Pullback attractors of nonautonomous reaction–diffusion equations

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Abstract

In this paper, firstly we introduce the concept of norm-to-weak continuous cocycle in Banach space and give a technical method to verify this kind of continuity, then we obtain some abstract results for the existence of pullback attractors about this kind of cocycle, using the measure of noncompactness. As an application, we prove the existence of pullback attractors in $H^1_0$ of the cocycle associated with the solutions for some nonlinear nonautonomous reaction–diffusion equations. The attractor pullback attracts all bounded subsets of $H^1_0$ in the norm of $H^1_0$.

Keywords: Attractor; Measure of noncompactness; Pullback $\omega$-limit compact; Pullback condition (PC); Reaction–diffusion equations

1. Introduction

A defining characteristic of an autonomous dynamical system is its dependence on the time that has elapsed only, and not on the absolute time itself. Consequently limiting objects, such as attractors, actually exist for all time as invariant sets under the evolution of the autonomous system. For general nonautonomous systems, the absolute starting time is as important as the time elapsed since starting. With developments in nonautonomous and random dynamical systems, a new type of attractor, called a pullback attractor, was proposed and investigated [9,12,14,17].
Essentially, it consists of a parametrized family of nonempty compact subsets of the state space. Pullback attraction describing this attractor to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. Traditionally nonautonomous dynamical systems can often be formulated in terms of a cocycle mapping for the dynamics in the state space. If a cocycle mapping with respect to group $\theta$ is continuous, and $\theta$ is continuous, the nonautonomous dynamical systems can be reduced to semigroup by constructing skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such nonautonomous dynamical systems via the associated skew-product flow \([1,3–6,10,19,21]\). Recently, by using the technique of measure of noncompactness that is put back forward first by K. Kuratowski \([13]\), a new concept of pullback $\omega$-limit compact has been proposed about the continuous cocycle, and the necessary and sufficient conditions for the existence of the pullback attractors of the nonautonomous infinite-dimensional dynamical system are proved in Y.J. Wang, C.K. Zhong, S.F. Zhou \([22]\), this method is an extension of the corresponding one in the autonomous framework (see Q.F. Ma, S.H. Wang, C.K. Zhong \([16]\)).

In this paper, we first introduce the concept of norm-to-weak continuous cocycle in Banach space and give a technical method to verify this kind of continuity. We obtain some abstract results about the existence of pullback attractors in term of the concept of measure of noncompactness, and apply our new method to nonautonomous reaction–diffusion equations.

Throughout this paper we use the following notation: $E$ is a Banach space with norm $\| \cdot \|$ and the metric is $d$. $B(E)$ is the set of all bounded subsets of $E$. Let $X, Y \subseteq E$; denote by $d_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$ the Hausdorff semidistance between $X$ and $Y$ and by $N(X, \varepsilon)$ the $\varepsilon$-neighborhood of $X$. Let $E_1, E_2$ be Banach spaces, $E_1 \hookrightarrow E_2$ means that $E_1$ is embedded in $E_2$. $\mathbb{R}_\tau = [\tau, +\infty)$ and $\mathbb{R}_+ = \mathbb{R}_0$. $\to$ means the convergence in strong topology and $\rightharpoonup$ means the convergence in weak topology.

2. Existence of pullback attractors

2.1. Preliminaries

Let $(E, d)$ be a complete metric space, $(P, \rho)$ be a metric space which will be called the parameter space, and let $T$, the time set, be $\mathbb{R}_+$. $\theta : \mathbb{R} \times P \to P$ is a mapping, $\theta_t = \theta(t, \cdot) : P \to P$ form a group, that is, $\theta$ satisfies

\[
\theta_{t+t'} = \theta_t \cdot \theta_{t'}, \quad \forall t, t' \in \mathbb{R},
\]

\[
\theta_0 = \text{Id}.
\]

**Definition 2.1.** A mapping $\phi : \mathbb{R}_+ \times P \times E \to E$ is said to be a cocycle on $E$ with respect to group $\theta$, if

1. $\phi(0, p, x) = x, \forall (p, x) \in P \times E$;
2. $\phi(t + \tau, p, x) = \phi(t, \theta_\tau(p), \phi(\tau, p, x)), \forall t, \tau \in \mathbb{R}_+, (p, x) \in P \times E$.

If $\phi : \mathbb{R}_+ \times P \times E \to E$ is continuous, $\phi$ is called a continuous cocycle on $E$ with respect to $\theta$. The mapping $\pi : \mathbb{R}_+ \times P \times E \to P \times E$ defined by

\[
\pi(t, p, x) := (\theta_t(p), \phi(t, p, x)) \quad \text{for all } t \in \mathbb{R}, (p, x) \in P \times E,
\]

forms a semigroup on $P \times E$ and is called a skew-product flow.
Definition 2.2. A family $\mathcal{A} = \{A_p\}_{p \in P}$ of nonempty compact sets of $E$ is called a pullback attractor of the cocycle $\phi$ if it is $\phi$-invariant, that is,

$$\phi(t, p, A_p) = A_{\theta_t(p)} \text{ for all } t \in \mathbb{R}_+, \ p \in P,$$

and pullback attracting, that is,

$$\lim_{t \to +\infty} d_H(\phi(t, \theta_{-t}(p), B), A_p) = 0 \text{ for all } B \in B(E), \ p \in P.$$ 

Theorem 2.1. Let $\phi$ be a continuous cocycle on $E$ with respect to a group $\theta$ of continuous mappings on $P$ and let $\pi = (\theta_t, \phi)$ be the corresponding skew-product flow on $P \times E$. In addition, suppose that there is a nonempty compact subset $B_0$ of $E$ and for every $B \in B(E)$ there exists a $T(B) \in \mathbb{R}_+$, which is independent of $p \in P$, such that

$$\phi(t, p, B) \subset B_0 \text{ for all } t > T(B).$$

Then

1. there exists a unique pullback attractor $\mathcal{A} = \{A_p\}_{p \in P}$ of the cocycle $\phi$ on $E$, where

$$A_p = \bigcap_{\tau \in \mathbb{R}_+} \bigcup_{t > \tau} \phi(t, \theta_{-t}(p), B_0);$$

