Topology Vol. 3, Suppl. 2, pp. 199–238. Pergamon Press, 1965. Printed in Great Britain

HIGHER ORDER GRASSMANN BUNDLES†

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(*Received* 1 *August* 1963)

INTRODUCTION

THE AIM of this paper is to discuss higher-order Grassmann bundles as a setting for (non-linear) partial differential equations (including systems of such equations). The kinds of equations we have in mind are those whose solutions are submanifolds of a given manifold M, e.g. the equation for a p-dimensional minimal surface in a d-dimensional Riemannian manifold. From a geometric point of view a system of kth order partial differential equations assigns at each point m of M some collection of kth order contact spaces there, a kth order contact space at m being a linear subspace of the kth order tangent vectors at m; a solution is then a submanifold N of M such that the kth order tangent space to N at each $n \in N$ is one of the given contact spaces at n. For example, in the minimal surface equation (usually called a system of equations) one is given at each $m \in M$ (M being assumed Riemannian) a family of second-order tangent spaces at each point of M, namely all those whose first-order part is p-dimensional and such that the trace of the second fundamental form of the second order space, relative to any first-order tangent vector which is perpendicular to this p-dimensional space, vanishes. (One can define a second fundamental form for a second-order tangent space at a point—a whole submanifold is unnecessary.)

We attempt to formulate systems of partial differential equations of this kind geometrically because they arise geometrically; a co-ordinate expression for such equations seems to be an extra complication, depending on an arbitrary choice of a co-ordinate system.

Now we indicate how the usual expression for a partial differential equation (or system—but hereafter we shall use the term 'partial differential equation' to include what are usually called 'systems') can be transcribed into geometrical language. First consider a single first order equation for a single unknown function, which is usually written as

$$f\left(x_1, \dots, x_p, u, \dots, \frac{\partial u}{\partial x_i}, \dots\right) = 0,$$

u being the 'unknown' function and $x_1, \dots, x_p$ the 'independent variables'. Consider the graph of u; at a point $(x_1, \dots, x_p, u) = (x, u)$ of that graph the $\dfrac{\partial u}{\partial x_i}$ are the slopes of the tangent plane to the graph at (x, u), and characterize that p-plane. So f may be considered

† This research was partially supported by the National Science Foundation.

as a function on p-planes at points in R^{p+1}. Then a solution of this equation is a function u such that f vanishes at every p-plane tangent to the graph of u. We may generalize this situation, replacing R^{p+1} by a manifold M, and taking f to be a function on $G_p(M)$, where $G_p(M)$ is the Grassmann bundle over M whose elements are all the (m, P) such that $m \in M$ and P is a p-dimensional subspace of the tangent space at m. Then a 'solution' will be any p-dimensional submanifold N of M such that at each point n of N, f vanishes on the tangent plane to N. So a solution is now a submanifold rather than a function; the possibility above of representing the submanifold as the graph of a function u was related to a particular co-ordinate system for R^{p+1} so we are willing to drop that feature.

Now suppose we have a single first order equation for a family of 'unknown' functions, written classically as

$$f\left(x_1, \dots, x_p, u_1, \dots, u_q, \dots, \frac{\partial u_r}{\partial x_i}, \dots\right) = 0.$$

Letting $u = (u_1, \dots, u_q)$ the graph of u is a submanifold of R^{p+q} and f can be considered as a function on p-planes at points of R^{p+q}; a solution is a u such that f vanishes on the p-planes tangent to the graph of u. Similarly, if we have a collection $f_1, \dots, f_s$ of such f's; a solution is still a $u = (u_1, \dots, u_q)$ whose graph lies in R^{p+q}, but such that all the f_k vanish on the graph of u. So to generalize this as above we replace R^{p+q} by a manifold M of dimension $d = p + q$, replace the f_k's by functions on $G_p(M)$, and define a solution to be a p-dimensional submanifold N of M such that at each point of N all the f_k vanish on the tangent plane to N. The only difference from the preceeding case is that now d–$p = q$ instead of d–$p = 1$. So in our general formulation we define the 'number of unknown functions' to be d–p.

Now we note that the only feature of f, or of the f_k, which was used above was the set of its zeros, or the set of common zeros in the case of more than one f. So letting E be this set of zeros, a solution is a submanifold N of M whose tangent space at each point is in E. For this reason we shall define a 'system of first-order partial differential equations', depending on a given manifold M and integer p, to be a subset E of $G_p(M)$. If N is any p-dimensional submanifold of M it has a natural lift ${}^{[1]}N$ which is a submanifold of $G_p(M)$, i.e. ${}^{[1]}N$ consists of all $(n, P) \in G_p(M)$ such that $n \in N$ and P is the tangent space to N at n. Then N is defined to be a *solution* of E if and only if ${}^{[1]}N \subseteq E$.

We have been discussing first-order systems. Now we turn to higher-order systems. The concepts here can be formulated as above, but using higher-order tangent vectors, higher-order spaces (i.e. spaces of these higher-order tangent vectors) and Grassmann bundles of these higher-order spaces. But at this point there arises something which is the main concern of this paper, namely, the relation between these higher-order Grassmann bundles and the iterated first-order bundles. By the iterated first-order bundles we mean $G_p(G_p(M))$, etc. This relation seems important to us for the following reasons: (1) Through it we can express in general the fact that every system of partial differential equations is equivalent to a first-order system, (2) In removing the co-ordinate systems from the notion of a partial differential equation one loses the fact that each higher order derivative is an iterate of lower order derivatives. This loss is restored, however, by a theorem which we call the Kuranishi factoring theorem, which says that every higher-order contact space is uniquely expressible as a 'product' of first-order integrable contact spaces; however the factors are first-order tangent spaces to successive first-order Grassmann bundles. Thus this theorem seems to us to restore the gradation of derivatives and to provide an important structural element to the higher-order Grassmann bundles.

This paper begins with a discussion of integrability conditions and leads up to the Kuranishi factoring theorem (Theorem 4.1). This theorem occurs in Kuranishi [3] in a purely co-ordinate form and is also intertwined with prolongations of differential systems. Our contribution is to give the theorem a geometric setting. It ends with a discussion of characteristics of (non-linear) partial differential equations.

We are greatly indebted to both James Simons and I. M. Singer, first for many discussions of matters considered here, but more importantly, for the very concepts on which this paper is based. The notion of an integrable element of an iterated Grassmann bundle was pointed out to us by Simons; he not only pointed out that there was an important notion here but he also explained to us that such elements were characterized by the vanishing of certain differential forms. Our characterization of these forms as 'lift-forms' and differentials of lift forms is our way of describing these forms. But the notion of these lift forms we owe to I. M. Singer, who pointed out to us that these were the essential feature of certain matters in [3]. We also owe to I. M. Singer the procedure for passing from an f defined on M to the related $e_a f$ on $S_p^z(M)$, used in section 1. We are also indebted to H. Wu for reading and criticizing this paper.

NOTATION

We use $u_1, \dots, u_n$ for the usual co-ordinate functions on R^n. Usually the integers i, j will satisfy $1 \leq i, j \leq p$, though sometimes they will be allowed to be 0; the integers r, s will usually satisfy $p + 1 \leq r, s \leq d$, and a, b will be integers satisfying $1 \leq a, b \leq d$.

§1. ITERATED STIEFEL BUNDLES AND THEIR INTEGRABLE POINTS

Let p be any integer with $1 \leq p \leq d$ and we now define the Stiefel bundle $S_p(M)$ over M. The elements of $S_p(M)$ shall be, by definition, all the $(m, e_1, \dots, e_p)$ where m is any point of M and $e_1, \dots, e_p$ is any ordered set of p linearly independent elements from M_m. We define the projection map $\pi : S_p(M) \to M$, by $\pi(m, e_1, \dots, e_p) = m$. Co-ordinate systems of $S_p(M)$ are defined as follows. For any co-ordinate system $\{x_a\}$ of M with domain Q we define a co-ordinate system for $S_p(M)$, with domain $\pi^{-1}(Q)$, consisting of the functions x_a^0 and x_a^i defined (for $1 \leq a \leq d$, $1 \leq i \leq p$) by

$$x_a^0 = x_a \circ \pi$$

$$x_a^i(m, e_1, \dots, e_p) = dx_a(e_i) = e_i x_a.$$

Thus $S_p(M)$ has dimension $d + pd$.

Now we define, for any non-negative integer z, the zth iterated Stiefel bundle $S_p^z(M)$ by $S_p^z(M) = S_p(S_p^{z-1}(M))$, making the convention that $S_p^0(M) = M$, $S_p^1(M) = S_p(M)$.

Iterating the procedure used above to define a co-ordinate system for $S_p(M)$ from a given co-ordinate system for M we obtain, starting with a co-ordinate system $\{x_a\}$ of M, a co-ordinate system of $S_p^z(M)$, consisting of functions x_a^α, where $1 \leq a \leq d$ and α runs through all $\alpha = (\alpha_1, \ldots, \alpha_z)$ such that the α_w are integers with $0 \leq \alpha_w \leq p$. That is, if such a co-ordinate system $\{x_a^\alpha\}$ has been defined for $S_p^{z-1}(M)$ (α' running through the $(\alpha_1, \ldots, \alpha_{z-1})$) we then obtain the co-ordinate system $\{x_a^\alpha\}$ of $S_p^z(M)$ by

$$x_a^{(\alpha',0)} = x_a^{\alpha'} \circ \pi$$
$$x_a^{(\alpha',i)}(m, e_1^1, \ldots, e_p^1, \ldots, e_1^z, \ldots, e_p^z) = e_i^z x_a^{\alpha'}.$$

Note that among these are functions $x_a^{(0,\ldots,0)}$ which we shall usually denote by x_a^0, this last superscript 0 denoting the zero element of R^z; for induction purposes we shall sometimes write x_a^0 also for x_a, considering this zero superscript as the zero element of R^0. The context will always show where the 0 really lies.

