Normal covers of infinite products and normality of \( \Sigma \)-products

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Abstract

We give some characterizations for normal covers of infinite products of generalized metric spaces such as \( M \)-spaces, \( \Sigma \)-spaces and \( \beta \)-spaces. We prove them simultaneously in terms of \( \beta \)-spaces and perfect maps. Next, we give affirmative answers to two questions concerning the normality of \( \Sigma \)-products, which were raised by the author and Yamazaki, respectively. These results are stated in terms of \( \Sigma \)-products of \( \beta \)-spaces.

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1. Introduction

Throughout this paper, all spaces are assumed to be Hausdorff. For a set \( \Lambda \), \( |\Lambda| \) denotes the cardinality of \( \Lambda \) and \( [\Lambda]^{\leq \omega} \) (respectively, \( [\Lambda]^{< \omega} \)) denotes the family of all finite (respectively, countable) subsets of \( \Lambda \).

Let \( X \) be a space. Recall that \( U \) is a cozero-set in \( X \) if there is a continuous function \( f : X \to [0, 1] \) such that \( U = \{ x \in X : f(x) > 0 \} \). The complement of a cozero-set is called a zero-set. A cover \( \mathcal{G} \) of \( X \) is called a cozero (respectively, zero-set) cover if each member of \( \mathcal{G} \) is a cozero-set (respectively, zero-set) in \( X \).

Recall that an open cover \( \mathcal{O} \) of \( X \) is normal if there is a sequence \( \{U_n\} \) of open covers of \( X \) such that \( U_{n+1} \) is a star-refinement of \( U_n \) for each \( n \in \omega \), where \( U_0 = \mathcal{O} \). For an open cover \( \mathcal{O} = \{O_\alpha : \alpha \in \Omega\} \) of \( X \), a cover \( \{S_\alpha : \alpha \in \Omega\} \) of \( X \) with the same index set is called a shrinking of \( \mathcal{O} \) if \( \overline{S_\alpha} \subset O_\alpha \) for each \( \alpha \in \Omega \).

Stone, Michael and Morita have given various equivalent conditions for normal covers of topological spaces as follows (ex., see [9, Theorem 1.2]).

Theorem 1.1 (Stone–Michael–Morita). Let \( X \) be a space and \( \mathcal{O} \) an open cover of \( X \). Then the following are equivalent:

(a) \( \mathcal{O} \) is normal.
(b) \( \mathcal{O} \) has a \( \sigma \)-locally finite cozero refinement.
(c) \( \mathcal{O} \) has a \( \sigma \)-discrete cozero refinement.
(d) \( \mathcal{O} \) has a locally finite cozero refinement.
(e) \( \mathcal{O} \) has a locally finite, \( \sigma \)-discrete, cozero refinement which has a zero-set shrinking.

Let \( X \times Y \) be a product space. A subset of the form \( U \times V \) in \( X \times Y \) is called a rectangle. A cover \( \mathcal{G} \) of \( X \times Y \) is rectangular if each member of \( \mathcal{G} \) is a rectangle in \( X \times Y \). The product \( X \times Y \) is said to be rectangular [12] if every finite cozero cover (or equivalently, every binary cozero cover) of \( X \times Y \) has a \( \sigma \)-locally finite rectangular cozero refinement.

In the previous paper [19], we have given an analogous characterization for normal covers of rectangular products as follows:

**Theorem 1.2.** [19] Let \( X \times Y \) be a rectangular product with a paracompact \( \sigma \)-space factor \( X \). Let \( \mathcal{O} \) be an open cover of \( X \times Y \). Then the following are equivalent:

(a) \( \mathcal{O} \) is normal.
(b) \( \mathcal{O} \) has a \( \sigma \)-locally finite rectangular cozero refinement.
(c) \( \mathcal{O} \) has a \( \sigma \)-discrete rectangular cozero refinement.
(d) \( \mathcal{O} \) has a locally finite rectangular cozero refinement.
(e) \( \mathcal{O} \) has a locally finite, \( \sigma \)-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.

The purpose of this paper to give the same characterizations as Theorem 1.2 for normal covers of infinite products (where the exact definitions of rectangularity for infinite products are stated in the next section). The most useful idea for our purpose was given by Filippov [2] to prove:

**Theorem 1.3.** [2] Let \( X = \prod_{\lambda \in \Lambda} X_{\lambda} \) be an infinite product of paracompact \( M \)-spaces (= paracompact \( p \)-spaces). Then every normal cover of \( X \) has a \( \sigma \)-locally finite rectangular cozero refinement.

The original form of Theorem 1.3 was given by Klebanov [6] for an infinite product of metric spaces. A similar result to Theorem 1.3 was pointed out in [16] for an infinite product of paracompact \( \Sigma \)-spaces with countable tight condition. Subsequently, Odinokov [11] has generalized these results for infinite products.

In the next section, we give some characterizations of normal covers of infinite products, which strengthen not only Theorem 1.3 but also all other results stated above. The class of \( \beta \)-spaces is fairly broad as a class of generalized metric spaces, because it contains many classes such as \( \Sigma \)-spaces and semi-stratifiable spaces. These characterizations are actually proved simultaneously in terms of \( \beta \)-spaces and perfect maps.

In the third section, we recall a question for the normality of \( \Sigma \)-products of \( \beta \)-spaces, which was raised in [18, 4]. Here we give an affirmative answer to this question. This result is an extension of almost all results, which have been known, concerning the normality of \( \Sigma \)-products with countable tight condition.

On the other hand, Yamazaki [21] introduced the concept of base-normality which is stronger than normality, and proved that a \( \Sigma \)-product of metric spaces is base-normal. Moreover, she [22] has asked whether a \( \Sigma \)-product of paracompact \( M \)-spaces is base-normal if it is normal. As an application of our results obtained here, we also give an affirmative answer to this question.

2. A main theorem and corollaries

A space \( S \) is called a \( \beta \)-space [5] (respectively, strong \( \beta \)-space [20]) if there is a function \( g : S \times \omega \to \text{Top}(S) \), where \( \text{Top}(S) \) denotes the topology of \( S \), satisfying

(i) \( x \in \bigcap_{n \in \omega} g(x, n) \) for each \( x \in S \),
(ii) if \( \bigcap_{n \in \omega} g(x_n, n) \neq \emptyset \), then \( \bigcap_{k \in \omega} \{x_n : n \geq k\} \) is non-empty (respectively, non-empty and compact).

For convenience, we call the function \( g \) a \( \beta \)-function (respectively, strong \( \beta \)-function) for \( S \).
Remark. Note that “$\bigcap_{n \in \omega} g(x_n, n) \neq \emptyset$” can be replaced by “$\bigcap_{n \geq m} g(x_n, n) \neq \emptyset$” for some $m \in \omega$” in the above (ii).

We can use the property of strong $\beta$-spaces instead of that of $\beta$-spaces in the class of paracompact spaces, which is assured by the following.

**Lemma 2.1.** [20] Every paracompact $\beta$-space is a strong $\beta$-space.

A continuous map $f$ from $S$ onto $T$ is called perfect (respectively, quasi-perfect) if $f$ is a closed map such that $f^{-1}(t)$ is compact (respectively, countably compact) for each $t \in T$. For a perfect (respectively, quasi-perfect) map $f : S \to T$, $S$ is a strong $\beta$-space (respectively, $\beta$-space) iff so is $T$ (see [20]).

A space $S$ has countable tightness if for each $A \subset S$ and each $x \in A$, there is $B \subset A$ such that $|B| \leq \omega$ and $x \in B$.

Let $X = \prod_{x \in A} X_\lambda$ be an infinite product. For each $\theta \in [A]^{<\omega}$, $X_\theta = \prod_{\lambda \in \theta} X_\lambda$ is called a finite subproduct of $X$, and let $\pi_\theta$ denote the projection of $X$ onto $X_\theta$. A subset of the form $\pi_\theta^{-1}(T)$ in $X$ is called a cylinder, where $\theta \in [A]^{<\omega}$ and $T \subset X_\theta$. The cylinder $\pi_\theta^{-1}(T)$ is also called $\theta$-distinguished in $X$. Note that a cylinder $\pi_\theta^{-1}(T)$ is open (respectively, closed) in $X$ iff $T$ is open (respectively, closed) in $X_\theta$. A cover $G$ of the product $X$ is said to be cylindrical if each member of $G$ is a cylinder in $X$.