2. there exists a global compact attractor $\hat{A}$ of the autonomous semiflows $\phi$ on $P \times E$, where

$$\hat{A} = \bigcap_{\tau \in \mathbb{R}_+} \bigcup_{t > \tau} \pi(t, P \times B_0);$$

3. assertions (1) and (2) above are equivalent, and

$$\hat{A} = \bigcup_{p \in P} \{p\} \times A_p.$$

See Crauel and Flandoli [8] and Schmalfuß [18] for the proof of assertion (1) and Cheban and Fakeeh [6] and Hale [11] for the proof of assertion (2). Assertion (3) has been proved by Cheban [7].

Let $B \in B(E)$. Its Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B) = \inf\{\delta \mid B \text{ admits a finite cover by set of diameter } \leq \delta\}. \quad (2.1)$$

It has the following properties (see Hale [11], Sell and You [20]).

Lemma 2.1. Let $B, B_1, B_2 \in B(E)$, then

1. $\alpha(B) = 0 \iff \alpha(N(B, \varepsilon)) \leq 2\varepsilon \iff \overline{B}$ is compact;
2. $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;
3. $\alpha(B_1) \leq \alpha(B_2)$ whenever $B_1 \subset B_2$;
4. $\alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;
5. $\alpha(\overline{B}) = \alpha(B)$;
6. if $B$ is a ball of radius $\varepsilon$ then $\alpha(B) \leq 2\varepsilon$.

Lemma 2.2. Let $\cdots \supset F_n \supset F_{n+1} \supset \cdots$ be a sequence of nonempty closed subsets of $E$ such that $\alpha(F_n) \to 0$, as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.
2.2. The existence of pullback attractor for norm-to-weak continuous cocycle

We will characterize the existence of pullback attractor for a cocycle in term of the concept of measure of noncompactness.

**Definition 2.3.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. We say that $\phi$ is a norm-to-weak continuous cocycle on $E$ if $\phi$ satisfies

1. $\phi(0, p, x) = x, \forall (p, x) \in P \times E$;
2. $\phi(t + \tau, p, x) = \phi(t, \theta_\tau(p), \phi(\tau, p, x))$;
3. $\phi(t, p, x_n) \rightharpoonup \phi(t, p, x)$, if $x_n \rightarrow x$ in $E$, $\forall t \in R_+, p \in P$.

**Definition 2.4.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. A set $B_0 \subset E$ is said to be uniformly absorbing set for $\phi$, if for any $B \in B(E)$ there exists $T_0 = T_0(B) \in R_+$ such that $\phi(t, p, B) \subset B_0$ for all $t \geq T_0, p \in P$.

**Definition 2.5.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. $\phi$ is said to be pullback $\omega$-limit compact if for any $B \in B(E), p \in P$,

$$\lim_{t \rightarrow +\infty} \alpha \left( \bigcup_{s \geq 0} \phi \left( t, \theta_{-t}(p), B \right) \right) = 0. \quad (2.2)$$

**Definition 2.6.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. Define the pullback $\omega$-limit set $\omega_p(B)$ of $B$ by the following:

$$\omega_p(B) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi \left( t, \theta_{-t}(p), B \right). \quad (2.3)$$

**Remark 2.1.** It is easy to see that $y \in \omega_p(B)$ if and only if there are sequences $\{x_n\} \subset B$, $\{t_n\} \subset R_+, t_n \rightarrow \infty$, such that $\phi(t_n, \theta_{-t_n}(p), x_n) \rightarrow y(n \rightarrow \infty)$.

**Lemma 2.3.** If a cocycle on $E$ with respect to group $\theta$ $\phi$ is pullback $\omega$-limit compact, then for any $\{x_n\} \subset B \in B(E), p \in P$, $\{t_n\} \subset R_+, t_n \rightarrow +\infty$ as $n \rightarrow \infty$, there exists a convergent subsequence of $\{\phi(t_n, \theta_{-t_n}(p), x_n)\}$ whose limit lies in $\omega_p(B)$.

**Proof.** For any $\varepsilon > 0$, it derives from Definition 2.5 and Lemma 2.1 that for a sufficiently large $N_0$,

$$\alpha \left\{ \phi(t_n, \theta_{-t_n}(p), x_n) \right\} = \alpha \left\{ \phi(t_n, \theta_{-t_n}(p), x_n) \mid n \geq N_0 \right\} \leq \varepsilon.$$  

Let $\varepsilon \rightarrow 0$, then by (1) of Lemma 2.1, $\{\phi(t_n, \theta_{-t_n}(p), x_n)\}$ is precompact. Remark 2.1 informs limit of the subsequence lies in $\omega_p(B)$. \quad $\square$

**Theorem 2.2.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. If $\phi$ is norm-to-weak continuous and possesses a uniformly absorbing set $B_0$, then $\phi$ possesses a pullback attractor $A = \{A_p\} \subset P$, satisfying

