On Adaptive Information with Varying Cardinality for Linear Problems with Elliptically Contoured Measures

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Adaptive information is not more powerful than nonadaptive information for solving linear problems with elliptically contoured measures provided that the cardinality of information is fixed (see G. W. Wasilkowski and H. Woźniakowski, 1984, Numer. Math. 44, 169-190). Can adaptive information be essentially more powerful than nonadaptive information when cardinality is allowed to vary? The answer is negative if a Gaussian measure is considered (see G. W. Wasilkowski, 1986, J. Complexity 2, 204-228). This work generalizes the result to a class of elliptically contoured measures for which the answer is still negative. © 1989 Academic Press, Inc.

1. Basic Definitions

For given $F$ a separable Banach space and $G$ a normed linear space, let

$$S: F \to G$$

be a linear bounded operator. For every element $f \in F$ we wish to construct an approximation $x(f)$ to $S(f)$. The approximation $x(f)$ is constructed based on computed information $N(f)$, i.e.,

$$x(f) = \phi(N(f)),$$

where $\phi: N(F) \to G$ is an arbitrary mapping.

To define information we begin with a simple case of nonadaptive information. Namely, we say that $N$ is nonadaptive information iff $N: F \to \mathbb{R}^n$ is a linear bounded operator. Equivalently, there exist linear bounded

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functionals $L_i$ such that

$$N(f) = [L_1(f), \ldots, L_n(f)] \quad \forall f \in F.$$ 

Hence nonadaptive information consists of simultaneous evaluations of linear functionals, whose total number $n$ is called cardinality.

We now turn to adaptive information, which is more general than nonadaptive information. The essence of adaptation is that neither the cardinality $n$ nor functionals have to be fixed a priori. Instead, they are determined dynamically during the process of computing $N(f)$. More precisely, suppose we have already computed $y_1(f), \ldots, y_i(f)$. Based on these values, we decide whether another evaluation is to be performed. If no, $N(f) = [y_1(f), \ldots, y_i(f)]$ constitutes the final information about $f$. Otherwise, we select $(i + 1)$st functional $L_{i+1}(\cdot) = L_{i+1}(\cdot; y_1(f), \ldots, y_i(f))$, compute $y_{i+1}(f) := L_{i+1}(f)$, decide whether to terminate, and so on. Hence, formally, adaptive information is provided by an operator $N$ such that

$$N(f) = [L_1(f), L_2(f; y_1(f)), \ldots, L_n(f; y_1(f), \ldots, y_{n-1}(f))],$$

where $y_i(f) = L_i(f; y_1(f), \ldots, y_{i-1}(f))$, and for every fixed $z_1, \ldots, z_{i-1} \in \mathbb{R}$, $L_i(\cdot; z_1, \ldots, z_{i-1})$ is a linear bounded functional. The cardinality number $n(f)$ at $f$ equals

$$n(f) = \min\{i : \text{ter}_i(y_1(f), \ldots, y_{i-1}(f)) = 1\},$$

where $\text{ter}_i$, called termination functions, are arbitrary Boolean functions. If $n(f) = \text{const}$, $N$ is said to have a fixed cardinality. Otherwise, it has a varying cardinality.

Adaptive information need not be more powerful than nonadaptive information, even though adaptive information is much more general than nonadaptive information. This is the case for linear problems in the worst case setting (see Gal and Micchelli, 1980; Traub and Woźniakowski, 1980), as well as for linear problems in the average case setting with Gaussian measures (see Wasilkowski, 1986). It is also known that adaptation with fixed cardinality is no more powerful than nonadaptation on the average with elliptically contoured measures (see Wasikowski and Woźniakowski, 1984). In this note we continue the study of the power of adaptive information with varying cardinality over nonadaptive information for linear problems with elliptically contoured measures.

To study the power of one information over another, we compare the intrinsic errors caused by them and their cardinalities, respectively.
More specifically, assume that the space $F$ is equipped with a probability measure $\mu$ defined on the Borel $\sigma$-field of $F$. We assume that $N$ is measurable. Then the intrinsic error caused by $N$, called the *average radius* of $N$, is defined by

$$r(N; \mu) = \inf_{\phi} \int_F \|S(f) - \phi(N(f))\|^2 \mu(df)$$

with the infimum taken over all measurable mappings $\phi: N(F) \to G$. The *average cardinality* of $N$ is defined by

$$\text{card}(N; \mu) = \int_F n(f) \mu(df).$$

### 2. MAIN RESULT

From now on we assume that the probability measure is *elliptically contoured* (see Crawford, 1977); i.e., for every Borel set $A \subseteq F$

$$\mu(A) = \int_0^{+\infty} \gamma(t^{-1/2}A) \alpha(dt),$$

where $\gamma$ is a zero-mean Gaussian measure on $F$ and $\alpha$ is a probability measure on the Borel $\sigma$-field of subsets of $(0, +\infty)$ such that

$$\int_0^{+\infty} t \alpha(dt) = 1. \quad (2.1)$$

Given adaptive information with varying cardinality,

$$N(f) = [L_1(f), L_2(f; y_1(f)), \ldots, L_{n(f)}(f; y_1(f), \ldots, y_{n(f)-1}(f))],$$

by fixing $n(f) = k$ and $y_i(f) = z_i \in \Re$ we get nonadaptive information $N_{k,z}$,

$$N_{k,z}(f) = [L_1(f), L_2(f; z_1), \ldots, L_k(f; z_1, \ldots, z_{k-1}(f))].$$

Obviously, $N_{k,z}$ uses some of the functionals that are used by $N$ and has the cardinality equal to $k$. Let

$$r(k; \mu) = \inf_{z \in \Re^{k-1}\cap N(F)} r(N_{k,z}; \mu)$$
be the minimal average radius among all such nonadaptive information of cardinality $k$. We stress that $r(k; \mu)$ does not depend on the measure $\alpha$, i.e., $r(k; \mu) = r(k; \gamma)$ (see Wasilkowski and Woźniakowski, 1984), and the minimal average radii are known for a number of problems with Gaussian measures $\gamma$.

