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Which topologies are quasimetrizable?

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Abstract

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A characterization of quasimetrizable spaces is given; they are those whose topologies arise from a σ-self-cocushioned pairbase whose dual is σ-self-cocushioned.

This is closely related to the known characterization of γ-spaces as those whose topologies arise from a σ-self-cocushioned pairbase (with no dual condition).

The last section of the paper discusses to what extent this is a *topological* characterization of quasimetrizability, and notes the absence of a "Bing-style" characterization.

Keywords: Quasimetrizable; Conjugate; Enclosing set relation; Pairbase; Pairgenerator; (Self-) cocushioned; (Local) quasiuniformity; σ-Alexandroff space.

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1. Introduction and notation

The question of which topologies arise from a quasimetric has been open since Wilson's publication of the first paper on quasimetrics, [25], in 1931. In its Section 7, Wilson explored relations between quasimetric and topological spaces, and proved that second countable topological spaces are quasimetrizable. Study of this question was spurred by discovery of the best-known characterizations of metrizable spaces in 1950-1 by Bing [1], Smirnov [24], and Nagata [18]. Sufficient conditions for quasimetrizability of topologies were given in the 1960's by Doitchinov, Nedev [19], Norman [20], and Sion and Zelmer [23], and in the early 1970's by Fletcher and Lindgren [2]; none of the conditions they considered was necessary.

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More recently the last authors devoted the beginning of the last chapter of their 1982 monograph, [3], to a summary of the state of the question.

Results leading toward a characterization of quasimetrizability for bitopological spaces began in the early 1960's with the first papers in the field, by Kelly [9], and Lane [16]. Unlike the search for a characterization of quasimetrizable topological spaces, this one came to a conclusion. The first satisfactory characterization was due to Fox in [4]; this paper also includes more of the history of these related subjects. The author wishes to thank Ivan Reilly for bringing to his attention the key Fox and Wilson references and Peter Collins for several useful discussions.

We now state Fox's result. Some key definitions are given later.

Theorem 1.1. *A biotopological space* $\mathscr{X}_B=(X, \mathscr{T}^0, \mathscr{T}^1)$ *is quasimetrizable iff for each* $i \in 0, 1$, $\mathscr{T}^i$ *has a* σ-$\mathscr{T}^{1-i}$*-cushioned,* σ-$\mathscr{T}^i$*-cocushioned pairbase.*

The following definitions and notations differ from Fox's as indicated:

Definition 1.2. A *quasimetric* on X is a $q: X\times X \to [0, \infty)$ such that whenever $x, y, z \in X$, $q(x, z) \leqslant q(x, y)+q(y, z)$, and $q(x, x)=0$. Most authors, including Fox, follow Wilson in also assuming an axiom we call (c_1): $q(x, y)=0 \Rightarrow x=y$. We also call a quasimetric (c_0) if $q(x, y)+q(y, x)=0 \Rightarrow x=y$.

Given a quasimetric q: its *conjugate* is the quasimetric $q^*: X\times X\to[0,\infty)$ defined by $q^*(x, y)=q(y, x)$ (Fox uses the notation $\tilde{q}$ here), $B_r=\{(x, y): q(x, y)<r\}$, $N_r=\{(x, y): q(x, y)\leqslant r\}$, the *topology arising from* q, $\mathscr{T}(q)$, is defined by: $P\in\mathscr{T}(q) \Leftrightarrow$ for each $x\in P$ there is an $r>0$ such that $B_r(x)\subseteq P$ ($\Leftrightarrow$ if $x\in P$ there is an $r>0$ such that $N_r(x)\subseteq P$), the *bitopological space arising from* q is $\mathscr{X}_B(q)=(X, \mathscr{T}(q), \mathscr{T}(q^*))$. A topology $\mathscr{T}$ on X is *quasimetrizable* if there is a quasimetric q on X such that $\mathscr{T}=\mathscr{T}(q)$; a bitopological space $\mathscr{X}_B$ is *quasimetrizable* if there is a quasimetric q on X such that $\mathscr{X}_B=\mathscr{X}_B(q)$.

