \mathbf{B} &= (B_2, \ldots, B_{k-1}) \\ &= (B_{k-1}^*, \ldots, B_2^*) = \mathbf{B}^*;\ X, Y, B_2, \ldots, B_{k-1} \in \mathbb{F}(\mathscr{A}) \\ &\quad \textit{with } \|X\| \leqslant 1 \textit{ and } \|B_j\| \leqslant 1 \\ &\quad \textit{for } 2 \leqslant j \leqslant k-1;\ \gamma \in \mathbb{F}(\mathbb{C}); \\ &\quad \gamma^*(X^*X - Y^*Y)\gamma = A\}, \end{aligned}$$

and let

$$\theta_n(A) = \{\gamma^*\gamma - \gamma^*\Phi_\infty(Y^*, \mathbf{B}, Y)\gamma \colon (X, Y, \gamma; \mathbf{B}) \in \Gamma(A)\}.$$

Then (θ_n) is matricial sublinear.

Before giving the proof of this lemma perhaps some remarks will help to clarify these messy definitions. The factors $\gamma \in \mathbb{F}(\mathbb{C})$ in the definitions are required to ensure that property (d) of Definition 2.5 holds. The element $\mathbf{B}$ will eventually provide the completely bounded $(k-2)$-linear operator of Section 3. The X and Y are to be thought of as the square roots of the positive and negative parts of A. Though the X does not appear directly in the definition of $\theta_n(A)$ it controls the size of Y. With Lemma 2.2, the condition $\|X\| \leqslant 1$ plays a vital role in checking Definition 2.5(ii)(c).

Proof of Lemma 2.6. Note that $\Gamma(A) \neq \emptyset$ for each $A \in M_n(\mathscr{A})_h$, because

$$(\alpha^{-1}A_+^{1/2}, \alpha^{-1}A_-^{1/2}, \alpha^2 I_n; \mathbf{B}) \in \Gamma(A)$$

for $\alpha > \|A\|^{1/2}$, where $A = A_+ - A_-$ with A_+ the positive part of A and A_- the negative part, and $\mathbf{B} = \mathbf{B}^*$ with $\|\mathbf{B}\| \leqslant 1$.

Let $A, B \in M_n(\mathscr{A})_h$ and let $(X, Y, \alpha; \mathbf{C}) \in \Gamma(A)$ and $(U, V, \beta; \mathbf{D}) \in \Gamma(B)$. We now proceed as in [15, p. 242, Proof of Theorem 3.1] by defining

$$S = \begin{pmatrix} \begin{pmatrix} X & 0 \\ 0 & U \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}(\mathscr{A}),$$

$$T = \begin{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & V \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}(\mathscr{A}),$$

$$\gamma = \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}(\mathbb{C}),$$

and

$$E_j \in \begin{pmatrix} \begin{pmatrix} C_j & 0 \\ 0 & D_j \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{F}(\mathscr{A})$$

for $2 \leqslant j \leqslant k-1$. The diagonal nature of the definition of S, T, $\mathbf{E} = \mathbf{E}^*$, and the "block column" definition of γ shows that $(S, T, \gamma; \mathbf{E}) \in \Gamma(A+B)$ and

$$\begin{aligned} \gamma^*\gamma - \gamma^*\Phi_\infty(T^*, \mathbf{E}, T)\,\gamma &= \alpha^*\alpha - \alpha^*\Phi_\infty(Y^*, \mathbf{C}, Y)\,\alpha \\ &\quad + \beta^*\beta - \beta^*\Phi_\infty(V^*, \mathbf{D}, V)\,\beta. \end{aligned}$$

This shows that the subadditivity condition (b) of Definition 2.5 holds.

Now let $(X, Y, \alpha; \mathbf{B})$ be in $\Gamma(0)$; then $\alpha^*X^*X\alpha = \alpha^*Y^*Y\alpha$. By Lemma 2.2 there is a sequence (Z_j) in $\mathbb{F}(\mathscr{A})$, regarded as a subalgebra of a C^*-algebra such that $\|Z_j\| \leqslant 1$ and $\|Z_jX\alpha - Y\alpha\| \to 0$. Lemma 2.4 now gives

$$\begin{aligned} \alpha^*\Phi_\infty(Y^*, \mathbf{B}, Y)\,\alpha &= \Phi_\infty(\alpha^*Y^*, \mathbf{B}, Y\alpha) \\ &= \lim_j \Phi_\infty(\alpha^*X^*Z_j^*, \mathbf{B}, Z_jX\alpha) \\ &= \lim_j \alpha^*\Phi_\infty(X^*Z_j^*, \mathbf{B}, Z_jX)\,\alpha \\ &\leqslant \alpha^*\alpha \end{aligned}$$

because $\|Z_jX\| \leqslant 1$, $\|\mathbf{B}\| \leqslant 1$, and $\|\Phi\|_{\text{scb}} \leqslant 1$. Hence $0 \prec \theta_n(0)$.

Let $\gamma \in M_{n\times k}(\mathbb{C})$, $A \in M_n(\mathscr{A})_h$, and $(X, Y, \alpha; \mathbf{B}) \in \Gamma(A)$; then

$$\gamma^*(\alpha^*(X^*X - Y^*Y)\,\alpha)\,\gamma = \gamma^*A\gamma$$

so that $(X, Y, \alpha\gamma; \mathbf{B}) \in \Gamma(\gamma^*A\gamma)$. Further

$$\begin{aligned}&\gamma^*(\alpha^*\alpha - \alpha^*\Phi_\infty(Y^*, \mathbf{B}, Y)\,\alpha)\,\gamma\\ &\qquad = (\alpha\gamma)^*\,\alpha\gamma - (\alpha\gamma)^*\,\Phi_\infty(Y^*, \mathbf{B}, Y)(\alpha\gamma) \in \theta_k(\gamma^*A\gamma);\end{aligned}$$

hence (iv) follows.

When k is even the application of Wittstock's matricial Hahn–Banach Theorem provides domination above only. To dominate the expression below requires either a second completely positive linear operator, which again eventually leads to an annoying factor 2 in certain estimates, or a trick which reduces the even case to the odd one. The following lemma contains the details of this and will be required in the subsequent proof.

2.7. LEMMA. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. Let $k = 2m$ be even, let Φ be a symmetric completely bounded k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$, and let f be a state on $\mathscr{A}$. If $f\Phi\colon \mathscr{A}^{k+1} \to \mathscr{B}$ is defined by*

$$\begin{aligned}&f\Phi(a_1,..., a_m, a_{m+1}, a_{m+2},..., a_{k+1})\\ &\qquad = f(a_{m+1})\,\Phi(a_1,..., a_m, a_{m+2},..., a_{k+1}),\end{aligned}$$

then $f\Phi$ is a symmetric completely bounded $k+1$ linear operator with $\|f\Phi\|_{\mathrm{scb}} = \|\Phi\|_{\mathrm{scb}}$.

Proof. Let $A_1,..., A_{k+1}$ be in $M_n(\mathscr{A})$; then the (i, j)th component is

$$\begin{aligned}&(f\Phi)_n(A_1,..., A_{k+1})_{ij}\\ &\quad = \sum_{l,...,r,s,t,u,...,v} f(a_{m+1,st})\,\Phi(a_{1il},..., a_{mrs}, a_{m+2,tu},..., a_{k+1,vj})\\ &\quad = \sum_{l,...,r,t,u,...,v} \Phi\left(a_{1il},..., \sum_s a_{mrs} f(a_{m+1,st}),...\right)\\ &\quad = \Phi_n(A_1,..., A_m\gamma, A_{m+2},..., A_{k+1})_{ij},\end{aligned}$$

where $\gamma = (f(a_{m+1,ij})) \in M_n(\mathbb{C})$. Because f is a state on $\mathscr{A}$, f is completely bounded with $\|f\|_{\mathrm{scb}} = 1$ so that

$$\|A_m\gamma\| \leqslant \|A_m\|\;\|\gamma\| \leqslant \|A_m\|\;\|A_{m+1}\|.$$

From this and the relationship between $(f\Phi)_n$ and Φ_n it follows that

$\|f\Phi\|_{\mathrm{scb}} \leqslant \|\Phi\|_{\mathrm{scb}}$. The reverse inequality may be obtained by letting γ approximate the identity operator (or equal it if $\mathscr{A}$ has an identity) by a suitable choice of A_{m+1}.