$$A_p = \omega_p(B_0), \quad \forall p \in P,$$

if and only if it is pullback $\omega$-limit compact.
Proof. The proof of the necessity is similar to that of Theorem 3.2 in [22], so we omit it here.
Now, we only need to prove the sufficiency. Set
\[ A_p = \omega_p(B_0) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi(t, \theta_{-t}(p), B_0) = \bigcap_{n=1}^{\infty} \bigcup_{t \geq n} \phi(t, \theta_{-t}(p), B_0). \]
Since \( \phi \) is pullback \( \omega \)-limit compact and \( B_0 \) is bounded, for any \( \varepsilon > 0 \), \( p \in P \), there exists \( t_\varepsilon > 0 \) such that
\[ \alpha \left( \bigcup_{t \geq t_\varepsilon} \phi(t, \theta_{-t}(p), B_0) \right) \leq \varepsilon. \]
Take \( \varepsilon = \frac{1}{n}, n = 1, 2, \ldots \), we find a sequence \( \{t_n\}, 0 \leq t_1 < t_2 < \cdots \), and \( t_n \to +\infty \) such that
\[ \alpha \left( \bigcup_{t \geq t_n} \phi(t, \theta_{-t}(p), B_0) \right) \leq \frac{1}{n}, \quad n = 1, 2, \ldots. \]
Using property of the measure of noncompactness, we obtain
\[ \alpha \left( \bigcup_{t \geq t_n} \phi(t, \theta_{-t}(p), B_0) \right) \leq \frac{1}{n}, \quad n = 1, 2, \ldots. \]
By Lemma 2.2, we know that \( \bigcap_{n=1}^{\infty} \bigcup_{t \geq t_n} \phi(t, \theta_{-t}(p), B_0) \) is a nonempty compact, is the pullback \( \omega \)-limit set of \( B_0 \). Therefore \( A_p \) is compact.
We now prove \( \{A_p\}_{p \in P} \) is \( \phi \)-invariant. In fact, if \( y \in \phi(t, p, A_p) \), then \( y = \phi(t, p, x) \), for some \( x \in A_p \). Hence there exist sequences \( \{x_n\} \subset B_0 \) and \( \{t_n\} \subset R_+, t_n \to +\infty \) such that \( \phi(t_n, \theta_{-t_n}(p), x_n) \to x \). Since \( \phi \) is norm-to-weak continuous, we have
\[ \phi(t_n, \phi(t_n, \theta_{-t_n}(p), x_n)) \to \phi(t, p, x) = y. \]
Using the property of the cocycle, we have
\[ \phi(t_n, \phi(t_n, \theta_{-t_n}(p), x_n)) = \phi(t + t_n, \theta_{-t_n}(p), x_n) \]
\[ = \phi(t + t_n, \theta_{-t_n}(-\theta_{-t_n}(\theta_{t_n}(p)), x_n)). \]
By Lemma 2.3, there exists a convergent subsequence \( \phi(t + t_{n_j}, \theta_{-t_{n_j}}(-\theta_{-t_{n_j}}(\theta_{t_{n_j}}(p)), x_{n_j})) \to y' \in \omega_{\theta_{t_n}(p)}(B_0) \). Obviously, \( y' = y \). Namely,
\[ \phi(t + t_{n_j}, \theta_{-t_{n_j}}(-\theta_{-t_{n_j}}(\theta_{t_{n_j}}(p)), x_{n_j})) \to y. \]
By Remark 2.1, we know that
\[ y = \phi(t, p, x) \in A_{\theta_{t_n}(p)}, \quad (2.4) \]
which implies that
\[ \phi(t, p, A_p) \subset A_{\theta_{t_n}(p)}. \quad (2.5) \]
Conversely, if \( y \in A_{\theta_{t_n}(p)} \), by Remark 2.1, there exist sequences \( \{y_n\} \subset B_0 \), \( t_n \to +\infty \) such that \( \phi(t_n, \theta_{-t_n}(\theta_{t_n}(p)), y_n) \to y \) \((n \to +\infty)\). Since \( \phi \) is pullback \( \omega \)-limit compact, using Lemma 2.3, we know \( \{\phi(t_n - t, \theta_{-t_n + t}(p), y_n)\} \) has a subsequence \( \{\phi(t_{n_j} - t, \theta_{-t_{n_j} + t}(p), y_{n_j})\} \) which converges to some point \( x \) in \( E \), that is,
\[ \phi(t_{n_j} - t, \theta_{-t_{n_j} + t}(p), y_{n_j}) \to x. \]
which induces $x \in A_p$. Using the norm-to-weak continuity of the cocycle $\phi$ again, we obtain

$$\phi\left(t, p, \phi\left(t_n - t, \theta_{-t} \theta_{t_n + t}(p), y_n\right)\right) \rightharpoonup \phi(t, p, x).$$

Using property of the cocycle $\phi$, we find

$$\phi\left(t, p, \phi\left(t_n - t, \theta_{-t} \theta_{t_n + t}(p), y_n\right)\right) = \phi\left(t_n, \theta_{-t} \theta_{t_n + t}(p), y_n\right) = \phi\left(t_n, \theta_{-t} \theta_{t_n}\left(\theta_t(p)\right), y_n\right) \rightarrow y.$$

Hence, $y = \phi(t, p, x)$, which implies that $A_{\theta_t}(p) \subset \phi(t, p, A_p), \quad \forall t \geq 0$.

Combining with (2.5), we have

$$\phi(t, p, A_p) = A_{\theta_t}(p) \quad \text{for all } t \in R_+, \quad p \in P.$$

At last, we prove that

$$\lim_{t \to +\infty} d_H\left(\phi\left(t, \theta_{-t} \theta_{t_n}(p), B\right), A_p\right) = 0,$$

for any $B \in B(E)$.

We argue it by contradiction, assume that there exists $C \in B(E)$, such that

$$d_H\left(\phi\left(t, \theta_{-t} \theta_{t_n}(p), C\right), A_p\right) \nrightarrow 0 \quad (t \to +\infty).$$

Namely, there exist $\varepsilon_0 > 0, \{x_n\} \subset C$ and $t_n \to \infty$ ($n \to \infty$) such that

$$d_H\left(\phi\left(t_n, \theta_{-t_n}(p), x_n\right), A_p\right) \geq \frac{\varepsilon_0}{2}.$$  \hspace{1cm} (2.6)

Since $B_0$ is uniformly absorbing set, there exists $T_0 > 0$ such that

$$\phi(t, p, C) \subset B_0 \quad \forall t \geq T_0, \quad p \in P.$$  \hspace{1cm} (2.7)