It is shown in [14] that $\mathscr{T}(q)$ is T_i iff q satisfies (c_i), for $i=0, 1$ (the proofs are straightforward). Thus each quasimetrization result given below holds in three forms: one with no separation assumptions, one assuming T_0, and one assuming T_1. Below, Definition 1.3 helps in defining pairbase, Definition 1.4 defines the remaining terms in Fox's result:

Definition 1.3. Given an $R\subseteq X\times 2^X$, $\mathscr{T}(R)$ is the topology $\{P$: if $x\in P$ then for some finite $F\subseteq R(x)$, $\bigcap F\subseteq P\}$, and R is *basic* if whenever $x\in P\in T(R)$ there is an $A\in R(x)$ such that $A\subseteq P$. (R is basic if whenever $B, C\in R(x)$ there is an $A\in R(x)$ such that $A\subseteq B\cap C$; the converse fails.)

Definition 1.4. A *set relation* on X is a relation on the power set of X, and such a relation G is *enclosing* if $(A, B)\in G\Rightarrow A\subseteq B$. For an enclosing G and a topology

$\mathcal{T}$ on X: $dG=\{(x, A)\colon x\in A\in \mathrm{Dom}(G)\}$, $rG=\{(x, B)\colon \exists(A, B)\in G,\ x\in A\}(=G\circ dG)$, the *topology arising from* G is $\mathcal{T}(dG)$ (notice that this is the topology generated by $\mathrm{Dom}(G)$). G is

- a *pairgenerator* (for $\mathcal{T}$) if $\mathcal{T}(dG)=\mathcal{T}(rG)(=\mathcal{T})$,
- a *pairbase* (for $\mathcal{T}$) if a pairgenerator (for $\mathcal{T}$) and dG, rG are basic,
- $\mathcal{T}$-*cushioned* if $\mathrm{Cl}(\bigcup \mathrm{Dom}(H))\subseteq \bigcup \mathrm{Rg}(H)$ whenever $H\subseteq G$,
- $\mathcal{T}$-*cocushioned* if $\bigcap \mathrm{Dom}(H)\subseteq \mathrm{Int}(\bigcap \mathrm{Rg}(H))$ whenever $H\subseteq G$,
- σ-$\mathcal{T}$-*(co)cushioned* if it is a countable union of $\mathcal{T}$-(co)cushioned sets, $\bigcup_{n\geqslant 0}G(n)$.

G is *σ-self-(co)cushioned* if it is σ-$\mathcal{T}(dG)$-(co)cushioned. In this case $G(n)$ is a *(self-)(co)cushioning* sequence (for G). $G(n)$ is *increasing* if $n<p\Rightarrow G(n)\subseteq G(p)$.

In his corresponding definitions, Fox did not require G to be enclosing, but cushioned or cocushioned set relations are enclosing, and this requirement simplifies our work below.

Lemma 1.5. (a) *If G is an enclosing set on X, then $\mathcal{T}(rG)\subseteq\mathcal{T}(dG)$.*

(b) *An enclosing G on X is a pairbase iff (A', B'), $(A'', B'')\in G$ and $x\in A'\cap A''\Rightarrow$ for some $(A, B)\in G$, $x\in A$ and $B\subseteq A'\cap A''$. It is a pairgenerator iff for each finite $F\subseteq G$, if $x\in\bigcap\mathrm{Dom}(F)$, there is a finite $F'\subseteq G$ such that $x\in\bigcap\mathrm{Dom}(F')$ and $\bigcap\mathrm{Rg}(F')\subseteq\bigcap\mathrm{Dom}(F)$.*

(c) *If G' and G'' are (co)cushioned, then so is $G\cup G'$.*

A sequence of set relations $G(n)$ on X is a self-cocushioning sequence for a pairbase G iff for each m, whenever $x\in\bigcap\mathrm{Dom}(H)$, $H\subseteq\bigcup_{n<m}G(n)$, there is an $(A, B)\in G$ such that $x\in A$ and $B\subseteq\bigcap\mathrm{Rg}(H)$.