2.8. THEOREM. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with $\mathscr{B}$ injective, and let k be a positive integer with $k \geqslant 2$. If Φ is a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$, then there is a completely positive linear operator ψ from $\mathscr{A}$ into $\mathscr{B}$ such that*

$$-\psi_n(X^*X) \leqslant \Phi_n(X^*, A_2, \ldots, A_{k-1}, X) \leqslant \psi_n(X^*X)$$

for all $X \in M_n(\mathscr{A})$ and $\mathbf{A} = \mathbf{A}^* = (A_2, \ldots, A_{k-1}) \in M_n(\mathscr{A})^{k-1}$ (not occuring when $k = 2$) with $\|\mathbf{A}\| \leqslant 1$ and for all n, and that

$$\|\psi\| = \|\psi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{scb}}.$$

This theorem is proved for k odd by an application of Lemma 2.6 and Wittstock's matricial Hahn–Banach Theorem. However, to prove the result for k even we shall require Lemma 2.7. The reason for this is that the technique initially provides a bound on one side only, the bound on the other side being obtained by reversing the sign of A_m, when k is odd and $m = [(k+1)/2]$; then A_m occurs only once as a linear variable.

Proof of Theorem 2.8. Normalise Φ so that $\|\Phi\|_{\mathrm{cb}} = 1$. Apply Wittstock's matricial Hahn–Banach Theorem [14, Kor. 2.2.4] to the matricial sublinear (θ_n) defined in Lemma 2.6; then there exists a symmetric linear operator $\varphi \colon \mathscr{A} \to \mathscr{B}$ such that

$$\{\varphi_n(A)\} \prec \theta_n(A)$$

for all $n \in \mathbb{N}$ and $A \in M_n(\mathscr{A})_h$. Now let $A \in M_n(\mathscr{A})_h$ with $\|A\| = 1$, let $A = A_+ - A_-$ be the decomposition of A into positive and negative parts, and let $X = A_+^{1/2}$ and $Y = A_-^{1/2}$. Then $(X, Y, I_n; \mathbf{B}) \in \Gamma(A)$ for $\mathbf{B} = \mathbf{B}^*$, $\|\mathbf{B}\| \leqslant 1$. Hence

$$\begin{aligned} \varphi_n(A) &\leqslant 1 - \Phi_n(Y^*, \mathbf{B}, Y), \\ \varphi_n(A) &\leqslant 1, \end{aligned}$$

provided $k \geqslant 3$ by choosing $\mathbf{B} = \mathbf{0}$. Replacing A by $-A$ and the symmetry of φ show that $\|\varphi\|_{\mathrm{scb}} \leqslant 1$ for $k \geqslant 3$. When $k = 2$ this calculation just yields $\|\varphi\|_{\mathrm{scb}} \leqslant 2$, because there is no $\mathbf{B}$ term present; however we shall circumvent this difficulty later in the proof.

Let $Y \in M_n(\mathscr{A})_h$ and let $A = -Y^*Y$. If $t > 0$, then

$$(0, t^{-1}Y, tI_n; \mathbf{B}) \in \Gamma(A)$$

with the same restrictions on $\mathbf{B}$ as before; thus

$$\varphi_n(-Y^*Y) \leqslant 2t^2 I_n - \Phi_\infty(Y^*, \mathbf{B}, Y)$$

so that

$$\Phi_\infty(Y^*, \mathbf{B}, Y) \leqslant \varphi_n(Y^*Y).$$

By Wittstock's decomposition of a symmetric completely bounded operator as a difference of completely positive operators for $\mathscr{B}$ injective [14, 10] there is a completely positive linear operator $\psi\colon \mathscr{A} \to \mathscr{B}$ such that $\psi - \varphi$ is completely positive and

$$\|\psi\| = \|\psi\|_{\text{cb}} \leqslant \|\varphi\|_{\text{cb}} \leqslant 1.$$

Thus $\Phi_\infty(Y^*, \mathbf{B}, Y) \leqslant \psi_n(Y^*Y)$ for all $Y \in M_n(\mathscr{A})$ and all $\mathbf{B} = \mathbf{B}^*$, $\|\mathbf{B}\| \leqslant 1$. If k is odd, then we obtain

$$-\psi_n(Y^*Y) \leqslant \Phi_\infty(Y^*, \mathbf{B}, Y)$$

just by replacing the single linear term B_m by $-B_m$.

If k is even apply Lemma 2.7 to obtain a completely bounded symmetric $(k+1)$-linear operator $f\Phi$ from $\mathscr{A}^{k+1}$ into $\mathscr{B}$ with $\|f\Phi\|_{\text{scb}} = \|\Phi\|_{\text{scb}}$. The above working is now applied to $f\Phi$, and the resulting ψ does for Φ as well, because the matrix $(f(b_{k/2+1,ij}))$ approximates I_n for suitable $B_{k/2+1}$. This completes the proof of the theorem.

3. A Technical Stinespring Construction

The main theorem of the paper is proved by induction on k, the number of variables on which the multilinear operator acts, reducing k by 2 at each stage of the proof. The mathematical argument is a combination of the following lemma and Theorem 2.8. Lemma 3.1 has hypotheses, which are the conclusion of Theorem 2.8, and has a conclusion that are the hypotheses of Theorem 2.8 with k reduced to $k-2$.

3.1. Lemma. *Let H be a Hilbert space, let $\mathscr{A}$ be a C^*-algebra, and let Φ be a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into* $\mathrm{BL}(H)$ *with $k \geqslant 2$. Let $\varphi\colon \mathscr{A} \to \mathrm{BL}(H)$ be a completely positive linear operator such that*

$$-\varphi_n(X^*X) \leqslant \Phi_n(X^*, A_2, ..., A_{k-1}, X) \leqslant \varphi_n(X^*X)$$

for all $X \in M_n(\mathscr{A})$ and all $\mathbf{A}^* = \mathbf{A} = (A_2, ..., A_{k-1}) \in M_n(\mathscr{A})^{k-2}$ with $\|\mathbf{A}\| \leqslant 1$. Then there is a Hilbert space K, a $*$-representation θ of $\mathscr{A}$ on K, a *continuous linear operator* $V \in \mathrm{BL}(H, K)$ *with* $\|V\| = \|\varphi\|^{1/2}$, *and a completely bounded symmetric* $(k-2)$*-linear operator* Ψ *from* $\mathscr{A}^{k-2}$ *into* $\mathrm{BL}(K)$ *with* $\|\Psi\|_{\mathrm{scb}} \leqslant 1$ (*when* $k=2$, Ψ *is just a fixed selfadjoint element of* $\mathrm{BL}(K)$) *such that*

$$\Phi(a_1,..., a_k) = V^*\theta(a_1)\, \Psi(a_2,..., a_{k-1})\, \theta(a_k)\, V$$

for all $a_1,..., a_k \in \mathscr{A}$.

In addition if Φ is completely positive, so is Ψ.

Proof. The proof falls naturally into two distinct parts. First, the Stinespring construction is given, which produces the Hilbert space K, the *-representation θ, and the operator V. Though the Stinespring construction is well known we sketch it for completeness and to establish notation. At the same time as the space K is constructed a sesquilinear form $F_{\mathbf{a}}$ is defined on K for $\mathbf{a} = (a_2,..., a_{k-1}) \in \mathscr{A}^{k-2}$ in terms of Φ. This form is shown to be continuous, and so to arise from a continuous linear operator on K denoted by $\Psi(\mathbf{a})$. In the second part of the proof this operator Ψ is shown to be a completely bounded symmetric $(k-2)$-linear operator from $\mathscr{A}^{k-2}$ into $\mathrm{BL}(K)$.

We begin as in Stinespring [12] (see also Arveson [1]) by defining a positive semidefinite inner product $\langle \cdot, \cdot \rangle$ on the algebraic tensor product $\mathscr{A} \otimes H$ of $\mathscr{A}$ and H. The sesquilinear forms $\langle \cdot, \cdot \rangle$ and $F_{\mathbf{a}}$ are defined on $\mathscr{A} \otimes H$ by

$$\langle x \otimes \xi, y \otimes \eta \rangle = (\varphi(y^*x)\, \xi, \eta),$$

and

$$F_{\mathbf{a}}(x \otimes \xi, y \otimes \eta) = (\Phi(y^*, a_2,..., a_{k-1}, x)\, \xi, \eta)$$

for all $x, y \in \mathscr{A}$, $\xi, \eta \in H$, and $\mathbf{a} \in \mathscr{A}^{k-1}$. Here $(\cdot, \cdot)$ denotes the inner product in the Hilbert space H.