We know $\{\phi(t_n, \theta_{-t_n}(p), x_n)\}$ is precompact, then there exists a subsequence $\{\phi(t_n, \theta_{-t_n}(p), x_n)\}$ which converges to some point $x_0 \in E$, that is,

$$x_0 = \lim_{n \to \infty} \phi(t_n, \theta_{-t_n}(p), x_n) = \lim_{n \to \infty} \phi\left(t_n - T_0, \theta_{-t_n + T_0}(p), \phi\left(T_0, \theta_{-t_n}(p), x_n\right)\right).$$

Using (2.7), $\phi(T_0, \theta_{-t_n}(p), x_n) \subset B_0$, hence $x_0 \in A_p = \omega_p(B_0)$. By (2.6), we know

$$d_H(x_0, A_p) \geq \frac{\varepsilon_0}{2} > 0,$$

which contradicts with $x_0 \in A_p$. The proof is complete. \hfill \Box

**Definition 2.7.** Let $\phi$ be a cocycle on $E$ with respect to group $\theta$. A cocycle $\phi$ is said to be satisfying pullback condition (PC) if for any $p \in P, \quad B \in B(E)$ and $\varepsilon > 0$, there exist $t_0 = t_0(p, B, \varepsilon) \geq 0$ and a finite dimensional subspace $E_1$ of $E$ such that

1. $P(\bigcup_{t \geq t_0} \phi(t, \theta_{-t}(p), B))$ is bounded; and
2. $\|(I - P)(\bigcup_{t \geq t_0} \phi(t, \theta_{-t}(p), x))\| \leq \varepsilon, \quad \forall x \in B,$

where $P : E \to E_1$ is a bounded projector.
Theorem 2.3. Let $E$ be a Banach space and let $\phi$ be a cocycle on $E$ with respect to group $\theta$. If cocycle $\phi$ satisfies pullback condition (PC), then $\phi$ is pullback $\omega$-limit compact. Moreover, let $E$ is a uniformly convex Banach space, then $\phi$ is pullback $\omega$-limit compact if and only if pullback condition (PC) holds true.

See Y.J. Wang, C.K. Zhong, S.F. Zhou [22] for the proof of the theorem, and the theorem will be used in our consideration.

Now, we introduce a new method to verify that a cocycle is norm-to-weak continuous.

Theorem 2.4. Let $X, Y$ be two Banach spaces, $X^*, Y^*$ be respectively their dual spaces. $X$ is dense in $Y$, the injection $i: X \rightarrow Y$ is continuous and its adjoint $i^*: Y^* \rightarrow X^*$ is dense, and $\phi$ is a norm-to-weak continuous cocycle on $Y$. Then $\phi$ is a norm-to-weak continuous cocycle on $X$ if and only if for $p \in P$, $t \in R_+$, $\phi(t, p, x)$ maps the compact set of $X$ to be a bounded set of $X$.

Proof. If $\phi(t, p, x)$ is a norm-to-weak continuous cocycle on $X$, $B$ is compact subset of $X$, then $\phi(t, p, B)$ is a weakly compact set. Namely $\phi(t, p, B)$ is bounded in $X$. The necessity is proved.

Now, we prove the sufficiency. For any $p \in P$, $t \in R_+$, let $x_n \rightarrow x$ in $X$ as $n \rightarrow \infty$, then we need to prove that for any given $x^* \in X^*$,

$$\langle x^*, \phi(t, p, x_n) - \phi(t, p, x) \rangle_{X^*} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$  

In fact, $x_n \rightarrow x$ in $X$, $\{x_n\} \cup \{x\}$ is compact set, then $\{\phi(t, p, x_n)\}$ is bounded in $X$, that is,

$$\|\phi(t, p, x_n) - \phi(t, p, x)\|_X \leq M, \quad \forall n \in N,$$

where $M$ is a constant.

From the assumption that $i^*: Y^* \rightarrow X^*$ is dense, we know the for any $\varepsilon \geq 0$ and $x^* \in X^*$, there exists $y^*_\varepsilon \in Y^*$, such that

$$\|i^*(y^*_\varepsilon) - x^*\|_{X^*} \leq \frac{\varepsilon}{2M}.\tag{2.9}$$

Since $\phi(t, p, x)$ is norm-to-weak continuous in $Y$, for $y^*_\varepsilon$ given above, there exists $N_0 > 0$, such that for any $n \geq N_0$, we have

$$\|\langle y^*_\varepsilon, i(\phi(t, p, x_n) - \phi(t, p, x))\rangle_{Y^*}\| < \frac{\varepsilon}{2}.\tag{2.10}$$

So combining (2.8)–(2.10), we have for any $n \geq N_0$

$$\|\langle x^*, \phi(t, p, x_n) - \phi(t, p, x)\rangle_{X^*}\| \leq \|i^*(y^*_\varepsilon) - x^*, \phi(t, p, x_n) - \phi(t, p, x)\|_{X^*} \|i^*(y^*_\varepsilon), \phi(t, p, x_n) - \phi(t, p, x)\|_{X^*}$$

$$\leq \|i^*(y^*_\varepsilon) - x^*\|_{X^*} \|\phi(t, p, x_n) - \phi(t, p, x)\|_X$$

$$\leq \|\langle y^*_\varepsilon, i(\phi(t, p, x_n) - \phi(t, p, x))\rangle_{Y^*}\| \leq \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} \leq \varepsilon.$$ 

Therefore the proof is complete.  \(\square\)
3. Attractor of nonautonomous reaction–diffusion equation

3.1. Setting of the problem

Consider now the nonautonomous partial differential equation:

\[
\frac{\partial u}{\partial t} - \Delta u + f(u) = g(t),
\]

\[
u|_{\partial \Omega} = 0,
\]

\[
u(0) = u_0.
\]

(3.1)

where \(f \in C^1(\mathbb{R}, \mathbb{R})\), \(g(\cdot) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))\), \(\Omega\) is a bounded open subset of \(\mathbb{R}^n\) and there exist \(p \geq 2\), \(c_i > 0, i = 1, \ldots, 5\), such that

\[
C_1|u|^p - C_2 \leq f(u)u \leq C_3|u|^p - C_4,
\]

(3.2)

\[
f_u(u) \geq -C_5,
\]

(3.3)

for all \(u, t \in \mathbb{R}\).