(d) *If G', G'' are enclosing, we use the following notations*: $G'\odot G''=\{(A'\cap A'', B'\cup B'')\colon (A', B')\in G',\ (A'', B'')\in G''\}$, $G'\otimes G''=\{(A'\cap A'', B'\cap B'')\colon (A', B')\in G',\ (A'', B'')\in G''\}$, $G'\oplus G''=\{(A'\cup A'', B'\cup B'')\colon (A', B')\in G',\ (A'', B'')\in G''\}$. *If either G' or G'' is (co)cushioned, then so is $G'\odot G''$; if both are then so are $G'\otimes G''$ and $G'\oplus G''$.*

(e) *If G is a σ-$\mathcal{T}$-cocushioned, σ-$\mathcal{T}'$-cushioned pairgenerator, then there is an increasing sequence of enclosing set relations $\{H(n)\}$ such that $H=\bigcup_{n\geqslant 0}H(n)$ is a pairbase for $\mathcal{T}(dG)$, and each $H(n)$ is $\mathcal{T}$-cocushioned and $\mathcal{T}'$-cushioned.*

Proof. (a) If $x\in P\in\mathcal{T}(rG)$, then for some finite $F\subseteq G$, $x\in\bigcap\mathrm{Dom}(F)$ and $\bigcap\mathrm{Rg}(F)\subseteq P$; since G is enclosing, $x\in\bigcap\mathrm{Dom}(F)\subseteq P$, so $P\in\mathcal{T}(dG)$.

(b) We show the assertion about pairbases and leave pairgenerators to the reader. Let $x\in P\in\mathcal{T}(dG)$ and assume our condition; then for some finite $F\subseteq G$, $x\in\bigcap\mathrm{Dom}(F)\subseteq P$. By induction, there is an $(A, B)\in G$ such that $x\in A$ and $B\subseteq\bigcap\mathrm{Dom}(F)$, so $x\in A$, $B\subseteq P$. This shows that $\mathcal{T}(dG)\subseteq\mathcal{T}(rG)$, thus by (a) $\mathcal{T}(dG)=\mathcal{T}(rG)$; it also shows dG, rG to be basic. Conversely, if (A', B'), $(A'', B'')\in G$, then $A'\cap A''\in\mathcal{T}(dG)$, so if $x\in A'\cap A''$, rG is basic, and $\mathcal{T}(dG)=\mathcal{T}(rG)$ then for some $(A, B)\in G$, $x\in A$ and $B\subseteq A'\cap A''$.

(c) We show the cocushioned case; the other is similar. Thus suppose $H\subseteq G'\cup G''$, $x\in\bigcap\mathrm{Dom}(H)$. Let $H'=G'\cap H$, $H''=G''\cap H$; then $H=H'\cup H''$, so the following completes the proof: $\bigcap \mathrm{Dom}(H) = (\bigcap \mathrm{Dom}(H')) \cap (\bigcap \mathrm{Dom}(H'')) \subseteq \mathrm{Int}(\bigcap \mathrm{Rg}(H')) \cap \mathrm{Int}(\bigcap \mathrm{Rg}(H'')) = \mathrm{Int}((\bigcap \mathrm{Rg}(H')) \cap (\bigcap \mathrm{Rg}(H''))) = \mathrm{Int}(\bigcap \mathrm{Rg}(H))$.

For the other assertion here, if $G(n)$ is a self-cocushioning sequence for the pairbase G then $\bigcup_{n<m} G(n)$ is $\mathscr{T}(dG)$-cocushioned. Thus if $H \subseteq \bigcup_{n<m} G(n)$, $x \in \bigcap \mathrm{Dom}(H)$ then for some $(A, B) \in G$, $x \in A$ and $B \subseteq \bigcap \mathrm{Rg}(H)$. Conversely if this condition holds, the special cases in which H has two elements show G to be a pairbase and each $G(n)$ is $\mathscr{T}(dG)$-cocushioned because each $A \in \mathrm{Dom}(G)$ is in $\mathscr{T}(dG)$.