Let $E$ be a set of finite sequences and $\emptyset$. We introduce the partial order $< \in E$ defined as $\mu = (\beta_0, \ldots, \beta_m) < \xi = (\alpha_0, \ldots, \alpha_n)$ means that $m < n$ and $\beta_0 = \alpha_0, \ldots, \beta_m = \alpha_m$. that is, $\mu = \xi \upharpoonright (m + 1)$. In particular, for each $\xi = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in E$, let $\xi_\mu = (\alpha_0, \ldots, \alpha_{n-1})$ and $\xi \upharpoonright (\alpha_n) = (\alpha_0, \ldots, \alpha_{n-1}, \alpha_n, \alpha)$.

In the proof of the next theorem, we will partially use an idea from [2, Theorem 4.2]. However, Filippos’ paper may not be easily available. So, for convenience of the readers, we will describe the proof rather in full detail.

**Theorem 2.2.** For each $\lambda \in A$, let $f_\lambda : X_\lambda \to Y_\lambda$ be a perfect map. If each finite subproduct of $Y = \prod_{\lambda \in A} Y_\lambda$ is a paracompact $\beta$-space having countable tightness, then every binary cozero cover of $X = \prod_{\lambda \in A} X_\lambda$ has a locally finite, $\sigma$-discrete, cylindrical open refinement $U$ which has a cylindrical shrinking $E$.

**Proof.** Let $A$ and $B$ be any disjoint zero-sets in $X$. There is a continuous function $h : X \to [0, 1]$ such that $h \upharpoonright A \equiv 0$ and $h \upharpoonright B \equiv 1$. Take an $\varepsilon$ with $0 < \varepsilon < 1$ and fix it. Let $f = \prod_{\lambda \in A} f_\lambda$, that is, $f(x) = (f_\lambda(x(\lambda)))$ for each $x = (x(\lambda)) \in X$. Then $f$ is a perfect map from $X$ onto $Y$. It should be noted that, for each $y \in Y$, there is some $\varphi(y) \in [A]^{<\omega}$ such that $x, x' \in X$ with $x(\lambda) = x'(\lambda) \in f_\lambda^{-1}(y(\lambda))$ for each $\lambda \in \varphi(y)$ implies $|h(x) - h(x')| < \varepsilon/3$.

For each $\theta \in [A]^{<\omega}$, let $X_\theta = \prod_{\lambda \in \theta} X_\lambda$ and $Y_\theta = \prod_{\lambda \in \theta} Y_\lambda$, and let $\pi_\theta$ and $\varphi_\theta$ denote the projections of $X$ onto $X_\theta$ and $Y$ onto $Y_\theta$, respectively. For each $\theta \in [A]^{<\omega}$, $X_\theta$, $Y_\theta$, $\pi_\theta$, and $\varphi_\theta$ are abbreviated by $X_\xi$, $Y_\xi$, $\pi_\xi$, and $\varphi_\xi$, respectively. By Lemma 2.1, note that each $Y_\xi$ is a strong $\beta$-space.

Now, for each $n \in \omega$, we will construct an index sets $\Delta_n$ of $n$-tuple sequences and a subset $E_n$ of $\Delta_n$ such that each $\xi \in \Delta_n$ assigns $F(\xi)$, $W(\xi)$, $V(\xi)$ to $X$ and each $\xi \in E_n$ assigns $\theta_\xi \in [A]^{<\omega}$, $y_\xi \in Y_\xi$, $\{a_{\xi, k}, b_{\xi, k}\} \subset X$ and a strong $\beta$-function $\varphi_\xi$ for $Y_\xi$, satisfying the following conditions:

(a) $\{V(\xi) : \xi \in \Delta_n\}$ is locally finite and $\sigma$-discrete in $Y$.
(b) For each $\xi \in \Delta_n$,
   (1) $\xi_- \in E_n$.
   (2) $V(\xi)$ and $W(\xi)$ are $\theta_\xi$-distinguished open cylinders in $X$ and $F(\xi)$ is a $\theta_\xi$-distinguished closed cylinder in $Y$ such that $F(\xi) \subset W(\xi) \subset \overline{W(\xi)} \subset V(\xi)$.
(c) $E_n = \{\xi : \xi \in \Delta_n\}$: There are two points $a, b \in f^{-1}(V(\xi))$ such that $\pi_\xi(a) = \pi_\xi(b)$ and $|h(a) - h(b)| \geq \varepsilon$.
(d) For each $\xi \in E_n$,
   (3) $\bigcup\{F(\eta) : \eta \in \Delta_{n+1} \text{ with } \eta_- = \xi\} \subset V(\xi)$ and $\bigcup\{F(\eta) : \eta \in \Delta_{n+1} \text{ with } \eta_- = \xi\} = F(\xi)$, where $F(\emptyset) = Y$.
   (4) $V(\xi)$ meets at most finitely many members of $\{V(\mu) : \mu \in \bigcup_{\xi \in \Delta_n} \Delta_\mu\}$.
   (5) $p_{\xi_-}(g_{\xi}(z, k)) \subset g_{\xi}(p_{\xi_-}(z), k)$ and $g_{\xi}(z, k + 1) \subset g_{\xi}(z, k)$ for each $z \in Y_\xi$ and each $k \in \omega$.
   (6) $p_{\xi_-}(V(\xi)) \subset g_{\xi_-}(y_\xi, n)$.
(7) $\forall_{\xi} \left( \{ p_{\xi}(f(a_{\xi}, k)) : k \in \omega \} \cap p_{\xi}(F(\xi)) \right)$.
(8) $\pi_{\xi}(a_{\xi}, k) = \pi_{\xi}(b_{\xi}, k)$ and $| h(a_{\xi}, k) - h(b_{\xi}, k) | \geq \varepsilon$ for each $k \in \omega$.
(9) $\theta_{\xi} \subset \theta_{\xi}$ and $\varphi(f(a_{j}, j)) \subset \varphi(f(a_{\mu}, j)) \subset \varphi$ for each $\mu \leq \xi$ and $j \leq n$, where $\varphi(y) \in [\Lambda]^{<\omega}$ has been described above.
(10) $\mu \in \bigcup_{i \leq n} (\Delta_i \setminus \mathcal{E}_i)$ implies $W(\mu) \cap V(\xi) = \emptyset$.
(11) $\mu \in \bigcup_{i \leq n} (\Delta_i \setminus \mathcal{E}_i)$ implies $W(\mu) \cap V(\xi) = \emptyset$.

Let $\Delta_0 = \mathcal{E}_0 = \{ \emptyset \}$ and let $F(\emptyset) = W(\emptyset) = V(\emptyset) = Y$. Take any $a \in A$ and $b \in B$. Let $a_{\theta_{\xi}, k} = a_{\theta_{\xi}}$ and $b_{\theta_{\xi}, k} = b_{\theta_{\xi}}$ for each $k \in \omega$. Let $\theta_{\emptyset} = \varphi(f(a_{\emptyset})) \cup \varphi(f(b_{\emptyset}))$. Take any $y_{\emptyset} \in Y_{\emptyset}$ and a strong $\beta$-function $g_{\emptyset}$ for $Y_{\emptyset}$. Assume that the construction above has already been performed for no greater than $n$.

Take any $\xi \in \mathcal{E}_n$ and fix it.

**Claim 1.** $\mathcal{V}_{\xi} = \{ p_{\xi}(V(\mu)) : \mu \in \bigcup_{i \leq n} (\Delta_i \setminus \mathcal{E}_i) \}$ is locally finite at each point of $p_{\xi}(V(\xi))$.

**Proof.** Let $\mathcal{V}_{\xi}^* = \{ p_{\xi}(V(\mu)) : \mu \in \bigcup_{i \leq n} (\Delta_i \setminus \mathcal{E}_i) \}$ with $p_{\xi}(V(\mu)) \cap p_{\xi}(V(\xi)) = \emptyset$. Take any $p_{\xi}(V(\mu)) \in \mathcal{V}_{\xi}^*$. By (2), $V(\mu)$ and $V(\xi)$ are $\theta_{\mu}$-distinguished and $\theta_{\xi}$-distinguished, respectively, in $\xi$. Since $V(\xi) \in \theta_{\xi}$-distinguished in $\xi$, we have $V(\mu) \cap V(\xi) = \emptyset$. It follows from (10) that $\theta_{\mu} \subset \theta_{\xi}$. Hence $V(\mu)$ is $\theta_{\xi}$-distinguished in $\xi$. It follows from (a) that $\mathcal{V}_{\xi}^*$ is locally finite in $\mathcal{V}_{\xi}$. This shows Claim 1 is true. $\square$

Let $\Phi = \{ p_{\xi}(y) \in \mathcal{V}_{\xi} : \text{ There are two points } a, b \in X \text{ such that } y = f(a), \pi_{\xi}(a) = \pi_{\xi}(b) \text{ and } | h(a) - h(b) | \geq \varepsilon \}$. Let $\mathcal{W}_{\xi} = \{ p_{\xi}(W(\mu)) : \mu \in \bigcup_{i \leq n} (\Delta_i \setminus \mathcal{E}_i) \}$.