**Theorem 1.** Let $c = \text{card}(N; \mu)$ be finite. Then

$$r(N; \mu) \geq \sup_{x > 1, \alpha > 0} r(|cx|; \mu) a(\alpha([a, +\infty)) - x^{-1}).$$

**Proof.** For $x > 1$, let

$$B_x = \{ f \in F : \nu(f) \leq cx \}.$$

Since $c = \text{card}(N; \mu) = \int_F \nu(f) \mu(df) \geq \int_{F \setminus B_x} \nu(f) \mu(df) > cx(1 - \mu(B_x))$, we get

$$\mu(B_x) > 1 - 1/x.$$  \hfill (2.2)

Furthermore,

$$r(N; \mu) \geq \inf_{\phi} \int_{B_x} \|S(f) - \phi(N(f))\| \nu(df)$$

$$= \inf_{\phi} \int_0^{1/x} \left[ \int_{x^{-1/2}B_x} \|S(\sqrt{t}f) - \phi(N(\sqrt{t}f))\| \gamma(df) \right] \alpha(dt).$$

Since $\gamma$ is Gaussian and $N$ restricted to $B_x$ has the cardinality number $n(f) \leq cx$, the form of the conditional measure $\gamma(\cdot | y = N(f))$ (see Lee and Wasilkowski, 1986; Wasilkowski, 1986) implies that

$$\int_{x^{-1/2}B_x} \|S(\sqrt{t}f) - \phi(N(\sqrt{t}f))\| \gamma(df) \geq r(|cx|, \gamma) t \gamma(t^{-1/2}B_x).$$

Hence

$$r(N; \mu) \geq r(|cx|; \gamma)a_x$$

with

$$a_x = \int_0^{1/x} t \gamma(t^{-1/2}B_x) \alpha(dt).$$
Furthermore, as mentioned before, \( r(k; \gamma) = r(k; \mu) \). Hence,

\[
r(N; \mu) \geq r(\|cx\|; \mu)a_x. \tag{2.3}
\]

To complete the proof we need to show that

\[
a_x \geq a(\alpha([a, +\infty)) - x^{-1}) \quad \forall a \geq 0. \tag{2.4}
\]

Note that

\[
a_x \geq a \int_{a}^{+\infty} \gamma(t^{-1/2}B_x) \alpha(dt)
\]

and

\[
\int_{a}^{+\infty} \gamma(t^{-1/2}B_x) \alpha(dt) = \int_{0}^{a} \gamma(t^{-1/2}B_x) \alpha(dt) - \int_{0}^{a} \gamma(t^{-1/2}B_x) \alpha(dt)
= \mu(B_x) - \int_{0}^{a} \gamma(t^{-1/2}B_x) \alpha(dt) \geq \mu(B_x) - \int_{0}^{a} \alpha(dt)
= \mu(B_x) - \alpha([0, a)) = \mu(B_x) - 1 + \alpha([a, \infty)).
\]

This and (2.2) prove (2.4) and together with (2.3) complete the proof of the theorem.

We end this note by the following remarks.

**Remark 1.** Note that for an elliptically contoured measure \( \mu \)

\[
\lim_{a \to 0^+} \alpha([a, +\infty)) = 1.
\]

Thus, for a fixed number \( x > 1 \), \( a_x \) is positive. From Theorem 1 we get \( r(N; \mu) \geq a_x r(\|cx\|; \mu) \). Obviously, there exists nonadaptive information of cardinality \( \|cx\| \) with the average radius equal to \( r(\|cx\|; \mu) \). Hence nonadaptive information that uses roughly only \( x \) more evaluations than \( N \) has the average radius not greater than \( a_x^{-1}r(N; \mu) \).

For some operators \( S \) and measures \( \mu \) (or \( \gamma \), equivalently), \( r(\|cx\|; \mu) = \Theta(r(\|c\|; \mu)) \) for any fixed \( x \). This is the case, for instance, for function approximation and integration problems with \( F = C^{m}([0, 1]) \) and with \( \gamma \) an \( m \)-fold Wiener measure (see e.g., Sachs and Ylvisaker, 1970, or Lee and Wasilkowski, 1986), or more generally, if \( G \) is a Hilbert space and the covariance operator of the induced measure \( \gamma(S^{-1}(\cdot)) \) has the eigenvalues \( \lambda_k \) that converge to zero no faster than \( k^{-s} \) for some positive real \( s \). Then,
as \( c = \text{card}(N; \mu) \) approaches infinity, \( r(N; \mu) = \Omega(r([c]; \mu)) \). Thus, in this case, adaption is essentially no more powerful than nonadaption.

**Remark 2.** Theorem 1 can be easily generalized for different definitions of the average radius. For instance, if

\[
r(N; \mu) = \inf_{\phi} \int_F \|S(f) - \phi(N(f))\|^p \mu(df)
\]

for some \( p > 0 \), then the following holds:

\[
r(N; \mu) \geq \sup_{x > 1, \alpha > 0} r([cx]; \mu) a^{p/2} \frac{\alpha([a, +\infty)) - x^{-1}}{\int_F t^{p/2} \alpha(dt)}.
\]

Here we assume that \( \int_F t^{p/2} \alpha(dt) < \infty \) since otherwise any information with finite \( \text{card}(N; \mu) \) has infinite average radius.

**References**