(d) We show the cushioned case for $\odot$. By the commutativity of $\odot$, it will do to assume G' is cushioned. If $H \subseteq G' \odot G''$ and $(A, B) \in H$, choose $(A', B') \in G'$, $(A'', B'') \in G''$ such that $(A, B) = (A' \cap A'', B' \cup B'')$ and let $H' = \{(A', B'): (A, B) \in H\}$; $\mathrm{Cl}(\bigcup \mathrm{Dom}(H)) \subseteq \mathrm{Cl}(\bigcup \mathrm{Dom}(H')) \subseteq \bigcup \mathrm{Rg}(H') \subseteq \bigcup \mathrm{Rg}(H)$. The remaining assertions are proved similarly.

(e) Suppose $G = \bigcup_{n \geqslant 0} G(n) = \bigcup_{n \geqslant 0} G'(n)$, where $G(n)$ is $\mathscr{T}$-cocushioned, $G'(n)$ is $\mathscr{T}'$-cushioned. Replacing $G(n)$ by $\bigcup_{m \leqslant n} G(m) \cup \{(X, X)\}$ if necessary (and using (c)), we can assume that the $G(n)$ and similarly, the $G'(n)$ contain (X, X) and are increasing. Let $H(0) = \emptyset$, $H(n+1) = (H(n) \cup (G(n) \odot G'(n))) \otimes (H(n) \cup (G(n) \odot G'(n)))$, $H = \bigcup_{n \in \omega} H(n)$. Since $(X, X) \in K \Rightarrow K \subseteq K \otimes K$, $\{H(n)\}$ is increasing; by a slight variant of the proof used to show this, if $F \subseteq H(n)$ has at most 2^k elements, then $(\bigcap \mathrm{Dom}(F), \bigcap \mathrm{Rg}(F)) \in H(n+k)$. Since $(X, X) \in G(n)$, $G'(n)$, we have $\mathrm{Dom}(G(n))$, $\mathrm{Dom}(G'(n)) \subseteq \mathrm{Dom}(G(n) \odot G'(n)) \subseteq \mathrm{Dom}(H(n+1)) \subseteq \mathscr{T}(dH)$. By induction $\mathrm{Dom}(H(n)) \subseteq \mathscr{T}(dG)$, thus $\mathscr{T}(dG) = \mathscr{T}(dH)$. Induction using (c), (d) shows that the $H(n)$ are $\mathscr{T}$-cocushioned and $\mathscr{T}'$-cushioned.

To see that H is a pairbase, let $x \in P \in \mathscr{T}(dH) = \mathscr{T}(dG)$; since $G(n)$, $G'(n)$ are increasing, find some n and some $F \subseteq G(n)$, $F' \subseteq G'(n)$, finite, with $x \in \bigcap \mathrm{Dom}(F)$, $\bigcap \mathrm{Dom}(F')$, $\bigcap \mathrm{Rg}(F)$, $\bigcap \mathrm{Rg}(F') \subseteq P$. Thus $F'' = F \odot F'$ is a finite subset of $H(n+1)$, so for some m, $(\bigcap \mathrm{Dom}(F''), \bigcap \mathrm{Rg}(F'')) \in H(m)$, $x \in (\bigcap \mathrm{Dom}(F)) \cap (\bigcap \mathrm{Dom}(F')) = \bigcap \mathrm{Dom}(F'')$, and by distributivity, $\bigcap \mathrm{Rg}(F'') = (\bigcap \mathrm{Rg}(F)) \cup (\bigcap \mathrm{Rg}(F')) \subseteq P$. □

There have been subsequent characterizations of quasimetrizable bitopological spaces (e.g., see Raghavan and Reilly [21]). Below we use Fox's result to obtain a topological characterization of quasimetrizable spaces.

