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Spectral characters of finite-dimensional representations of affine algebras

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Introduction

In this paper we study the category $\mathcal{C}$ of finite-dimensional representations of affine Lie algebras. The irreducible objects of this category were classified and described explicitly in [1,2]. It was known, however, that $\mathcal{C}$ was not semisimple. In such a case a natural problem is to describe the blocks of the category. The blocks of an abelian category are themselves abelian subcategories, each of which cannot be written as a proper direct sum of abelian categories and such that their direct sum is equal to the original category. Block decompositions of representations of algebras are often given by a character, usually a central character, namely, a homomorphism from the center of the algebra to $\mathbf{C}$, as, for instance, in the case of modules from the BGG category $\mathcal{O}$ for a simple Lie algebra. In our case, however, the center of the universal algebra of the affine algebra acts trivially on all representations in the category $\mathcal{C}$ and the absence of a suitable notion of character has been an obstacle to determining the blocks of $\mathcal{C}$.

In recent years the study of the corresponding category $\mathcal{C}_q$ of modules for quantum affine algebras has been of some interest [3,4,9,10,12,14,15]. In [6] the authors defined the notion of an elliptic character for objects of $\mathcal{C}_q$ when $|q| \neq 1$ and showed that for $|q| < 1$, the character could be used to determine the blocks of $\mathcal{C}_q$. The original definition of the elliptic character used convergence properties of the (non-trivial) action of the R-matrix on the tensor product of finite-dimensional representations. Of course, in the $q = 1$ case, the action of the R-matrix on a tensor product is trivial. However, the combinatorial part of the proof given in [6] suggests that an elliptic character can be viewed as a function $\chi : E \to \mathbf{Z}^m$ with finite support, where E is the elliptic curve $\mathbf{C}^\times/q^{2\mathbf{Z}}$ and $m \in \mathbf{N}^+$ depends

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on the underlying simple Lie algebra. This then motivated our definition when $q = 1$ of a spectral character of $L(\mathfrak{g})$ as a function $\mathbf{C}^\times \to \Gamma$ with finite support, where Γ is the quotient of the weight lattice of $\mathfrak{g}$ by the root lattice of $\mathfrak{g}$.

The other ingredient used in [6] to prove that two modules with the same elliptic character belonged to the same block, was a result proved in [1,12] that a suitable tensor product of irreducible representations was indecomposable but reducible on certain natural vectors. In the classical case, however, it was known from the work of [2] that a tensor product of irreducible representations was either irreducible or completely reducible. However, it was shown in [4] that the tensor product of the irreducible representations of the quantum affine algebra specialized to indecomposable, but usually reducible representations of the classical affine algebra. This led to the definition of the Weyl modules as a family of universal indecomposable modules. The Weyl modules are in general not well-understood; see [4,7,8] for several conjectures about them. However, in this paper, we are still able to identify a large family of quotients of the Weyl modules, which allows us to effectively use them as a substitute for the methods of [6]. Although we work with the affine Lie algebra, our results and proofs work for the current algebra, $\mathfrak{g} \otimes \mathbf{C}[t]$, but with the spectral character being defined as functions from $\mathbf{C}$ to Γ with finite support.

The paper is organized as follows: Section 1 is devoted to preliminaries and Section 2 to the definition of the spectral character and the statement of the main theorem. In Section 3, we recall the definition of the Weyl modules and give an explicit realization of certain indecomposable but reducible quotients of these modules and the parametrization of the irreducible objects of $\mathcal{C}$. The theorem is proved in the remaining two sections. We prove that to every indecomposable object of $\mathcal{C}$, one can associate a spectral character. To do this we show that if two modules V_j, $j = 1, 2$ have distinct spectral characters, then the corresponding $\mathrm{Ext}^1(V_1, V_2) = 0$. Finally, we prove that any two modules with the same spectral character must be in the same block of $\mathcal{C}$ and hence we get a parametrization of the blocks of $\mathcal{C}$ analogous to the one in [6].

1. Preliminaries

Throughout this paper $\mathbf{N}$ (respectively $\mathbf{N}^+$) denotes the set of non-negative (respectively positive) integers.

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank n with a Cartan subalgebra $\mathfrak{h}$. Set $I = \{1, 2, \ldots, n\}$ and let $\{\alpha_i\colon i \in I\} \subset \mathfrak{h}^*$ (respectively $\{\omega_i\colon i \in I\} \subset \mathfrak{h}^*$) be the set of simple roots (respectively fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Define a non-degenerate bilinear form $\langle\,,\rangle$ on $\mathfrak{h}$ by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ and let $h_i \in \mathfrak{h}$ be defined by requiring $\omega_i(h_j) = \delta_{ij}$, $i, j \in I$. We shall assume that the nodes of the Dynkin diagram are numbered as shown in Table 1 and we let $I_\bullet \subset I$ be the indices of the shaded nodes in the diagram.

Let R^+ be the corresponding set of positive roots and denote by θ the highest root of $\mathfrak{g}$. As usual, Q (respectively P) denotes the root (respectively weight) lattice of $\mathfrak{g}$ and we let $\Gamma = P/Q$. It is known that [11]

$$\Gamma \cong \mathbf{Z}_{n+1}, \quad \mathfrak{g} \text{ of type } A_n, \qquad \Gamma \cong \mathbf{Z}_2, \quad \mathfrak{g} \text{ of type } B_n,\ C_n,\ E_7,$$

$$\Gamma \cong \mathbf{Z}_4, \quad \mathfrak{g} \text{ of type } D_{2m+1}, \qquad \Gamma \cong \mathbf{Z}_2 \times \mathbf{Z}_2, \quad \mathfrak{g} \text{ of type } D_{2m},$$
$$\Gamma \cong \mathbf{Z}_3, \quad \mathfrak{g} \text{ of type } E_6, \qquad \Gamma \cong 0, \quad \mathfrak{g} \text{ of type } E_8,\ F_4,\ G_2.$$

The group Γ is generated by the images of the elements $\{\omega_i\colon\ i \in I_\bullet\}$ and hence any $\gamma \in \Gamma$ defines a unique element $\lambda_\gamma = \sum_{i=1}^n r_i\omega_i \in P^+$ where $r_i \in \mathbf{Z}$ are the minimal non-negative integers such that λ_γ is a representative of γ. In particular, $r_i = 0$ if $i \notin I_\bullet$.

Let W be the Weyl group of $\mathfrak{g}$ and assume that w_0 is the longest element of W. The group W acts on $\mathfrak{h}^*$ and preserves the root and weight lattice. Let $P^+ = \sum_{i\in I} \mathbf{N}\omega_i$ be the set of dominant integral weights and set $Q^+ = \sum_{i\in I} \mathbf{N}\alpha_i$. We shall assume that P has the usual partial ordering, given $\lambda, \mu \in P$ we say that $\lambda \geqslant \mu$ if $\lambda - \mu \in Q^+$. For $\alpha \in R^+$, let $\mathfrak{g}_{\pm\alpha}$ denote the corresponding root spaces, and fix elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$, $h_\alpha \in \mathfrak{h}$, such that they span a subalgebra of $\mathfrak{g}$ which is isomorphic to sl_2. For $i \in I$, set $x_i^\pm = x_{\alpha_i}^\pm$, $h_{\alpha_i} = h_i$. Set $\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{\pm\alpha}$. Given any Lie algebra $\mathfrak{a}$, let $L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbf{C}[t, t^{-1}]$ be the loop algebra associated with $\mathfrak{a}$ and let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. Clearly, we have

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \qquad L(\mathfrak{g}) = L(\mathfrak{n}^+) \oplus L(\mathfrak{h}) \oplus L(\mathfrak{n}^-),$$

and a corresponding decomposition

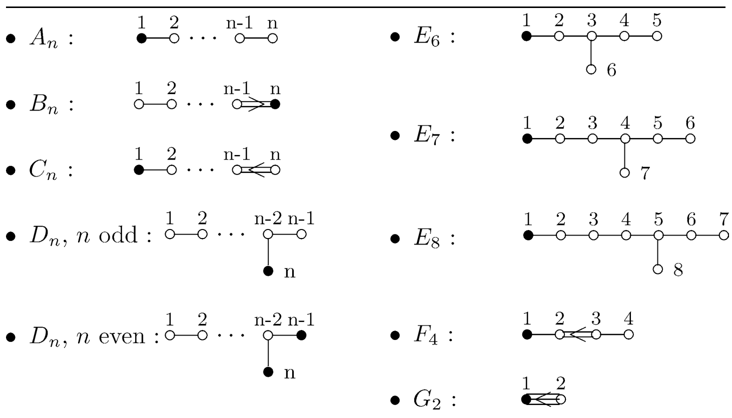
$$\mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{n}^-)\mathbf{U}(\mathfrak{h})\mathbf{U}(\mathfrak{n}^+), \qquad \mathbf{U}\big(L(\mathfrak{g})\big) = \mathbf{U}\big(L(\mathfrak{n}^-)\big)\mathbf{U}\big(L(\mathfrak{h})\big)\mathbf{U}\big(L(\mathfrak{n}^+)\big).$$

Given $a \in \mathbf{C}^\times$, let $\mathrm{ev}_a : L(\mathfrak{g}) \to \mathfrak{g}$ be the evaluation homomorphism, $\mathrm{ev}_a(x \otimes t^n) = a^n x$.