Denote \(H = L^2(\Omega)\) with norm \(|\cdot|\) and scalar product \((\cdot, \cdot)\), \(V = H^1_0(\Omega)\) with norm \(\|\cdot\|\) and scalar product \((\cdot, \cdot)\). For a norm in another space \(X\) we shall use the notion \(\|\cdot\|_X\).

Theorem 3.1. For any \(T \in \mathbb{R}\), \(T > 0\), \(u_0 \in L^2(\Omega)\) there exists a unique solution \(u(\cdot) \in C([0, T], H) \cap L^2(0, T; V) \cap L^p(0, T; L^p(\Omega))\). Moreover, for all \(u_\tau, v_\tau \in L^2(\Omega)\), \(t \in [0, T]\) it holds

\[
|u(t) - v(t)| \leq \exp(C_5t)|u_\tau - v_\tau|.
\]

(3.4)

Proof. The existence of solution for any \(u_0 \in L^2(\Omega)\) was proved in [2, Theorem 2.1]. The unique property and (3.4) can be obtained exactly in the same way as in [15, Theorem 1.1] or [2, Theorem 3.1].

Lemma 3.1. Let \(f \in C^1(\mathbb{R}, \mathbb{R})\) and \(f(0) = 0\), then for every \(u \in V \cap H^2\) the following inequality holds:

\[
\int_\Omega (f(u(x)), \Delta u) dx \leq C_5\|u\|^2.
\]

Proof. Since \(f \in C^1(\mathbb{R}, \mathbb{R})\), using this assumption, we can integrate by parts. Since \(f(u(x))|_{\partial \Omega} = 0\), we obtain

\[
\int_\Omega f(u(x))\Delta u(x) dx = \int_\Omega \sum_{i=1}^n f(u(x)) \frac{\partial^2 u}{\partial x_i^2} dx
\]

\[
= \sum_{i=1}^n - \int_\Omega f_u(u) \left(\frac{\partial u}{\partial x_i}\right)^2 dx
\]

\[
\leq C_5 \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx
\]

\[
= C_5\|u\|^2;
\]

we have used inequality \(f_u(u) \geq -C_5\).
Theorem 3.2. Suppose that the function \( g(t) \) is translation bounded in \( L_{\text{loc}}^2(\mathbb{R}; H) \), that is,

\[
|g|^2_b = \sup_{h \in \mathbb{R}} \int_h^{h+1} |g(s)|^2 \, ds < +\infty.
\]

Let \( u(x, t) \in L^p_{\text{loc}}(\mathbb{R}_+; L^p(\Omega)) \cap L_{\text{loc}}^2(\mathbb{R}_+; V) \) be a weak solution of (3.1), then for all \( t \geq \tau \), the following estimates hold:

\[
\|u(t)\|_2 \leq \|u(\tau)\|_2 e^{-\lambda(t-\tau)} + R_1^2;
\]

\[
R_1^2 = 2C_2|\Omega|\lambda^{-1} + \lambda^{-1}(1+\lambda^{-1})|g|^2_b;
\]

\[
\int_\tau^t \|u(s)\|^2 e^{\lambda(s-\tau)} \, ds \leq (1+\lambda(t-\tau))\|u(\tau)\|^2 + 2R_1^2 e^{\lambda(t-\tau)};
\]

\( \lambda \) is the first eigenvalue of \(-\Delta\) with zero boundary condition.

Proof. Weak solution \( u(x, t) \) satisfies:

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|f(u(t))\|^2 = (g(t), u(t)).
\]

Using (3.2), we have

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|u(t)\|^2 \leq \int_\Omega C_2 - C_1|u|^p \, dx + \int_\Omega (g(t), u(t)) \, dx.
\]

Using Young’s inequality, and the Poincaré inequality,

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \lambda\|u(t)\|^2 \leq 2C_2|\Omega| + \frac{1}{2\lambda} |g(t)|^2 + \frac{\lambda}{2} |u(t)|^2,
\]

\[
\frac{d}{dt} \|u(t)\|^2 + \lambda\|u(t)\|^2 \leq 2C_2|\Omega| + \frac{1}{\lambda} |g(t)|^2.
\]

Applying Gronwall’s lemma, we obtain

\[
\|u(t)\|^2 \leq \|u(\tau)\|^2 e^{-\lambda(t-\tau)} + \int_\tau^t e^{-\lambda(t-s)} \left( 2C_2|\Omega| + \frac{1}{\lambda} |g(s)|^2 \right) ds
\]

\[
\leq \|u(\tau)\|^2 e^{-\lambda(t-\tau)} + \frac{2}{\lambda} C_2|\Omega| \left( 1 - e^{-\lambda(t-\tau)} \right) + \int_\tau^t e^{-\lambda(t-s)} \frac{1}{\lambda} |g(s)|^2 \, ds
\]

\[
\leq \|u(\tau)\|^2 e^{-\lambda(t-\tau)} + \frac{2}{\lambda} C_2|\Omega| \left( 1 - e^{-\lambda(t-\tau)} \right)
\]

\[
+ \int_{t-1}^t e^{-\lambda(t-s)} \frac{1}{\lambda} |g(s)|^2 \, ds + \int_{t-2}^{t-1} e^{-\lambda(t-s)} \frac{1}{\lambda} |g(s)|^2 \, ds + \cdots
\]
\[
\leq |u(\tau)|^2 e^{-\lambda(t-\tau)} + \frac{2}{\lambda} C_2 |\Omega| (1 - e^{-\lambda(t-\tau)}) \\
+ \frac{1}{\lambda} (1 + e^{-\lambda} + e^{-2\lambda} + \cdots) \sup_{t \in \mathbb{R}} \int_t^{t+1} |g(s)|^2 \, ds \\
\leq |u(\tau)|^2 e^{-\lambda(t-\tau)} + \lambda^{-1} (1 - e^{-\lambda})^{-1} |g|^2 + 2C_2 |\Omega| \lambda^{-1} \\
\leq |u(\tau)|^2 e^{-\lambda(t-\tau)} + \lambda^{-1} (1 + \lambda^{-1}) |g|^2 + 2C_2 |\Omega| \lambda^{-1}.
\]

Inequality (3.5) is proved.