If $u = \sum_1^n x_j \otimes \xi_j \in \mathscr{A} \otimes H$, let $\xi = (\xi_1,..., \xi_n)^{\mathrm{T}}$, let $X \in M_n(\mathscr{A})$ have first row $x_1,..., x_n$ and all other rows zeros, and let $A_j = a_j \otimes I$ be the $n \times n$ diagonal matrix in $M_n(\mathscr{A})$ with a_j down the diagonal. From the definitions of φ_n, Φ_n, $\langle \cdot, \cdot \rangle$, and $F_{\mathbf{a}}$ it follows that

$$\langle u, u \rangle = (\varphi_n(X^*X)\, \xi, \xi)_n$$

and (1)

$$F_{\mathbf{a}}(u, u) = (\Phi_n(X^*, A_2,..., A_{k-1}, X)\, \xi, \xi)_n,$$

where $(\cdot, \cdot)_n$ denotes the inner product in $H^n = H \oplus \cdots \oplus H$. One of the reasons for the particular choice of definitions was so that these equalities would hold. The complete positivity of φ ensures immediately that $\langle \cdot, \cdot \rangle$ is positive semidefinite, and the hypothesis on φ_n and Φ_n implies that

$$|F_{\mathbf{a}}(u, u)| \leqslant \langle u, u \rangle \, \|a_2\| \cdots \|a_{k-1}\| \tag{2}$$

provided $\mathbf{a} = (a_2, ..., a_{k-1}) = (a_{k-1}^*, ..., a_2^*) = \mathbf{a}^*$ because $\|A_j\| = \|a_j\|$ for $2 \leqslant j \leqslant k-1$. The remainder of this part of the proof is a routine Hilbert space argument going back to the Gelfand–Naimark–Segal construction, and to the standard representation of continuous sesquilinear forms on a Hilbert space.

Using the polarization identity for sesquilinear forms yields

$$\begin{aligned} |F_{\mathbf{a}}(u, v)| &\leqslant \tfrac{1}{4} \sum_{r=0}^{3} |F_{\mathbf{a}}(u + i^r v, u + i^r v)| \\ &\leqslant \tfrac{1}{4} \, \|a_2\| \cdots \|a_{k-1}\| \sum_{r=0}^{3} \langle u + i^r v, u + i^r v \rangle \\ &= \|a_2\| \cdots \|a_{k-1}\| (\langle u, u \rangle + \langle v, v \rangle); \end{aligned}$$

minimizing $F_{\mathbf{a}}(tu, t^{-1}v)$ over $t > 0$ implies that

$$|F_{\mathbf{a}}(u, v)| \leqslant 2 \, \|a_2\| \cdots \|a_{k-1}\| \langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2}$$

for all $u, v \in \mathscr{A} \otimes H$, and all $\mathbf{a} = (a_2, ..., a_{k-1}) = \mathbf{a}^*$. Holding the variables $(a_3, ..., a_{k-2}) = (a_{k-2}^*, ..., a_3^*)$, u, v fixed and applying the polarization identity to the variables a_2, a_{k-1} yields as above that

$$|F_{\mathbf{a}}(u, v)| \leqslant 4 \, \|a_2\| \cdots \|a_{k-1}\| \langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2}$$

for $(a_3, ..., a_{k-2}) = (a_{k-2}^*, ..., a_3^*)$. Continuing this process repeatedly we obtain

$$|F_{\mathbf{a}}(u, v)| \leqslant 2^m \, \|a_2\| \cdots \|a_{k-1}\| \langle u, u \rangle^{1/2} \langle v, v \rangle^{1/2} \tag{3}$$

for all $\mathbf{a} = (a_2, ..., a_{k-1}) \in \mathscr{A}^{k-2}$ and $u, v \in H$. Now let

$$\mathscr{N} = \{u \in \mathscr{A} \otimes H: \langle u, u \rangle = 0\},$$

and let K be the Hilbert space completion of $\mathscr{A} \otimes H/\mathscr{N}$. We shall denote the inner product in K by $\langle \cdot, \cdot \rangle$ and identify elements $x \otimes \xi$ of $\mathscr{A} \otimes H$ with their images $x \otimes \xi + \mathscr{N}$ in K, in an effort to simplify the notation. Let θ be the *-representation of $\mathscr{A}$ on K defined by $\theta(a)(x \otimes \xi) = ax \otimes \xi$. If $\mathscr{A}$ has an identity, the linear operator V: $H \to K$ is defined by $V\xi = 1 \otimes \xi$. If $\mathscr{A}$ has no identity, then the set $\mathscr{A}_1^+$ of positive elements in $\mathscr{A}$ of norm less than 1 forms a bounded approximate identity for $\mathscr{A}$. If $x \in \mathscr{A}$ and $\xi, \eta \in H$, then

$$\begin{aligned}\lim\{\langle a \otimes \xi, x \otimes \eta\rangle : a \in \mathscr{A}_1^+\} \\ &= \lim\{(\varphi(x^*a)\,\xi, \eta) : a \in \mathscr{A}_1^+\} \\ &= (\varphi(x^*)\,\xi, \eta).\end{aligned}$$

In particular it is possible to define V in $\mathrm{BL}(H, K)$ by

$$V\xi = w^* - \lim\{a \otimes \xi : a \in \mathscr{A}_1^+\}.$$

In both the unital and nonunital cases it follows that

$$\varphi(a) = V^*\theta(a)\,V \qquad \text{and} \qquad a \otimes \xi + \mathscr{N} = \theta(a)\,V\xi$$

for all $a \in \mathscr{A}$ and $\xi \in H$, and that $\|V\|^2 = \|\varphi\|$.

Except for the introduction and properties of $F_{\mathbf{a}}$ the above is the standard Stinespring construction. Now inequality (3) implies that $F_{\mathbf{a}}$ lifts to K, and that for each $\mathbf{a} \in \mathscr{A}^{k-2}$ there is a unique

$$\Psi(\mathbf{a}) = \Psi(a_2, \ldots, a_{k-1})$$

in $\mathrm{BL}(K)$ such that

$$F_{\mathbf{a}}(u, v) = \langle \Psi(\mathbf{a})\,u, v\rangle$$

for all $u, v \in K$ and

$$\|\Psi(\mathbf{a})\| \leqslant 2^m\,\|a_2\| \cdots \|a_{k-1}\|$$

for all $\mathbf{a} \in \mathscr{A}^{k-2}$. The multilinearity of the operator Φ ensures that the map $\mathbf{a} \to F_{\mathbf{a}}$ is multilinear, and the uniqueness of $\Psi(\mathbf{a})$ then implies that Ψ is $(k-2)$-linear. Further Ψ is bounded with $\|\Psi\| \leqslant 2^m$ by (3). Now

$$\begin{aligned}\langle \Psi(a_{k-1}^*, \ldots, a_2^*)\,x \otimes \xi, y \otimes \eta\rangle \\ &= F_{\mathbf{a}^*}(x \otimes \xi, y \otimes \eta) \\ &= (\Phi(y^*, a_{k-1}^*, \ldots, a_2^*, x)\,\xi, \eta) \\ &= (\xi, \Phi(x^*, a_2, \ldots, a_{k-1}, y)\,\eta) \\ &= \langle x \otimes \xi, \Psi(a_2, \ldots, a_{k-1})\,y \otimes \eta\rangle\end{aligned}$$

so that

$$\Psi(a_2, \ldots, a_{k-1})^* = \Psi(a_{k-1}^*, \ldots, a_2^*);$$

hence $\Psi = \Psi^*$ is symmetric.

To complete the proof we have to show that Ψ is completely bounded with $\|\Psi\|_{\text{scb}} \leqslant 1$. This requires the following elementary intuitively reasonable equality whose proof depends on the definitions of Φ_n, $F_{\mathbf{a}}$ and V, and Ψ, and the close link between the definition of Φ_n and matrix multiplication. For $A=(a_{ij}) \in M_n(\mathscr{A})$, let $\theta_n(A)=(\theta(a_{ij}))$ and let $V_n = V \otimes I_n$ be the n-fold amplification of V. Then the equality

$$\Phi_n(A_1,..., A_k) = V_n^* \theta_n(A_1)\, \Psi_n(A_2,..., A_{k-1})\, \theta_n(A_k)\, V_n \tag{4}$$

holds for all $A_1,..., A_k \in M_n(\mathscr{A})$.