2. From bitopologies to topologies

A straightforward method of showing the existence of a quasimetric from which a given topology arises is to actually construct one. Each quasimetric gives rise to two topologies, $\mathscr{T}(q)$ and $\mathscr{T}(q^*)$; thus the construction of a second topology is implicit in such a proof. Fox's result makes it reasonable to reverse the logic here: first construct the second topology in a way to give a bitopological space satisfying

Fox's conditions, then use his result. Thus the key question here is how we get the second topology:

In [15] we constructed two topologies from a single set S of subsets of X: $\mathscr{T}_S$ generated by S, and $\mathscr{T}_{S*}$ generated by the complements of the sets in S. This technique appears earlier in the proof in [17] that if $\mathscr{T}$ is a T_1 topology on X, then $(X, \mathscr{T}, \mathscr{D})$ is a pairwise completely regular, $\mathscr{D}$ the discrete topology on X. The bitopological spaces which so arise are the *pairwise* 0-*dimensional* spaces, (see [6,7,22]). The topology of computer graphics (see [10] or [12]), like other topologies on finite spaces, are *Alexandroff*: arbitrary intersections of open sets are open. A topology is clearly Alexandroff iff the closed sets form a topology, $\mathscr{C}$, and again $(X, \mathscr{T}, \mathscr{C})$ is pairwise 0-dimensional.

But bitopological spaces which arise from quasimetrics are merely *pairwise regular* (see [8]): if $x \in P \in \mathscr{T}^i$ then for some $Q \in \mathscr{T}^i$, $x \in Q$ and $\mathrm{Cl}^{1-i}Q \subseteq P$. The following allows application of the above idea of obtaining the second topology using complements:

Definition 2.1. The *conjugate* of an enclosing set relation G on X is $G^* = \{(X - B, X - A)\colon (A, B) \in G\}$.

The reader should check the motivating fact that the pairwise regular bitopological spaces are those for which there is a pairbase G for $\mathscr{T}$ such that G^* is a pairbase for $\mathscr{T}^*$ (in fact, $G = \{(A, \mathrm{Cl}^*(A))\colon A \in \mathscr{T}\} \cup \{(X - \mathrm{Cl}(B), X - B)\colon B \in \mathscr{T}^*\}$ works in the pairwise regular case).

Certainly $G = G^{**}$, and for a topology $\mathscr{T}$ on X if $H \subseteq G$, then $\bigcap \mathrm{Dom}(H^*) = X - \bigcup \mathrm{Rg}(H)$, and $X - \mathrm{Cl}(\bigcup \mathrm{Dom}(H)) = \mathrm{Int}(\bigcap \mathrm{Rg}(H^*))$. Thus, an enclosing G on X is $\mathscr{T}$-cocushioned iff G^* is $\mathscr{T}$-cushioned, and G (G^*) is σ-$\mathscr{T}$-cushioned (σ-$\mathscr{T}$-cocushioned) iff G^* (G) is σ-$\mathscr{T}$-cocushioned (σ-$\mathscr{T}$-cushioned).

A fact not used below is that, applying complementation to the characterization of self-cocushioning sequences of Lemma 1.5(c), G^* is a self-cocushioning sequence iff whenever $x \notin \bigcup \mathrm{Rg}(H)$, $H \subseteq \bigcup_{n<m} G(n)$, there is an $(A, B) \in G$ such that $x \notin B$ and $\bigcup \mathrm{Dom}(H) \subseteq A$.

Theorem 2.2. *Let $\mathscr{Z}$ be a (T_i) topological space $(X, \mathscr{T})$ or a bitopological space $\mathscr{X}_B$ (with each topology T_i). The following are equivalent*:

(a) *$\mathscr{Z}$ is quasimetrizable (with (c_i) quasimetric),*

(b) *there is an enclosing set relation G on X such that $\mathscr{Z}$ arises from G and both G and G^* are σ-self-cocushioned pairbases,*

(c) *there is an enclosing set relation G on X such that $\mathscr{Z}$ arises from G and both G and G^* are σ-self-cocushioned pairgenerators.*