**Lemma 2.A.** $(p_{\xi}(F(\xi)) \cap \Phi) \cap \bigcup \mathcal{W}_{\xi} = \emptyset$.

**Proof.** Assume that there is $z_0 \in (p_{\xi}(F(\xi)) \cap \Phi) \cap \bigcup \mathcal{W}_{\xi}$. By (2) and Claim 1, $W_{\xi}$ is locally finite at $z_0$. So $z_0 \in p_{\xi}(W(\nu))$ for some $\nu \in \Delta_j \setminus \mathcal{E}_j$, where $j \leq n$. Since $p_{\xi}(V(\xi))$ is an open neighborhood of $z_0$, we have $p_{\xi}(V(\xi)) \cap p_{\xi}(W(\nu)) = \emptyset$. Since $V(\xi)$ is $\theta_{\xi}$-distinguished in $Y$, we also have $V(\xi) \cap W(\nu) = \emptyset$. By (10), $\theta_{\nu} \subset \theta_{\xi}$ is true. Hence $W(\nu)$ is also $\theta_{\xi}$-distinguished in $\xi$. So it follows that $z_0 \in p_{\xi}(W(\nu)) = p_{\xi}(W(\nu)) \subset p_{\xi}(V(\nu))$.

By $z_0 \in \Phi$, $p_{\xi}(W(\nu))$ meets $\Phi$. By the choice of $\Phi$, there are two points $a, b \in X$ such that $\pi_{\xi}(a) = \pi_{\xi}(b)$, $| h(a) - h(b) | \geq \varepsilon$ and $p_{\xi}(f(a)) \subset p_{\xi}(f(b)) \subset p_{\xi}(V(\nu)) \cap \Phi$. Since $V(\nu)$ is $\theta_{\xi}$-distinguished in $\mathcal{V}_{\xi}$, we have $a \in f^{-1}(V(\nu))$. Since $f^{-1}(V(\nu))$ is also $\theta_{\xi}$-distinguished in $X$, we have $b \in p_{\xi}^{-1}(\pi_{\xi}(b)) \subset p_{\xi}^{-1}(\pi_{\xi}(f^{-1}(V(\nu)))) = f^{-1}(V(\nu))$. By $\theta_{\nu} \subset \theta_{\xi}$, we have $p_{\nu, \nu}(a) = p_{\nu, \nu}(b)$. By (c), we conclude that $\nu \in \mathcal{E}_j$. This contradicts the choice of $\nu$. $\square$

Since $\mathcal{V}_{\xi}$ is a strong $\beta$-space, we can take a strong $\beta$-function $g_{\xi}$ for $\mathcal{V}_{\xi}$ satisfying (5). Moreover, by Claim 1 and Lemma 2.A, we can assume that $g_{\xi}$ satisfies the following conditions; for each $z \in p_{\xi}(F(\xi))$ and each $k \in \omega$,

(i) $g_{\xi}(z, k) \subset p_{\xi}(V(\xi))$,
(ii) $g_{\xi}(z, k)$ meets at most finitely many members of $\mathcal{V}_{\xi}$,
(iii) $z \notin \Phi$ implies that $g_{\xi}(z, k)$ does not meet $\Phi$,
(iv) $z \in \Phi$ implies that $g_{\xi}(z, k)$ does not meet any members of $\mathcal{W}_{\xi}$.

Since $\{ g_{\xi}(z, n + 1) : z \in p_{\xi}(F(\xi)) \} \cup \{ \mathcal{Y}_{\xi} \setminus p_{\xi}(F(\xi)) \}$ is an open cover of $Y_{\xi}$ which is paracompact, it has a locally finite and $\sigma$-discrete open refinement $\{ U_{\alpha}(\alpha) : \alpha \in \Omega(\xi) \}$. Moreover, there are an open cover $\{ U_{1}(\alpha) : \alpha \in \Omega(\xi) \}$ and a closed cover $\{ C(\alpha) : \alpha \in \Omega(\xi) \}$ of $Y_{\xi}$ such that $C(\alpha) \subset U_{1}(\alpha) \subset \overline{U_{1}(\alpha)} \subset U_{0}(\alpha)$ for each $\alpha \in \Omega(\xi)$. Let

$\Delta(\xi) = \{ \xi \cap \alpha : \alpha \in \Omega(\xi) \}$.

For each $\eta = \xi \cap \alpha \in \Delta(\xi)$, let $V(\eta) = V(\xi \cap \alpha) = p_{\xi}^{-1}(U_{0}(\alpha))$, $W(\eta) = W(\xi \cap \alpha) = p_{\xi}^{-1}(U_{1}(\alpha))$ and $F(\eta) = F(\xi \cap \alpha) = p_{\xi}^{-1}(C(\alpha)) \cap F(\xi)$.
Here, letting $\xi$ range over $\Sigma_n$, we put $\Delta_{n+1} = \bigcup \{ \Delta(\xi) : \xi \in \Sigma_n \}$. Then it is verified that the conditions (a), (1), (2) in (b) and (3)–(5) in (d) are satisfied. We define $\Sigma_{n+1} \subset \Delta_{n+1}$ such as it satisfies the condition (c).

Now, take an $\eta \in \Sigma_{n+1}$ and fix it. Let $\eta = \xi \in \Sigma_n$ such as for some $\alpha \in \Omega(\xi)$, and $V(\eta) = \pi_{\xi}^{-1}(U_0(\alpha))$. There is some $\eta \in \pi_{\xi}(F(\xi))$ with $\pi_{\xi}(V(\eta)) = U_0(\alpha) \subset g_{\xi}(y_n, n + 1)$.

Claim 2. $\eta \in \Sigma_{n+1}$ implies $\eta \in \Phi$.

Proof. By (c), there are two points $a, b \in f^{-1}(V(\eta))$ such that $\pi(\xi) = \pi(\xi \in V(\eta))$ and $|h(a) - h(b)| \geq \varepsilon / 3$. Assume $\eta \notin \Phi$. By (iii), we have $U_0(\alpha) \cap \Phi \subset g_{\xi}(y_n, n + 1) \cap \Phi = \emptyset$. Since $\pi_{\xi}(f(a)) \in \Phi$, we obtain $f(a) \notin \pi_{\xi}^{-1}(U_0(\alpha)) = V(\eta)$. Hence $\eta \notin \pi_{\xi}(V(\eta))$, which is a contradiction. □

Since $\eta \in \Phi$ and $Y_\varepsilon$ has countable tightness, there are two sequences $\{a_{\eta, k}\}$ and $\{b_{\eta, k}\}$ of points in $X$ such that $\eta \in \{\pi_{\xi}(f(a_{\eta, k})): k \in \omega\}$, $\pi_{\xi}(a_{\eta, k}) = \pi_{\xi}(b_{\eta, k})$ and $|h(a_{\eta, k}) - h(b_{\eta, k})| \geq \varepsilon$. Finally, we put

$$\theta_\eta = \theta_{\xi} \cup \left( \bigcup \left\{ \varphi(f(a_{\mu, j})) \cup \varphi(f(b_{\mu, j})) : \mu \notin \eta \text{ and } j \leq n + 1 \right\} \right)$$

$$\cup \left( \bigcup \left\{ \theta_{\mu_{\xi}} : \mu \in \bigcup_{i \leq n} \Sigma_i \text{ with } V(\mu) \cap V(\eta) \neq \emptyset \right\} \right) \setminus \Theta.$$

By (4), we have $\eta \in \Delta_{n+1}^\omega$. Then the conditions (6)–(10) in (d) are satisfied. We only check about (11). Take any $\eta \in \Delta_n \setminus \Sigma_n$ such that $W(\mu) \cap V(\eta) = \emptyset$. Hence it follows from (2), (iv) and Claim 2 that

$$W(\mu) \cap V(\eta) = \pi_{\xi}^{-1}(W(\mu)) \cap \pi_{\xi}^{-1}(U_0(\alpha)) \subset g_{\xi}(W(\mu)) \cap g(\eta, n + 1) = \emptyset.$$

Thus, our construction has been accomplished.