For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ and highest weight vector v_λ. Thus, $V(\lambda)$ is generated by v_λ as a $\mathfrak{g}$-module with defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \qquad h v_\lambda = \lambda(h) v_\lambda, \qquad \big(x_i^-\big)^{\lambda(h_i)+1} v_\lambda = 0, \quad \forall h \in \mathfrak{h},\ i \in I.$$

Table 1



Let V be a finite-dimensional representation of $\mathfrak{g}$. Then we can write V as a direct sum

$$V = \bigoplus_{\mu \in P} V_\mu, \qquad V_\mu = \{v \in v\colon\ hv = \mu(h)v,\ \forall h \in \mathfrak{h}\}, \tag{1.1}$$

and set

$$\mathrm{wt}(V) = \{\mu \in P\colon\ V_\mu \neq 0\}.$$

The following result is well-known [11].

Proposition 1.1. *Let V be a finite-dimensional representation of $\mathfrak{g}$.*

(i) *For all $w \in W$, $\mu \in P$, we have $\dim(V_\mu) = \dim(V_{w\mu})$.*
(ii) *The module V is isomorphic to a direct sum of $\mathfrak{g}$-modules of type $V(\lambda)$, $\lambda \in P^+$.*
(iii) *Let $V(\lambda)^*$ be the representation of $\mathfrak{g}$ which is dual to $V(\lambda)$. Then*

$$V(\lambda)^* \cong V(-w_0\lambda).$$

The following proposition is crucial for the proof of the main theorem.

Proposition 1.2. *Let $\mu, \lambda \in P^+$ be such that $\lambda - \mu \in Q$. Then, there exists a sequence of weights $\mu_l \in P^+$, $l = 0, \dots, m$, with*

(i) *$\mu_0 = \mu$, $\mu_m = \lambda$, and*
(ii) $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})) \neq 0$, $\forall 0 \leqslant l \leqslant m$.

Proof. Consider the module $V(\lambda) \otimes V(\mu)^*$. Since $\lambda - \mu \in Q$, it follows that $\lambda - w_0\mu \in Q$. In particular, this means that if $V(\nu)$ is an irreducible summand of $V(\lambda) \otimes V(\mu)^*$, then $\nu \in Q^+ \cap P^+$. It follows that $V(\nu)_0 \neq 0$. This implies by a result of Kostant [5,13], that there exists $m \geqslant 0$ such that $\mathrm{Hom}_{\mathfrak{g}}(S^m(\mathfrak{g}), V(\nu)) \neq 0$. It follows that $\mathrm{Hom}_{\mathfrak{g}}(S^m(\mathfrak{g}) \otimes V(\mu), V(\lambda)) \neq 0$.

We now proceed by induction on m. If $m = 1$, we are done for then $\mu_0 = \mu$. Otherwise there must exist $\mu_{m-1} \in P^+$ with $\mathrm{Hom}_{\mathfrak{g}}(V(\mu_{m-1}), \mathfrak{g}^{\otimes(m-1)} \otimes V) \neq 0$ such that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu_{m-1}), U) \neq 0$. Since the category of finite-dimensional representations of $\mathfrak{g}$ is semisimple, we see also that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^{\otimes(m-1)} \otimes V, V(\mu_{m-1})) \neq 0$. But now we are done by the inductive hypothesis. □

Remark. In Appendix A we construct the sequence $\mu_1, \dots, \mu_m$ in the special case when $\mu = \lambda_\gamma$ with the properties stated above. In particular, this gives a different and perhaps more elementary proof of this proposition.

2. Spectral characters and the block decomposition of $\mathcal{C}$

Let Ξ be the set of all functions $\chi : \mathbf{C}^\times \to \Gamma$ with finite support. Clearly, addition of functions defines a group structure on Ξ. Given $\lambda \in P^+$, $a \in \mathbf{C}^\times$, let $\chi_{\lambda,a} \in \Xi$ be defined by

$$\chi_{\lambda,a}(z) = \delta_a(z)\bar{\lambda},$$

where $\bar{\lambda}$ is the image of λ in Γ and $\delta_a(z)$ is the characteristic function of $a \in \mathbf{C}^\times$. We denote by $\mathcal{P}$ the space of n-tuples of polynomials with constant term one. Coordinatewise multiplication defines the structure of monoid on $\mathcal{P}$. Given $\lambda \in P^+$, $a \in \mathbf{C}^\times$, define $\boldsymbol{\pi}_{\lambda,a} = (\pi_1, \dots, \pi_n) \in \mathcal{P}$, by

$$\pi_i = (1 - au)^{\lambda(h_i)}, \quad 1 \leqslant i \leqslant n.$$

Any $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n) \in \mathcal{P}$ can be written uniquely as a product,

$$\boldsymbol{\pi} = \prod_{j=1}^{r} \boldsymbol{\pi}_{\lambda_j, a_j}, \tag{2.1}$$

where

(i) $\{a_j^{-1}\colon 1 \leqslant j \leqslant r\}$ is the set of distinct roots of $\prod_{i=1}^n \pi_i$,
(ii) $\lambda_j = \sum_{k=1}^n m_{kj}\omega_k \in P^+$, and m_{kj} is the multiplicity with which a_j^{-1} occurs as a root of π_k.

Define $\boldsymbol{\pi}^* \in \mathcal{P}$ by

$$\boldsymbol{\pi}^* = \prod_{j=1}^{r} \boldsymbol{\pi}_{-w_0\lambda_j, a_j},$$

where we recall that w_0 is the longest element of the Weyl group of $\mathfrak{g}$. Given $\boldsymbol{\pi} \in \mathcal{P}$, define $\chi_{\boldsymbol{\pi}} \in \Xi$ by

$$\chi_{\boldsymbol{\pi}} = \sum_{j=1}^{r} \chi_{\lambda_j, a_j},$$

where λ_i, a_i are as in (2.1). Obviously,

$$\chi_{\boldsymbol{\pi}\boldsymbol{\pi}'} = \chi_{\boldsymbol{\pi}} + \chi_{\boldsymbol{\pi}'},$$

for all $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}$.

To state our main result, we need to recall the parametrization of irreducible finite-dimensional modules of affine Lie algebras [1,4], and also the definition of blocks in an abelian category.

Proposition 2.1. *There exists a bijective correspondence between the isomorphism classes of irreducible finite-dimensional representations of affine Lie algebras and elements of $\mathcal{P}$.*

We denote by $V(\boldsymbol{\pi})$ an element of the isomorphism class corresponding to $\boldsymbol{\pi}$.

Definition 2.1. We say that a module $V \in \mathcal{C}$ has spectral character $\chi \in \Xi$ if $\chi = \chi_{\boldsymbol{\pi}}$ for every irreducible component $V(\boldsymbol{\pi})$ of V. Let $\mathcal{C}_\chi$ be the abelian subcategory consisting of all modules $V \in \mathcal{C}$ with spectral character χ.

Let $\mathcal{C} = \mathcal{C}_{\mathrm{fin}}(L(\mathfrak{g}))$ be the category of finite-dimensional representations of $L(\mathfrak{g})$. This category is not semisimple, i.e., there exist indecomposable reducible $L(\mathfrak{g})$-modules in $\mathcal{C}$. However, $\mathcal{C}$ is an abelian tensor category and every object in $\mathcal{C}$ has a Jordan–Holder series of finite length. This means that $\mathcal{C}$ has a block decomposition which is obtained as follows.

Definition 2.2. Say that two indecomposable objects $U, V \in \mathcal{C}$ are linked if there do not exist abelian subcategories $\mathcal{C}_k$, $k = 1, 2$ such that $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$ with $U \in \mathcal{C}_1$, $V \in \mathcal{C}_2$. If U and V are decomposable, then we say that they are linked if every indecomposable summand of U is linked to every indecomposable summand of V.