Let us verify inequality (3.6). Multiplying (3.5) by $\lambda e^{\lambda(t-\tau)}$ and integrating, we obtain

\[
\lambda \int_{\tau}^{t} |u(s)|^2 e^{\lambda(s-\tau)} \, ds \leq \lambda(t-\tau) |u(\tau)|^2 + R_2^2 e^{\lambda(t-\tau)}.
\]  

(3.8)

Inequality (3.7) implies that

\[
\frac{d}{dt} \left( |u(t)|^2 e^{\lambda(t-\tau)} + \|u(t)\|^2 e^{\lambda(t-\tau)} \right) \leq 2C_2 |\Omega| e^{\lambda(t-\tau)} + \lambda |u(t)|^2 e^{\lambda(t-\tau)} \\
+ \lambda^{-1} |g(t)|^2 e^{\lambda(t-\tau)},
\]

therefore using (3.8), we find

\[
|u(t)|^2 e^{\lambda(t-\tau)} + \int_{\tau}^{t} \|u(s)\|^2 e^{\lambda(s-\tau)} \, ds \\
\leq |u(\tau)|^2 + \int_{\tau}^{t} 2C_2 |\Omega| e^{\lambda(s-\tau)} \, ds + \lambda \int_{\tau}^{t} |u(s)|^2 e^{\lambda(s-\tau)} \, ds + \int_{\tau}^{t} \lambda^{-1} |g(t)|^2 e^{\lambda(s-\tau)} \, ds \\
\leq |u(\tau)|^2 + 2\lambda^{-1} C_2 |\Omega| e^{\lambda(t-\tau)} + \lambda^{-1} e^{\lambda(t-\tau)} |g|^2 + (1 + \lambda^{-1})^{-1} \\
+ \lambda(t-\tau) |u(\tau)|^2 + R_2^2 e^{\lambda(t-\tau)} \\
\leq (1 + \lambda(t-\tau)) |u(\tau)|^2 + 2R_2^2 e^{\lambda(t-\tau)}.
\]

The proof is completed. \(\square\)

**Theorem 3.3.** Suppose that the function $g(t)$ is translation bounded in $L^2_{\text{loc}}(\mathbb{R}; H)$, and $f(u)$ satisfies conditions (3.2), (3.3). Then for every weak solution $u(x, t) \in L^p_{\text{loc}}(\mathbb{R}^+; L^p(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}^+; V)$ of (3.1), the following inequality holds for $t > \tau$:

\[
(t-\tau) \|u(t)\|^2 \leq ((t-\tau)^2 + (t-\tau) + 1) C |u(\tau)|^2 e^{-\lambda(t-\tau)} + (1 + t-\tau) R_4^2,
\]  

(3.9)

where $R_4$ is monotone function of $|g|^2$.

**Proof.** We multiply (3.1) by $-\Delta u(t)$ and integrate over $x \in \Omega$. We obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 \leq \left( f(u(t)), \Delta u(t) \right) - (g(t), \Delta u(t)).
\]  

(3.10)
Without loss of generality, we can assume that $f(0) = 0$. Otherwise, we can replace $f(u)$ and $g(x, t)$ by $\tilde{f}(u) = f(u) - f(0)$ and $\tilde{g}(x, t) = g(x, t) - f(0)$, respectively. The functions $\tilde{f}$ and $\tilde{g}$ satisfy the same condition.

Using Young’s inequality,

$$-(g(t), \Delta u(t)) \leq \frac{1}{2} |\Delta u(t)|^2 + \frac{1}{2} |g(t)|^2.$$ 

Therefore by (3.10) and Lemma 3.1, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 \leq C_5 \|u(t)\|^2 + \frac{1}{2} |\Delta u(t)|^2 + \frac{1}{2} |g(t)|^2,$$

and using the Poincaré inequality, we have

$$\frac{d}{dt} \|u(t)\|^2 + \lambda \|u(t)\|^2 \leq 2C_5 \|u(t)\|^2 + |g(t)|^2.$$

Multiplying by $(t - \tau)e^{\lambda(t-\tau)}$, we find

$$\frac{d}{dt} \left( (t - \tau)e^{\lambda(t-\tau)} \|u(t)\|^2 \right) \leq \|u(t)\|e^{\lambda(t-\tau)} + 2C_5 \|u(t)\|^2 (t - \tau)e^{\lambda(t-\tau)}$$

$$+ (t - \tau)e^{\lambda(t-\tau)}|g(t)|^2$$

$$\leq (2C_5(t - \tau) + 1) \|u(t)\|^2 e^{\lambda(t-\tau)} + (t - \tau)e^{\lambda(t-\tau)}|g(t)|^2,$$

integrating from $\tau$ to $t$ ($t > \tau$), we obtain

$$(t - \tau)e^{\lambda(t-\tau)} \|u(t)\|^2 \leq (2C_5(t - \tau) + 1) \int_\tau^t \|u(s)\|^2 e^{\lambda(t-\tau)} ds$$

$$+ (t - \tau)|g|^2 (1 - e^{-\lambda})^{-1} e^{\lambda(t-\tau)}.$$

Now using estimate (3.6) we have

$$(t - \tau) \|u(t)\|^2 e^{\lambda(t-\tau)} \leq (2C_5(t - \tau) + 1) \left( (1 + \lambda(t - \tau)) \|u(\tau)\|^2 + (1 + \lambda)R_1^2 e^{\lambda(t-\tau)} \right)$$

$$+ (t - \tau)|g|^2 (1 - e^{-\lambda})^{-1} e^{\lambda(t-\tau)}$$

$$\leq (1 + (t - \tau) + (t - \tau)^2)C|u(\tau)|^2 + (1 + t - \tau)R_1^2 e^{\lambda(t-\tau)},$$