If ϕ is any non-singular map of an open Q in R^p into M then ϕ has a natural lift, which we denote by ϕ^1, mapping Q into $S_p(M)$. ϕ^1 is defined by

$$\phi^1(q) = \left(\phi(q), \phi_* \frac{\partial}{\partial u_1}(q), \ldots, \phi_* \frac{\partial}{\partial u_p}(q)\right).$$

Iterating this procedure we define the zth lift ϕ^z, a map of Q into $S_p^z(M)$ by

$$\phi^z(q) = \left(\phi^{z-1}(q), \phi_*^{z-1} \frac{\partial}{\partial u_1}(q), \ldots, \phi_*^{z-1} \frac{\partial}{\partial u_p}(q)\right)$$

and we clearly have

$$\pi \circ \phi^z = \phi^{z-1} \qquad \pi = \pi[S_p^z(M) \to S_p^{z-1}(M)]$$

If (m, e) is any point of $S_p(M)$ it is trivial that there exists such a ϕ with $\phi^1(q) = (m, e)$. However, the corresponding property for $S_p^z(M)$ is false if $z > 1$.

DEFINITION. *A point $(m, e) \in S_p^z(M)$ is* integrable *if and only if there is a non-singular map of an open Q in R^p into M and a point $q \in Q$ such that $\phi^z(q) = (m, e)$.*

The purpose of this section is to prove Theorem (1.1) below, which characterizes integrable points intrinsically, i.e. without referring to a ϕ as above. At this point we note that if $(m, e_1^1, \ldots, e_p^1, \ldots, e_1^z, \ldots, e_p^z)$ is an integrable point of $S_p^z(M)$ then

$$\text{(1.1)} \qquad \pi_* e_i^z = e_i^{z-1} \qquad (1 \leq i \leq p) \pi = \pi[S_p^{z-1}(M) \to S_p^{z-2}(M)]).$$

This is proved inductively from

$$\pi_* e_i^z = \pi_* \circ \phi_*^{z-1}\left(\frac{\partial}{\partial u_i}(q)\right) = \phi_*^{z-2}\left(\frac{\partial}{\partial u_i}(q)\right) = e_i^{z-1}.$$

Notation. We shall write $\alpha = (\alpha_1, \ldots, \alpha_z)$ and $\beta = (\beta_1, \ldots, \beta_z)$ where the α_w and β_w are integers with $0 \leq \alpha_w, \beta_w \leq p$; this meaning for the letters α and β will be fixed throughout this paper. We say β is a *permutation* of α if and only if there is a permutation π of $\{1, \ldots, z\}$ such that $\beta_w = \alpha_{\pi w}$ for all w, and in this case we write $\beta = \pi\alpha$. We define $|\alpha|$ to be the number of w for which $\alpha_w \neq 0$ and $\alpha! = n_1! \ldots n_p!$ where n_i is the number of w for which $\alpha_w = i$. We shall also use this notation $|\mu|$ and $\mu!$ below where $\mu = (\mu_1, \ldots, \mu_z)$ with $0 \leq \mu_w \leq d$, replacing p by d in the above definitions.

We now point out that each $(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ in $S_p^z(M)$ gives rise to a family $\{e_\alpha\}$ of tangent vectors of order $\leq z$ at m. To define the e_α we must define $e_\alpha f$ for f in C^∞ at m. We first note that each such f gives rise to $p+1$ functions, $f^0, f^1, \dots, f^p$ of $S_p(M)$ defined (on $\pi^{-1}(Q)$ where Q is the domain of f) by

$$f^0(m, e_1, \dots, e_p) = f(m)$$

$$f^i(m, e_1, \dots, e_p) = e_i f \qquad (1 \leq i \leq p).$$

It will be convenient also to write $e_0 f = f(m)$, so the preceding becomes

$$f^j(m, e_1, \dots, e_p) = e_j f \qquad (0 \leq j \leq p).$$

Now we iterate this procedure to define, for any such f, functions f^α of $S_p^z(M)$, i.e.

$$f^\alpha = f^{(\alpha_1, \dots, \alpha_z)} = (\dots (f^{\alpha 1})^{\alpha 2} \dots)^{\alpha z}$$

and in particular, we have

$$e_\alpha f = e_{\alpha_z}^z f^{(\alpha_1, \dots, \alpha_{z-1})} = f^\alpha(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z).$$

If $\{x_a\}$ is a co-ordinate system of M then this definition of x_a^α coincides with that used above in defining a co-ordinate system of $S_p^z(M)$.

It is easily verified that e_α is a tangent vector of order $|\alpha|$ at m. As such we have the usual representation

$$e_\alpha = \sum (e_\alpha x_\mu / \mu!) \frac{\partial}{\partial x_\mu} \tag{1.2}$$

where μ ranges through all $\mu = (\mu_1, \dots, \mu_z)$ such that $0 \leq \mu_w \leq d$, and we use the following conventions,

$$\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_{\mu_1}} \cdots \frac{\partial}{\partial x_{\mu_z}}, \frac{\partial f}{\partial x_0} = f.$$

If $(m, e) \in S_p^z(M)$ we let $H(m, e) = (m, \{e_\alpha\})$ where $\{e_\alpha\}$ is the family of tangent vectors (of various orders) at m obtained, as above, from (m, e), thus defining a map H from $S_p^z(M)$ into certain families of tangent vectors. H is $1:1$ because if $H(m, e) = H(m, e^*)$ then $e_\alpha = e_\alpha^*$, hence, for any co-ordinate system $\{x_a\}$ at m, $e_\alpha x_a = e_\alpha^* x_a$, i.e. $x_a^\alpha(m, e) = x_a^\alpha(m, e^*)$ for all a, α. Since the x_a^α are a co-ordinate system this shows $(m, e) = (m, e^*)$.

Now we show that (1.1) *is equivalent to*

$$e_i^z e_0^{z-1} e_{\alpha''} = e_0^z e_i^{z-1} e_{\alpha''} \qquad (0 \leq i \leq p). \tag{1.1'}$$

Proof. Let $\alpha = (i, 0, \alpha'')$, $\alpha^* = (0, i, \alpha'')$. If (1.1) holds then

$$\begin{aligned} e_i^z e_0^{z-1} e_{\alpha''} f &= e_i^z e_0^{z-1} f^{\alpha''} = e_i^z (f^{\alpha''} \circ \pi) = e_i^{z-1} f^{\alpha''} \\ &= f^{(\alpha'', i)} = e_0 f^{(\alpha'', i)} = e_0^z e_i^{z-1} e_{\alpha''} f \end{aligned}$$

On the other hand, if (1.1′) holds then the two ends of this string of equalities are equal, hence the middle equality must hold, since the others are true by definition. And the middle equality is (1.1).

By virtue of (1.1′) the condition (1.3) below includes (1.1). We shall prove in the following theorem that (1.3) characterizes integrable points.

LEMMA (1.1). *If* $(m, e_1^1, \ldots, e_p^1, \ldots, e_1^z, \ldots, e_p^z)$ *is an integrable point of* $S_p^z(M)$ *then*

$$e_\alpha = e_\beta \quad \text{if } \beta \text{ is a permutation of } \alpha. \tag{1.3}$$

In fact, if ϕ *is a non-singular map of an open* Q *in* R^p *into* M *with* $\phi^z(q) = (m, e_1^1, \ldots, e_p^1, \ldots, e_1^z, \ldots, e_p^z)$ *then*

$$\phi_* \frac{\partial}{\partial u_\alpha}(q) = e_\alpha. \tag{1.3'}$$

Proof. It is sufficient to prove (1.3′), i.e. that

$$e_\alpha f = \frac{\partial(f \circ \phi)}{\partial u_\alpha}(q)$$

for f in C^∞ at $m = \phi(q)$. For this it is sufficient to prove

$$f^\alpha \circ \phi^z = \frac{\partial(f \circ \phi)}{\partial u_\alpha} \tag{1.3''}$$

If $\alpha = (\alpha_1, \ldots, \alpha_z)$ we let $\alpha' = (\alpha_1, \ldots, \alpha_{z-1})$. Then (1.3″) follows by induction on z from

$$\begin{aligned} f^\alpha \circ \phi^z &= \phi_*^{z-1}\left(\frac{\partial}{\partial u_{\alpha_z}}\right) f^{\alpha'} = \frac{\partial}{\partial u_{\alpha_z}}(f^{\alpha'} \circ \phi^{z-1}) \\ &= \frac{\partial}{\partial u_{\alpha_z}}\left(\frac{\partial(f \circ \phi)}{\partial u_{\alpha'}}\right) = \frac{\partial}{\partial u_\alpha}(f \circ \phi). \end{aligned}$$

We remark that (1.3′) shows, on the zth order tangent spaces at points of R^p, and for ϕ as above, that

$$(H \circ \phi^z)(q) = \left(m, \left\{\phi_* \frac{\partial}{\partial u_\alpha}(q)\right\}\right). \tag{1.4}$$

We now state a generalized Leibnitz product rule for our derivatives e_α for which we need the following notation. Let E be any subset of $\{1, \ldots, z\}$. We define the *support* of α by $\operatorname{supp} \alpha = [w | \alpha_w \neq 0]$ and $E\alpha = \alpha^*$ where $\alpha_w^* = \alpha_w$ if $w \in E$, $\alpha_w^* = 0$ if $w \notin E$. Then the product rule, which is easily proved by induction, is: if $f_1, \ldots, f_w$ are functions in C^∞ at m, then

$$(f_1 \ldots f_w)^\alpha = \sum f_1^{E_1\alpha} \ldots f_w^{E_w\alpha} \tag{1.5}$$

where this sum is over all partitions of supp α into w subsets, i.e. over all choices of ordered families of subsets $E_1, \ldots, E_w$ of supp α such that each E_v is disjoint from E_u if $u \neq v$ and $\bigcup_v E_v = \operatorname{supp} \alpha$; in this we include those partitions in which any number of the E_v may be empty and we emphasize that for each $E_1, \ldots, E_w$ which occurs, each permutation of it will also occur. We note that in another notation (1.5) reads:

$$e_\alpha(f_1 \cdot \ldots \cdot f_w) = \sum ((e_{E_1\alpha})(f_1)) \ldots ((e_{E_w\alpha})(f_w)) \tag{1.5'}$$

and we point out that by previous conventions, $e_{(0, \ldots, 0)} f = f(m)$, and $f^{(0, \ldots, 0)} = f$.