This defines an equivalence relation on $\mathcal{C}$ and a block of $\mathcal{C}$ is an equivalence class for this relation, clearly $\mathcal{C}$ is a direct sum of blocks. The following lemma is trivially established.

Lemma 2.2. *Two indecomposable modules V_1 and V_2 are linked iff they contain submodules $U_k \subset V_k$, $k = 1, 2$ such that U_1 is linked to U_2.*

The main result of the paper is the following.

Theorem 1. *We have*

$$\mathcal{C} = \bigoplus_{\chi \in \Xi} \mathcal{C}_\chi .$$

Moreover, each $\mathcal{C}_\chi$ is a block. Equivalently, the blocks of $\mathcal{C}$ are in bijective correspondence with Ξ.

The theorem is obviously a consequence of the next two propositions.

Proposition 2.3. *Any two irreducible modules in $\mathcal{C}_\chi$, $\chi \in \Xi$, are linked.*

Proposition 2.4. *Every indecomposable $L(\mathfrak{g})$-module has a spectral character.*

We prove these propositions in Sections 4 and 5, respectively. We shall need several results on a certain family of indecomposable but generally reducible modules for $L(\mathfrak{g})$, the so-called Weyl modules, this is done in the next section.

We conclude this section with an equivalent definition of linked modules and with some general results on Jordan–Holder series.

Definition 2.3. Let $U, V \in \mathcal{C}$ be indecomposable $L(\mathfrak{g})$-modules. We say that U is strongly linked to V if there exists $L(\mathfrak{g})$-modules $U_1, \dots, U_\ell$, with $U_1 = U$, $U_\ell = V$ and either $\mathrm{Hom}_{L(\mathfrak{g})}(U_k, U_{k+1}) \neq 0$ or $\mathrm{Hom}_{\mathrm{L}(\mathfrak{g})}(U_{k+1}, U_k) \neq 0$ for all $1 \leqslant k \leqslant \ell$. We extend this to all of $\mathcal{C}$ by saying that two modules U and V are strongly linked iff every indecomposable component of U is strongly linked to every indecomposable component of V.

It is clear that the notion of strongly linked defines an equivalence relation on $\mathcal{C}$ which induces a decomposition of $\mathcal{C}$ into a direct sum of abelian categories. If two modules U and V are strongly linked, then they must be linked. For otherwise, suppose that U and V belong to different blocks. It suffices to consider the case $\mathrm{Hom}_{L(\mathfrak{g})}(U, V) \neq 0$. This means that U and V have an irreducible constituent say M in common. Then, since each block is an abelian subcategory, M must belong to both blocks which is a contradiction. Conversely, suppose that U and V are linked but not strongly linked. Then, there is obviously a splitting of $\mathcal{C}$ into abelian subcategories coming from the strong linking, such that U and V belong to different subcategories. We have proved the following lemma.

Lemma 2.5. *Two modules U and V are linked iff they are strongly linked.*

Lemma 2.6. *Suppose that $U \in \mathcal{C}_{\chi_1}$ and $V \in \mathcal{C}_{\chi_2}$ are strongly linked. Then $\chi_1 = \chi_2$.*

Proof. It suffices to check this when $\mathrm{Hom}_{L(\mathfrak{g})}(U, V) \neq 0$. But this means that U and V have an irreducible constituent say M in common and hence $\chi_1 = \chi_2$. □

We shall make use of the following simple proposition repeatedly without further mention.

Proposition 2.7.

(i) *Any sequence $0 \subset V_1 \cdots \subset V_k \subset V$ of $L(\mathfrak{g})$-modules in $\mathcal{C}$ can be refined to a Jordan–Holder series of V.*
(ii) *Suppose that $0 \subset U_1 \subset \cdots \subset U_r = U$ and $0 \subset V_1 \cdots \subset V_s \subset V$ are Jordan–Holder series for modules U, V in $\mathcal{C}$. Then the irreducible constituents of $U \otimes V$ occur as constituents of $U_k \otimes V_\ell$ for some $1 \leqslant k \leqslant r$ and $1 \leqslant \ell \leqslant s$.*
(iii) *Suppose that $U_k \in \mathcal{C}$, $1 \leqslant k \leqslant 3$ and that U_1 and U_2 are strongly linked. Then $U_1 \otimes U_3$ is strongly linked to $U_2 \otimes U_3$.*

3. Weyl modules

In this section we recall from [4] the definition and some results on Weyl modules. We also study further properties of these modules.

Let $V \in \mathcal{C}$. Regarding V as a finite-dimensional module for $\mathfrak{g}$, we can write V as a direct sum as in Section 1, (1.1), $V = \bigoplus_{\mu \in P} V_\mu$. Let $\mathrm{wt}(V)$ be the set of weights of V. Notice that $L(\mathfrak{h})V_\mu \subset V_\mu$. Since $L(\mathfrak{h})$ is an abelian Lie algebra, we get a further decomposition

$$V_\mu = \bigoplus_{\mathbf{d} \in L(\mathfrak{h})^*} V_\mu^{\mathbf{d}},$$

where

$$V_\mu^{\mathbf{d}} = \left\{ v \in V_\mu \colon \left(h \otimes t^k - \mathbf{d}\left(h \otimes t^k\right)\right)^r v = 0,\ \forall r \geqslant r(h,k) \gg 0 \right\},$$

are the generalized eigenspaces for the action of $L(\mathfrak{h})$ on V_μ. Clearly, if $U, V \in \mathcal{C}$, then any $L(\mathfrak{g})$ homomorphism from U to V maps $U_\mu^{\mathbf{d}}$ to $V_\mu^{\mathbf{d}}$. Since V_μ is finite-dimensional, we see that if $V_\mu^{\mathbf{d}} \neq 0$, then there exists $0 \neq v \in V_\mu^{\mathbf{d}}$ such that

$$\left(h \otimes t^k\right) v = \mathbf{d}\left(h \otimes t^k\right) v, \quad h \in \mathfrak{h},\ k \in \mathbf{Z}.$$

We say that $\mathbf{d}$ is of type $\boldsymbol{\pi} \in \mathcal{P}$, if

$$\mathbf{d}\left(h \otimes t^k\right) = \left(\sum_{j=1}^{r} \lambda_j(h) a_j^k \right),$$

where $\lambda_j \in P^+$ and $a_j \in \mathbf{C}^\times$ are as in (2.1) and we denote the corresponding generalized eigenspace by $V_\mu^{\boldsymbol{\pi}}$.

Definition 3.1. Given an n-tuple of polynomials with constant term 1, we denote by $W(\boldsymbol{\pi})$ the $L(\mathfrak{g})$-module generated by an element $w_{\boldsymbol{\pi}}$ and the following relations:

$$L(\mathfrak{n}^+) w_{\boldsymbol{\pi}} = 0, \qquad \left(h \otimes t^k\right) w_{\boldsymbol{\pi}} = \left(\sum_{j=1}^{r} \lambda_j(h) a_j^k \right) w_{\boldsymbol{\pi}},$$
$$\left(x_i^- \otimes t^\ell\right)^{\sum_{j=1}^r \lambda_j(h_i) + 1} w_{\boldsymbol{\pi}} = 0, \tag{3.1}$$

for all $i \in I$, $k, \ell \in \mathbf{Z}$, $\alpha \in R^+$, $h \in \mathfrak{h}$ and where we assume that $\boldsymbol{\pi}$ is written as in (2.1). Set $\lambda_{\boldsymbol{\pi}} = \sum_{j=1}^r \lambda_j$.

The following properties of $W(\boldsymbol{\pi})$ are standard and easily established.

Lemma 3.1. *With the notation as above, we have*

(i) $W(\boldsymbol{\pi}) = \mathbf{U}(L(\mathfrak{n}^-))w_{\boldsymbol{\pi}}$ *and so* $\operatorname{wt}(W(\boldsymbol{\pi})) \subset \lambda_{\boldsymbol{\pi}} - Q^+$.
(ii) $\dim W(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}} = 1$, *and so* $W(\boldsymbol{\pi})^{\boldsymbol{\pi}}_{\lambda_{\boldsymbol{\pi}}} = W_{\lambda_{\boldsymbol{\pi}}}$.
(iii) *Let V be any finite-dimensional $L(\mathfrak{g})$-module generated by an element $v \in V$ satisfying*

$$L(\mathfrak{n}^+)v = 0, \qquad \left(h \otimes t^k\right)v = \left(\sum_{j=1}^{r} \lambda_j(h) a_j^k\right)v.$$

Then V is a quotient of $W(\boldsymbol{\pi})$.
(iv) *$W(\boldsymbol{\pi})$ is an indecomposable $L(\mathfrak{g})$-module with a unique irreducible quotient.*

Let $\beta_1, \dots, \beta_N$ be an enumeration of the elements of R^+. Given $r \in \mathbf{Z}$, set $x^-_{\beta_j,r} = x_{\beta_j} \otimes t^r$. The following result was proved in [4].