(3.9) is proved. □

3.2. Existence of pullback attractor

For problem (3.1), we now give a fixed symbol $g_0(t)$ and take the symbol space $P = \{g_0(t + h) | h \in \mathbb{R}\}$, $\theta : P \to P$, $\theta_t(p) := p(t + \cdot, \cdot)$. By Theorem 3.1 we define a cocycle on $V$,

$$\phi(t, p, u_0) = u(t), \quad \forall (t, p, u_0) \in R_+ \times P \times V,$$

where $u(t)$ is the solution of (3.1).

Lemma 3.2. The cocycle defined by (3.1) is norm-to-weak continuous in $V$. 
Proof. We know \( V \hookrightarrow H, H^* \hookrightarrow V^* \) and \( i, i^* \) are dense. From Theorem 3.1, we obtain that for any \( p \in P, t \in R_+, \phi(t, p, u_0) : H \to H \) is continuous, therefore \( \phi(t, p, u_0) : H \to H \) is norm-to-weak continuous.

Now let us verify that for any \( p \in P, t \in R_+, \phi(t, p, u_0) \) maps a compact subset of \( V \) to be a bounded set of \( V \). In fact, multiplying (3.1) by \(-\Delta u \) and integrating over \( x \in \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 = \int_{\Omega} f(u) \Delta u \, dx - \int_{\Omega} g(t) \Delta u \, dx.
\]

Using Hölder’s inequality and Young’s inequality,

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + |\Delta u(t)|^2 \leq C_5 \|u(t)\|^2 + \frac{1}{2} |g(t)|^2
\]

\[
\frac{d}{dt} \|u(t)\|^2 \leq 2C_5 \|u(t)\|^2 + |g(t)|^2.
\]

Using Gronwall’s lemma,

\[
\|u(t)\|^2 \leq \|u(0)\|^2 e^{2C_5 t} + \int_0^t |g(s)|^2 e^{2C_5 (t-s)} \, ds
\]

\[
\leq \|u(0)\|^2 e^{C_5 t} + e^{2C_5} \int_0^t |g(s)|^2 \, ds.
\]

Namely, for any \( t \in R_+, p \in P, \phi(t, p, u_0) \) maps a bounded set to be a bounded set, therefore \( \phi(t, p, u_0) \) maps a compact set to be a bounded set. By Theorem 2.4, the proof is complete. \( \square \)

Theorem 3.4. If \( g_0(x, t) \) is translation bounded in \( L^2_{\text{loc}}(\mathbb{R}; H), f(u) \) satisfies conditions (3.2) and (3.3) where \( 2 \leq p < +\infty \) (\( n \leq 2 \)), \( 2 \leq p \leq \frac{n}{n-2} + 1 \) (\( n \geq 3 \)), then the cocycle \( \{\phi(t, p, x)\} \) corresponding to problem (3.1) possesses a compact pullback attractor \( \mathcal{A} = \{A_p\}_{p \in P} = \{\omega_p(B_0)\}_{p \in P} \), where \( B_0 \) is the uniformly (w.r.t. \( p \in P \)) absorbing set in \( V \).

Proof. For any \( g \in P, |g|^2_b = |g_0|^2_b \), using (3.5) we obtain

\[
B_0 = \{u \in H \mid |u| \leq 2R_1^2\}
\]

is the uniformly (w.r.t. \( g \in P \)) absorbing set in \( H \), i.e., for any \( B \in B(H) \), there exists \( t_0 = t_0(B) \geq 0 \) such that

\[
\phi(t, p, B) \subset B_0 \quad \text{for all } t \geq t_0, \ g \in P.
\]

Let

\[
B_1 = \bigcup_{g \in P} \bigcup_{t > t_0 + 1} \phi(t_0 + 1, g, B_0).
\]

Using (3.9), \( B_1 \) is bounded,

\[
\|u\|^2 \leq \rho^2, \quad \forall u \in B_1,
\]

and \( B_1 \) is the uniformly (w.r.t. \( g \in P \)) absorbing set in \( V \).
Let us verify pullback condition (PC). Since \((-\Delta)^{-1}\) is continuous compact operator in \(H\), by classical spectral theorem, there exists a sequence \(\{\lambda_j\}_{j=1}^{\infty}\),

\[0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots \leq \lambda_j \to \infty, \quad \text{as} \; j \to \infty, \quad (3.11)\]

and a family of elements \(\{\omega_j\}_{j=1}^{\infty}\) of \(D(-\Delta)\), which are orthonormal in \(H\) such that

\[-\Delta \omega_j = \lambda_j \omega_j, \quad \forall j \in \mathbb{N}.\]

Let \(V_m = \text{span}\{\omega_1, \omega_2, \ldots, \omega_m\}\) in \(V\) and \(P_m : V \to V_m\) is an orthogonal projector.