THEOREM (1.1). *The following conditions on a point* $(m, e) = (m, e_1^1, \ldots, e_p^1, \ldots, e_1^z, \ldots, e_p^z)$ *of* $S_p^z(M)$ *are equivalent:*

(a) (m, e) *is integrable,*

(b) $e_\alpha = e_\beta$ *if* β *is a permutation of* α,

(c) *there exists a co-ordinate system* $\{x_a\}$ *of* M *at* m *such that all* $x_a(m) = 0$ *and*

$$e_\alpha = \frac{\partial}{\partial x_\alpha}(m) \qquad \text{for all } \alpha, \tag{1.6}$$

(d) *there exists a co-ordinate system* $\{x_a\}$ *of* M *at* m *such that* $x_a^\alpha(m, e) = x_a^\beta(m, e)$ *whenever* β *is a permutation of* α,

(e) *for all co-ordinate systems* $\{x_a\}$ *of* M *at* m, $x_a^\alpha(m, e) = x_a^\beta(m, e)$ *whenever* β *is a permutation of* α.

Proof. Lemma (1.1) says (a) implies (b). It is trivial that (c) implies (a) because, restricting the homeomorphism that defines the x_a to its first p co-ordinates gives a non-singular ϕ as above for which

$$\phi_* \frac{\partial}{\partial u_\alpha}(q) = \frac{\partial}{\partial x_\alpha}(m) = e_\alpha.$$

Using (1.4) and the fact that H is $1:1$ we see that $\phi^z(q) = (m, e)$, proving (a). Clearly (b) implies (e) since $x_a^\alpha(m, e) = e_\alpha x_a$, and (e) contains (d). Hence it will be sufficient to prove (d) implies (b) and (b) implies (c). Proof that (d) implies (b): From (d) we have $e_\alpha x_a = e_\beta x_a$ whenever β is a permutation of α. Let $x_\mu = x_{\mu_1} \dots x_{\mu_w}$ with $1 \le \mu_v \le d$ and we shall show

(i) $e_\beta x_\mu = e_\alpha x_\mu$ if β is a permutation of α.

By (1.2) this will prove (b).

To prove (i) we first observe

(ii) $\pi(E\alpha) = (\pi^{-1}E)(\pi\alpha)$

for all α and all subsets E of $\{1, \dots, z\}$, π being any permutation of $\{1, \dots, z\}$. We also observe that when $E_1, \dots, E_w$ run through all partitions of supp α then $\pi^{-1}E_1, \dots, \pi^{-1}E_w$ run through all partitions of supp β, if $\beta = \pi\alpha$. The following calculation, using (1.5), (ii) and (d), now proves (i):

$$\begin{aligned}
e_\beta x_\mu &= \sum (e_{(\pi^{-1}E_1)\beta})x_{\mu_1} \dots (e_{(\pi^{-1}E_w)\beta})x_{\mu_w} \\
&= \sum (e_{(\pi^{-1}E_1)(\pi\alpha)})x_{\mu_1} \dots (e_{(\pi^{-1}E_w)(\pi\alpha)})x_{\mu_w} \\
&= \sum (e_{\pi(E_1\alpha)}x_{\mu_1}) \dots (e_{\pi(E_w\alpha)}x_{\mu_w}) \\
&= \sum (e_{E_1\alpha}x_{\mu_1}) \dots (e_{E_w\alpha}x_{\mu_w}) \\
&= e_\alpha x_\mu.
\end{aligned}$$

Proof that (b) implies (c): We induct on z. This is trivial for $z = 1$ and we now show it for z assuming it for $z - 1 \le 1$. Let (m, e) be any point of $S_p^z(M)$. By the induction assumption there is a co-ordinate system x_a^* at m such that all $x_a^*(m) = 0$ and

(i) $e_{\alpha'} = \dfrac{\partial}{\partial x_{\alpha'}^*}(m)$

for all $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$. For any numbers c_a^μ with $|\mu| = z$ we can define functions x_a by

(ii) $x_a = x_a^* + \sum_{|\mu| = z} c_a^\mu x_\mu^*$

and these x_a will form a co-ordinate system with all $x_a(m) = 0$. We now determine the c_a^μ so these will satisfy (1.6). By (1.2) we see that (1.6) is equivalent to

$$\text{(iii)} \quad e_\alpha x_\mu = \begin{cases} \alpha! & \text{if } \alpha \text{ is a permutation of } \mu \\ 0 & \text{if not} \end{cases}$$

for all $\mu = (\mu_1, \dots, \mu_z)$ such that $0 \leq \mu_w \leq d$ and all α. We first determine the c_a^μ so that

$$\text{(iv)} \quad e_\alpha x_a = 0 \qquad \text{if} \quad |\alpha| = z, 1 \leq a \leq d$$

and then show that (i) plus (ii) plus (iv) implies (iii).

Using (1.5) and that all $x_a^*(m) = 0$ we have, if $|\mu| = |\alpha| = z$,

$$e_\alpha x_\mu^* = \sum (e_{\alpha_{\pi 1}} x_{\mu_1}^*) \dots (e_{\alpha_{\pi z}} x_{\mu_z}^*)$$

which is clearly either $\alpha!$ or 0 according as α is or is not a permutation of μ. By (ii) then,

$$e_\alpha x_a = e_\alpha x_a^* + c_a^\alpha \alpha!.$$

Hence if we choose the $c_a^\alpha = -(e_\alpha x_a^*)/\alpha!$ we shall have (iv), no matter how the remaining c_a^μ are chosen.

We now show that (i) plus (ii) plus (iv) implies (iii). If $\alpha_z = 0$, thus $\alpha = (\alpha', 0)$ we have, by (i),

$$e_\alpha x_\mu = e_{\alpha'} x_\mu = e_{\alpha'}\left(\prod_{w=1}^{z} (x_{\mu_w}^* + \sum_v c_{\mu_w}^v x_v^*)\right)$$
$$= \frac{\partial}{\partial x_{\alpha'}^*}\left(\prod_{w=1}^{z} (x_{\mu_w}^* + \sum_v c_{\mu_w}^v x_v^*)\right) = \frac{\partial}{\partial x_{\alpha'}^*} x_\mu^*$$

and this shows (iii) in case $\alpha_z = 0$. If $|\alpha| < z$ then there is a permutation β of α with $\beta_z = 0$, and $e_\alpha = e_\beta$, hence (iii) also holds whenever $|\alpha| < z$. If $|\alpha| = z$ then, by (1.5),

$$e_\alpha x_\mu = \sum ((e_{E_1 \alpha}) x_{\mu_1}) \dots ((e_{E_z \alpha}) x_{\mu_z}).$$

Any product here containing an $(e_{E_w \alpha}) x_{\mu_w}$ with $1 < |E_w \alpha| < z$ will be 0 by the preceeding case and any containing such a term with $|E_\alpha w| = z$ will be 0 by (iv). Since $E_1 \cup \dots \cup E_z = \{1, \dots, z\}$ and $|\alpha| = z$, the only products here which do not contain an $(e_{E_w \alpha}) x_{\mu_w}$ with $|E_w \alpha| > 1$ are those in which all $|E_w \alpha| = 1$. Hence

$$e_\alpha x_\mu = \sum (e_{\alpha_{\pi 1}} x_{\mu_1}) \dots (e_{\alpha_{\pi z}} x_{\mu_z})$$

and this, by the result for the case $|\alpha| < z$, equals $\alpha!$ or 0 according as μ is a permutation of α or not. This proves (iii) and hence the theorem.

The preceding theorem shows that *the integrable points of $S_p^z(M)$ form a submanifold* of dimension $d(p_z)$ where p_w is the dimension of the linear space of polynomials in p variables of degree $\leq w$. The integrable points form a submanifold because in any co-ordinate region with co-ordinates $\{x_a^\alpha\}$ as above, the integrable points are those for which $x_a^\alpha = x_a^\beta$ whenever β is a permutation of α; and the dimension is $d(p)_z$ because this is the number of equivalence classes of α's if we make two equivalent if and only if they differ by a permutation.

Notation. We shall denote the submanifold of integrable points of $S_p^z(M)$ by $IS_p^z(M)$.

Now we give an alternative to (1.3) which is important because it enables us to reduce certain considerations to the case where $z = 2$.

If $(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) \in S_p^z(M)$ then $e_j^w e_i^{w-1}$ is defined from the above, for $0 \le i, j \le p$ and is a second order tangent vector to $S_p^{w-2}(M)$ at $(m, e_1^1, \dots, e_p^1, \dots, e_1, \dots, e_p^{w-2})$. Our alternative to (1.3) is

$$(1.3^*) \qquad e_j^w e_i^{w-1} = e_i^w e_j^{w-1} \qquad \text{for} \qquad 0 \le i, j \le p, 2 \le w \le z.$$

LEMMA (1.2). *If $(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ is any point of $S_p^z(M)$ then the conditions* (1.3) *and* (1.3*) *are equivalent.*

Proof. If (1.3*) holds then for each C^∞ function h of $S_p^{w-2}(M)$ which is defined at $(m, e_1^1, \dots, e_p^1, \dots, e_1^{w-2}, \dots, e_p^{w-2})$ we have $e_j^w e_i^{w-2} h = e_i^w e_j^{w-1} h$. In particular, if $\{x_a\}$ is any co-ordinate system of M such that the $x_a^{\alpha''}$, for $\alpha'' = (\alpha_1, \dots, \alpha_{w-2})$, are defined there then $x_a^{(\alpha'', i, j)} = x_a^{(\alpha'', j, i)}$. Hence $x_a^\alpha = x_a^\beta$ if $\alpha = (\alpha'', i, j, 0, \dots, 0)$ and $\beta = (\alpha'', j, i, 0, \dots, 0)$. Repeated application of this shows $x_a^\alpha = x_a^\beta$ whenever β is a permutation of α, hence $e_\alpha = e_\beta$ by Theorem (1.1).