Here, we put

$$\mathcal{V} = \{ V(\xi) : \xi \in \Delta_n \setminus \Sigma_n \text{ and } n \in \omega \} \quad \text{and} \quad \mathcal{F} = \{ F(\xi) : \xi \in \Delta_n \setminus \Sigma_n \text{ and } n \in \omega \}.$$

For each $Q \subset \Lambda$, let $X_Q = \prod_{k \in Q} X_k$ and $Y_Q = \prod_{k \in Q} Y_k$. Moreover, $\pi_Q$ and $\pi_Q$ denote the projections of $X$ onto $X_Q$ and $Y$ onto $Y_Q$, respectively.

Lemma 2.B. $\mathcal{F}$ covers $Y$.

Proof. Assume that there is a point $y_0 \in \Sigma \setminus \bigcup \mathcal{F}$. By (1) and (3), we can inductively choose a sequence $\{\xi_n\}$ of indices such that $\xi_n \in \Sigma_n$, $\xi_n < \xi_{n+1}$ and $y_0 \in F(\xi_n)$ for each $n \in \omega$. We abbreviate $\pi_{\xi_n}, p_{\xi_n}$ and $p_{\xi_n}^k$ to $\pi_n, p_n$ and $p_n^k$, respectively, in the proof of Lemma 2.B. Let $D_{n,m} = \{ p_n^k(y_{\xi_{k+1}}) : k \geq m \}$ for each $n, m \in \omega$ with $m \geq n$. Let $C_n = \bigcap_{m \geq n} D_{n,m}^\wedge$ for each $n \in \omega$. Since $p_n^{n+1}(D_{n+1,m}) = D_{n,m}$, it follows that $p_n^{n+1}(C_{n+1}) \subset C_n$ for each $n \in \omega$.

Claim 3. Each $C_n$ is non-empty and compact.

Proof. Take any $n, k \in \omega$ with $k \geq n$. By (5), note that $p_n^k \circ g_{\xi_n}(z, k) \subset g_{\xi_n}(p_n^k(z), k)$ for each $z \in Y_{\xi_n}$. By (6), we have

$$p_k(y_n) \in p_k(F(\xi_{k+1})) \subset p_k(V(\xi_{k+1})) \subset g_{\xi_n}(y_{\xi_{k+1}}, k + 1).$$

So it follows that $p_n(y_n) = p_n^k \circ p_k(y_n) \in p_n^k \circ g_{\xi_n}(y_{\xi_{k+1}}, k + 1) \subset g_{\xi_n}(p_n^k(y_{\xi_{k+1}}), k + 1)$. This means that $\bigcap_{k \geq n} g_{\xi_n}(p_n^k(y_{\xi_{k+1}}), k) \neq \emptyset$. Since $g_{\xi_n}$ is a strong $\beta$-function for $X_{\xi_n}$, it follows that $C_n$ is non-empty and compact. □

Since $\{C_n, p_n^{n+1}\}$ is an inverse sequence of non-empty compact spaces, there is some $z_n \in C_n$ such that $p_n^{n+1}(z_{n+1}) = z_n$ for each $n \in \omega$. Let $R = \bigcup_{n \in \omega} \theta_{\xi_n}$. We can define the point $z \in Y_R$ by $z(\lambda) = z_n(\lambda)$ for each $\lambda \in \theta_{\xi_n}$ and each $n \in \omega$. Take the point $y_1 \in Y$ defined by $p_R(y_1) = z$ and $p_{\Lambda \setminus R}(y_1) = p_{\Lambda \setminus R}(y_0)$. Moreover,
pick a point \( x_0 \in f^{-1}(y_1) \). For each \( n, k \in \omega \), take the two points \( a_{n,k}, b_{n,k} \in X \) defined by \( \pi_R(a_{n,k}) = \pi_R(b_{\varepsilon_n,k}), \pi_R(b_{n,k}) = \pi_R(b_{\varepsilon_n,k}) \) and \( \pi_{A \setminus R}(a_{n,k}) = \pi_{A \setminus R}(b_{n,k}) = \pi_{A \setminus R}(x_0) \).

Claim 4. \( y_1 \) is a cluster point of \( \{ f(a_{n,k}) \} \).

**Proof.** Let \( O \) be any basic open neighborhood of \( y_1 \) in \( Y \). Take an \( m \in \omega \) such that \( O = p_{m-1}(O) \times Y_R \setminus \eta_{\varepsilon_m-1} \times p_{A \setminus R}(O) \). Since \( p_{m-1}(O) \) is an open neighborhood of \( p_{m-1}(y_1) = z_{m-1} \) in \( Y_{\varepsilon_m-1} \) and \( z_{m-1} \in C_{m-1} \subset \{ p_{m-1}(y_{\varepsilon_m}) \colon i \geq m - 1 \} \), there is an \( n \geq m \) with \( p_{m-1}^{-1}(y_{\varepsilon_n}) \in p_{m-1}(O) \). So we have \( y_{\varepsilon_n} \in p_{n-1}(O) \). By (7), there is a \( k \in \omega \) with \( p_{n-1}(f(a_{\varepsilon_n,k})) \in p_{n-1}(O) \). Hence we obtain

\[
f(a_{n,k}) = (p_{R}(f(a_{\varepsilon_n,k})), p_{A \setminus R}(f(x_0))) \in p_{n-1}(O) \times Y_{R} \setminus \eta_{\varepsilon_n-1} \times p_{A \setminus R}(O) = O.
\]

Since \( f \) is a perfect map, it follows from Claim 4 that the sequence \( \{ a_{n,k} \colon k \in \omega \} \) of points in \( X \) has a cluster point \( v \in f^{-1}(y_1) \).

Claim 5. Every open neighborhood of \( v \) in \( X \) contains some \( a_{n,k} \) and \( b_{n,k} \).

**Proof.** Let \( N \) be any basic open neighborhood of \( v \) in \( X \). Take an \( m \in \omega \) such that \( N = \pi_{m-1}(N) \times X_R \setminus \eta_{\varepsilon_m-1} \times \pi_{A \setminus R}(N) \). Choose some \( n, k \geq m \) with \( a_{n,k}, b_{n,k} \in N \). By (8), we have \( \pi_{n-1}(b_{n,k}) = \pi_{n-1}(b_{\varepsilon_n,k}) = \pi_{n-1}(a_{\varepsilon_n,k}) = \pi_{n-1}(a_{n,k}) \in \pi_{n-1}(N) \). Note that \( \pi_{A \setminus R}(v) = \pi_{A \setminus R}(x_0) \). So we have \( \pi_{A \setminus R}(a_{n,k}) = \pi_{A \setminus R}(b_{n,k}) = \pi_{A \setminus R}(v) \in \pi_{A \setminus R}(N) \). Hence we obtain \( b_{n,k} \in N \). \( \square \)

Claim 6. \( |h(a_{n,k}) - h(b_{n,k})| \geq \varepsilon/3 \) for each \( n, k \in \omega \).

**Proof.** Take any \( n, k \in \omega \). By (8), observe \( |h(a_{\varepsilon_n,k}) - h(b_{\varepsilon_n,k})| \geq \varepsilon \). By (9) and \( \pi_R(a_{\varepsilon_n,k}) = \pi_R(b_{\varepsilon_n,k}) \), we have \( \pi_{\psi}(f(a_{\varepsilon_n,k}))(a_{\varepsilon_n,k}) = \pi_{\psi}(f(a_{\varepsilon_n,k}))(b_{\varepsilon_n,k}) \). It follows from the choice of \( \psi(f(a_{\varepsilon_n,k})) \) that \( |h(a_{\varepsilon_n,k}) - h(b_{\varepsilon_n,k})| < \varepsilon/3 \). Similarly, we have \( |h(b_{\varepsilon_n,k}) - h(b_{n,k})| < \varepsilon/3 \). Hence these inequalities yield the inequality of Claim 6. \( \square \)

Since \( h \) is continuous at \( v \) in \( X \), it follows from Claim 5 that we can take some \( n_0, k_0 \in \omega \) such that \( |h(a_{n_0,k_0}) - h(v)| < \varepsilon/6 \) and \( |h(b_{n_0,k_0}) - h(v)| < \varepsilon/6 \). These yield the inequality \( |h(a_{n_0,k_0}) - h(b_{n_0,k_0})| < \varepsilon/3 \), which contradicts Claim 6. \( \square \)

It follows from (a), (2) and Lemma 2.B that \( \mathcal{V} \) is \( \sigma \)-discrete cylindrical open cover of \( Y \) and that \( \mathcal{F} \) is a cylindrical shrinking of \( \mathcal{V} \).