Computing part of the expression on the right acting on $\xi = (\xi_1,..., \xi_n) \in H^n$ yields

$$\begin{aligned}
&\Psi_n(A_2,..., A_{k-1})\, \theta_n(A_k)\, V_n \xi \\
&\quad = \Psi_n(A_2,..., A_{k-1}) \left(\sum_j \theta(a_{k\ell j})\, V\xi_j \right) \\
&\quad = \left(\sum_{r,...,t,\ell,j} \Psi(a_{2ir},..., a_{(k-1)t\ell})\, \theta(a_{k\ell j})\, V\xi_j \right);
\end{aligned}$$

so that

$$\begin{aligned}
&\langle V_n^* \theta_n(A_1)\, \Psi_n(A_2,..., A_{k-1})\, \theta_n(A_k)\, V_n \xi, \eta \rangle \\
&\quad = \left\langle \left(\sum_{i,r,...,t,\ell,j} V^* \theta(a_{1si})\, \Psi(a_{2ir},..., a_{(k-1)t\ell})\, \theta(a_{k\ell j})\, V\xi_j \right), (\eta_s) \right\rangle \\
&\quad = \sum_{s,i,r,...,t,\ell,j} \langle V^* \theta(a_{1si})\, \Psi(a_{2ir},..., a_{(k-1)}t\ell)\, \theta(a_{k\ell j})\, V\xi_j, \eta_s \rangle \\
&\quad = \sum (\Phi(a_{1si}, a_{2ir},..., a_{(k-1)t\ell}, a_{k\ell j})\, \xi_j, \eta_s)
\end{aligned}$$

from the definition of Ψ in terms of $F_{\mathbf{a}}$ and $F_{\mathbf{a}}$ in terms of Φ and $\langle \cdot, \cdot \rangle$. Finally the definition of Φ_n ensures the last series is equal to

$$\langle \Phi_n(A_1,..., A_k)\, \xi, \eta \rangle$$

as required.

To show that $\|\Psi\|_{\text{scb}} \leqslant 1$ it is sufficient to show that

$$\|\Psi_n(A_2,..., A_{k-1})\| \leqslant 1$$

for $(A_2,..., A_{k-1}) = (A_{k-1}^*,..., A_2^*)$ in $M_n(\mathscr{A})^{k-2}$ with $\|A_j\| \leqslant 1$, when $2 \leqslant j \leqslant k-1$. The equality of the norm and the numerical radius for a selfadjoint element of $\mathrm{BL}(K)$ implies that it is sufficient to show that the numerical radius of $\Psi_n(A_2,..., A_{k-1})$ is less than or equal to 1. The set of elements in K^n of the form

$$\eta = \left(\sum_{j=1}^{m} x_{ij} \otimes \xi_j \right) \qquad (i = 1, \ldots, n) \quad \text{with } \|\eta\| \leqslant 1$$

with $m \in \mathbb{N}$, $x_{ij} \in \mathscr{A}$ and $\xi_j \in H$ is dense in the unit ball of K^n. Thus to show that the numerical radius of $\Psi_n(A_2, \ldots, A_k)$ is less than or equal to 1, we have to show that

$$|\langle \Psi_n(A_2, \ldots, A_k)\, \eta, \eta \rangle| \leqslant 1$$

for all η in K^n of the above form. Fix such an η; in general m will be much greater than n, and we may assume this by taking some of the ξ_j to be zero if necessary. Let $\xi = (\xi_1, \ldots, \xi_m)$ be in H^m, which has inner product $(\cdot, \cdot)_m$, and let X be in $M_m(A)$ with the first n ($\leqslant m$) rows equal to x_{ij} ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$) and the last $m - n$ rows zero. From the definitions (1) of $\langle \cdot, \cdot \rangle$ and φ_m it follows that

$$\begin{aligned}(\varphi_m(X^*X)\, \xi, \xi)_m &= \left\langle \sum_j x_{ij} \otimes \xi_j, \sum_j x_{ij} \otimes \xi_j \right\rangle_n \\ &= \langle \eta, \eta \rangle_n,\end{aligned}$$

where $\langle \cdot, \cdot \rangle_n$ is the inner product in K^n. From the Stinespring construction we have

$$\begin{aligned}(\theta_m(X)\, V_m \xi)_i &= ((\theta(x_{ij}))\, V_m \xi)_i \\ &= \sum_j \theta(x_{ij})\, V \xi_j \\ &= \sum_j x_{ij} \otimes \xi_j\end{aligned}$$

for $1 \leqslant i \leqslant n$, and 0 for $n + 1 \leqslant i \leqslant m$.

The hypotheses of the theorem that link Φ_n and φ_n, and equation (4) imply that for $(A_2, \ldots, A_{k-1}) = (A_{k-1}^*, \ldots, A_2^*)$ in $M_n(\mathscr{A})^{k-2}$ with $\|A_j\| \leqslant 1$, when $2 \leqslant j \leqslant k - 1$,

$$\begin{aligned}&|\langle \Psi_n(A_2, \ldots, A_{k-1})\, \eta, \eta \rangle_n| \\ &\quad = \left| \left\langle \Psi_n(A_2, \ldots, A_{k-1}) \left(\sum_j x_{ij} \otimes \xi_j \right), \left(\sum_j x_{ij} \otimes \xi_j \right) \right\rangle_n \right| \\ &\quad = |\langle \Psi_m(A_2, \ldots, A_{k-1})\, \theta_m(X)\, V_m \xi, \theta_m(X)\, V_m \xi \rangle_m| \\ &\quad = |(\Phi_m(X^*, A_2, \ldots, A_{k-1}, X)\, \xi, \xi)_m| \\ &\quad \leqslant (\varphi_m(X^*X)\, \xi, \xi)_m \\ &\quad = \langle \eta, \eta \rangle_n.\end{aligned}$$

Here the $n \times n$ matrices A_j are sometimes regarded as being in $M_m(\mathscr{A})$ by placing them in the top left hand corner and filling out the matrix with zeros; the standard embedding of $M_n(\mathscr{A})$ in $M_m(\mathscr{A})$. This shows $\|\Psi\|_{\text{scb}} \leqslant 1$.

When $k=2$, Ψ is just a fixed selfadjoint element of $\text{BL}(K)$.

If Φ is completely positive then the complete positivity of Ψ follows directly from the definition of completely positive and equality (4). This completes the proof.

3.2. *Remark.* Let $\mathscr{C}$ be the commutant of $\Phi(\mathscr{A}, \mathscr{A},..., \mathscr{A})$ in $\text{BL}(H)$ in the situation of Lemma 3.1. The symmetry of Φ ensures that $\mathscr{C}$ is a von Neumann subalgebra of $\text{BL}(H)$. Then in addition to the other conclusions of Lemma 3.1 there is a faithful normal representation ρ of $\mathscr{C}$ on K such that $\rho(x) V = Vx$ for all $x \in \mathscr{C}$ and $\rho(\mathscr{C}) \subseteq \Psi(\mathscr{A},..., \mathscr{A})'$. Further the representation θ, the map V, the representation ρ of $\mathscr{C}$, and the completely bounded $(k-2)$-linear operator Ψ are unique upto unitary equivalence in a similar way to that which holds in Stinespring's Theorem (see [12, Theorem 3.6]). The techniques are standard.

4. Representation Theorems

The detailed representation of a completely bounded symmetric k-linear operator Φ from $\mathscr{A}^k$ into $\text{BL}(H)$ turns out to be slightly different in the k odd and k even cases. This is because of the special linear role the central variable has in the k-odd case, as compared with the quadratic type role of the other variables.