If (1.3) holds then we have $x_a^\alpha = x_a^\beta$ for any $\alpha = (\alpha'', i, j, 0, \dots, 0)$ and $\beta = (\alpha'', j, i, 0, \dots, 0)$, where $\alpha'' = (\alpha_1, \dots, \alpha_{w-2})$. Hence for any such α'', $e_j^w e_i^{w-1} x_a^{\alpha''} = e_i^w e_j^{w-1} x_a^{\alpha''}$. Then, by Theorem (1.1), $e_j^w e_i^{w-1} = e_i^w e_j^{w-1}$.

LEMMA (1.3). *If $(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) \in S_p^z(M)$ and for each $w \le z-2$ the point $(m, e_1^1, \dots, e_p^1, \dots, e_1^{w+2}, \dots, e_p^{w+2})$ is integrable over $S_p^w(M)$ then $(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ is integrable over M, i.e. lies in $IS_p^z(M)$.*

Proof. Immediate from Theorem (1.1) and Lemma (1.2).

§2. ITERATED GRASSMANN BUNDLES, THEIR INTEGRABLE POINTS, AND A REDUCTION THEOREM

We begin by defining the Grassmann bundle, $G_p(M)$. The elements of $G_p(M)$ are (by definition) all the (m, P) where m is any point of M and P any p-plane at m, i.e. P is any p-dimensional subspace of M_m $(1 \le p \le d)$. Given a co-ordinate system $\{x_a\}$ of M with domain Q we now define a co-ordinate system of $G_p(M)$ consisting of some functions that we denote by $y_1^0, \dots, y_p^0, \dots, y_a^j, \dots$ where $1 \le a \le d$ and $0 \le j \le p$. To define the domain of these functions we consider, for each $m \in Q$, the projection ρ_m (depending on $\{x_a\}$) of $M_m - M_m$ defined by

$$\rho_m \textstyle\sum_{a=1}^d c_a \dfrac{\partial}{\partial x_a}(m) = \textstyle\sum_{a=1}^p c_a \dfrac{\partial}{\partial x_a}(m)$$

We define the subset Q^* of $G_p(M)$ by $Q^* = [(m, P) | m \in Q$ and ρ_m is non-singular on $P]$. We also denote by π the projection of $G_p(M) \to M$, defined by: $\pi(m, P) = m$. We now define the functions y_i^0 and y_a^j on Q^* by

$$y_a^0 = x_a \circ \pi$$
$$y_r^i(m, P) = dx_r(e_i) = e_i x_r \qquad (1 \le i \le p,\ p+1 \le r \le d)$$

where e_i is the (unique) element of P such that $\rho_m e_i = \dfrac{\partial}{\partial x_i}(m)$. By a previous convention we have also

$$y_r^0(m, P) = e_0 x_r \qquad (p+1 \le r \le d).$$

Note that for the above $e_i (1 \leq i \leq p)$ we have

$$e_i = \frac{\partial}{\partial x_i}(m) + \sum_r y_r^i(m, P) \frac{\partial}{\partial x_r}(m).$$

Clearly, $\dim G_p(M) = d + p(d-p)$ and $S_p(M)$ is a bundle over $G_p(M)$ whose fibre is the group of non-singular $p \times p$ matrices (with real entries), the projection map π of this bundle being defined by $\pi(m, e_1, \dots, e_p) = (m, sp\{e_1, \dots, e_p\})$, where $sp\{v_1, \dots, v_p\}$ denotes the span of the vectors $v_1, \dots, v_p$—a notation that will be used frequently below.

We now define the iterated Grassmann bundles, $G_p^z(M)$, for each integer $z \geq 0$. They are defined inductively, by $G_p^z(M) = G_p(G_p^{z-1}(M))$, with the conventions that $G_p^0(M) = M$, $G_p^1(M) = G_p(M)$. Iterating the procedure used above for defining a co-ordinate system for $G_p(M)$ from a given co-ordinate system for M we obtain, for each z, starting from a co-ordinate system $\{x_a\}$ of M, a co-ordinate system for $G_p^z(M)$ consisting of functions that we denote by $y_i^{(0, \dots, 0)}$ and y_r^α where $1 \leq i \leq p$, $p+1 \leq r \leq d$, and $\alpha = (\alpha_1, \dots, \alpha_z)$ as usual (i.e. the α_w integers with $0 \leq \alpha_w \leq p$). However, if $z > 1$, not every point of $G_p^z(M)$ is contained in the domain of such a co-ordinate system so we call these *special* co-ordinate systems.

If A is a non-singular map of a p-dimensional manifold N into M then it has a lift, that we denote by $A^{[1]}$, into $G_p(M)$, defined by $A^{[1]}(n) = (n, A_* N_n)$, and a zth lift, $A^{[z]}$, defined inductively by $A^{[z]} = (A^{[z-1]})^{[1]}$. Since we shall be concerned here with conditions for integrability at a single point we shall consider only non-singular maps of an open Q in R^p into M. It is trivial that if $(m, P) \in G_p(M)$ then there exists such an A with $A^{[1]}(q)$, (m, P) (for some $q \in Q$) but, as in the Stiefel case, the corresponding statement for $z \geq 2$ is false. We define an *integrable point* of $G_p^z(M)$ to be a point $(m, P_1, \dots, P_z) \in G_p^z(M)$ for which there exists such an A with $A^{[z]}(q) = (m, P_1, \dots, P_z)$. We seek, as in the Stiefel case, an intrinsic characterization of an integrable point, and we find such a characterization in terms of the 'lift forms' discussed below. Before discussing these however we reduce the problem of higher order lifts to the problem of second order lifts by the reduction theorem given below. We now develop some lemmas necessary for the reduction theorem and for other considerations below.

As with $S_p^z(M)$, we write y_a^0 for $y_a^{(0, \dots, 0)}$ and sometimes, for induction purposes, write $x_a = y_a^0$. Writing $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$ we have, with $\pi = \pi[G_p^z(M) \to G_p^{z-1}(M)]$,

$$(2.1) \qquad \begin{aligned} & y_i^0 = y_i^{0'} \circ \pi \qquad (0 \in R^z, 0' \in R^{z-1}) \\ & y_r^{(\alpha', 0)} = y_r^{\alpha'} \circ \pi \\ & \frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1}) + \sum_{r,\alpha'} y_r^{(\alpha', i)}(m, P_1, \dots, P_z) \frac{\partial}{\partial y_r^{\alpha'}}(m, P_1, \dots, P_{z-1}) \in P_z \end{aligned}$$

where $1 \leq i \leq p$, $p+1 \leq r \leq d$, and $(m, P_1, \dots, P_z) \in G_p^z(M)$, so $\pi(m, P_1, \dots, P_z) = (m, P_1, \dots, P_{z-1})$. Consequently we have, at points in the domain of y_i^0, y_r^α,

$$\pi_* \frac{\partial}{\partial y_i^0} = \frac{\partial}{\partial y_i^{0'}}$$

$$\pi_* \frac{\partial}{\partial y_r^{(\alpha',0)}} = \frac{\partial}{\partial y_r^{\alpha'}} \tag{2.2}$$

$$\pi_* \frac{\partial}{\partial y_r^{(\alpha',i)}} = 0$$

for i and r as above.

Hence if $\rho_{(m,P_1,\dots,P_w)}$ denotes the projection on $G_p^w(M)_{(m,P_1,\dots,P_w)}$ given by the co-ordinate system $\{y_i^0, y_r^{\alpha''}\}$, onto the span of the $\frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_w)$ then we have, on tangent spaces to $G_p^w(M)$ at points in the domain of this co-ordinate system,

$$\pi_* \circ \rho_{(m,P_1,\dots,P_w)} = \rho_{(m,P_1,\dots,P_{w-1})} \circ \pi_* \tag{2.3}$$

provided, of course, that $\rho_{(m,P_1,\dots,P_{w-1})}$ is defined from the co-ordinate system of $G_p^{w-1}(M)$ obtained from the same $\{x_a\}$.

The domains of the special co-ordinate systems do not cover $G_p^z(M)$ (if $z > 1$) but they cover the only part of $G_p^z(M)$ that will interest us so we shall be able to make all our co-ordinate computations with such systems. We now characterize intrinsically that open subset of $G_p^z(M)$ that is covered by the domains of special co-ordinate systems and shall denote this open submanifold of $G_p^z(M)$ by $G_p^z(M)^0$. We shall now write π_z^G for $\pi[G_p^z(M) \to G_p^{z-1}(M)]$. We assign to each $(m, P_1, \dots, P_z) \in G_p^z(M)$ a sequence $P_1^0, \dots, P_z^0$ of subspaces of M_m, defined by

$$P_1^0 = P_1, \qquad P_w^0 = \pi_{1*}^G \dots \pi_{w-1*}^G P_w$$

LEMMA (2.1). *$G_p^z(M)^0$ consists of those $(m, P_1, \dots, P_z)$ in $G_p^z(M)$ for which* $\dim P_1^0 = \dots = \dim P_z^0 = p$. *If $\{x_a\}$ is any co-ordinate system of M at m for which the associated ρ_m is non-singular on each P_w^0 $(1 \le w \le z)$ then $(m, P_1, \dots, P_z)$ is in the domain of the special co-ordinate system y_i^0, y_r^α obtained from this $\{x_a\}$.*

Proof. Suppose $(m, P_1, \dots, P_z) \in G_p^z(M)$ is in the domain of the special co-ordinate system $\{y_i^0, y_r^\alpha\}$ obtained from the co-ordinate system $\{x_a\}$ of M. Then P_w is spanned by a set of vectors of the form: $\frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{w-1}) + t_i$, where each t_i is a linear combination of the $\frac{\partial}{\partial y_r^{\alpha''}}(m, P_1, \dots, P_{w-1})$ and $p + 1 \le r \le d$. Hence, by (2.1), $\pi_{1*}^G \dots \pi_{w-1*}^G P_w$ is spanned by the $\frac{\partial}{\partial x_i}(m) + t_i'$ where t_i' is a linear combination of the $\frac{\partial}{\partial x_r}(m)$, showing that $\dim P_w^0 = p$.