**Lemma 2.C.** \( \mathcal{V} \) is locally finite in \( Y \).

**Proof.** Pick any \( y \in Y \). By Lemma 2.B, there is a \( \rho \in \bigcup_{n \in \omega}(\Delta_n \setminus \Xi_n) \) with \( y \in F(\rho) \). Let \( \rho \in \Delta_m \setminus \Xi_m \). Take any \( \eta \in \bigcup_{n \geq m+2}(\Delta_n \setminus \Xi_n) \). Let \( \xi = \eta \upharpoonright (m+2) \). It follows from (11) that \( W(\rho) \cap V(\xi) = \emptyset \). By (3), we have \( V(\eta) \subset V(\xi) \). Hence the open neighborhood \( W(\rho) \) of \( y \) does not meet any member of \( \{ V(\eta) \colon \eta \in \bigcup_{n \geq m+2}(\Delta_n \setminus \Xi_n) \} \). By (a), \( \mathcal{V} \) is locally finite at \( y \). \( \square \)

Now, let us proceed the final stage of the proof of Theorem 2.2. Take any \( \xi \in \bigcup_{n \in \omega}(\Delta_n \setminus \Xi_n) \). Pick an \( x_\xi \in f^{-1}(F(\xi)) \) and fix it. Let

\[
U^*_\xi = \pi_{\varepsilon_\xi}(f^{-1}(V(\xi))) \times X_{\varepsilon_\xi} \setminus \eta_{\varepsilon_\xi} \quad \text{and} \quad E^*_\xi = \pi_{\varepsilon_\xi}(f^{-1}(F(\xi))) \times X_{\varepsilon_\xi} \setminus \eta_{\varepsilon_\xi}.
\]

Moreover, we let

\[
U_0^*\xi = \{ x' \in U^*_\xi \colon h(x', \pi_{A \setminus \eta_{\varepsilon_\xi}}(x_\xi)) < 5/6 \},
\]

\[
U_1^*\xi = \{ x' \in U^*_\xi \colon h(x', \pi_{A \setminus \eta_{\varepsilon_\xi}}(x_\xi)) > 1/6 \},
\]

\[
E_0^*\xi = \{ x' \in E^*_\xi \colon h(x', \pi_{A \setminus \eta_{\varepsilon_\xi}}(x_\xi)) \leq 2/3 \} \quad \text{and} \quad E_1^*\xi = \{ x' \in E^*_\xi \colon h(x', \pi_{A \setminus \eta_{\varepsilon_\xi}}(x_\xi)) \geq 1/3 \}.
\]
Here we put $U_j(\xi) = \pi_\xi^{-1}(U_j^*(\xi))$ and $E_j(\xi) = \pi_\xi^{-1}(E_j^*(\xi))$ for $j = 0, 1$. Then note that $U_0(\xi) \cup U_1(\xi) = f^{-1}(V(\xi))$ and $E_0(\xi) \cup E_1(\xi) = f^{-1}(F(\xi))$ and that $E_j(\xi) \subset U_j(\xi)$ for $j = 0, 1$. Then $U_j(\xi)$ and $E_j(\xi)$ are an open cylinder and a closed cylinder, respectively, in $X$ for $j = 0, 1$. Here, letting $\xi$ range over $\bigcup_{n \in \mathbb{N}}(\Delta_n \setminus \mathcal{E}_n)$, we put

$$
U = \{U_j(\xi) : \xi \in \Delta_n \setminus \mathcal{E}_n, \ n \in \omega \text{ and } j = 0, 1\} \quad \text{and}
$$

$$
\mathcal{E} = \{E_j(\xi) : \xi \in \Delta_n \setminus \mathcal{E}_n, \ n \in \omega \text{ and } j = 0, 1\}.
$$

By Lemma 2.B, $\mathcal{F}$ covers $Y$. So $\mathcal{E}$ covers $X$. Hence $\mathcal{E}$ is a cylindrical shrinking of $\mathcal{U}$. By (a) and Lemma 2.C, $\mathcal{V}$ is locally finite and $\sigma$-discrete in $Y$. Hence $\mathcal{U}$ is a locally finite, $\sigma$-discrete cylindrical open cover of $X$.

Take any $\xi \in \bigcap_{n \in \mathbb{N}}(\Delta_n \setminus \mathcal{E}_n)$ again and put $\varepsilon = 1/6$.

**Claim 7.** $U_0(\xi) \cap B = \emptyset$ and $U_1(\xi) \cap A = \emptyset$.

**Proof.** Let $\xi \in \Delta_n \setminus \mathcal{E}_n$ and $j = 0$. Pick any $x \in U_0(\xi)$. Take the point $x^* \in X$ defined by $\pi_\xi(x^*) = \pi_\xi(x)$ and $\pi_{A^i} \theta_\xi(x^*) = \pi_{A^i} \theta_\xi(x_\xi)$. By $\pi_\xi(x^*) = \pi_\xi(x) \in U_0^*(\xi)$, we have $x, x^* \in U_0(\xi) \subset f^{-1}(V(\xi))$. Since $\pi_\xi(x) = \pi_{A^i} \pi_\xi(x^*)$ and $\xi \notin \mathcal{E}_n$, it follows from (c) that $|h(x) - h(x^*)| < 1/6$. By $\pi_\xi(x^*) \in U_0^*(\xi)$, we have $h(x^*) = h(\pi_\xi(x^*), \pi_{A^i} \theta_\xi(x_\xi)) < 5/6$. This implies $h(x) < 1$, that is, $x \notin B$. Similarly, it follows that $x \in U_1(\xi)$ implies $x \notin A$. □

By Claim 7, no member of $\mathcal{U}$ meets both $A$ and $B$. Thus we have proved that $\mathcal{U}$ and $\mathcal{E}$ are our desired covers. The proof of Theorem 2.2 is completed. □

Let $X = \prod_{\lambda \in A} X_\lambda$ be an infinite product. A cylinder $\pi_\theta^{-1}(T)$ in $X$ is called a cozero cylinder (respectively, a zero-set cylinder) if $T$ is a cozero-set (respectively, a zero-set) in $X_\theta$. A cover $\mathcal{G}$ of the product $X$ is said to be cylindrical cozero (respectively, cylindrical zero-set) if each member of $\mathcal{G}$ is a cozero cylinder (respectively, zero-set cylinder) in $X$.

**Lemma 2.3.** [20] The class of strong $\beta$-spaces is countably productive.

In the aid of Lemmas 2.1 and 2.3, Theorem 2.2 yields the following.

**Corollary 2.4.** Let $X = \prod_{\lambda \in A} X_\lambda$ be an infinite product of $\beta$-spaces, each finite subproduct of which is paracompact and has countable tightness. Let $\mathcal{O}$ be an open cover of $X$. Then the following are equivalent:

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite cylindrical cozero refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete cylindrical cozero refinement.
(d) $\mathcal{O}$ has a locally finite cylindrical cozero refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cylindrical cozero refinement which has a cylindrical zero-set shrinking.

**Proof.** It suffices to show (a) $\Rightarrow$ (e). By Lemmas 2.1 and 2.3, note that each finite subproduct of $X$ is a strong $\beta$-space. First, assume that $\mathcal{O}$ is a binary cozero cover of $X$. It follows from Theorem 2.2 that there are a locally finite, $\sigma$-discrete, cylindrical open refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Omega\}$ of $\mathcal{O}$ and a cylindrical shrinking $\mathcal{E} = \{E_\alpha : \alpha \in \Omega\}$ of $\mathcal{V}$. Take an $\alpha \in \Omega$. Since $E_\alpha \subset V_\alpha$, we may consider that $E_\alpha$ and $V_\alpha$ are both $\theta_\alpha$-distinguished in $X$ for some $\theta_\alpha \in [\lambda]^c$. Since $X_{\theta_\alpha}$ is normal, there are a cozero-set $W_\alpha$ and a zero-set $F_\alpha$ in $X_{\theta_\alpha}$ such that $\pi_{\theta_\alpha}(E_\alpha) \subset F_\alpha \subset W_\alpha \subset \pi_{\theta_\alpha}(V_\alpha)$. Then $\{\pi_{\theta_\alpha}(W_\alpha) : \alpha \in \Omega\}$ and $\{\pi_{\theta_\alpha}^{-1}(F_\alpha) : \alpha \in \Omega\}$ are our desired covers for $\mathcal{O}$.