Proof. The topological and bitopological cases are similar. For (a) $\Rightarrow$ (b): Given a quasimetric q, recall that for $r > 0$, $B_r[P] = \{z\colon$ for some $y \in P$, $q(y, z) < r\}$ and notice that $B_r[P] \subseteq Q \Leftrightarrow B_r^*[X - Q] \subseteq X - P$. Let $G(q, n) = \{(P, Q)\colon P \in \mathscr{T}(q),$

$X-Q\in\mathscr{T}(q^*)$, and $B_{1/n}[P]\subseteq Q\}$, and notice that $G(q,n)$ is $\mathscr{T}(q)$-cocushioned because if $H\subseteq G(q,n)$ then $\bigcap\text{Dom}(H)\subseteq B_r[\bigcap\text{Dom}(H)]\subseteq\text{Int}(\bigcap\text{Rg}[H])$. By duality, $G(q^*,n)$ is $\mathscr{T}(q^*)$-cocushioned and since $G(q^*,n)=G(q,n)^*$, $G(q,n)$ is $\mathscr{T}(q^*)$-cushioned. Let $G=\bigcup_{n\in\omega}G(q,n)$; G is then σ-$\mathscr{T}(q)$-cocushioned and G^* is σ-$\mathscr{T}(q^*)$-cocushioned. Further, G is a pairbase for $(X,\mathscr{T}(q),\mathscr{T}(q^*))$, for if $x\in P\in\mathscr{T}(q)$, there is some $r>0$ such that $B_r(x)\subseteq P$; choosing n such that $2/n<r$, we have $x\in B_{1/n}(x)$, $N_{2/n}(x)\subseteq P$, and $(B_{1/n}(x), N_{2/n}(x))\in G(q,n)$ (check (or see in [15]) that each $N_r(x)$ is $\mathscr{T}(q^*)$-closed); dually, the same holds for $\mathscr{T}(q^*)$. The last needed assertions, that G and G^* are σ-self-cocushioned, now result from the fact that $\mathscr{T}(G)=\mathscr{T}(q)$ and $\mathscr{T}(G^*)=\mathscr{T}(q^*)$.

Clearly, (b) ⇒ (c); further (c) ⇒ (b) by Lemma 1.5(e).

Proof of (b) ⇒ (a) using Fox's result (Theorem 1.1): From our hypotheses and the comments after Definition 2.1, it is clear (in both the topological and bitopological cases) that for the bitopological space $(X,\mathscr{T}(G),\mathscr{T}(G^*))$, G is a σ-$\mathscr{T}(G^*)$-cushioned, σ-$\mathscr{T}(G)$-cocushioned pairbase for $\mathscr{T}(G)$, and G^* is a σ-$\mathscr{T}(G)$-cushioned, σ-$\mathscr{T}(G^*)$-cocushioned pairbase for $\mathscr{T}(G^*)$. By Theorem 1.1 there is a quasimetric q such that $(X,\mathscr{T}(G),\mathscr{T}(G^*))=(X,\mathscr{T}(q),\mathscr{T}(q^*))$. □

Corollary 2.3. *A topology $\mathscr{T}$ is pseudometrizable (a T_0 topology $\mathscr{T}$ is metrizable) iff there is a self-conjugate enclosing set relation G from which $\mathscr{T}$ arises, and which additionally satisfies* (i) *and* (ii):

(i) *G is σ-self-cocushioned or σ-self-cushioned,*

(ii) *G is a pairbase or pairgenerator.*

Proof. By Theorem 2.2, $\mathscr{T}$ is pseudometrizable iff:

(α) there is an enclosing set relation H such that H and H^* are σ-self-cocushioned and $\mathscr{T}=\mathscr{T}(H)=\mathscr{T}(H^*)$.

Further, (α) surely holds if H is a self-conjugate, σ-self-cocushioned pairbase. But if (α), set $G(n)=H(n)\cup H(n)^*$, where $\{H(n)\}$ is an increasing cocushioning sequence for H. Then G is a self-conjugate pairbase which is both σ-self-cocushioned and σ-self-cushioned, and $\mathscr{T}$ arises from G. □

3. Local quasiuniformities

Since [4] is not easily accessible, we give an alternative proof of Theorem 2.2, (b) ⇒ (a) using local quasiuniformities.