Now suppose that $(m, P_1, \dots, P_z)$ is a point of $G_p^z(M)$ for which all the P_w^0 have dimension p. Choose a p-dimensional linear subspace Q of M_m and a linear complement Q' of Q such that the projection of M_m onto Q given by this decomposition is non-singular on each P_w^0. We can choose a co-ordinate system $\{x_a\}$ at m such that the $\frac{\partial}{\partial x_1}(m), \dots, \frac{\partial}{\partial x_p}(m)$ span Q and the $\frac{\partial}{\partial x_r}(m)$ span Q'. We shall finish the proof of both statements of the lemma by showing $(m, P_1, \dots, P_z)$ is in the domain of the special co-ordinate system $\{y_i^0, y_r^\alpha\}$ obtained from such an $\{x_a\}$, whether the $\{x_a\}$ are obtained after Q as above, or whether Q and Q' are defined from the $\{x_a\}$.

By the definition of Q, $P_1 = P_1^0$ is in the domain of the co-ordinate system $\{y_i^0, y_r^j\}$ of $G_p(M)$. Now we show by induction on w that $(m, P_1, \dots, P_w)$ is in the domain of the co-ordinate system $\{y_i^0, y_r^{\alpha''}\}$ of $G_p^w(M)$. Assuming this for $w-1$ we wish to prove $\rho_{(m,P_1,\dots,P_{w-1})}$ is non-singular on P_w. By assumption $P_w^0 = \pi_{1*}^G \dots \pi_{w-1*}^G P_w$ has dimension p and by the choice of Q and the x_a, ρ_m is non-singular on P_w^0, hence $\rho_m \circ \pi_{1*}^G \dots \pi_{w-1*}^G$ is non-singular on P_w. Using (2.3) and iterating we have

$$\rho_m \circ \pi_{1*}^G \dots \pi_{w-1*}^G = \pi_{1*} \dots \pi_{w-1*} \circ \rho_{(m,P_1,\dots,P_{w-1})}$$

hence the right side must be non-singular on P_w, thus $\rho_{(m,P_1,\dots,P_{w-1})}$ is non-singular on P_w. This proves Lemma (2.1).

We now define a subset $S_p^z(M)^0$ of $S_p^z(M)$ analogous to $G_p^z(M)^0$ in $G_p^z(M)$. Let π_z^S be the projection of $S_p^z(M)$ into $S_p^{z-1}(M)$: $\pi_z^S(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) = (m, e_1^1, \dots, e_p^1, \dots, e_1^{z-1}, \dots, e_p)$. We define, for each $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ a sequence $P_1^0, \dots, P_z^0$ of subspaces of M_m by

$$P_1^0 = sp\{e_1^1, \dots, e_p^1\}$$
$$P_w^0 = sp\{\pi_{1*}^S \dots \pi_{w-1*}^S e_1^w, \dots, \pi_{1*}^S \dots \pi_{w-1*}^S e_p^w\}.$$

Then we define $S_p^z(M)^0 = [(m, e) \in S_p^z(M)|$ for the associated sequence of subspaces, $P_1^0, \dots, P_z^0$, all have dimension $p]$.

$S_p^z(M)^0$ is an open subset of $S_p^z(M)$ for the following reason. In $S_p^z(M)$ (unlike $G_p^z(M)$) every point is in the domain of a co-ordinate system obtained from a co-ordinate system $\{x_a\}$ of M. One verifies easily that in the domain of each such co-ordinate system of $S_p^z(M)$ the points of $S_p^z(M)^0$ are those for which each of the matrices $x_a^{i\delta_w}$ has rank p, i.e. for each w $(1 \le w \le z)$ we have such a $p \times a$ matrix $1 \le i \le p$, $1 \le a \le d$; we are using the notation here: $\delta_w = (0, \dots, 0, 1, 0, \dots, 0)$, so $i\delta_w = (0, \dots, i, \dots, 0)$.

We now define the projection map $\pi_z : S_p^z(M)^0 \to G_p^z(M)^0$, under which $S_p^z(M)^0$ will be a bundle over $G_p^z(M)^0$. π_z is defined inductively by:

$$\pi_1 : S_p^1(M)^0 \to G_p^1(M)^0 : \pi_1(m, e_1, \dots, e_p) = (m, sp\{e_1, \dots, e_p\}),$$

and if π_{z-1} has been defined then π_z is defined by

$$\pi_z : S_p^z(M)^0 \to G_p^z(M)^0 : \pi_z(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) = (m, P_1, \dots, P_z),$$

where the P_i^w are defined by

$$\pi_{z-1}(m, e_1^1, \dots, e_p^1, \dots, e_1^{z-1}, \dots, me_p^{z-1}) = (m, P_1, \dots, P_{z-1})$$
$$P_z = sp\{\pi_{z-1*}e_1^z, \dots, \pi_{z-1*}e_p^z\}$$

or, more briefly,

$$\pi_z(m, e) = (\pi_{z-1}(m, e'), sp\{\pi_{z-1*}e_1^z, \dots, \pi_{z-1*}e_p^z\})$$

if $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ and $(m, e') = (m, e_1^1, \dots, e_p^1, \dots, e_1^{z-1}, \dots, e_p^{z-1})$; we shall frequently use the notation (m, e) and (m, e') in this way below. For inductive purposes we also define π_0 to be the identity map of M onto M; then the definition of π_1 above is given, in the inductive process, from π_0. One verifies trivially that the sequence of subspaces $P_1^0, \dots, P_z^0$ associated with (m, e) is the same as that associated with $\pi_z(m, e)$ and this shows that π_z maps $S_p^z(M)^0$ onto $G_p^z(M)^0$ (and not just into $G_p^z(M)$). The reason for introducing $S_p^z(M)^0$ and $G_p^z(M)^0$ is that π_z does not exist from $S_p^z(M)$ to $G_p^z(M)$ since the spans used above are not p-dimensional for a general point of $S_p^z(M)$. One verifies easily that

(2.3) $$\pi_{z-1} \circ \pi_z^S = \pi_z^G \circ \pi_z.$$

For the purpose of seeing how $S_p^z(M)^0$ 'lies over' $G_p^z(M)^0$ in terms of the special co-ordinate systems we are using, and thus for showing that $S_p^z(M)^0$ is a bundle over $G_p^z(M)^0$ we now determine the *ranges* of the co-ordinate systems $\{y_i^0, y_r^\alpha\}$ and $\{x_a^\alpha\}$ obtained from a given co-ordinate system x_a of M. We shall now use the following notation. $\{x_a\}$, $\{y_i^0, y_r^\alpha\}$ and $\{x_a^\alpha\}$ shall be as just described. Q, $Q^{[z]}$, Q^z shall be the domains of these co-ordinate systems, and 0, $0^{[z]}$, 0^z shall be the ranges of these co-ordinate systems, i.e. 0, $0^{[z]}$, 0^z are the images of Q, $Q^{[z]}$, Q^z under the homeomorphisms onto Euclidean spaces which define the co-ordinate systems; we also write $Q = Q^{[0]} = Q^0$ and $0 = 0^{[0]} = 0^0$.

LEMMA (A). *The range of* $\{y_i^0, y_r^\alpha\}$ *is* $0 \times R^{p(d-p)} \times R^{(1+p)p(d-p)} \times \dots \times R^{(1+p)^{z-1}p(d-p)}$. *More precisely, for each choice of real numbers* $\{b_i^0, b_r^\alpha\}$ *such that* $(b_1^0, \dots, b_d^0) \in 0$ *there is a unique* $(m, P_1, \dots, P_z) \in Q^{[z]}$ *such that* $y_i^0(m, P_1, \dots, P_z) = b_i^0$ *and* $y_r^\alpha(m, P_1, \dots, P_z) = b_r^\alpha$, *for all* i, r, α.

Remark. The product $R^{p(d-p)} \times R^{(1+p)p(d-p)} \times \dots \times R^{(1+p)^{z-1}p(d-p)}$ above is just the Euclidean space of dimension $(d-p)((1+p)^z - 1)$ but it will be convenient below to consider it decomposed as above. We note that the dimension of $G_p^z(M)$ (and $G_p^z(M)^0$) is $(d-p)(p+1)^z + p$.

Proof. One observes for any manifold N with co-ordinate system $v_1, \dots, v_e$ that for any real numbers $b_1, \dots, b_e, \dots, b_s^i, \dots$ $(1 \le i \le p; p+1 \le s \le e)$ such that $(b_1, \dots, b_e)$ is in the range of $v_1, \dots, v_e$, say $b_c = v_c(n)$ for all c, that there is a unique p-plane P at n with these co-ordinates, namely, the P spanned by the

$$\frac{\partial}{\partial v_i}(n) + \textstyle\sum_{s=p+1}^{e} b_s^i \frac{\partial}{\partial v_s}(n)$$

Iteration of this remark yields Lemma (A).

If p and q are integers with $p \le q$ we shall write $R^{p\times q}$ for the open set in R^{pq} consisting of all matrices $(a_{u,i})$ of rank p, where $1 \le i \le p$ and u ranges through some set with q elements. In particular then, $R^{p\times p}$ is the full linear group.