Theorem 2. Let $\boldsymbol{\pi}$ be an n-tuple of polynomials with constant term one and assume that $\boldsymbol{\pi}$ has a factorization as in (2.1).

(i) The $L(\mathfrak{g})$-module $W(\boldsymbol{\pi})$ is spanned by monomials of the form

$$x^-_{\beta_{j_1},r_1} x^-_{\beta_{j_2},r_2} \cdots x^-_{\beta_{j_\ell},r_\ell} w\boldsymbol{\pi},$$

where $\ell \in \mathbf{N}^+$, $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_\ell$, and $0 \leqslant r_k < \lambda_{\boldsymbol{\pi}}(h_{\beta_{j_k}})$ for all $1 \leqslant k \leqslant \ell$. In particular, $\dim W(\boldsymbol{\pi}) < \infty$.
(ii) As $L(\mathfrak{g})$-modules,

$$W(\boldsymbol{\pi}) \cong W(\boldsymbol{\pi}_{\lambda_1,a_1}) \otimes \cdots \otimes W(\boldsymbol{\pi}_{\lambda_r,a_r}).$$

We can now elaborate on the parametrization of the irreducible finite-dimensional modules stated in Section 2 of this paper.

Proposition 3.2. *The irreducible finite-dimensional $L(\mathfrak{g})$-module $V(\boldsymbol{\pi})$ is the irreducible quotient of $W(\boldsymbol{\pi})$ and we have*

$$V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}_{\lambda_1,a_1}) \otimes \cdots \otimes V(\boldsymbol{\pi}_{\lambda_r,a_r}).$$

Further, the module $V(\boldsymbol{\pi}_{\lambda,a})$ is the $L(\mathfrak{g})$-module obtained by pulling back the $\mathfrak{g}$-module $V(\lambda)$, by the evaluation homomorphism $\operatorname{ev}_a : L(\mathfrak{g}) \to \mathfrak{g}$. Finally, as $L(\mathfrak{g})$-modules we have

$$V(\boldsymbol{\pi})^* \cong V(\boldsymbol{\pi}^*).$$

The structure of $W(\boldsymbol{\pi})$ is not well-understood in general, although it is known that $W(\boldsymbol{\pi})$ is in general not isomorphic to $V(\boldsymbol{\pi})$, a necessary and sufficient condition for $W(\boldsymbol{\pi})$ to be isomorphic to $V(\boldsymbol{\pi})$ can be found in [4]. In what follows, we establish further properties of the Weyl modules which we need in this paper, and also identify natural indecomposable reducible quotients of $W(\boldsymbol{\pi})$.

Proposition 3.3. *Let* $\lambda = \sum_{i=1}^{n} r_i \omega_i \in P^+$, $a \in \mathbf{C}^\times$.

(i) *For all* $\alpha \in R^+$ *we have*

$$\left(x_\alpha^- \otimes (t-a)^{\lambda(h_\alpha)}\right) w_{\boldsymbol{\pi}_{\lambda,a}} = 0.$$

In particular, $W(\boldsymbol{\pi}_{\lambda,a})$ *is spanned by elements of the form*

$$\left(x_{\beta_{j_1}}^- \otimes (t-a)^{r_1}\right)\left(x_{\beta_{j_2}}^- \otimes (t-a)^{r_2}\right)\cdots\left(x_{\beta_{j_\ell}}^- \otimes (t-a)^{r_\ell}\right) w\boldsymbol{\pi}_{\lambda,a},$$

where $\ell \in \mathbf{N}^+$, $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_\ell$, *and* $0 \leqslant r_k < \lambda_{\boldsymbol{\pi}}(h_{\beta_{j_k}})$ *for all* $1 \leqslant k \leqslant \ell$.

(ii) *For all* $h \in \mathfrak{h}$, $k \in \mathbf{Z}$, $\mu \in P$, *and* $w \in W(\boldsymbol{\pi}_{\lambda,a})_\mu$, *we have,*

$$\left(h \otimes \left(t^k - a^k\right)\right)^r w = 0, \quad \forall r \gg 0.$$

(iii) *There exists a bijective correspondence between irreducible* $\mathfrak{g}$*-submodules of* $W(\boldsymbol{\pi}_{\lambda,a})$ *and the irreducible* $L(\mathfrak{g})$*-constituents of* $W(\boldsymbol{\pi}_{\lambda,a})$.

Proof. The relation $(x_\alpha^- \otimes (t-a)^{\lambda(h_\alpha)}) w_{\boldsymbol{\pi}_{\lambda,a}} = 0$ was proved in [4, Section 6]. This immediately implies the second assertion of (i). To prove (ii) one just uses commutation relations once we know from (i) that

$$w = \left(x_{\beta_{j_1}}^- \otimes (t-a)^{r_1}\right)\left(x_{\beta_{j_2}}^- \otimes (t-a)^{r_2}\right)\cdots\left(x_{\beta_{j_\ell}}^- \otimes (t-a)^{r_\ell}\right) w\boldsymbol{\pi}_{\lambda,a}.$$

To prove (iii) first notice that, from (ii), it follows that the irreducible constituents of $W(\boldsymbol{\pi}_{\lambda,a})$ are all of the form $V(\boldsymbol{\pi}_{\mu,a})$ for some $\mu \in P^+$. Then, since $V(\boldsymbol{\pi}_{\mu,a}) \cong_{\mathfrak{g}} V(\mu)$, it follows that all $\mathfrak{g}$-constituents of $W(\boldsymbol{\pi}_{\mu,a})$ must also be $L(\mathfrak{g})$-constituents with the same multiplicity. □

We now prove the following proposition.

Proposition 3.4. *Let* $\lambda, \mu \in P^+$. *Assume that there exists a non-zero homomorphism* $p : \mathfrak{g} \otimes V(\lambda) \to V(\mu)$ *of* $\mathfrak{g}$*-modules. The following formulas define an action of* $L(\mathfrak{g})$*-module on* $V(\lambda) \oplus V(\mu)$:

$$xt^r(v, w) = \left(a^r xv, a^r xw + ra^{r-1} p(x \otimes v)\right),$$

where $x \in \mathfrak{g}$, $r \in \mathbf{Z}$, $v \in V(\lambda)$, *and* $w \in V(\mu)$. *Denoting this module by* $V(\lambda, \mu, a)$, *we see that*

$$0 \to V(\boldsymbol{\pi}_{\mu,a}) \to V(\lambda, \mu, a) \to V(\boldsymbol{\pi}_{\lambda,a}) \to 0$$

is a non-split short exact sequence of $L(\mathfrak{g})$*-modules. Finally, if* $\lambda > \mu$, *there exists a canonical surjective homomorphism of* $L(\mathfrak{g})$*-modules* $W(\boldsymbol{\pi}_{\lambda,a}) \to V(\lambda, \mu, a)$.

Proof. To check that the formulas give a $L(\mathfrak{g})$-module structure is a straightforward verification. Since $L(\mathfrak{g})V(\mu) \subset V(\mu)$, it follows that $V(\boldsymbol{\pi}_{\mu,a})$ is a $L(\mathfrak{g})$-submodule of $V(\lambda,\mu,a)$. Since $p:\mathfrak{g}\otimes V(\lambda)\to V(\mu)$ is non-zero, it follows that the module $V(\lambda,\mu,a)$ is indecomposable and we have the desired short exact sequence of $L(\mathfrak{g})$-modules. Note that if $\lambda > \mu$, we have

$$L(\mathfrak{n}^+)(v_\lambda, 0) = 0, \qquad h \otimes t^k(v_\lambda, 0) = \big(a^k v_\lambda, 0\big).$$

Also, since $V(\lambda) = \mathbf{U}(\mathfrak{g})v_\lambda$, we see that $(V(\lambda), 0) \subset \mathbf{U}(L(\mathfrak{g}))v_\lambda$, and hence it follows that $V(\lambda,\mu,a) = \mathbf{U}(L(\mathfrak{g}))v_\lambda$. But now Lemma 3.1(iii) implies that $V(\lambda,\mu,a)$ must be a quotient of $W(\boldsymbol{\pi}_{\lambda,a})$. □

Remark. One can view the modules $V(\lambda,\mu,a)$ as generalizations of the modules $V(\boldsymbol{\pi}_{\lambda,a})$ as follows. Thus, while $V(\boldsymbol{\pi}_{\lambda,a})$ is a module for $L(\mathfrak{g})$ on which $x \otimes (f - f(a))$ acts trivially for all $f \in \mathbf{C}[t,t^{-1}]$ and $x \in \mathfrak{g}$, the modules $V(\lambda,\mu,a)$ are modules on which $x \otimes (f - (t-a)f'(a) - f(a))$ acts trivially for all $f \in \mathbf{C}[t,t^{-1}]$, where f' is the first derivative of f with respect to t.