4.1. Theorem. *Let $\mathscr{A}$ be a C^*-algebra, let H be a Hilbert space, and let Φ be a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into* $\text{BL}(H)$. *Let $m = [(k+1)/2]$.*

(i) k odd. *There are $*$-representations $\theta_1,..., \theta_{m-1}$, ψ_1, ψ_2 on Hilbert spaces $H_1,..., H_{m-1}$, K_1, K_2, linear operators $V_j \in \text{BL}(H_j, H_{j+1})$ for $0 \leqslant j \leqslant m-2$ where $H_0 = H$ with*

$$\|V_0\| \cdot \|V_1\| \cdot \cdots \cdot \|V_{m-2}\| = \|\Phi\|_{\text{scb}}^{1/2}$$

and $W_1 \in \text{BL}(H_{m-1}, K_1)$ and $W_2 \in \text{BL}(H_{m-1}, K_2)$ with $\|W_1^* W_1 + W_2^* W_2\| = 1$ such that

$$\begin{aligned}\Phi(a_1,..., a_k) = {} & V_0^* \theta_1(a_1) V_1^* \theta_2(a_2) \cdots \theta_{m-1}(a_{m-1}) \\ & \times \{W_1^* \psi_1(a_m) W_1 - W_2^* \psi_2(a_m) W_2\} \\ & \times \theta_{m-1}(a_{m+1}) \cdots V_1 \theta_1(a_k) V_0\end{aligned}$$

for all $a_1,\dots,a_k \in \mathscr{A}$. *If in addition* Φ *is completely positive, then the* W_2 *expression is zero.*

(ii) k even. *There are *-representations* $\theta_1,\dots,\theta_m$ *of* $\mathscr{A}$ *on Hilbert spaces* $H_1,\dots,H_m$, *continuous linear operators* $V_j \in \mathrm{BL}(H_j, H_{j+1})$ *for* $0 \leqslant j \leqslant m-2$, *where* $H_0 = H$, *and* $W = W^* \in \mathrm{BL}(H_{m-1})$ *with* $\|W\| = 1$ *and* $\|V_0\| \cdot \|V_1\| \times \cdots \times \|V_{m-2}\| = \|\Phi\|_{\mathrm{scb}}^{1/2}$ *such that*

$$\Phi(a_1,\dots,a_k) = V_0^*\theta_1(a_1)\, V_1^* \cdots V_{m-2}^*\theta_{m-1}(a_{m-1})\, W\theta_m(a_m)\, V_{m-2} \times \cdots \times V_1\theta_1(a_k)\, V_0$$

for all $a_1,\dots,a_k \in \mathscr{A}$.

If in addition Φ is completely positive, then $W \geqslant 0$.

Proof. The proof is by induction on k with step length 2.

(i) k *odd.* The case $k=1$ is just the Wittstock decomposition theorem for a completely bounded symmetric linear operator from a C^*-algebra into an injective C^*-algebra (see [14, Satz 4.5; 10, Corollary 2.6; 7, Theorem 2.1]) followed by an application of Stinespring's Theorem [12, 1] to represent each of the two completely positive operators involved in the decomposition of Φ. Note that though Wittstock's results are stated for completely bounded symmetric operators φ with the completely bounded norm $\|\varphi\|_{\mathrm{cb}}$, the results hold with the symmetric completely bounded norm $\|\varphi\|_{\mathrm{scb}}$ because all calculations are done with elements of $M_n(\mathscr{A})_h$.

Let Φ be a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$ $(k \geqslant 3)$. By Theorem 2.8 there is a completely positive linear operator $\psi\colon \mathscr{A} \to \mathrm{BL}(H)$ such that

$$\begin{aligned} -\psi_n(X^*X)\,\|A_2\| \cdots \|A_{k-1}\| &\leqslant \Phi_n(X^*, A_2,\dots, A_{k-1}, X) \\ &\leqslant \psi_n(X^*X)\,\|A_2\| \cdots \|A_{k-1}\| \end{aligned}$$

for all $X, A_2,\dots, A_{k-1} \in M_n(\mathscr{A})$ with $(A_2,\dots,A_{k-1}) = (A_{k-1}^*,\dots,A_2^*)$, and that $\|\psi\| \leqslant \|\Phi\|_{\mathrm{scb}}$. Lemma 3.1 now implies that there is a Hilbert space H_1, a *-representation θ_1 of $\mathscr{A}$ on H_1, a continuous linear operator $V_0 \in \mathrm{BL}(H, H_1)$, and a completely bounded symmetric $(k-2)$-linear operator Ψ from $\mathscr{A}^{k-2}$ into $\mathrm{BL}(H_1)$ such that

$$\Phi(a_1,\dots,a_k) = V_0\,\theta_1(a_1)\,\Psi(a_2,\dots,a_{k-1})\,\theta_1(a_k)\,V_0$$

for all $a_1,\dots,a_k \in \mathscr{A}$, $\|V_0\|^2 \leqslant \|\Phi\|_{\mathrm{scb}}$, and $\|\Psi\|_{\mathrm{scb}} \leqslant 1$. Equality (4) of Lemma 3.1, which is just the same matrix calculation as Lemma 5.1, shows that $\|\Phi\|_{\mathrm{scb}} \leqslant \|\Psi\|_{\mathrm{scb}} \cdot \|V_0\|^2$ so that $\|V_0\|^2 = \|\Phi\|_{\mathrm{scb}}$ and $\|\Psi\|_{\mathrm{scb}} = 1$. The inductive assumption completes the proof of (i).

(ii) *k even.* If Φ is a completely bounded symmetric 2-linear operator from $\mathscr{A}^2$ into $\mathrm{BL}(H)$, then Theorem 2.8 provides a completely positive linear operator from $\mathscr{A}$ into $\mathrm{BL}(H)$ dominating Φ and from this domination Lemma 3.1 produces a suitable representation of Φ. Thus $\Phi(a, b) = V^*\theta(a)\, W\theta(b)\, V$, where $W = W^* \in \mathrm{BL}(K)$ and $V \in \mathrm{BL}(H, K)$ with $\|W\| \leqslant 1$ and $\|V\|^2 \leqslant \|\Phi\|_{\mathrm{scb}}$. The equalities of the norms and the induction proceed as in (i).

The following result shows that the completely bounded and symmetric completely bounded norms coincide for completely bounded symmetric k-linear operators; this corresponds to the result that the norm of a hermitian operator on a Hilbert space is its numerical radius, which was used in the proof of Lemma 3.1.

4.2. Corollary. *Let $\mathscr{A}$ be a C^*-algebra, and let H be a Hilbert space. If Φ is a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into* $\mathrm{BL}(H)$, *then Φ is symmetrically representable and*

$$\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{scb}} = \|\Phi\|_{\mathrm{rep}} = \|\Phi\|_{\mathrm{srep}};$$

further $\|\Phi\|_{\mathrm{srep}}$ is attained.

Proof. The inequalities

$$\|\Phi\|_{\mathrm{scb}} \leqslant \|\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|_{\mathrm{rep}} \leqslant \|\Phi\|_{\mathrm{srep}}$$

are all elementary and have been discussed in the Introduction. Theorem 4.1 shows that the symmetric representation norm is attained and that $\|\Phi\|_{\mathrm{srep}} \leqslant \|\Phi\|_{\mathrm{scb}}$.

The following result is motivated by Haagerup's discussion [7] of Wittstock's decomposition theorem [14, Satz 4.5].

4.3. Corollary. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras with $\mathscr{B}$ injective, and let Φ be a completely bounded symmetric k-linear operator from $\mathscr{A}^k$ into $\mathscr{B}$. Then there are completely bounded, completely positive k-linear operators Φ_+ and Φ_- from $\mathscr{A}^k$ into $\mathscr{B}$ such that*

$$\Phi = \Phi_+ - \Phi_- \qquad \textit{and} \qquad \|\Phi\|_{\mathrm{cb}} = \|\Phi_+ + \Phi_-\|_{\mathrm{cb}}.$$

Proof. Represent $\mathscr{B}$ on a Hilbert space H so that there is a completely positive projection E from $\mathrm{BL}(H)$ onto $\mathscr{B}$. Regard Φ as being a completely bounded k-linear operator from $\mathscr{A}^k$ into $\mathrm{BL}(H)$, and represent Φ as in Theorem 4.1. When k is odd take Φ_+ to be E acting on that part of the representation of Φ given by placing the W_2 term to zero; let $\Phi_- = \Phi_+ - \Phi$ be E acting on the negative of that part of the representation obtained by placing W_1 to be zero. Then a straightforward calculation yields the result $\|\Phi\|_{\text{cb}} \geqslant \|\Phi_+ + \Phi_-\|_{\text{cb}}$ using $\|\Phi\|_{\text{scb}} = \|\Phi\|_{\text{srep}}$ and the observation on $W_1^* \psi_1(a) W_1 + W_2^* \psi_2(a) W_2$ that the norm and completely bounded norm coincide for a completely positive linear operator. When k is even the operator $W = W^*$ in the representation of Theorem 4.1 is split into its positive W_+ and negative W_- parts, $W = W_+ - W_-$. Using W_+ and W_- in place of W in the representation of Φ leads via E to the representations of Φ_+ and Φ_- with $\Phi = \Phi_+ - \Phi_-$. Further $\|\Phi\|_{\text{cb}} \geqslant \|\Phi_+ + \Phi_-\|_{\text{cb}}$ follows as before. On the other hand $-(\Phi_+ + \Phi_-) \leqslant_{\text{cp}} \Phi \leqslant_{\text{cp}} (\Phi_+ + \Phi_-)$ so $\|\Phi\|_{\text{cb}} = \|\Phi\|_{\text{scb}} \leqslant \|\Phi_+ + \Phi_-\|_{\text{scb}} \leqslant \|\Phi\|_{\text{cb}}$.