LEMMA (B). *The range of* $\{x_a^\alpha\}$ *is* $0 \times R^{p\times d} \times R^{p\times((p+1)d)} \times \dots \times R^{p\times((p+1)^{z-1}d)}$. *More precisely, the following is true. Considering any set of real numbers* c_a^α *satisfying the following two conditions*:

1. $(c_1^0, \dots, c_d^0) \in 0$

2. *For each integer* w *the matrix* (c_{ui}) *defined as follows has rank* p. *Let* $1 \le i \le p$ *and* u *run through all pairs* (a, α) *such that* $1 \le a \le d$ *and* $\alpha = (\alpha_1, \dots, \alpha_w, 0, \dots, 0)$ *and let* $c_{ui} = c_a^{(\alpha,i)}$.

Then there exists a unique $(m, e) \in S_p^z(M)$ *with* $x_a^\alpha(m, e) = c_a^\alpha$ *for all* a *and* α, *and the set of all such* $\{c_a^\alpha\}$ *is the range of* $\{x_a^\alpha\}$.

Proof. This is proved by iterating the following fact. Consider any manifold N with co-ordinate system $v_1, \dots, v_e$. Consider any real numbers $c_1, \dots, c_e, \dots, c_b^i, \dots$ $(1 \le i \le p, 1 \le b \le e)$. Suppose there is an $n \in N$ with $v_b(n) = c_b$ for all b and that the matrix (c_b^i) has rank p. Then there exist unique linearly independent $f_1, \dots, f_p$ in N_n for which $v_b^i(n, f_1, \dots, f_p) = c_b^i$, namely,

$$f_i = \sum_{b=1}^e c_b^i \frac{\partial}{\partial v_b}(n)$$

Iteration of this shows that for any c_a^α as in the Lemma there is a unique $(m, e) \in S_p^z(M)$ with all $x_a^\alpha(m, e) = c_a^\alpha$. Conversely, the c_b, c_b^i satisfying the above clearly form the range of the co-ordinate system v_b^0, v_b^i of $S_p(N)$ obtained from the given v_b and iterating this fact gives the statement in the lemma about the range of the x_a^α.

Remark. The dimension of $S_p^z(M)$ is $d(1 + p)^z$.

It is clear that

$$\pi_z^{-1}(Q^{[z]}) \subseteq Q^z$$

and we wish to determine the range of the x_a^α when restricted to $\pi_z^{-1}(Q^{[z]})$. For this and other reasons we wish to obtain formulas expressing the $y_i^0 \circ \pi_z$ and $y_r^\alpha \circ \pi_z$ in terms of the x_a^α, these formulas to hold on $\pi_z^{-1}(Q^{[z]})$. From (2.3) we have

$$y_r^{(\alpha',0)} \circ \pi_z = (y_r^{\alpha'} \circ \pi_{z-1})^0 \tag{2.4a}$$

where $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$ and the zero superscript on the right denotes the lift of the function in the parenthesis from $S_p^{z-1}(M)^0$ to $S_p^z(M)^0$.

In the following we shall often give special consideration to the functions $x_i^{(0,\dots,0,j,0,\dots,0)}$; letting $\delta_w = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 in the wth spot) so $j\delta_w = (0, \dots, j, \dots, 0)$; then these functions are denoted by $x_i^{j\delta_w}$, in our usual notation. We note that for each integer w with $1 \le w \le z$ we have a $p \times p$ matrix $(x_i^{j\delta_w})$ when $0 \le i, j \le p$. Now we prove

$$\begin{aligned} (y_r^{\alpha'} \circ \pi_{z-1})^j &= \textstyle\sum_i x_i^{j\delta_z}(y_r^{(\alpha',i)} \circ \pi_z) \\ (y_i^0 \circ \pi_{z-1})^j &= x_i^{j\delta_z}. \end{aligned} \tag{2.4b}$$

We are using here our previous notation f^j for the functions of $S_p(N)$ induced from a function f of N and it is understood in this formula that the y_i^0, y_r^α of $G_p^z(M)$ and the x_a^α of $S_p^z(M)$ are the co-ordinate functions obtained from the same co-ordinate system $\{x_a\}$ of M. The range of the indices appearing in this formula is: $1 \le i, j \le p, p + 1 \le r \le d, \alpha' = (\alpha_1, \dots, \alpha_{z-1}), 0 \le \alpha_w \le p$.

Proof of (2.4b). Let $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ and $\pi_z(m, e) = (m, P) = (m, P_1, \dots, P_z) \in Q^{[z]}$, $(m, P') = (m, P_1, \dots, P_{z-1})$. Then, by definition of π_z and the co-ordinates y_i^0, y_r^α,

$$\pi_{z-1*}e_j^z = \sum_i a_{ij}\left[\frac{\partial}{\partial y_i^0}(m, P') + \sum_{\alpha',r} y_r^{(\alpha',i)}(m, P)\frac{\partial}{\partial y_r^{\alpha'}}(m, P')\right]$$

where (a_{ij}) is a non-singular $p \times p$ matrix. We determine the a_{ij} by

$$a_{ij} = (\pi_{z-1*}e_j^z)y_i^0 = e_j^z(y_i^0 \circ \pi_{z-1}) = e_j^z x_i^0 = x_i^{j\delta_z}.$$

Then the preceeding formula becomes

$$\pi_{z-1*}e_j^z = \textstyle\sum_i x_i^{j\delta_z}(m, e) \dfrac{\partial}{\partial y_i^0}(m, P') + \sum_{i,\alpha',r} x_i^{j\delta_z}(m, e) y_r^{(\alpha',i)}(m, P) \dfrac{\partial}{\partial y_r^{\alpha'}}(m, P');$$

Hence

$$(\pi_{z-1*}e_j^z)y_r^{\alpha'} = \textstyle\sum_i x_i^{j\delta_z}(m, e) y_r^{(\alpha',i)}(m, P)$$

$$(\pi_{z-1*}e_j^z)y_i^0 = x_i^{j\delta_z}(m, e)$$

and these are just (2.4b) in a different notation.

We now determine the range of the co-ordinate system $\{x_a^\alpha\}$ when restricted to $\pi_z^{-1}(Q^{[z]})$; we also find, for any fixed $(m, P) = (m, P_1, \dots, P_z) \in Q^{[z]}$, the range of $\{x_a^\alpha\}$ when restricted to the 'fibre' $\pi_z^{-1}(m, P)$. These facts will be useful in several ways including: (1) obtaining the local product representation needed to show $S_p^z(M)^0$ is a bundle over $G_p^z(M)^0$, (2) determining the fibre of this bundle, (3) obtaining the previous two facts for the manifolds of integrable points of $S_p^z(M)$ and $G_p^z(M)$.

For the determination of these ranges it would be convenient to have explicit formulas for the $y_i^0 \circ \pi_z$ and $y_r^\alpha \circ \pi_z$ in terms of the x_a^α. We could obtain such formulas from (2.4) but the explicit formulas would be complicated; the information obtained about them in the next lemma will be sufficient for our purposes.

We now introduce certain functions v_i^α on Q^z, where $1 \le i \le p$. These v_i^α will depend *only* on the x_i^α (*not* on the x_r^α). We define the v_i^α as follows:

(a) $v_i^0 = x_i^0$;

(b) $(v_i^{j\delta_w})$ = the inverse matrix of $(x_i^{j\delta_w})$;

(c) for general $\alpha = (\alpha_1, \dots, \alpha_z)$ let α_w be its first non-zero co-ordinate; each $x_i^{j\delta_w}$ is the lift of a function x_{ij} of $S_p^w(M)$, hence each $v_i^j\delta w$ is the lift of a function v_{ij} of $S_p^w(M)$; we define

$$v_i^\alpha(m, e) = e_{\alpha_z} \dots e_{\alpha_{w+1}} v_{i\alpha_w} = (\dots (v_{i\alpha_w})^\alpha w + 1) \dots)^\alpha z(m, e)$$

where, as usual, $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$.

LEMMA (C). *If* $1 \le u < w \le z$ *then*

$$v_m^{j\delta_u + l\delta_w} = -\textstyle\sum_{k,i} (v_m^{i\delta_u}) x_i^{k\delta_u + l\delta_w} (v_k^{j\delta_u}) \tag{2.5}$$

where $1 \le i, j, k, l, m \le p$.

Proof. If x_{ij}, v_{ij} are functions of $S_p^u(M)$ related to $x_i^{j\delta_u}$, $v_i^{j\delta_u}$ as in (c) above (but with u in place of w) and if x'_{ij}, v'_{ij} are the lifts of x_{ij}, v_{ij} to $S_p^{w-1}(M)$ then, because $\sum_k x'_{ik}v =_{kj}' \delta_{ij}$,

$$\textstyle\sum_k (x'_{ik})^l (v'_{kj})^0 + \sum_k (x'_{ik})^0 (v'_{kj})^l = 0$$

hence, multiplying on the left by $(v_{mi})^0$ and summing on i,

$$-\textstyle\sum (v'_{mi})^0 (x'_{ik})^l (v'_{kj})^0 = (v'_{mj})^l.$$

This is (2.5) in a different notation, hence (2.5) is proved.

LEMMA (D). *For each r, α $(p+1 \le r \le d)$ there exists a polynomial Q_r^α such that*

$$y_r^\alpha \pi_z = Q_r^\alpha(v_i^{j\delta_w}, x_k^\beta, x_r^\gamma) + \sum_\omega v_{\omega_1}^{\alpha_1\delta_1} \dots v_{\omega_z}^{\alpha_z\delta_z} x_r^\omega \tag{2.6}$$

where the $w, i, j, k, \beta, \gamma, \omega$ occurring in Q_r^α satisfy:

(a) $1 \le i, j, k \le p$;

(b) $w \in [t|\alpha_t \ne 0]$;

(c) $\beta_t = \gamma_t = \omega_t = 0$ *if* $\alpha_t = 0$;

(d) $0 < |\beta| \le |\alpha|$;

(e) $|\gamma| < |\alpha|$;

(f) $|\omega| = |\alpha|$;

(g) *every term of Q_r^α contains an x_r^γ as a factor.*

The sum in (2.6) *is over all $\omega = (\omega_1, \dots, \omega_z)$ satisfying* (*c*) *and* (*f*). *We use the convention here that $v_0^0 = 1$.*

Proof. We induct on z. If $z = 0$ we have $y_r^0 = x_r^0 = x_r$ so (2.6) holds with $Q_r^0 = 0$. Now suppose it holds for all $z' < z$ and we prove it for z. Let $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$ so $\alpha = (\alpha', \alpha_z)$. If $\alpha_z = 0$ then (2.6) follows from (2.4a) and the induction assumption. So now suppose $\alpha_z \ne 0$.