Next, let $\mathcal{O} = \{O_\xi : \xi \in \mathcal{F}\}$ be a normal cover of $X$. By Theorem 1.1, we may consider that $\mathcal{O}$ is a locally finite, $\sigma$-discrete, cozero cover of $X$, and that there is a zero-set shrinking $\{S_\xi : \xi \in \mathcal{F}\}$ of $\mathcal{O}$. For each $\xi \in \mathcal{F}$, since $\{O_\xi, X \setminus S_\xi\}$ is a binary cozero cover of $X$, there is a locally finite, $\sigma$-discrete, cylindrical cozero refinement $\mathcal{U}_\xi$ which has a cylindrical zero-set shrinking $C_\xi = \{C_\xi^U : U \in \mathcal{U}_\xi\}$. Let $\mathcal{U}_\xi^+ = \{U \in \mathcal{U}_\xi : \xi \subset O_\xi\}$ and $\mathcal{C}_\xi^+ = \{C_\xi^U : U \in \mathcal{U}_\xi^+\}$ for each $\xi \in \mathcal{F}$. Then we have $S_\xi \subset \bigcup \mathcal{C}_\xi^+ \subset \bigcup \mathcal{U}_\xi^+ \subset O_\xi$. Here, we let $\mathcal{U} = \bigcup \{\mathcal{U}_\xi^+ : \xi \in \mathcal{F}\}$ and $\mathcal{C} = \bigcup \{\mathcal{C}_\xi^+ : \xi \in \mathcal{F}\}$. It is easy to check that $\mathcal{U}$ and $\mathcal{C}$ are our desired covers. □
A regular space $S$ is called a strong $\Sigma$-space if there are a $\sigma$-locally finite closed cover $F$ of $S$ and a cover $K$ of $S$ by compact sets such that, whenever $K \in K$ and $U$ is open in $S$ with $K \subset U$, one can find $F \in F$ with $K \subset F \subset U$.

It follows from [3, Theorem 7.8(ii)] and [20] that every $\Sigma$-space is a $\beta$-space, and that every strong $\Sigma$-space is a strong $\beta$-space. Obviously, paracompact $\Sigma$-spaces are strong $\Sigma$-spaces, and the class of all paracompact $\Sigma$-spaces is countably productive (see [10]).

Let $X = \prod_{\lambda \in A} X_{\lambda}$ be an infinite (or a finite) product. For each $\lambda \in A$, $\pi_{\lambda}$ denotes the projection of $X$ onto $X_{\lambda}$. A subset of the form $\bigcap_{\lambda \in \theta} \pi_{\lambda}^{-1}(U_{\lambda})$ in $X$ is called a rectangle if $\theta \in [A]^{<\omega}$, where $U_{\lambda} \subset X_{\lambda}$ for each $\lambda \in \theta$. A rectangle $\bigcap_{\lambda \in \theta} \pi_{\lambda}^{-1}(U_{\lambda})$ in $X$ is called a cozero rectangle (respectively, zero-set rectangle) if $U_{\lambda}$ is a cozero-set (respectively, zero-set) for each $\lambda \in \theta$. A cover $\mathcal{G}$ of the product $X$ is said to be rectangular (respectively, rectangular cozero, rectangular zero-set) if each member of $\mathcal{G}$ is a rectangle (respectively, cozero rectangle, zero-set rectangle) in $X$.

For a finite product $X = \prod_{i \leq n} X_{i}$, note that a cozero (respectively, zero-set) rectangle in $X$ is a subset of the form $\prod_{i \leq n} U_{i}$ such that each $U_{i}$ is a cozero-set (respectively, zero-set) in $X_{i}$.

**Lemma 2.5.** Let $X = \prod_{i \leq n} X_{i}$ be a finite product of paracompact $\Sigma$-spaces. Then every open cover of $X$ has a locally finite, $\sigma$-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.

**Proof.** By [10, Theorem 2.7], every $\Sigma$-space is a $P$-space. So it is assured by [17, Theorem 4.3] that this is true for the case of $X = X_{0} \times X_{1}$. Assume that this is true for each $k \leq n - 1$. Since $X = (\prod_{i \leq n-1} X_{i}) \times X_{n}$ and $\prod_{i \leq n-1} X_{i}$ is a paracompact $\Sigma$-space, it is easily verified by induction that this is true for the product $X$. □

**Corollary 2.6.** Let $X = \prod_{\lambda \in A} X_{\lambda}$ be an infinite product of paracompact $\Sigma$-spaces, each finite subproduct of which has countable tightness. Let $\mathcal{O}$ be an open cover of $X$. Then the following are equivalent:

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite rectangular cozero refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete rectangular cozero refinement.
(d) $\mathcal{O}$ has a locally finite rectangular cozero refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.

**Proof.** We only need to show (a) $\Rightarrow$ (e). Let $\mathcal{O}$ be a normal cover of $X$. As in the proof of Corollary 2.4, we may assume that $\mathcal{O}$ is a binary cozero cover of $X$. By Theorem 2.2, $\mathcal{O}$ has a locally finite, $\sigma$-discrete, cylindrical open refinement $\mathcal{V} = \{V_{\alpha} \in \mathcal{O} : \alpha \in \Omega\}$ of $\mathcal{O}$ and a cylindrical shrinking $\mathcal{E} = \{E_{\alpha} \in \mathcal{O} : \alpha \in \Omega\}$ of $\mathcal{V}$. Moreover, for each $\alpha \in \Omega$, we may consider that $V_{\alpha} = \pi_{\theta_{\alpha}}^{-1}(W_{\alpha})$ and $E_{\alpha} = \pi_{\theta_{\alpha}}^{-1}(F_{\alpha})$ for some $\theta_{\alpha} \in [A]^{<\omega}$, where $W_{\alpha}$ is open and $F_{\alpha}$ is closed in $X_{\theta_{\alpha}}$ with $W_{\alpha} \subset F_{\alpha}$. It follows from Lemma 2.5 that there are a locally finite, $\sigma$-discrete, rectangular cozero refinement $\{U_{\alpha \gamma} : \gamma \in \Gamma_{\alpha}\}$ of $\{W_{\alpha}, X_{\theta_{\alpha}} \setminus F_{\alpha}\}$ and its rectangular zero-set shrinking $\{C_{\alpha \gamma} : \gamma \in \Gamma_{\alpha}\}$. Let $\Gamma_{\alpha}^{+} = \{\gamma \in \Gamma_{\alpha} : C_{\alpha \gamma} \cap F_{\alpha} \neq \emptyset\}$. Now, let $\mathcal{U} = \{\pi_{\theta_{\alpha}}^{-1}(U_{\alpha \gamma}) : \gamma \in \Gamma_{\alpha}^{+} \text{ and } \alpha \in \Omega\}$ and $\mathcal{C} = \{\pi_{\theta_{\alpha}}^{-1}(C_{\alpha \gamma}) : \gamma \in \Gamma_{\alpha}^{+} \text{ and } \alpha \in \Omega\}$. Then $\mathcal{U}$ is a locally finite, $\sigma$-discrete, rectangular cozero refinement of $\mathcal{O}$ and $\mathcal{C}$ is a rectangular zero-set shrinking of $\mathcal{U}$. □

Recall that a space $S$ is an $M$-space if $S$ is a quasi-perfect preimage of a metric space. So every paracompact $M$-space (= paracompact $p$-space) is a perfect preimage of a metric space. The class of all paracompact $M$-spaces is countably productive.

The following is a strengthening of Theorem 1.3 due to Filipov.

**Corollary 2.7.** Let $X = \prod_{\lambda \in A} X_{\lambda}$ be an infinite product of paracompact $M$-spaces and $\mathcal{O}$ an open cover of $X$. Then the following are equivalent:

(a) $\mathcal{O}$ is normal.
(b) $\mathcal{O}$ has a $\sigma$-locally finite rectangular cozero refinement.
(c) $\mathcal{O}$ has a $\sigma$-discrete rectangular cozero refinement.
(d) $\mathcal{O}$ has a locally finite rectangular cozero refinement.
(e) $\mathcal{O}$ has a locally finite, $\sigma$-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.
Since every $M$-space is a $\Sigma$-space (see [10, Theorem 2.6]), this is obtained from Theorem 2.2 and Lemma 2.5 in the same way as Corollary 2.6.