4. Proof of Proposition 2.3

We begin with the following lemma.

Lemma 4.1.

(i) *Assume that $\lambda,\mu \in P^+$ and that there exists a non-zero homomorphism $p:\mathfrak{g}\otimes V(\lambda)\to V(\mu)$ of $\mathfrak{g}$-modules. Then the modules $V(\boldsymbol{\pi}_{\lambda,a})$ and $V(\boldsymbol{\pi}_{\mu,a})$ are strongly linked.*

(ii) *Let $\gamma \in \Gamma$ be such that $\lambda = \lambda_\gamma \bmod Q$. Then, $V(\boldsymbol{\pi}_{\lambda,a})$ and $V(\boldsymbol{\pi}_{\lambda_\gamma,a})$ are strongly linked.*

Proof. The first part of the lemma is immediate from Proposition 3.4. The second part is now immediate from (i) and Proposition 1.2. □

Proposition 4.2. *Let $V(\boldsymbol{\pi}_k) \in \mathcal{C}_{\chi_k}$ for some $\chi_k \in \Xi$, $k = 1,2$. Then $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2) \in \mathcal{C}_{\chi_1+\chi_2}$.*

Proof. By Proposition 3.2, we can write $V(\boldsymbol{\pi}_1) = \bigotimes_{j=1}^k V(\boldsymbol{\pi}_{\lambda_j,a_j})$ with $a_j \neq a_l$ for all $1 \leqslant l \neq j \leqslant k$ and $\lambda_1,\dots,\lambda_k \in P^+$. Similarly, write $V(\boldsymbol{\pi}_2) = \bigotimes_{j=1}^\ell V(\boldsymbol{\pi}_{\mu_j,b_j})$. We proceed by induction on the cardinality of S, where

$$S = \{a_1,\dots,a_k\} \cap \{b_1,\dots,b_\ell\}.$$

If S is empty, then $V \otimes U$ is irreducible and the result is clear. Suppose then that $S \neq \emptyset$ and assume without loss of generality that $a_1 = b_1$. Write

$$V(\boldsymbol{\pi}_{\lambda_1,a_1}) \otimes V(\boldsymbol{\pi}_{\mu_1,a_1}) = \bigoplus_{\nu \in P^+} m_\nu V(\boldsymbol{\pi}_{\nu,a_1}),$$

where m_ν is the multiplicity with which $V(\nu)$ occurs inside the tensor product of the $\mathfrak{g}$-modules $V(\lambda_1) \otimes V(\mu_1)$. Since $\lambda + \mu - \nu \in Q^+$, it follows from the definition of spectral characters that $\chi_{\boldsymbol{\pi}_{\nu,a_1}} = \chi_{\boldsymbol{\pi}_{\lambda_1,a_1}} + \chi_{\boldsymbol{\pi}_{\mu_1,a_1}}$. The inductive step follows by noting that,

$$V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2) = \left(\bigoplus_{\nu} m_\nu V(\boldsymbol{\pi}_{\nu,a_1})\right) \bigotimes_{s=2}^{k} V(\boldsymbol{\pi}_{\lambda_s,a_s}) \bigotimes_{j=2}^{\ell} V(\boldsymbol{\pi}_{\mu_j,b_j}). \qquad \square$$

Corollary 4.3.

(i) *For all* $\chi_k \in \Xi$, $k = 1, 2$, *we have*

$$\mathcal{C}_{\chi_1} \otimes \mathcal{C}_{\chi_2} \subset \mathcal{C}_{\chi_1+\chi_2}.$$

(ii) *Let* $V \in \mathcal{C}_\chi$, *then* $V^* \in \mathcal{C}_{-\chi}$.

Proof. Let $V_k \in \mathcal{C}_{\chi_k}$, $k = 1, 2$. Since every irreducible constituent of V_k is in $\mathcal{C}_{\chi_k}$, part (i) is immediate from the proposition. For part (ii), suppose that $V^* \in \mathcal{C}_{\chi'}$ for some $\chi' \in \Xi$. Since $V \otimes V^*$ contains the trivial representation of $L(\mathfrak{g})$, it follows that $V \otimes V^* \in \mathcal{C}_0$ and the lemma is proved. $\square$

Proof of Proposition 2.3. Suppose that $V(\boldsymbol{\pi}_\ell)$, $\ell = 1, 2$, are irreducible $L(\mathfrak{g})$ modules with the same spectral character χ. By Proposition 4.2, there exist $\lambda_{1,\ell}, \ldots, \lambda_{s,\ell} \in P^+$, $\ell = 1, 2$ and $a_1, \ldots, a_s \in \mathbf{C}^\times$ such that $\lambda_{j,1} - \lambda_{j,2} \in Q$ and

$$\boldsymbol{\pi}_\ell = \prod_{j=1}^{s} \boldsymbol{\pi}_{\lambda_{j,\ell,a_j}}.$$

If $s = 1$, then the proposition follows from Lemma 4.1. If $\chi = \sum_{j=1}^{s} \chi_{\lambda_j,a_j}$, then it follows from Proposition 2.7 and Lemma 4.1 that $V(\boldsymbol{\pi}_\ell)$ is strongly linked to $\bigotimes_{j=1}^{s} V(\boldsymbol{\pi}_{\bar{\lambda}_{j,\ell},a_j})$. The result follows. $\square$

5. Proof of Proposition 2.4

We begin with the following lemma.

Lemma 5.1. *We have* $W(\boldsymbol{\pi}) \in \mathcal{C}_{\chi_\pi}$.

Proof. In view of Corollary 4.3, it suffices to prove the lemma when $\boldsymbol{\pi} = \boldsymbol{\pi}_{\lambda,a}$. It follows from Proposition 3.3 that every irreducible component of $W(\boldsymbol{\pi})$ is of the form $V(\boldsymbol{\pi}_{\mu,a})$ for some $\mu \in \lambda - Q^+$. The result is now immediate. □

Lemma 5.2.

(i) *Let* $U \in \mathcal{C}_\chi$. *Let* $\boldsymbol{\pi}_0 \in \mathcal{P}$ *be such that* $\chi \neq \chi_{\boldsymbol{\pi}_0}$. *Then* $\mathrm{Ext}^1_{L(\mathfrak{g})}(U, V(\boldsymbol{\pi}_0)) = 0$.
(ii) *Assume that* $V_j \in \mathcal{C}_{\chi_j}$, $j = 1, 2$ *and that* $\chi_1 \neq \chi_2$. *Then* $\mathrm{Ext}^1_{L(\mathfrak{g})}(V_1, V_2) = 0$.

Proof. Since Ext^1 is (bi)additive, to prove (i) it suffices to consider the case when U is indecomposable. Consider an extension,

$$0 \to V(\boldsymbol{\pi}_0) \to V \to U \to 0.$$

We prove by induction on the length of U that the extension is trivial. Suppose first that $U = V(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in \mathcal{P}$ and that $\chi_{\boldsymbol{\pi}} \neq \chi_{\boldsymbol{\pi}_0}$. Then, either

(i) $\lambda_{\boldsymbol{\pi}} < \lambda_{\boldsymbol{\pi}_0}$, or
(ii) $\lambda_{\boldsymbol{\pi}_0} - \lambda_{\boldsymbol{\pi}} \notin (Q^+ - \{0\})$.