From (2.4b) we have, if we multiply by $v_1^\alpha z^\delta z$ and sum on 1,

$$y_r^\alpha \circ \pi_z = \sum_l v_l^{\alpha_z\delta_z}(y_r^{\alpha'} \circ \pi_{z-1})^l. \tag{2.7}$$

If $|\alpha| = 1$ then, since $\alpha_z \ne 0$, $\alpha' = 0$ and (2.7) gives

$$y_r^\alpha \circ \pi_z = \sum_l v_l^{\alpha_z\delta_z} x_r^{l\delta} z$$

proving (2.6) in this case, with $Q_r^\alpha = 0$. Henceforth we suppose that $|z^\alpha| \ge 2$. Now using (2.7) and the induction assumption,

$$\begin{aligned} y_r^\alpha \circ \pi_z = {} & \sum_l v_l^{\alpha_z\delta_z} Q_r^{\alpha'}(v_i^{j\delta_w}, x_k^{\beta'}, x_r^{\gamma'})^l \\ & + \sum_l v_l^{\alpha_z\delta_z}\left(\sum_{\omega'} v_{\omega'}^{\alpha_1\delta_1} \dots v_{\omega'_{z-1}}^{\alpha_{z-1}\delta_{z-1}} x_r^{\omega'}\right)^l \end{aligned}$$

where

(a′) $1 \le i, j, k, l \le p$;

(b′) $w \in [t|\alpha_t' \ne 0]$;

(c′) $\beta_t' = \gamma_t' = \omega_t' = 0$ if $\alpha_t' = 0$;

(d′) $0 < |\beta'| \le |\alpha'| = |\alpha| - 1$;

(e′) $|\gamma'| < |\alpha'| = |\alpha| - 1$;

(f′) $|\omega'| = |\alpha'| = |\alpha| - 1$,

(g′) every term of $Q_r^{\alpha'}$ contains an $x_r^{\gamma'}$ as a factor.

Repeated use of the differentiation rule for products then gives

(i) $Q_r^{\alpha'}(v_i^{j\delta_w}, x_k^{\beta'}, x_r^{\gamma'})^l =$ a polynomial in the $v_i^{j\delta_w}$, $v_i^{j\delta_w + l\delta_z}$, $x_k^{\beta'}$, $x_k^{\beta' + l\delta_z}$, $x_r^{\gamma'}$, $x_r^{\gamma' + l\delta_z}$

(ii) $(\sum_{\omega'} v_{\omega_1}^{\alpha_1\delta_1} \dots v_{\omega_{z-1}}^{\alpha_{z-1}\delta_{z-1}} x_r^{\omega'})^l = \sum_{\omega'} v_{\omega_1}^{\alpha_1\delta_1} \dots v_{\omega_{z-1}}^{\alpha_{z-1}\delta_{z-1}} x_r^{\omega'+l\delta_z}$ plus a polynomial in the $v_{\omega_i}^{\alpha_i\delta_i}$, $v_{\omega_i}^{\alpha_i\delta_i+\delta_z}$, $x_r^{\omega'}$,

where the $i, j, k, l, w, \alpha', \beta', \gamma', \omega'$, satisfy $(a') - (e')$.
Using (2,5) we then see

(i′) $Q_r^{\alpha'}(v_i^{j\delta_w}, x_k^{\beta'}, x_r^{\gamma'})$ = a polynomial in the $v_i^{j\delta_w}, x_k^{\beta'}, x_k^{\beta'+m\delta_z}, x_r^{\gamma'}, x_r^{\gamma'+l\delta_z}$,

(ii′) $(\sum_{\omega'} v_{\omega_1}^{\alpha_1\delta_1} \dots v_{\omega_{z-1}}^{\alpha_{z-1}\delta_{z-1}} x_r^{\omega'})^l = \sum_{\omega'} v_{\omega'_1}^{\alpha_1\delta_1} \dots v_{\omega'_{z-1}}^{\alpha_{z-1}\delta_{z-1}} x_r^{\omega'+l\delta_z}$ plus a polynomial in the $v_{\omega'_1}^{\alpha_i\delta_i}$, $x_i^{\alpha_i\delta_i+m\delta_z}$, $x_r^{\omega'}$,

where the $i, j, k, l, w, \beta', \gamma', \omega'$, satisfy (a′) – (g′) and $1 \le m \le p$. Substituting these expressions in (2.7) we see that we obtain the desired form for $y_r^\alpha \circ \pi_z$ and that the indices satisfy (a)—(g).

The following lemma shows that $S_p^z(M)^0$ is locally the product of $G_p^z(M)^0$ by $(L_p)^z \times R^q$ where L_p denotes the full linear group of $p \times p$ matrices, $(L_p)^z$ denotes the product of L_p by itself z times, and $q = p(p+1)^z - p - zp^2$. It follows, except for showing that such local product representations are properly related (a step we omit because it is not relevant to what we are doing, and is only tedious to carry out), that $S_p^z(M)^0$ is a fibre bundle over $G_p^z(M)^0$ with fibre $(L_p)^z \times R^q$ (q as above).

We let $0(z)$ be the open set in Euclidean space of dimension $p(p+1)^z - p$ defined in the following way. Consider all i, α with $\alpha \neq 0$ (i and α as usual). Define

$$0(z) = [(c_i^\alpha) \mid \text{for each } w \text{ in } 1 \le w \le z \text{ the } p \times p \text{ matrix } (c_i^{j\delta_w}) \text{ is non-singular}].$$

Clearly $0(z)$ is diffeomorphic to $(L_p)^z \times R^q$ (q as above).

We also define $\pi_z'\colon Q^{[z]} \times 0(z) \to Q^{[z-1]} \times 0(z-1)$, by

$$\pi_z'((c_i^\alpha), (m, P)) = ((c_i^{\alpha'}), (m, P'))$$

where $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$, $(m, P) = (m, P_1, \dots, P_z)$, and $(m, P') = (m, P_1, \dots, P_{z-1})$.

LEMMA (2.2). *For each $(m, P) \in Q^{[z]}$ and $(c_i^\alpha) \in 0(z)$ there exists a unique $(m, e) \in \pi_z^{-1}(Q^{[z]})$ such that*

(a) $\pi_z(m, e) = (m, P)$;

(b) $x_i^\alpha(m, e) = c_i^\alpha$ *(for all i, α as above).*

This defines a map $A(z)$ of $Q^{[z]} \times 0(z)$ into $\pi_z^{-1}(Q^{[z]})$. This $A(z)$ is a diffeomorphism of $Q^{[z]} \times 0(z)$ onto $\pi_z^{-1}(Q^{[z]})$. The set of such maps $A(z)$, as z varies (but all depending on the same co-ordinate system $\{x_a\}$) is consistent in the sense that

(c) $A(z-1) \circ \pi_z' = \pi_z^S \circ A(z)$ *on $Q^{[z]} \times 0(z)$.*

Proof. We induct on z. For $z = 0$ there is nothing to prove so we now assume the lemma for all $z' < z$ and prove it for z. First we prove the existence of $A(z)$, i.e. given $(m, P) \in Q^{[z]}$ and $(c_i^\alpha) \in 0(z)$, we prove the existence of an $(m, e) \in 0^z$ which satisfies (a) and (b). We obtain (m, e) by determining its co-ordinates $x_a^\alpha(m, e)$, then use Lemma (B) to ensure the existence of an (m, e) with these co-ordinates.

So let $(m, P) = (m, P_1, \dots, P_z) \in Q^{[z]}$, $(c_i^\alpha) \in 0(z)$, and let $(m, P') = (m, P_1, \dots, P_{z-1})$

and $(c_i^{\alpha'})$ be the point of $0(z-1)$ whose co-ordinates are the numbers $(c_i^{(\alpha',0)})$ $(\alpha' = (\alpha_1, \ldots, \alpha_{z-1})$, i.e.

$$\pi_z'((c_i^\alpha), (m, P)) = ((c_i^{\alpha'}), (m, P')).$$

Using the induction assumption we choose $(m, e')\varepsilon\pi_{z-1}^{-1}(Q^{[z-1]})$ such that

(a') $\pi_{z-1}(m, e') = (m, P')$;

(b') $x_i^{\alpha'}(m, e') = c_i^{\alpha'}$.

We can now define certain of the co-ordinates of the desired (m, e), denoting them by b_a^α, by

$$\text{(i)}\quad \begin{aligned} b_i^{(\alpha',0)} &= c_i^{\alpha'} = x_i^{\alpha'}(m, e') \\ b_r^{(\alpha',0)} &= x_r^{\alpha'}(m, e') \\ b_i^\alpha &= c_i^\alpha \qquad \text{if } \alpha_z \neq 0. \end{aligned}$$

It remains to determine those b_r^α with $p + 1 \leq r \leq d$ and $\alpha_z \neq 0$. For this we shall use (2.6).