Definition 3.1. A *local quasiuniformity* on X is a filter L of reflexive relations on X such that if $U\in L$, $x\in X$ then for some $V\in L$, $(V\circ V)(x)\subseteq U(x)$. A *quasiuniformity* is a local quasiuniformity also satisfying: if $U\in L$ then for some $V\in L$, $V\circ V\subseteq U$.

If G is an enclosing relation on X, then $N_G = \{(x, y)\colon x \in A,\ y \in X$ and $(A, B) \in G \Rightarrow y \in B\}$ (a reflexive relation on X).

For a filter L:

A *base* is a collection $B \subseteq L$ such that if $U \in L$ then $V \subseteq U$ for some $V \in B$.

The *inverse* of L is $L^{-1} = \{V^{-1}\colon V \in L\}$.

The *topology induced by* L is $\mathcal{T}[L] = \{P\colon x \in P \Rightarrow$ for some $V \in L,\ V(x) \subseteq P\}$ $(= \mathcal{T}(\{(x, V(x))\colon x \in X,\ V \in L\}))$.

Theorem 3.2. *A topology arises from a quasiuniformity with countable base iff it arises from a local quasiuniformity with countable base whose inverse is a local quasiuniformity.*

This is Theorem 7.15 of [3] (from [4]), and its proof is sketched on p. 162 of [3].

Applying Kelley's metrization lemma [8, p. 185], we have a result, stated in the T_1-case in [3]:

Theorem 3.3. *A topology which arises from a local quasiuniformity with countable base, whose inverse is a local quasiuniformity, is quasimetrizable.*

Lemma 3.4. *Let* $\{G(n)\colon n \in \omega\}$ *be an increasing cocushioning sequence for the* σ*-self-cocushioned pairbase* G. *Then the filter generated by* $\{N_{G(n)}\colon n \in \omega\}$ *is a local quasiuniformity* L *which gives rise to the topology* $\mathcal{T}(dG)$.

Proof. First notice that if $G \subseteq H$, G, H enclosing sets, then $N_H \subseteq N_G$; thus $L = \{V \subseteq X \times X\colon$ for some n, $N_{G(n)} \subseteq V\}$ is a filter of reflexive sets. Now suppose $P \in \mathcal{T}[L]$, $V \in L$, $x \in X$. For some n, $N_{G(n)} \subseteq V$, so $N_{G(n)}(x) \subseteq V(x) \subseteq P$. Since $G(n)$ is cocushioned, $N_{G(n)}(x)$ is a $\mathcal{T}(dG)$ neighborhood of x, so $P \in \mathcal{T}(dG)$. For the reverse inclusion, if $x \in P \in \mathcal{T}(dG)$, find m, $(A, B) \in G(m)$, such that $x \in A$ and $B \subseteq P$. But then $N_{G(m)}(x) \subseteq B \subseteq P$, so $P \in \mathcal{T}[L]$. To see that L is a local quasiuniformity, let $x \in X$, $V \in L$. Then for some n, $N_{G(n)} \subseteq V$; since $N_{G(n)}(x)$ is a neighborhood of x, we can find some m and $(A, B) \in G(m)$ such that $x \in A$ and $B \subseteq N_{G(n)}(x)$. But since A is also a neighborhood of x, there is a p and an $(A', B') \in G(p)$ such that $x' \in A'$ and $B' \subseteq A$. Since the $G(k)$ are increasing, we may also assume $m, n \leqslant p$. If $z \in N_{G(p)} \circ N_{G(p)}(x)$ then for some y, (x, y), $(y, z) \in N_{G(p)}$. But then $y \in B'$, so $y \in A$; thus $z \in N_{G(p)}(y) \subseteq B \subseteq N_{G(n)}(x) \subseteq V(x)$, completing the proof. □