Since dualizing the exact sequence above takes us from (i) to (ii), we can assume without loss of generality that we are in case (ii). This implies immediately that

$$L(\mathfrak{n}^+) V_{\lambda_{\boldsymbol{\pi}}} = 0,$$

since $\mathrm{wt}(V(\boldsymbol{\pi}_0)) \subset \lambda_{\boldsymbol{\pi}_0} - Q^+$. On the other hand, since $V_{\lambda_{\boldsymbol{\pi}}}$ maps onto $V(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}}$, we see that $\dim V^{\boldsymbol{\pi}}_{\lambda_{\boldsymbol{\pi}}} \neq 0$. Thus there exists an element $0 \neq v \in V_{\lambda_{\boldsymbol{\pi}}}$ which is a common eigenvector for the action of $L(\mathfrak{h})$ with eigenvalue $\boldsymbol{\pi}$. Since V has length two, it follows that either $V = \mathbf{U}(L(\mathfrak{g}))v$ or that $\mathbf{U}(L(\mathfrak{g}))v = V(\boldsymbol{\pi}_0)$. But the submodule $\mathbf{U}(L(\mathfrak{g}))v$ of V is a quotient of $W(\boldsymbol{\pi})$ and hence has spectral character $\chi_{\boldsymbol{\pi}}$. Since $\chi_{\boldsymbol{\pi}} \neq \chi_{\boldsymbol{\pi}_0}$, we get $V(\boldsymbol{\pi}_0) \cap \mathbf{U}(L(\mathfrak{g}))v = 0$. Hence

$$V \cong V(\boldsymbol{\pi}_0) \oplus \mathbf{U}\big(L(\mathfrak{g})\big)v.$$

This proves that induction begins.

Now assume that U is indecomposable but reducible and that we know the result for all modules of length strictly less than that of U. Let U_1 be a proper non-trivial submodule of U and consider the short exact sequence,

$$0 \to U_1 \to U \to U_2 \to 0.$$

Since $\mathrm{Ext}^1_{L(\mathfrak{g})}(U_j, V(\boldsymbol{\pi}_0)) = 0$ for $j = 1, 2$, by the induction hypothesis, the result follows by using the exact sequence

$$\mathrm{Ext}^1_{L(\mathfrak{g})}\big(U_2, V(\boldsymbol{\pi}_0)\big) \to \mathrm{Ext}^1_{L(\mathfrak{g})}\big(U, V(\boldsymbol{\pi}_0)\big) \to \mathrm{Ext}^1_{L(\mathfrak{g})}\big(U_1, V(\boldsymbol{\pi}_0)\big).$$

Part (ii) is now immediate by using a similar induction on the length of V_2. □

The proof of Proposition 2.4 is now completed as follows. Let V be an indecomposable $L(\mathfrak{g})$-module. We prove that there exists $\chi \in \Xi$ such that $V \in \mathcal{C}_\chi$ by an induction on the length of V. If V is irreducible, follows from the definition of spectral characters that $V \in \mathcal{C}_{\chi_{\boldsymbol{\pi}}}$ for some $\boldsymbol{\pi} \in \mathcal{P}$. If V is reducible, let $V(\boldsymbol{\pi}_0)$ be an irreducible subrepresentation of V and let U be the corresponding quotient. In other words, we have an extension

$$0 \to V(\boldsymbol{\pi}_0) \to V \to U \to 0.$$

Write $U = \bigoplus_{j=1}^r U_j$ where each U_j is indecomposable. By the inductive hypothesis, there exist $\chi_j \in \Xi$ such that $U_j \in \mathcal{C}_{\chi_j}$, $1 \leqslant j \leqslant r$. Suppose that there exists j_0 such that $\chi_{j_0} \neq \chi_{\boldsymbol{\pi}_0}$. Lemma 5.2 implies that

$$\mathrm{Ext}^1_{L(\mathfrak{g})}\big(U, V(\boldsymbol{\pi}_0)\big) \cong \bigoplus_{j=1}^r \mathrm{Ext}^1_{L(\mathfrak{g})}\big(U_j, V(\boldsymbol{\pi}_0)\big) \cong \bigoplus_{j \neq j_0} \mathrm{Ext}^1_{L(\mathfrak{g})}\big(U_j, V(\boldsymbol{\pi}_0)\big).$$

In other words, the exact sequence $0 \to V(\boldsymbol{\pi}_0) \to V \to U \to 0$ is equivalent to one of the form

$$0 \to V(\boldsymbol{\pi}_0) \to U_{j_0} \oplus V' \to U_{j_0} \bigoplus_{j \neq j_0} U_j \to 0,$$

where

$$0 \to V(\boldsymbol{\pi}_0) \to V' \to \bigoplus_{j \neq j_0} U_j \to 0$$

is an element of $\bigoplus_{j \neq j_0} \mathrm{Ext}^1_{L(\mathfrak{g})}(U_j, V(\boldsymbol{\pi}_0))$. But this contradicts the fact that V is indecomposable. Hence $\chi_j = \chi_{\boldsymbol{\pi}_0}$ for all $1 \leqslant j \leqslant r$ and $V \in \mathcal{C}_{\chi_{\boldsymbol{\pi}_0}}$.

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Appendix A

We give an alternate elementary proof of Proposition 1.2. This has the advantage of computing the sequence μ_ℓ of weights explicitly, which is useful in determining precisely the irreducible representations in each block. Further, it also makes precise the algorithm for determining the blocks in the quantum case studied in [6]. We proceed in two steps, namely,

(i) Let $\mu \in P^+$. There exists a sequence of weights $\mu_l \in P^+$, $l = 0, \ldots, m$, with $\mu_0 = \mu$, $\mu_m = \sum_{i \in I_\bullet} s_i \omega_i$, $s_i \in \mathbf{N}^+$, satisfying

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})\big) \neq 0, \quad \forall 1 \leqslant l \leqslant m.$$

(ii) Assume that $\mu = \sum_{i \in I_\bullet} s_i \omega_i \in P^+$. Then, there exists a sequence of weights $\mu_l \in P^+$, $l = 0, \ldots, m$, with $\mu_0 = \lambda_\gamma$, $\mu_m = \mu$ satisfying

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})\big) \neq 0, \quad \forall 0 \leqslant l \leqslant m.$$

We also need the following result proved in [16].

Proposition A.1. Suppose that $\lambda, \mu \in P^+$. Fix a non-zero element $v_{w_0\mu} \in V(\mu)_{w_0\mu}$. Then $V(\lambda) \otimes V(\mu)$ is generated as a $\mathfrak{g}$-module by the element $v_\lambda \otimes v_{w_0\mu}$ and the following defining relations:

$$\big(x_i^+\big)^{-w_0(\mu)(h_i)+1}(v_\lambda \otimes v_{w_0\mu}) = 0, \qquad \big(x_i^-\big)^{\lambda(h_i)+1}(v_\lambda \otimes v_{w_0\mu}) = 0, \quad \forall i \in I.$$

Assume that $\mathfrak{g}$ is of type A_n or C_n. Write $\mu = \sum_{i=1}^n r_i \omega_i$. To prove the first step, we proceed by induction on $k_0 = \max\{1 \leqslant k \leqslant n\colon\ r_k > 0\}$, and show that such a sequence exists and further that $\mu_m = \big(\sum_{i=1}^n i r_i\big)\omega_1$. Clearly, induction starts when $k_0 = 1$. Assume now that we know the result for all $k < k_0$. To complete the inductive step, we proceed by a further induction on r_{k_0}. Defining $\mu_1 = \mu + \sum_{i=1}^{k_0-1} \alpha_i$, it is easily seen that $\mu_1 \in P^+$ and, using Proposition A.1, we have

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu), V(\mu_1)\big) \neq 0.$$

Since

$$\mu_1 = (r_1 + 1)\omega_1 + \sum_{i=2}^{k_0-2} r_k \omega_k + (r_{k_0-1} + 1)\omega_{k_0-1} + (r_{k_0} - 1)\omega_{k_0},$$

the proof of step (1) is now immediate by the inductive hypothesis. To prove the second step, it is enough to show that there exists a sequence of the desired form if $\mu = k\omega_1$ and $\mu_0 = r\omega_1$ are such that $(k - r)\omega_1 \in Q^+$. In the case of C_n it suffices to consider the case $k - r = 2$. Noting that $2\omega_1 = \theta$, we see that by Proposition A.1,

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(r\omega_1), V\big((r+2)\omega_1\big)\big) \neq 0,$$

and the result follows. For A_n, we have to consider the case when $k - r = n + 1$. Consider $\mu_1 = \mu_0 + \theta$ so that

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(r\omega_1), V\big((r+1)\omega_1 + \omega_n\big)\big) \neq 0.$$

By the first step we know that there exists a sequence $\mu_1, \ldots, \mu_m$ with $\mu_1 = (r+1)\omega_1 + \omega_n$ and $\mu_m = (r+n+1)\omega_1$ with

$$\operatorname{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu_k), V(\mu_{k+1})\big) \neq 0$$

and the proof is now complete for A_n.