5. Representability of Completely Bounded Operators

This section begins with the elementary lemma showing that if $\Phi: \mathscr{A}^k \to \mathrm{BL}(H)$ is representable, then so is $\Phi_n: M_n(\mathscr{A})^k \to M_n(\mathrm{BL}(H))$. This lemma leads to the complete boundedness of representable operators.

5.1. Lemma. *Let $\mathscr{A}$ be a C^*-algebra, let $H, H_1, ..., H_k$ be Hilbert spaces, let $\theta_1, ..., \theta_k$ be $*$-representations of $\mathscr{A}$ on $H_1, ..., H_k$, and let $V_j \in \mathrm{BL}(H_{j+1}, H_j)$ for $j = 0, ..., k$, where $H_0 = H_{k+1} = H$. If*

$$\Phi(a_1, ..., a_k) = V_0 \theta_1(a_1) V_1 \theta_2(a_2) \cdots V_{k-1} \theta_k(a_k) V_k$$

for all $a_1, ..., a_k \in \mathscr{A}$, then

$$\Phi_n(A_1, ..., A_k) = V_{0,n} \theta_{1,n}(A_1) V_{1,n} \cdots V_{k-1,n} \theta_{k,n}(A_k) V_{k,n}$$

for all $A_1, ..., A_k \in M_n(\mathscr{A})$ and all $n \in \mathbb{N}$, where $V_{\ell,n} = V_\ell \otimes I_n$ is the diagonal operator with V_ℓ down the diagonal, and

$$\theta_{\ell,n}(a_{ij}) = (\theta_\ell(a_{ij})).$$

Proof. The matrix multiplication nature of the definition of Φ_n implies that the (i, j)th component is

$$\begin{aligned} &\Phi_n(A_1, ..., A_k)_{ij} \\ &\quad = \sum_{r, ..., t} V_0 \theta_1(a_{1ir}) V_1 \cdots V_{k-1} \theta_k(a_{ktj}) V_k \\ &\quad = (V_{0,n} \theta_{1,n}(A_1) V_{1,n} \cdots \theta_{k,n}(A_k) V_{k,n})_{ij} \end{aligned}$$

as required.

The main result is

5.2. THEOREM. *Let $\mathcal{A}$ be a C^*-algebra, and let H be a Hilbert space. A k-linear operator Φ from $\mathcal{A}^k$ into* $\mathrm{BL}(H)$ *is representable if and only if it is completely bounded; when Φ is completely bounded, $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{rep}}$, and the representable norm is attained.*

A fairly standard 2×2 matrix symmetrization trick combined with the results of Section 4 provides the proof of Theorem 5.2 and Corollary 5.4. In the proof of Theorem 5.2 we could use the decomposition $\Phi = \Phi_1 + i\Phi_2$ with Φ_1 and Φ_2 symmetric except then equality of the norms would not follow.

5.3. *Proof of Theorem* 5.2. If Φ is a representable k-linear operator from $\mathcal{A}^k$ into $\mathrm{BL}(H)$, then so is Φ_n, by Lemma 5.1, and $\|\Phi_n\| \leqslant \|\Phi_n\|_{\mathrm{rep}} = \|\Phi\|_{\mathrm{rep}}$ because $\|V_{j,n}\| = \|V_j\|$ for all j, n. Hence Φ is completely bounded with $\|\Phi\|_{\mathrm{cb}} \leqslant \|\Phi\|_{\mathrm{rep}}$.

If Φ is completely bounded, let $S\Phi$ be the k-linear operator from $\mathcal{A}^k$ into $M_2(\mathrm{BL}(H)) \approx \mathrm{BL}(H \oplus H)$ defined by

$$S\Phi = \begin{pmatrix} 0 & \Phi^* \\ \Phi & 0 \end{pmatrix}.$$

Note that $S\Phi$ is k-linear symmetric and that $(S\Phi)_n = S(\Phi_n)$ for all $n \in \mathbb{N}$. Because the norm of the symmetrization operator S from the space of continuous k-linear maps from $\mathcal{A}^k$ into $\mathrm{BL}(H)$ into the space of continuous symmetric k-linear maps from $\mathcal{A}^k$ into $\mathrm{BL}(H \oplus H)$ is 1, $S\Phi$ is completely bounded and $\|S\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}}$. By Corollary 4.2 the completely bounded symmetric k-linear operator $S\Phi$ is symmetrically representable attaining $\|S\Phi\|_{\mathrm{cb}} = \|S\Phi\|_{\mathrm{rep}} = \|S\Phi\|_{\mathrm{srep}}$. Restricting attention to the lower left corner of the 2×2 matrix defining $S\Phi$ gives the result (see the proof of Corollary 5.4 for further details).

The following result is a corollary of representability and a 2×2 matrix trick as above. However, it seems desirable to deduce it directly from Theorem 2.8 and Lemma 3.1. The result is intuitively reasonable as regards symmetry once the representability and the 2×2 matrix trick are known. It brings out yet again the vital importance of the outer two variables in a completely bounded k-linear operator.

5.4. COROLLARY. *Let $\mathcal{A}$ be a C^*-algebra, let H be a Hilbert space, and let $k \geqslant 3$ be a positive integer. If Φ is a completely bounded k-linear operator from $\mathcal{A}^k$ into* $\mathrm{BL}(H)$*, then there is a *-representation θ of $\mathcal{A}$ on a Hilbert space K, a completely bounded symmetric $(k-2)$-linear operator Ψ from*

$\mathscr{A}^{k-2}$ into BL(K), *and continuous linear operators* $V: K \to H$ *and* $W: H \to K$ *such that*

$$\Phi(a_1,..., a_k) = V\theta(a_1)\, \Psi(a_2,..., a_{k-1})\, \theta(a_k)\, W$$

for all $a_1,..., a_k \in \mathscr{A}$ *and that*

$$\|\Phi\|_{\text{cb}} = \|V\| \cdot \|W\| \cdot \|\Psi\|_{\text{cb}}.$$

Proof. Using the symmetrization

$$S\Phi = \begin{pmatrix} 0 & \Phi^* \\ \Phi & 0 \end{pmatrix}$$

of the operator Φ gives a completely bounded symmetric k-linear operator $S\Phi$ from $\mathscr{A}^k$ into BL$(H \oplus H)$ with $\|S\Phi\|_{\text{scb}} = \|\Phi\|_{\text{cb}}$; we apply Theorem 2.8 and Lemma 3.1 as in the proof of Theorem 4.1. By Theorem 2.8 there is a completely positive linear operator φ from $\mathscr{A}$ into BL$(H \oplus H)$ dominating $S\Phi$ as in 2.8. Then Lemma 3.1 implies that there is a *-representation θ of $\mathscr{A}$ on a Hilbert space K, a completely bounded symmetric $(k-2)$-linear operator Ψ from $\mathscr{A}^{k-2}$ into BL(K), and a continuous linear operator $V: H \to K$ such that

$$S\Phi(a_1,..., a_k) = V^*\theta(a_1)\, \Psi(a_2,..., a_{k-1})\, \theta(a_k)\, V$$

for all $a_1,..., a_k \in \mathscr{A}$, and that $\|S\Phi\|_{\text{scb}} = \|V\|^2\, \|\Psi\|_{\text{scb}}$. Let P be the orthogonal projection from $H \oplus H$ onto $H \oplus 0$. Letting $V = (I-P)\, U^*$ and $W = U \,|\, H \oplus 0$ proves the result.

The following corollary of Theorem 5.2 is a consequence of being able to lift *-representations to the enveloping von Neumann algebra of a C^*-algebra, so ensuring that representable k-linear operators lift without change of norm.