Alternative proof of Theorem 2.2, (b) ⇒ (a): Suppose G is enclosing on X, $\mathcal{T}$ is the topology arising from G and both G and G^* are σ-self-cocushioned pairbases. By the comment following Definition 2.1, we write $G = \bigcup_{n \in \omega} G'(n) = \bigcup_{n \in \omega} G''(n)$, where the $G'(n)$, $G''(n)$ are increasing, each $G'(n)$ is $\mathcal{T}(dG)$-cocushioned, and each $G''(n)$ is $\mathcal{T}(dG^*)$-cushioned. By Lemma 1.5(e) there is an increasing pairbase $\{H(n)\}$ for $\mathcal{T}(dG)$ which is both $\mathcal{T}(dG)$-cocushioned and $\mathcal{T}(dG^*)$-cushioned. Similarly obtain $J(n)$ from G^*, and let $G(n) = H(n) \cup J(n)^*$.

$G(n)$ is increasing and $\mathscr{T}(dG)$-cocushioned by Lemma 1.5(c), and by the comment following Definition 2.1, $G(n)^*$ is $\mathscr{T}(dG^*)$-cocushioned. Thus the filters of subsets of $X \times X$ generated by $\{N_{G(n)}\}$ and $\{N_{G^*(n)}\}$ are local quasiuniformities with countable bases, and $\mathscr{T}(dG)$ arises from the first, $\mathscr{T}(dG^*)$ from the second. Our proof is completed by showing that for each enclosing set H, $N_{H^*} = N_H^{-1}$, so these two local quasiuniformities are inverse to each other. For this, simply note that if $(x, y) \in N_H$ and $(A, B) \in H^*$, $y \in A$, then $(X - B, X - A) \in H$ and $y \notin X - A$, so $x \notin X - B$, $x \in B$, thus $(y, x) \in N_{H^*}$.

4. Discussion and open questions

A key question about the above characterization is whether it is topological. In particular, three issues arise:

(a) it gives rise not to one topology, but to a bitopological space,

(b) it uses pairs of neighborhoods, rather than single ones, much like a marked ruler rather than unmarked straightedge,

(c) it explicitly uses duality.

I felt at one time that the first of these comments was definitive. However, the following, which is usually taken to be a purely topological characterization (see [3]), also gives rise to a bitopological space:

A topology arises from a quasimetric which is *nonarchimedean* ($\forall x, y, z$, $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$) iff it arises from a *$\sigma$-interior-preserving base* (a base which is a countable union of *interior-preserving collections*; sets with the property that arbitrary intersections from them are open).

Notice first that a topology arises from a σ-interior-preserving base iff it is *σ-Alexandroff*, that is, the join of a countable set of Alexandroff topologies. ([13], in which this characterization of σ-interior-preserving base was introduced, also discusses an open problem to which our characterization of quasimetrizable spaces may be applicable.) Thus $\mathscr{C} = \bigvee_{n \in \omega} \mathscr{C}_n$, where $\mathscr{C}_n$ is the collection of closed sets of $\mathscr{T}_n$ (a topology, since $\mathscr{T}_n$ is Alexandroff), is itself a σ-Alexandroff topology uniquely determined by that base.

Of course, this $\mathscr{C}$ is not uniquely determined by the fact that $\mathscr{T}$ is σ-Alexandroff, since $\mathscr{T}$ may be expressed many ways as a countable join of Alexandroff topologies, but similarly, neither is the $\mathscr{T}(G^*)$ of the proof of Theorem 2.2.

Issue (b) above is equally raised by the characterization of γ-spaces (in [2]) as those arising from a σ-cocushioned pairbase (with no dual condition). But issue (c) seems unique to the present characterization of quasimetrizability, and seems at the heart of bitopology. Whether our characterization of quasimetrizability is eventually seen as bitopological or topological, the following remains a key open question:

In light of the usefulness of the Bing and Nagata–Smirnov characterizations of metrizability, find an appropriate weakening of their conditions which characterizes quasimetrizability (i.e., a "Bing-style" characterization of quasimetrizability).

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