Suppose that $\mathfrak{g}$ is of type B_n and $\mu = \sum_{i=1}^n r_i\omega_i$. If $r_i = 0$ for $i \neq n$, the first step is obvious. Otherwise, we have $r_k \neq 0$ for some $k < n$. We prove by induction on $k_0 = \min\{1 \leqslant k < n\colon r_{k_0} \neq 0\}$ that we can find the sequence $\mu_1, \ldots, \mu_m$ with $\mu_m = (r_m + 2\sum_{i=1}^n r_i)\omega_n$. When $k_0 = n-1$, consider $\mu_1 = \mu + \alpha_n$. Then, Proposition A.1 implies that

$$\operatorname{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(r_{n-1}\omega_{n-1} + r_n\omega_n), V\big((r_{n-1}-1)\omega_{n-1} + (r_n+2)\omega_n\big)\big) \neq 0,$$

and now an obvious induction on r_{n-1} gives the result. Assume now that $k_0 < n-1$ and that we know the result for all $k > k_0$. We proceed by a further induction on r_{k_0}. Set $\mu_1 = \mu + (\alpha_{k_0+1} + 2(\alpha_{k_0+2} + \cdots + \alpha_n))$. We now proceed as in the case of A_n to complete the first step. For the second step it suffices to prove the existence of the sequence when $\mu = k\omega_n$ and $\mu_0 = r\omega_n$ and $k - r = 2$. To do this observe that if we take $\mu_1 = \mu + \theta$, then

$$\operatorname{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu), V(\mu_1)\big) \neq 0,$$

and the proof of the first step shows that we can connect μ_1 and μ by a sequence of the appropriate form.

Suppose next that $\mathfrak{g}$ is of type D_n with n even and that $\mu = \sum_{i=1}^n r_i\omega_i$. If $r_i = 0$, $i \neq n, n-1$ there is nothing to prove. Otherwise, we have $r_k \neq 0$ for some $k < n-1$. We prove by induction on $k_0 = \min\{1 \leqslant k < n-1\colon r_{k_0} \neq 0\}$ that we can find two sequences $\mu_1, \ldots, \mu_m$, one where

$$\mu_m = \left(r_{n-1} + \sum_{j=0}^{(n-4)/2} r_{2j+1} + 2\sum_{j=1}^{(n-2)/2} r_{2j}\right)\omega_{n-1} + \left(r_n + \sum_{j=0}^{(n-4)/2} r_{2j+1}\right)\omega_n$$

and another where,

$$\mu_m = \left(r_{n-1} + \sum_{j=0}^{(n-4)/2} r_{2j+1}\right)\omega_{n-1} + \left(r_n + \sum_{j=0}^{(n-4)/2} r_{2j+1} + 2\sum_{j=1}^{(n-2)/2} r_{2j}\right)\omega_n.$$

When $k_0 = n-2$ take $\mu_1 = \mu + \alpha_{n-1}$ (respectively $\mu_1 = \mu + \alpha_n$) and proceed by induction on r_{n-2}. To complete the inductive step for $k_0 < n-2$, we take $\mu_1 = \mu + \alpha_{k_0+1} + 2(\alpha_{k_0+2} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$, we omit further details. For the second step, we must prove that $k\omega_i$ and $(k-2)\omega_i$ are connected by an appropriate sequence of elements of P^+ for $i = n, n-1$. As before, we take $\mu_1 = (k-2)\omega_i + \theta$ and use the first step to get the result.

Now consider the case of D_n with n odd and let $\mu = \sum_{i=1}^n r_i\omega_i$. If $r_i = 0$, $i \neq n$ there is nothing to prove. In the general case we proceed in two further steps:

(a) There exists a sequence of weights $\mu_l \in P^+$, $l = 0, \dots, m$, with $\mu_0 = \mu$, $\mu_m = \sum_{i\,\mathrm{odd}} s_i \omega_i$, $s_i \in \mathbf{N}^+$, satisfying

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})\big) \neq 0, \quad \forall 1 \leqslant l \leqslant m.$$

(b) Assume that μ is supported only on the odd nodes. Then, there exists a sequence of weights $\mu_l \in P^+$, $l = 0, \dots, m$, with $\mu_0 = \mu$, $\mu_m = \sum_{i \in I_\bullet} s_i \omega_i$, $s_i \in \mathbf{N}^+$, satisfying

$$\mathrm{Hom}_{\mathfrak{g}}\big(\mathfrak{g} \otimes V(\mu_l), V(\mu_{l+1})\big) \neq 0, \quad \forall 1 \leqslant l \leqslant m.$$

To prove step (a), we assume that $r_k > 0$ for some k even and proceed by induction on $k_0 = \min\{k \text{ even: } r_k > 0\}$. First, assume that $k_0 = n-1$ and proceed by a further induction on r_{n-1} as usual. Setting

$$\begin{aligned}\mu_1 &= \mu + (\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n) \\ &= \sum_{j=1}^{(n-3)/2} r_{2j+1}\omega_{2j+1} + (r_1+1)\omega_1 + (r_n+1)\omega_n + (r_{n-1}-1)\omega_{n-1},\end{aligned}$$

and using the induction on $r_{n-1} - 1$ completes this case. Next, suppose that $k_0 = n-3$ and take

$$\begin{aligned}\mu_1 &= \mu + (\alpha_{n-2} + \alpha_{n-1} + \alpha_n) \\ &= \sum_{j=0}^{(n-3)/2} r_{2j+1}\omega_{2j+1} + (r_{n-1}+1)\omega_{n-1} + (r_n+1)\omega_n + (r_{n-3}-1)\omega_{n-3}\end{aligned}$$

and the result follows by induction on r_{n-3}. Now assume that $k_0 < n-3$ and that we know the result for all $k > k_0$. Taking $\mu_1 = \mu + (\alpha_{k_0+1} + 2(\sum_{i=k_0+2}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_{n-2})$. Then

$$\mu_1 = \sum_{j=0}^{(n-1)/2} r_{2j+1}\omega_{2j+1} + \sum_{j=(k_0+4)/2}^{(n-1)/2} r_{2j}\omega_{2j} + (r_{k_0+2}+1)\omega_{k_0+2} + (r_{k_0}-1)\omega_{k_0}$$

completes the inductive step. Observe that when $k_0 = 2$, we have

$$\mu_m = \sum_{j=1}^{(n-3)/2} r_{2j+1}\omega_{2j+1} + \left(r_1 + \sum_{j=1}^{(n-1)/2} r_{2j}\right)\omega_1 + \left(r_n + r_{n-1} + 2\sum_{j=1}^{(n-3)/2} r_{2j}\right)\omega_n.$$

Now we prove step (b), i.e., $r_j = 0$ for all $1 \leqslant j \leqslant n$ with j even. We proceed by induction on $k_0 = \min\{k\colon r_k > 0\}$ and on r_{k_0}. If $k_0 = n$, there is nothing to prove. If $k_0 = n-2$, then taking $\mu_1 = \mu + \alpha_n = (r_n+2)\omega_n + (r_{n-2}-1)\omega_{n-2}$ completes the induction.

Now assume that $k_0 < n-2$. Taking $\mu_1 = \mu + (\alpha_{k_0+1} + 2(\sum_{i=k_0+2}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_{n-2})$, we see that

$$\mu_1 = \sum_{j=(k_0+3)/2}^{(n-1)/2} r_{2j+1}\omega_{2j+1} + (r_{k_0+2}+1)\omega_{k_0+2} + (r_{k_0}-1)\omega_{k_0}.$$

This completes the proof of the first step, notice that following this procedure gives

$$\mu_m = \left(r_n + 3r_{n-1} + 2\sum_{j=0}^{(n-3)/2} r_{2j+1} + 4\sum_{j=1}^{(n-3)/2} r_{2j}\right)\omega_n.$$

The second step is completed by the usual method and we omit all details.