5.5. Corollary. *Let $\mathscr{A}$ be a C^*-algebra with enveloping von Neumann algebra $\mathscr{A}^{**}$. If Φ is a completely bounded k-linear operator from $\mathscr{A}^k$ into* BL(H), *then there is a completely bounded k-linear operator Ψ from $(\mathscr{A}^{**})^k$ into* BL(H) *such that* $\Psi|_{\mathscr{A}} = \Phi$ *and* $\|\Psi\|_{\text{cb}} = \|\Phi\|_{\text{cb}}$.

The study in this paper was partly motivated by Grothendieck's inequality and its relationship to the representation of continuous bilinear forms on a C^*-algebra [9, Theorem 5.2] and on a commutative C^*-algebra [2, 6, 9], and an attempt to obtain corresponding results for suitable bilinear operators. It seems worth noting yet one more equivalent version of Grothendieck's inequality.

5.6. COROLLARY. *Let $\mathscr{A}$ be a commutative C^*-algebra. Each continuous bilinear form Φ on $\mathscr{A}$ is completely bounded as a bilinear operator from $\mathscr{A}^2$ into $\mathbb{C}$ and*

$$\|\Phi\| \leqslant \|\Phi\|_{\mathrm{cb}} \leqslant K_{\mathrm{G}}\, \|\Phi\|,$$

where K_{G} is Grothendieck's constant; further K_{G} is the least such constant.

Proof. The "folklore" representation theorem for continuous bilinear forms on commutative C^*-algebras, gives for Φ a $*$-representation θ of $\mathscr{A}$ on a Hilbert space H, vectors $\xi, \eta \in H$, and a continuous linear operator $T \in \mathrm{BL}(H)$ such that

$$\Phi(x, y) = \langle \theta(x)\, T\theta(y)\, \xi, \eta \rangle$$

for all $x, y \in \mathscr{A}$ and $\|T\| \cdot \|\xi\| \cdot \|\eta\| \leqslant K_{\mathrm{G}} \|\Phi\|$, see [9, Remark 4.5(c)]. Further K_{G} is the least such constant. Regarding $\mathbb{C}$ as $\mathrm{BL}(\mathbb{C})$, this shows that Φ is representable with

$$\|\Phi\| \leqslant \|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{rep}} \leqslant K_G\, \|\Phi\|.$$

The condition that a multilinear form is completely bounded from $\mathscr{A}^k$ into $\mathbb{C}$ leads to a representation (Theorem 5.2) that is related to Grothendieck's inequality. There is a vague relationship between this result and the multilinear versions of Grothendieck-like inequalities (see [2] and [5] for these results).

If Φ is a continuous bilinear form on a C^*-algebra $\mathscr{A}$, then Φ may be written as a sum of four continuous bilinear forms,

$$\Phi = \Phi_{\mathrm{LL}} + \Phi_{\mathrm{LR}} + \Phi_{\mathrm{RL}} + \Phi_{\mathrm{RR}}$$

where the four forms Φ_{LL}, Φ_{LR}, Φ_{RL}, Φ_{RR} are of restricted type by [9, Remark 5.3(b)]. These forms may each be characterized by being completely bounded for the algebra $\mathscr{A}$ or the algebra $\mathscr{A}^0$ with reversed product; for example, the class RL corresponds to the completely bounded bilinear forms on $\mathscr{A}$. Note that a bilinear form Φ on a C^*-algebra $\mathscr{A}$ is completely bounded if and only if it satisfies one of the equivalent conditions of [4, Theorem 2.1], and that then $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{h}}$ in the notation of [4].

A continuous k-linear operator Φ from a von Neumann algebra $\mathscr{M}$ into a von Neumann algebra $\mathscr{N}$ is called *normal* if Φ is separately σ-weakly continuous in each variable (see [8, Proposition 2.3; 13, Definition 2.15]). If $\mathscr{M}$ is a von Neumann algebra on a Hilbert space K, let K^∞ be the countable infinite direct sum of K and let π be the amplification of the natural representation of $\mathscr{M}$ from K to K^∞. The following result corresponds to [8, Proposition 2.3; 4, Corollary 2.3].

5.7. COROLLARY. *Let $\mathscr{M}$ be a von Neumann algebra and let Φ be a normal completely bounded k-linear operator from $\mathscr{M}$ into* BL(H). *Then there are continuous linear operators $W_k: H \to K^\infty$, $W_j: K^\infty \to K^\infty$ $(1 \leqslant j \leqslant k-1)$, and $W_0: K^\infty \to H$ such that*

$$\Phi(a_1,..., a_k) = W_0 \pi(a_1) W_1 \cdots W_{k-1} \pi(a_k) W_k$$

for all $a_1,..., a_k \in \mathscr{M}$, and that

$$\|W_0\| \cdots \|W_k\| = \|\Phi\|_{\text{cb}}.$$

Proof. Let $\theta_1,..., \theta_k$ be *-representations of $\mathscr{M}$ on $H_1,..., H_k$, and let V_j $(0 \leqslant j \leqslant k)$ be continuous linear operators representing Φ with

$$\|V_0\| \cdots \|V_k\| = \|\Phi\|_{\text{cb}}.$$

By standard theory (see [13, p. 127]) each representation θ_j may be decomposed into a normal part π_j and a singular part ψ_j with $\theta_j = \pi_j \oplus \psi_j$. Let $K_j = \pi_j(1) H_j$, and let $U_j = P_j V_j | K_{j+1}$, where P_j is the orthogonal projection from H_j onto K_j. The proof proceeds by induction on j replacing θ_j by π_j one by one eventually obtaining the representation

$$\Phi(a_1,..., a_k) = U_0 \pi_1(a_1) U_1 \cdots U_{k-1} \pi_k(a_k) U_k$$

for all $a_1,..., a_k \in \mathscr{M}$. The first replacement of θ_1 by π_1 will be done here; the rest are similar. Let f be a σ-weakly continuous linear functional on BL(H). The linear functional g on $\mathscr{M}$ defined by $g(x) = f\Phi(x, a_2,..., a_k)$ is σ-weakly continuous. The uniqueness of the decomposition of a continuous linear functional into its normal and singular parts [13, Proposition III.2.14, p. 127, Eq. (10)] ensures that the singular part

$$f(V_0 \psi_1(x) V_1 \theta_2(a_2) \cdots \theta_k(a_k) V_k)$$

of g is zero. Thus

$$\Phi(a_1,..., a_k) = U_0 \pi_1(a_1) U_1 \theta_2(a_2) \cdots \theta_k(a_k) V_k$$

for all $a_1,..., a_k$. Replacing θ_j by π_j one by one leads to

$$\Phi(a_1,..., a_k) = U_0 \pi_1(a_1) U_1 \pi_2(a_2) \cdots \pi_k(a_k) U_k,$$

where a slight change of domain and range of the U_j has been ignored. Each of the representations π_j is a subrepresentation of π. Now W_{j-1} is taken as the projection from K^∞ onto the domain space of π_j followed by U_j. This completes the proof.

Corollary 5.7 may be used to give a multilinear generalization of an unpublished result of Uffe Haagerup [Stinespring type decompositions of completely bounded maps]; for a discussion of this result of Haagerup's and related results see [4]. Haagerup proves that if R is a von Neumann algebra on a Hilbert space H and if Φ: $\mathrm{BL}(H) \to \mathrm{BL}(H)$ is a normal completely bounded R' module mapping, then there are $x_k, y_k \in R$ such that

$$\Phi(b) = \sum x_k b y_k \qquad \text{for all } b \in \mathrm{BL}(H).$$

5.8. Definition. A k-linear mapping Φ from $\mathrm{BL}(H)$ into $\mathrm{BL}(H)$ is called an $\mathcal{N}$ module mapping, where $\mathcal{N} \subseteq \mathrm{BL}(H)$, if

$$a\Phi(b_1, b_2,..., b_k) = \Phi(ab_1, b_2,..., b_k),$$

$$\Phi(b_1,..., b_j a, b_{j+1},..., b_k) = \Phi(b_1,..., b_j, ab_{j+1},..., b_k),$$

and

$$\Phi(b_1,..., b_{k-1}, b_k)\, a = \Phi(b_1,..., b_{k-1}, b_k a)$$

for all $b_1,..., b_k \in \mathrm{BL}(H)$, all $a \in \mathcal{N}$, and $j = 1, 2,..., k-1$.