3. Normality of $\Sigma$-products

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be an infinite product of spaces $X_\lambda$, $\lambda \in \Lambda$, where we may assume that each factor $X_\lambda$ contains at least two points. Fix a point $s = (s_\lambda) \in X$. For each $x \in X$, we let $\text{Supp}(x) = \{ \lambda \in \Lambda : x(\lambda) \neq s(\lambda) \}$. Then the subspace

$$
\Sigma = \{ x = (x(\lambda)) \in X : |\text{Supp}(x)| \leq \omega \}
$$

of $X$ is called a $\Sigma$-product of spaces $X_\lambda, \lambda \in \Lambda$. The $s \in \Sigma$ is called a base point of $\Sigma$. The mention of the base point $s$ is often omitted.

For each $\theta \in [\Lambda]^{< \omega}$, $X_\theta = \prod_{\lambda \in \theta} X_\lambda$ is called a finite subproduct of $\Sigma$. For each $R \in [\Lambda]^{< \omega}$, $X_R = \prod_{\lambda \in R} X_\lambda$ is called a countable subproduct of $\Sigma$.

Let $R \in [\Lambda]^{< \omega}$. We denote by $p_R$ the projection of $X$ onto $X_R$. We also denote by $p_R^{-1}$ the projection of $X_R$ onto $X_{R'}$, where $R' \subset R$. We say that $U$ is $R$-distinguished in $\Sigma$ if $p_R^{-1} p_R(U) = U$. Note that $U$ is $R$-distinguished open (respectively, closed) in $\Sigma$ iff $U = p_R^{-1}(W)$ for some open (respectively, closed) set $W$ in $X_R$.

We have often used the following to show collectionwise normality.

**Lemma 3.1.** Let $S$ be a space and $\mathcal{D}$ a discrete collection of closed sets in $S$. If there is a $\sigma$-locally finite collection $\mathcal{U}$ of open sets in $S$ such that $\bigcup \mathcal{D} \subset \bigcup \mathcal{U}$ and for each $U \in \mathcal{U}$, $\overline{U}$ meets at most one member of $\mathcal{D}$, then there is a pairwise disjoint collection $\{ V_D : D \in \mathcal{D} \}$ of open sets in $S$ such that $D \subset V_D$ for each $D \in \mathcal{D}$.

**Lemma 3.2.** [15] Let $S$ be a space with countable tightness. Let $\mathcal{D}$ be a collection of subsets of a space $T$. Let $p$ be a continuous map from $T$ to $S$. If $p(\mathcal{D})$ is non-discrete at $x$, then there is a countable subset $M$ of $\bigcup \mathcal{D}$ such that $\{ p(D \cap M) : D \in \mathcal{D} \}$ is non-discrete at $x$.

**Lemma 3.3.** [8] Let $\Sigma$ be a $\Sigma$-product of spaces. Then the following are equivalent.

(a) $\Sigma$ has countable tightness.
(b) Every finite subproduct of $\Sigma$ has countable tightness.
(c) Every countable subproduct of $\Sigma$ has countable tightness.

Moreover, we make use of the following.

**Lemma 3.4.** [20] Let $X_i$ be a (strong) $\beta$-space for each $i \in \omega$. If $\prod_{i \in \omega} X_i$ is paracompact for each $n \in \omega$, then $\prod_{i \in \omega} X_i$ is a paracompact $\beta$-space.

Making use of the above lemmas, we can obtain an affirmative answer to [18, Question 4], which was restated in Gruenhage’s survey [4, Question 12.2] as follows.

**Theorem 3.5.** Let $\Sigma$ be a $\Sigma$-product of $\beta$-spaces. If each finite subproduct of $\Sigma$ is paracompact and has countable tightness, then $\Sigma$ is collectionwise normal.

**Proof.** We modify the proof of [17, Theorem 1]. Let $\Sigma$ be a $\Sigma$-product of spaces $X_\lambda, \lambda \in \Lambda$, with a base point $s \in \Sigma$. It follows from Lemmas 2.1, 3.3 and 3.4 that each countable subproduct of $\Sigma$ is a paracompact, strong $\beta$-space with countable tightness. Let $\mathcal{D}$ be any discrete collection of closed sets in $\Sigma$.

Now, for each $n \in \omega$, we construct a collection $\mathcal{U}_n$ of open sets in $\Sigma$ and an index set $\mathcal{E}_n$ of $n$-tuple sequences such that for each $\xi \in \mathcal{E}_n$ one can assign $R_\xi \in [\Lambda]^{< \omega}$, $E(\xi) \subset \Sigma$, $H(\xi) \subset \Sigma$, $x_\xi \in X_{E(\xi)}$, $\{ a_{\xi,k} : k \in \omega \} \subset \bigcup \mathcal{D}$ and a strong
\( \beta \)-function \( g_\xi \) for \( X_\xi \) where \( X_\xi \) and \( X_{\xi^\prime} \) are the abbreviations of \( X_{R_\xi} \) and \( X_{R_{\xi^\prime}} \), respectively, satisfying the following conditions for each \( n \in \omega \):

(a) \( U_n = \bigcup \{ U(\mu) : \mu \in \Sigma_{n-1} \} \) is locally finite in \( \Sigma \).

(b) Each \( U \in \mathcal{U}(\mu), \mu \in \Sigma_{n-1}, \) is a \( R_\mu \)-distinguished open set in \( \Sigma \) such that \( \overline{U} \) meets at most one member of \( \mathcal{D} \).

(c) \( \{ H(\xi) : \xi \in \Sigma_n \} \) is locally finite in \( \Sigma \) such that \( \bigcup \mathcal{U}(\xi) \subset H(\xi) \) for each \( \xi \in \Sigma_n \).

(d) For each \( \xi \in \Sigma_n, \)

1. \( \xi_- \in \Sigma_{n-1}, \)
2. \( E(\xi) \) is a \( R_{\xi_-} \)-distinguished closed set in \( \Sigma \) and \( H(\xi) \) is a \( R_{\xi_-} \)-distinguished open set in \( \Sigma \) such that \( E(\xi) \subset H(\xi) \), where \( E(\emptyset) = \Sigma \).
3. \( p_\xi(E(\xi)) \subset p_\xi(\mathcal{U}(\xi)) \cup p_\xi(\bigcup \{ E(\eta) : \eta \in \Sigma_{n+1} \text{ with } \eta_- = \xi \}) \),
4. \( g_\xi(x, k+1) \subset g_\xi(x, k) \) and \( p_\xi(x, k) \subset g_\xi(p_\xi(x, k), k) \) for each \( x \in X_\xi \) and each \( k \in \omega \),
5. \( p_\xi(E(\xi)) \subset g_\xi(x_\xi, n) \),
6. \( p_\xi(D \upharpoonright \{ a_{\xi, k} : k \in \omega \}) \) is not discrete at \( x_\xi \),
7. \( R_\xi = R_{\xi_-} \cup \{ \text{Supp}(a_{\xi, k}) : k \in \omega \} \),

where \( p_\xi, p_{\xi_-} \) and \( p_\xi \) are the abbreviations of \( p_{R_\xi}, p_{R_{\xi_-}} \) and \( p_{R_{\xi_-}} \), respectively.

The inductive construction is similar to that in the proof of [17, Theorem 1]. However, we use 

\[ \Phi = \{ x \in p_\xi(E(\xi)) : p_\xi(D) \text{ is not discrete at } x \} \]

instead of \( \Phi = \overline{p_\xi(A)} \cap \overline{p_\xi(B)} \) in there. Since every countable subproduct of \( \Sigma \) has countable tightness, it follows from Lemma 3.2, one can choose a sequence \( \{ a_{\xi, k} : k \in \omega \} \) of points in \( \bigcup \mathcal{D} \) for each \( x_\xi \), where \( \emptyset \in \Sigma_{n+1} \text{ with } \eta_- = \xi \), as it satisfies (6). The detail is left to the readers.