$\mathfrak{g} = E_6$. Consider the following sequence of weights:

$$\lambda_1 = (r_1 + r_6)\omega_1 + r_2\omega_2 + r_3\omega_3 + r_4\omega_4 + (r_5 + r_6)\omega_5,$$
$$\lambda_2 = (r_1 + r_3 + r_6)\omega_1 + (r_2 + r_3)\omega_2 + r_4\omega_4 + (r_5 + r_6)\omega_5,$$
$$\lambda_3 = (r_1 + r_3 + r_6)\omega_1 + (r_2 + r_3)\omega_2 + (2r_4 + r_5 + r_6)\omega_5,$$
$$\lambda_4 = (r_1 + r_3 + r_6)\omega_1 + (r_2 + r_3 + 2r_4 + r_5 + r_6)\omega_2,$$
$$\lambda_5 = (r_1 + 2r_2 + 3r_3 + 4r_4 + 2r_5 + 3r_6)\omega_1.$$

Setting $\mu = \lambda_0$, it suffices to show that λ_k and λ_{k+1} are connected by a sequence of weights as in (i) above. But this is clear from Proposition A.1, by noting that

$$\lambda_1 - \lambda_0 = r_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5),$$
$$\lambda_2 - \lambda_1 = r_3(\alpha_1 + \alpha_2),$$
$$\lambda_3 - \lambda_2 = r_4\alpha_5,$$
$$\lambda_4 - \lambda_5 = (2r_4 + r_5 + r_6)(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6),$$
$$\lambda_5 - \lambda_6 = (r_2 + r_3 + 2r_4 + r_5 + r_6)\alpha_1.$$

To prove the second step, we can assume that $\mu_0 = r\omega_1$, $\mu = k\omega_1$, and $k - r = 3$. Take

$$\mu_1 = \mu_0 + \theta = \mu + \omega_6.$$

Then by Proposition A.1, we have $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\mu), V(\mu_1)) \neq 0$. On the other hand, we see from step (i) that there exists an appropriate sequence connecting μ_1 and $(r+3)\omega_1$. The result is proved for E_6.

$\mathfrak{g} = E_7$. Consider the following sequence of weights:

$$\lambda_1 = (r_1 + r_7)\omega_1 + r_2\omega_2 + r_3\omega_3 + r_4\omega_4 + r_5\omega_5 + (r_6 + r_7)\omega_6,$$
$$\lambda_2 = (r_1 + r_4 + r_7)\omega_1 + r_2\omega_2 + (r_3 + r_4)\omega_3 + r_5\omega_5 + (r_6 + r_7)\omega_6,$$
$$\lambda_3 = (r_1 + r_4 + r_7)\omega_1 + r_2\omega_2 + (r_3 + r_4)\omega_3 + (r_6 + r_7 + 2r_5)\omega_6,$$
$$\lambda_4 = (r_1 + r_4 + r_7)\omega_1 + (r_2 + r_6 + r_7 + 2r_5)\omega_2 + (r_3 + r_4)\omega_3,$$
$$\lambda_5 = (r_1 + r_3 + 2r_4 + r_7)\omega_1 + (r_2 + r_3 + r_4 + 2r_5 + r_6 + r_7)\omega_2,$$
$$\lambda_6 = (r_1 + 2r_2 + 3r_3 + 4r_4 + 4r_5 + 2r_6 + 3r_7)\omega_1.$$

Setting $\mu = \lambda_0$, we see again that λ_k and λ_{k+1} are connected by an appropriate sequence. For the second step, we can assume that $\mu_0 = r\omega_1$, $\mu = k\omega_1$ with $k - r = 2$. Taking $\mu_1 = \mu_0 + \theta = \mu + \omega_6$, we find from step (i) that μ_1 and $(k + 2)\omega_1$ are connected and we are done.

$\mathfrak{g} = E_8$. Consider the following sequence of weights:

$$\lambda_1 = (r_1 + r_8)\omega_1 + r_2\omega_2 + r_3\omega_3 + r_4\omega_4 + r_5\omega_5 + r_6\omega_6 + (r_7 + r_8)\omega_7,$$
$$\lambda_2 = (r_1 + r_5 + r_8)\omega_1 + r_2\omega_2 + r_3\omega_3 + (r_4 + r_5)\omega_4 + r_6\omega_6 + (r_7 + r_8)\omega_7,$$
$$\lambda_3 = (r_1 + r_5 + r_8)\omega_1 + r_2\omega_2 + r_3\omega_3 + (r_4 + r_5)\omega_4 + (r_7 + r_8 + 2r_6)\omega_7,$$
$$\lambda_4 = (r_1 + r_4 + r_7)\omega_1 + (r_2 + 2r_6 + r_7 + r_8)\omega_2 + r_3\omega_3 + (r_4 + r_5)\omega_4,$$
$$\lambda_5 = (r_1 + 2r_4 + r_5 + r_7)\omega_1 + (r_2 + 2r_6 + r_7 + r_8)\omega_2 + (r_3 + r_4 + r_5)\omega_3,$$
$$\lambda_6 = (r_1 + r_3 + 3r_4 + 2r_5 + r_7)\omega_1 + (r_2 + r_3 + r_4 + r_5 + 2r_6 + r_7 + r_8)\omega_2,$$
$$\lambda_7 = (r_1 + 2r_2 + 3r_3 + 5r_4 + 4r_5 + 4r_6 + 3r_7 + 2r_8)\omega_1.$$

Setting $\mu = \lambda_0$, we see again that λ_k and λ_{k+1} are connected by an appropriate sequence. For the second step, we can assume that $\mu_0 = r\omega_1$, $\mu = k\omega_1$ with $k - r = 1$. Taking $\mu_1 = \mu_0 + \theta = \mu_0 + \omega_1 = \mu$ and we are done.

$\mathfrak{g} = F_4$. Consider the following sequence of weights:

$$\lambda_1 = (r_1 + 2r_2)\omega_1 + r_3\omega_3 + r_4\omega_4,$$
$$\lambda_2 = (r_1 + 2r_2)\omega_1 + (r_4 + 2r_3)\omega_4,$$
$$\lambda_3 = (r_1 + 2r_2 + 4r_3 + 2r_4)\omega_1.$$

Setting $\mu = \lambda_0$, we see again that λ_k and λ_{k+1} are connected by an appropriate sequence. For the second step we can assume that $\mu = r\omega_1$, with $r \neq 0$. Then we define

$$\mu_1 = \mu + (\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4),$$
$$\mu_2 = \mu_1 + \alpha_1,$$
$$\mu_3 = \mu_2 - (2\alpha_1 + 2\alpha_2 + \alpha_3),$$
$$\mu_4 = \mu_3 - \theta$$

and the result is proved by induction on r, noting that $\mu_4 = (r-1)\omega_1$.

$\mathfrak{g} = G_2$. Here we define $\lambda_1 = \mu + r_2(3\alpha_1 + \alpha_2)$ to see that μ and $(r_1 + 3r_2)\omega_1$ are connected as in step (i). To prove step (ii), we use the fact that $r\omega_1 + (2\alpha_1 + \alpha_2) = (r+1)\omega_2$ to get the result.

References

[1] V. Chari, Integrable representations of affine Lie algebras, Invent. Math. 85 (2) (1986) 317–335.
[2] V. Chari, A. Pressley, New unitary representations of loop groups, Math. Ann. 275 (1) (1986) 87–104.
[3] V. Chari, A. Pressley, Quantum affine algebras, Comm. Math. Phys. 142 (2) (1991) 261–283.
[4] V. Chari, A. Pressley, Weyl modules for classical and quantum affine algebras, Represent. Theory 5 (2001) 191–223.
[5] J. Dixmier, Algebres envelopantes, Cahiers Sci. 37 (1974).
[6] P. Etingof, A. Moura, Elliptic central characters and blocks of finite-dimensional representations of quantum affine algebras, Represent. Theory 7 (2003) 346–373.
[7] B. Feigin, S. Loktev, On generalized Kostka polynomials and quantum Verlinde rule, D. Fuchs 60th anniversary volume, preprint, math.QA/9812093, in press.
[8] B. Feigin, S. Loktev, Multi-dimensional Weyl modules and symmetric functions, preprint, math.QA/0212001.
[9] E. Frenkel, E. Mukhin, Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras, Comm. Math. Phys. 216 (1) (2001) 23–57.
[10] E. Frenkel, N. Reshetikhin, The q-characters of representations of quantum affine algebras and deformations of $\mathcal{W}$-algebras, in: Recent Developments in Quantum Affine Algebras and Related Topics, Raleigh, NC, 1998, in: Contemp. Math., vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 163–205.
[11] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, in: Grad. Texts in Math., Springer-Verlag, Berlin, 1970.
[12] M. Kashiwara, On level-zero representation of quantized affine algebras, Duke Math. J. 112 (1) (2002) 117–195.
[13] B. Kostant, Lie group representations on polynomial rings, Amer. J. Math. 85 (1963) 327–404.
[14] H. Nakajima, t-analogue of the q-characters of finite dimensional representations of quantum affine algebras, in: Physics and Combinatorics, Nagoya, 2000, World Scientific, River Edge, NJ, 2001, pp. 196–219.
[15] H. Nakajima, Extremal weight modules of quantum affine algebras, preprint, math.QA/0204183.
[16] K.R. Parthasarathy, R. Ranga Rao, V.S. Varadarajan, Representations of complex semi-simple Lie groups and Lie algebras, Ann. of Math. 85 (1967) 38–429.