For any \(u \in D(-\Delta)\), write

\[u = P_mu + (I - P_m)u = u_1 + u_2.\]

In fact, \(\phi(s, \theta_{-s}(g), u_0)\) satisfies

\[
\frac{du}{dt} - \Delta u + f(u) = \theta_{-s}g(t) = g(t - s). \quad (3.12)
\]

Taking the inner product in \(H\) of Eq. (3.12) with \(-\Delta u_2\), we have

\[
\frac{1}{2} \frac{d}{dt} \|u_2\|^2 + |\Delta u_2|^2 = (f(u), \Delta u_2) - (g(t - s), \Delta u_2). \quad (3.13)
\]

Using Young’s inequality, the Poincaré inequality and the Sobolev embedding theorem, and

\[|f(u)| \leq \beta(|u|^{p-1} + 1), \quad (3.14)\]

from (3.2), (3.3) we have

\[
|f(u), \Delta u_2| = \left| \int f(u) \Delta u_2 \, dx \right|
\leq \int \left| f(u) \Delta u_2 \right| \, dx
\leq |f(u)||\Delta u_2|
\leq \frac{1}{4}|\Delta u_2|^2 + |f(u)|^2
\leq \frac{1}{4}|\Delta u_2|^2 + \int \beta^2(|u|^{p-1} + 1)^2 \, dx
\leq \frac{1}{4}|\Delta u_2|^2 + 2\beta^2|\Omega| + 2\beta^2 \int |u|^{2(p-1)} \, dx
\leq \frac{1}{4}|\Delta u_2|^2 + 2\beta^2|\Omega| + 2\beta^2 \|u\|_2^{2(p-1)}
\leq \frac{1}{4}|\Delta u_2|^2 + 2\beta^2|\Omega| + 2\beta^2 C \|u\|_2^{2(p-1)}, \quad (3.15)
\]

and

\[
|g(t - s), -\Delta u_2| = \left| \int g(t - s)(-\Delta u_2) \, dx \right|
\leq |g(t - s)||\Delta u_2|
\leq \frac{1}{4}|\Delta u_2|^2 + |g(t - s)|^2. \quad (3.16)
\]
Using (3.15) and (3.16), we obtain
\[
\frac{d}{dt} \| u_2 \|^2 + |\Delta u_2 |^2 \leq 4 \beta^2 |\Omega | + 4 \beta^2 C \| u \|^{2(p-1)} + 2 |g(t-s)|^2 .
\]

Using the Poincaré inequality, we have
\[
\frac{d}{dt} \| u_2 \|^2 + \lambda_{m+1} \| u_2 \|^2 \leq 4 \beta^2 |\Omega | + 4 \beta^2 C \| u \|^{2(p-1)} + 2 |g(t-s)|^2 .
\]

Using Gronwall’s lemma, let \( \tau = t_0 + 1 \), we have
\[
\| u_2(s) \|^2 \leq \| u_2(\tau) \|^2 e^{-\lambda_{m+1}(s-\tau)} + \int_\tau^s e^{-\lambda_{m+1}(s-t)} (4 \beta^2 |\Omega | + 4 \beta^2 C \rho^{2(p-1)}) \| g(t-s) \|^2 dt 
\]

Using the continuity of the integral, for any \( \varepsilon > 0 \), there exists \( \eta > 0 \), such that
\[
\int_{s-\eta}^s |g(t-s)|^2 dt \leq \frac{\varepsilon}{12} \quad \text{and} \quad 2 \int_{s-\eta}^s e^{-\lambda_{m+1}} |g(t-s)|^2 dt \leq \frac{\varepsilon}{6}.
\]
We know
\[
2 \int_\tau^s e^{-\lambda_{m+1}(s-t)} |g(t-s)|^2 dt 
\]
\[
\leq 2 \int_{s-\eta}^s e^{-\lambda_{m+1}(s-t)} |g(t-s)|^2 dt + 2 \int_{s-\eta-1}^{s-\eta} e^{-\lambda_{m+1}} |g(t-s)|^2 dt + \ldots
\]
\[
\leq 2 \int_{s-\eta}^s e^{-\lambda_{m+1}(s-t)} |g(t-s)|^2 dt + 2 e^{-\lambda_{m+1} \eta} (1 + e^{-\lambda_{m+1}} + e^{-2\lambda_{m+1}} + \ldots)
\]
\[
\times \sup_{t \in \mathbb{R}} \int_{t-1}^t |g(t)|^2 dt
\]
\[
\leq 2 \int_{s-\eta}^s e^{-\lambda_{m+1}(s-t)} |g(t-s)|^2 dt + \frac{2 e^{-\lambda_{m+1} \eta}}{1 - e^{-\lambda_{m+1}}} \sup_{t \in \mathbb{R}} \int_{t-1}^t |g(t)|^2 dt.
\]
For any \( \varepsilon > 0 \) we can take \( m + 1 \) large enough such that
\[
\frac{1}{\lambda_{m+1}} (4\beta^2 |\Omega| + 4\beta^2 C \rho^{2(p-1)}) \leq \frac{\varepsilon}{3},
\]
\[
\frac{2e^{-\lambda_{m+1} \eta}}{1 - e^{-\lambda_{m+1}}} \sup_{t \in \mathbb{R}} \int_{t-1}^{t} |g(t)|^2 \, dt \leq \frac{\varepsilon}{6}.
\]
(3.18)

Let \( t_1 = \frac{1}{\lambda_{m+1}} \ln (3\rho^2/\varepsilon) + \tau \), then \( s \geq t_1 \) implies
\[
\rho^2 e^{-\lambda_{m+1}(s-\tau)} \leq \frac{\varepsilon}{3},
\]
using (3.17), we have
\[
\|u_2(s)\|^2 \leq \varepsilon, \quad \forall s \geq t_1 \quad \forall g \in P,
\]
which indicates \( \phi(s, \theta_{-s}(g), u_0) \) satisfying pullback condition (PC) in \( V \), that is, for any \( g \in P, \ B \in B(V) \), there exist \( t_1 \) and a finite dimensional subspace \( V_m \) such that
\[
P\left( \bigcup_{s \geq t_1} \phi(s, \theta_{-s}(g), B) \right)
\]
is bounded
and
\[
\left\| (I - P)\left( \bigcup_{s \geq t_1} \phi(s, \theta_{-s}(g), u_0) \right) \right\| \leq \varepsilon, \quad \forall u_0 \in B.
\]
Applying Theorems 2.3, 2.2 and Lemma 3.2, the proof is complete. \( \square \)

References