We let \( \mathcal{U} = \bigcup_{n \in \omega} U_n \). By (a) and (b), \( \mathcal{U} \) is a \( \sigma \)-locally finite collection of open sets in \( \Sigma \) such that \( \overline{\mathcal{U}} \) meets at most one member of \( \mathcal{D} \). It suffices from Lemma 3.1 to show that \( \mathcal{U} \) covers \( \Sigma \). Assuming the contrary, we pick some \( y \in \Sigma \setminus \bigcup \mathcal{D} \). By \( E(\emptyset) = \Sigma \) and (3), we can inductively choose a sequence \( \{ \xi_n \} \) of indices such that \( \xi_n \in \Sigma_n \), \( \xi_{n+1} = \xi_n \) and \( y \in E(\xi_n) \) for each \( n \in \omega \). Hereafter, we abbreviate \( p_{\xi_n}, p_{\xi_n}^{R_\xi}, x_{\xi_n} \) to \( p_n, p_n^{R_\xi}, \) and \( x_n \), respectively.

Take an \( n \in \omega \). Let \( D_{n,m} = \{ p_n^{k}(x_{\xi_n+1}) : k \geq m \} \) for each \( m \in \omega \) with \( m \geq n \), and let \( C_n = \bigcap_{m \geq n} \overline{D_{n,m}} \). Then we have \( p_n^{k}(C_{n+1}) \subset C_n \). It follows from (4) that \( p_n^{k} \circ g_{\xi_n}(x, j) \subset g_{\xi_n}(p_n^{k}(x_n), j) \) for each \( x \in X_{\xi_n} \) and each \( k, j \in \omega \), where \( k \geq m \). Using this, by (5), we have \( p_n(y) \in \bigcap_{k \geq n} g_{\xi_n}(p_n^{k}(x_{\xi_n+1}), k+1) \). Since \( g_{\xi_n} \) is a strong \( \beta \)-function for \( X_{\xi_n} \), \( C_n \) is non-empty and compact. Since \( \{ C_n, p_n^{k}(C_{n+1}) \} \) is an inverse sequence of non-empty compact spaces, there is some \( z_n \in C_n \) such that \( p_n^{k}(z_{n+1}) = z_n \) for each \( n \in \omega \). Let \( R = \bigcup_{n \in \omega} R_n \). Then \( |R| \leq \omega \) and we can take the point \( z \in \Sigma \) defined by \( p_n(z) = z_n \) for each \( n \in \omega \) and \( z(\lambda) = s(\lambda) \) for each \( \lambda \in \Lambda \setminus R \). By (7), note that \( R = \bigcup \{ \text{Supp}(a_{\xi_n,k}) : n \in \omega \} \). As in the proof of [7, Theorem 1], it is verified by (6) that \( \mathcal{D} \) is not discrete at \( z \). This contradicts the discreteness of \( \mathcal{D} \) in \( \Sigma \). \( \square \)

**Remark.** Daniel and Gruenhage [1] proved that there is a non-normal \( \Sigma \)-product \( \Sigma \) such that each finite (countable) subproduct of \( \Sigma \) is a first countable, paracompact and perfectly normal. So we cannot exclude the assumption of “\( \beta \)-space” in Theorem 3.5.

Since \( \Sigma \)-spaces and semi-stratifiable spaces are both \( \beta \)-spaces, the following are immediate consequences of Theorem 3.5.

**Corollary 3.6.** [15] Let \( \Sigma \) be a \( \Sigma \)-product of paracompact \( \Sigma \)-spaces. If \( \Sigma \) has countable tightness, then \( \Sigma \) is normal.

**Corollary 3.7.** [17] Let \( \Sigma \) be a \( \Sigma \)-product of semi-stratifiable spaces. If each finite subproduct of \( \Sigma \) is paracompact and has countable tightness, then \( \Sigma \) is normal.

A normal space \( S \) is said to be base-normal [21] (respectively, base-paracompact [13]) if there is a base \( \mathcal{B} \) of \( S \) such that \( |\mathcal{B}| = w(S) \) and every binary open cover (respectively, every open cover) of \( S \) has a locally finite refinement...
by members of $\mathcal{B}$, where $w(S)$ denotes the weight of $S$. Note that a space $S$ is base-paracompact iff $S$ is paracompact and base-normal.

**Theorem 3.8.** Let $\Sigma$ be a $\Sigma$-product of the spaces. If each finite subproduct of $\Sigma$ is base-paracompact and has countable tightness, then $\Sigma$ is base-paracompact.

**Proof.** Let $\Sigma$ be a $\Sigma$-product of the spaces $X_{\lambda}, \lambda \in \Lambda$. Let $X = \prod_{\lambda \in \Lambda} X_{\lambda}$. For each $\theta \in \{\Lambda\}^{<\omega}$, there is a base $\mathcal{B}_\theta$ of $X_{\theta} = \prod_{\lambda \in \theta} X_{\lambda}$ which witnesses the base-paracompactness of $X_{\theta}$. Note that $|\mathcal{B}_\theta| = \sup_{\lambda \in \theta} w(X_{\lambda})$. Let

$$\mathcal{B} = \{\pi_\theta^{-1}(B) \cap \Sigma : B \in \mathcal{B}_\theta \text{ and } \theta \in \{\Lambda\}^{<\omega}\},$$

where $\pi_\theta$ denotes the projection of $X$ onto $X_{\theta}$. Then $\mathcal{B}$ is a base of $\Sigma$ with $|\mathcal{B}| = |\Lambda| \cdot \sup_{\lambda \in \Lambda} w(X_{\lambda}) = w(\Sigma)$. We show that $\mathcal{B}$ witnesses the base-normality of $\Sigma$.

Let $\mathcal{G} = \{G_0, G_1\}$ be a binary open cover of $\Sigma$. It follows from Theorem 3.5 that $\Sigma$ is normal. So we may assume that $\mathcal{G}$ is a binary cozero cover of $\Sigma$. Since each finite subproduct of $\Sigma$ has countable tightness, it follows from [14, Theorem 1] that $\Sigma$ is $C$-embedded (hence $C^*$-embedded) in $X$. So there is a binary cozero cover $\mathcal{O} = \{O_0, O_1\}$ of $X$ such that $O_j \cap \Sigma \subseteq G_j$ for $j = 0, 1$. Since $\mathcal{O}$ is normal and each finite subproduct of $X$ has countable tightness, it follows from Corollary 2.4 that there are a locally finite cozero cylindrical refinement $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ of $\mathcal{O}$ and a cylindrical zero-set shrinking $\mathcal{E} = \{E_\alpha : \alpha \in \Omega\}$ of $\mathcal{U}$. Take an $\alpha \in \Omega$. We may assume that $U_\alpha = \pi_{\theta_\alpha}^{-1}(V_\alpha)$ and $E_\alpha = \pi_{\theta_\alpha}^{-1}(F_\alpha)$ for some $\theta_\alpha \in \{\Lambda\}^{<\omega}$, where $V_\alpha$ and $F_\alpha$ are a cozero-set and a zero-set, respectively, in $X_{\theta_\alpha}$ such that $F_\alpha \subseteq V_\alpha$. Then there is a locally finite refinement $\mathcal{A}_\alpha$ of $\{V_\alpha, X_{\theta_\alpha} \setminus F_\alpha\}$ with $\mathcal{A}_\alpha \subseteq \mathcal{B}_{\theta_\alpha}$, and let $\mathcal{A}_\alpha^+ = \{A \in \mathcal{A}_\alpha : A \cap F_\alpha \neq \emptyset\}$. Here we let

$$A = \{\pi_{\theta_\alpha}^{-1}(A) \cap \Sigma : A \in \mathcal{A}_\alpha^+ \text{ and } \alpha \in \Omega\}.$$

Then we have $\mathcal{A} \subseteq \mathcal{B}$. It is easily seen that $\mathcal{A}$ is a locally finite refinement of $\mathcal{G}$. $\square$

As is stated in the Introduction, the following is an affirmative answer to the question [22, Question 5.8].

**Corollary 3.9.** A $\Sigma$-product of paracompact $M$-spaces is base-normal if and only if it is normal.

**Proof.** Let $\Sigma$ be a $\Sigma$-product of paracompact $M$-spaces. Since the class of paracompact $M$-spaces is countably productive, it follows from [13, Corollary 3.8] that each finite subproduct of $\Sigma$ is base-paracompact. Assume that $\Sigma$ is normal. It follows from [7, Theorem 1] that $\Sigma$ has countable tightness. Since $M$-spaces are $\beta$-spaces, this is an immediate consequence of Theorem 3.8. $\square$

**References**