5.9. Corollary. *Let R be a von Neumann algebra and let Φ be a completely bounded R-module normal k-linear operator on* $\mathrm{BL}(K)$. *Let π be the amplification of the natural representation of* $\mathrm{BL}(K)$ *to the representation of* $\mathrm{BL}(K)$ *on $K^\infty = K \otimes \ell^2$. Then there are continuous linear operators $V_0, V_1,..., V_k$ with V_0: $K^\infty \to K$, V_j: $K^\infty \to K^\infty$ $(1 \leqslant j \leqslant k-1)$, and V_k: $K \to K^\infty$ such that*

$$\Phi(a_1,..., a_k) = V_0 \pi(a_1)\, V_1 \cdots V_{k-1} \pi(a_k)\, V_k$$

for all $a_1,..., a_k \in \mathrm{BL}(K)$,

$$\|\Phi\|_{\mathrm{cb}} = \|V_0\| \cdot \cdots \cdot \|V_k\|,$$

and all the entries in V_j are in R', where V_j is regarded as an operator matrix with respect to the natural direct sum decomposition of K^∞.

Proof. By Corollary 5.7 there exist continuous linear operators W_0: $K^\infty \to K$,

$$W_j\colon \quad K^\infty \to K^\infty \qquad (1 \leqslant j \leqslant k-1),$$

and W_k: $K \to K^\infty$ such that

$$\Phi(a_1,..., a_k) = W_0 \pi(a_1)\, W_1 \cdots W_{k-1} \pi(a_k)\, W_k$$

for all $a_1,..., a_k \in \mathrm{BL}(K)$ and that

$$\|\Phi\|_{\mathrm{cb}} = \|W_0\| \cdot \cdots \cdot \|W_k\|.$$

We shall show that each W_i can be replaced by a V_i in $\pi(R)'$ starting with V_0 and V_k. For $a_1,..., a_k$ in $\mathrm{BL}(K)$, let $e(a_1,..., a_k)$ denote the range projection of $\pi(a_1)\, W_1 \cdots W_{k-1} \pi(a_k)\, W_k$. Because Φ is an R-module operator, we have

$$rW_0 e(a_1,..., a_k) = W_0 \pi(r)\, e(a_1,..., a_k)$$

for all $r \in R$. If e is the least projection in $\mathrm{BL}(K^{\infty})$ majorizing all the $e(a_1,..., a_k)$ for $a_1,..., a_k$ in $\mathrm{BL}(K)$, then this equation shows that $rW_0e = W_0\pi(r)\,e$ for all $r \in R$. By construction eK^{∞} is a $\pi(\mathrm{BL}(K))$ invariant closed linear subspace of K^{∞} and so e is in $\pi(R)'$. Hence $rW_0e = W_0e\pi(r)$ for all $r \in R$, and

$$\Phi(a_1,..., a_k) = W_0e\pi(a_1)\, W_1 \cdots \pi(a_k)\, W_k$$

for all $a_1,..., a_k$ in $\mathrm{BL}(K)$. Defining $V_0 = W_0e$ carries out the replacement for the index $j = 0$.

A similar argument applies to W_k. If f is the least projection majorizing the support projections for all operators of the form $V_0\pi(a_1)\, W_1 \cdots W_{k-1}\pi(a_k)$, then $I - f$ is the projection onto the intersection of $\mathrm{Ker}\, V_0\pi(a_1)\, W_1 \cdots W_{k-1}\pi(a_k)$ for all $a_1,..., a_k \in \mathrm{BL}(K)$, f is in $\pi(R)'$, and $\pi(r)\, fW_k = f\pi(r)\, W_k = fW_k r$ because Φ is an R-module map. Now defining $V_k = fW_k$ replaces the last operator W_k by an operator V_k intertwining with R and satisfying

$$\Phi(a_1,..., a_k) = V_0\pi(a_1)\, W_1 \cdots W_{k-1}\pi(a_k)\, V_k.$$

Now suppose $1 \leqslant j \leqslant k-1$ and that $V_0,..., V_{j-1}$ have been found in $\pi(R)'$ such that

$$\Phi(a_1,..., a_k) = V_0\pi(a_1) \cdots V_{j-1}\pi(a_j)\, W_j \cdots W_{k-1}\pi(a_k)\, V_k$$

for all $a_1,..., a_k$ in $\mathrm{BL}(K)$. Then the projection e is defined to be the least projection majorizing the range projections of all operators

$$\pi(a_{j+1})\, W_{j+1} \cdots W_{k-1}\pi(a_k)\, V_k$$

and the projection f is the least projection majorizing the support projections of all the operators $V_0\pi(a_1) \cdots V_{j-1}\pi(a_j)$. As above e and f belong to $\pi(R)'$, and

$$\pi(r)\, eW_jf = e\pi(r)\, W_jf = eW_j\pi(r)\, f = eW_jf\pi(r)$$

for all $r \in R$, where the middle inequality holds because

$$V_0\pi(a_1)\, V_1 \cdots V_{j-1}\pi(a_{j-1})\{\pi(r)\, W_j - W_j\pi(r)\}$$
$$\times \pi(a_{j+1})\, W_{j+1} \cdots W_{k-1}\pi(a_k)\, V_k = 0$$

for $a_1,..., a_k \in \mathrm{BL}(K)$.

Letting $V_j = eW_jf$, we obtain $V_j \in \pi(R)'$ and

$$\Phi(a_1,..., a_k) = V_0\pi(a_1)\, V_1 \cdots V_j\pi(a_{j+1})\, W_{j+1} \cdots \pi(a_k)\, V_k,$$

so the required inequality follows by induction. Finally note that $\|V_j\| \leqslant \|W_j\|$ for $0 \leqslant j \leqslant k$ so that $\|V_0\| \cdots \|V_k\| \leqslant \|\Phi\|_{\mathrm{cb}}$, and the reverse inequality is the easy part of 5.2 following from Lemma 5.1.

5.10. *Remark.* If the von Neumann algebra R contains an infinite dimensional hyperfinite factor say N, then Φ is automatically completely bounded. Let n be in $\mathbb{N}$ and let F be a copy of the $n \times n$ matrices in N which contains I, then $\mathrm{BL}(H) \simeq F^C \otimes F \simeq \mathrm{BL}(H) \otimes M_n$. The N-module property of Φ shows that Φ is isomorphic to $\Phi \otimes id$ on $(\mathrm{BL}(H) \otimes M_n)^k$ and we see that $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|$.

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REFERENCES

1. W. B. ARVESON, Subalgebras of C^*-algebras, *Acta Math.* **123** (1969), 142–224.
2. T. K. CARNE, Banach lattices and extensions of Grothendieck's inequality, *J. London Math. Soc. (2)* **21** (1980), 496–516.
3. E. G. EFFROS, Amenability and virtual diagonals for von Neumann algebras, *J. Funct. Anal.*, in press.
4. E. G. EFFROS AND A. KISHIMOTO, Module maps and Hochschild–Johnson cohomology, preprint.
5. J. J. F. FOURNIER, Multilinear Grothendieck inequalities via the Schur algorithm, *Canad. Math. Soc. Conf. Proc.* **1** (1981), 189–211.
6. C. C. GRAHAM AND B. M. SCHREIBER, Bimeasure algebras on *LCA* groups, *Pacific J. Math.* **115** (1984), 91–127.
7. U. HAAGERUP, Injectivity and decomposition of completely bounded maps.
8. U. HAAGERUP, The Grothendieck inequality for bilinear forms on C^*-algebras, *Advan. in Math.* **56** (1985), 93–116.
9. S. KAIJSER AND A. M. SINCLAIR, Projective tensor products of C^*-algebras, *Math. Scand.* **55** (1984), 161–187.
10. V. I. PAULSEN, Completely bounded maps on C^*-algebras and invariant operator ranges, *Proc. Amer. Math. Soc.* **86** (1982), 91–96.
11. G. K. PEDERSEN, "C^*-Algebras and Their Automorphism Groups," Academic Press, New York, 1979.
12. W. F. STINESPRING, Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.* **6** (1955), 211–216.
13. M. TAKESAKI, "Theory of Operator Algebras I," Springer-Verlag, Berlin, 1979.
14. G. WITTSTOCK, Ein operatorwertiger Hahn–Banach Satz, *J. Funct. Anal.* **40** (1981), 127–150.
15. G. WITTSTOCK, Extensions of completely bounded C^*-module homomorphisms, *in* "Proc. Conference on Operator Algebras and Group Representations," Neptun 1980, Pitman, New York, 1983.