First we define numbers $(d_i^{j\delta_w})$ by: $(d_i^{j\delta_w})$ is, for each w, the inverse matrix of $(b_i^{j\delta_w})$. Now we use (2.6) to define the b_r^α for which $\alpha_z \neq 0$ by induction on $|\alpha|$. We define b_r^α by

$$y_r^\alpha(m, P) = Q_r^\alpha(d_i^{j\delta_w}, b_k^\beta, b_r^\gamma) + \sum_\omega d_{\omega_1}^{\alpha_1\delta_1} \ldots d_{\omega_z}^{\alpha_z\delta_z} b_r^\omega$$

i.e. having determined the b_r^γ for all $|\gamma| < |\alpha|$, this formula determines b_r^α, since the $(d_i^{j\delta_w})$ are non-singular. Using Lemma (B) (and our assumption about the c_i^α) it is clear that there exists a unique (m, e) in Q^z such that

(ii) $x_a^\alpha(m, e) = b_a^\alpha$ for all a, α,

Now we show this (m, e) satisfies (a) and (b). By (i) we know it satisfies (b). To show it satisfies (a) it is sufficient to prove both:

(iii) $\pi_z(m, e) \in Q^{[z]}$

$$\text{(iv)}\quad \begin{aligned} y_i^0(\pi_z(m, e)) &= y_i^0(m, P); \\ y_r^\alpha(\pi_z(m, e)) &= y_r^\alpha(m, P). \end{aligned}$$

Using (2.4b) we see that

$$\pi_{z-1*}e_j^z = \sum c_i^{j\delta_z} \frac{\partial}{\partial y_i^0}(m, P') + t_j^z$$

where t_j^z is a linear combination of the $\dfrac{\partial}{\partial y_r^{\alpha'}}(m, P')$; since the $(c_i^{j\delta_z})$ are assumed non-singular it follows that $\pi_z(m, e) \in Q^{[z]}$, proving (iii).

Proof of (d). We have the first statement in (iv) by

$$y_i^0(\pi_z(m, e)) = y_i^0(\pi_{z-1}(m, e')) = y_i^0(m, P') = y_i^0(m, P).$$

We have the second statement in (iv) for those α of the form $(\alpha', 0)$ by

$$\begin{aligned} y_r^{(\alpha',0)}(\pi_z(m, e)) &= y_r^{\alpha'} \circ \pi_z^G \circ \pi_z(m, e) \\ &= y_r^{\alpha'} \circ \pi_{z-1} \circ \pi_z^S(m, e) \\ &= y_r^{\alpha'} \circ \pi_{z-1}(m, e') \\ &= y_r^{\alpha'}(m, P') \\ &= y_r^{(\alpha',0)}(m, P). \end{aligned}$$

Finally, for the $\alpha = (\alpha', \alpha_z)$ with $\alpha_z \neq 0$ we have (iv) by the definition of the b_r^α, i.e. inducting on $|\alpha|$ and using (2.6) and that definition, we prove (iv).

It is now trivial that $A(z)$ is a diffeomorphism of $Q^{[z]} \times 0(z)$ onto $\pi_z^{-1}(Q^{[z]})$ and (c) is trivial; hence the lemma is proved.

REDUCTION THEOREM. *Let $(m, P_1, \dots, P_z)$ be in $G_p^z(M)$. Then $(m, P_1, \dots, P_z)$ is integrable if and only if, for each $w \leq z - 2$, the point $(m, P_1, \dots, P_{w+2})$ is integrable over $G_p^w(M)$.*

Proof. We first note that $(m, P_1, \dots, P_z) \in G_p^z(M)^0$ by showing

$$(*) \qquad \pi_{1*}^G \dots \pi_{w-1*}^G P_w = P_1 \qquad (1 \leq w \leq z).$$

This holds because integrability of $x(m, P_1, \dots, P_w)$ over $G_p^{w-2}(M)$ clearly gives

$$\pi_{w-1*}^G P_w = P_{w-1}$$

and iteration of this gives (*).

We now proceed by induction on z. For $z = 2$ this theorem is immediate so we now assume $z > 2$ and prove that if it is true for $z - 1$ then it is true for z. Consider our given $(m, P) = (m, P_1, \dots, P_z) \in G_p^z(M)^0$ as in the statement of the theorem. By the induction assumption there exists a non-singular map A of an open Q in R^p into M such that $A^{[z]}(q) = (m, P)$. We then define $(m, e') = (m, e_1^1, \dots, e_p^1, \dots, e_1^{z-1}, \dots, e_p^{z-1})$ by $A^z(q) = (m, e')$. Because $\pi_{z-1} A^{z-1} = A^{[z-1]}$ it is clear that $\pi_{z-1}(m, e') = (m, P')$. Since (m, e') is integrable, by the way it was defined, we have, for w with $w + 2 \leq z - 1$, that $(m, e_1, \dots, e_p, \dots, e_1^{w+2}, \dots, e_p^{w+2})$ is integrable over $S_p^w(M)$.

We now proceed as follows. We shall define $e_1^z, \dots, e_p^z \in S_p^{z-1}(M)_{(m,e')}$ such that

(a) $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z)$ is integrable over $S_p^{z-2}(M)$

hence, combined with the preceding, for each w with $w + 2 \leq z$, the point $(m, e_1, \dots, e_p, \dots, e_1^{w+2}, \dots, e_p^{w+2})$ is integrable over $S_p^w(M)$. By Lemma (1.3) this implies (m, e) is integrable over M. Furthermore, the $e_1^z, \dots, e_p^z$ will be so chosen that

(b) $\pi_z(m, e) = (m, P)$.

This plus integrability of (m, e) will prove (m, P) is integrable since if $A^z(q) = (m, e)$ then $A^{[z]}(q) = \pi_z(m, e) = (m, P)$. Hence it will be sufficient to obtain $e_1^z, \dots, e_p^z$ so that (a) and (b) hold.

Now choose B, a non-singular map of an open Q in R^p into $G_p^{z-2}(M)$ such that

$$(B^{[2]}(q) = (m, P)$$

We now wish to obtain a local cross-section χ_{z-2} of $S_p^{z-2}(M)^0$ such that both the following hold:

(i) $\chi_{z-2}(m, P_1, \dots, P_{z-2}) = (m, e_1, \dots, e_p, \dots, e_1^{z-2}, \dots, e_p^{z-2})$

(ii) $(\chi_{z-2} \circ B)^1(q) = (m, e')$.

It is clear from Lemma (2.2) (iterating it) that we can achieve (i) so we suppose we have that and now show how to modify a χ_{z-2} satisfying (i) to obtain one satisfying both (i) and (ii). Let $v_1, \dots, v_p$ be the elements of $G_p^{z-2}(M)\,(m, P_1, \dots, P_{z-2})$ such that $B_*\left(\dfrac{\partial}{\partial u_i}(q)\right) = v_i$.

Since $S_p^{z-2}(M)^0$ is a bundle over $G_p^{z-2}(M)^0$ we can choose another cross-section χ_{z-2} satisfying (i) and

(iii) $\chi_{z-2*}v_i = e_i^{z-1} \qquad 1 \leq i \leq p.$

It is then clear that (ii) holds.

We now define $e_1^z, \dots, e_p^z$ by

$$e_i^z = (\chi_{z-2} \circ B)^1{}_*\left(\frac{\partial}{\partial u_i}(q)\right)$$

i.e., by

$$(m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) = (\chi_{z-2} \circ B)^2(q)$$

so it is clear that (a) holds. It remains to prove that (b) holds.

To prove (b) we note the general fact: if χ is any local cross-section of $S_p^w(M)^0$ over $G_p^w(M)^0$ and B any non-singular map of an open Q in R^p into $G_p^w(M)^0$ then

(c) $B^{[1]} = \pi_{w+1} \circ (\chi_w \circ B)^1.$

This holds because if $(m, e'') \in S_p^w(M)^0$ and $e_i^w \in S_p^w(M)^0_{(m,e'')}$ then

$$B_*\left(\frac{\partial}{\partial u_i}(q)\right) = (\pi_w \circ \chi_w \circ B)_*\left(\frac{\partial}{\partial u_i}(q)\right)$$
$$= \pi_w{}^* (\chi_w B)_*\left(\frac{\partial}{\partial u_i}(q)\right).$$

Since $B^{[1]}(q) = \left(q, sp\left\{\frac{\partial}{\partial u_i}(q)\right\}\right)$ and $\pi_{w+1}(m, e_1^1, \dots, e_p^1, \dots, e_1^w, \dots, e_p^w) = (\pi_w(m, e_1^1, \dots, e_p^1, \dots, e_1^{w-1}, \dots, e_p^{w-1}), sp\{\pi_{w*}e_1^w, \dots, \pi_{w*}e_p^w\})$, this proves (c).

Applying (c) again to $B^{[1]}$ gives

$$\begin{aligned}(m, P) = B^{[2]}(q) &= \pi_z \circ (\chi_{z-1} \circ B^{[1]})^1(q)\\ &= \pi_z \circ (\chi_{z-1} \circ \pi_{z-1} \circ (\chi_{z-2} \circ B)^1)^1(q)\\ &= \pi_z \circ (\chi_{z-2} \circ B)^2(q)\\ &= \pi_z(m, e).\end{aligned}$$

This proves (b) and hence the theorem.

§3. INTEGRABILITY AND LIFT FORMS

The reduction theorem of the previous section reduces certain integrability considerations about points in $G_p^z(M)$ for general z to the case, $z = 2$. We now turn to the case, $z = 2$, and find an intrinsic condition for a point of $G_p^2(M)$ to be integrable. Combined with the reduction theorem this will give us an intrinsic criterion for a point of $G_p^z(M)$ to be integrable. This will enable us to show that the integrable points of $G_p^z(M)$ are a submanifold of $G_p^z(M)$, and that $IS_p^z(M)$ is a bundle over that submanifold.

Our intrinsic condition for integrability will be in terms of what we call 'lift forms' so we begin by discussing these. We shall use the following terminology. By a 1-form *on M* we shall mean a function ω on the tangent vectors to M, linear on each tangent space, and defined on all tangent vectors at all points of M; so ω is really a function on the tangent bundle to M. And we use the corresponding terminology for higher degree forms. If $m \in M$ then a 1-form *at m* will mean a linear function on M_m; a 1-form *of M* will mean a 1-form on an open submanifold of M. And similarly for higher degree forms.

DEFINITION. *Let $(m, P) \in G_p(M)$. A* lift form at (m, P) *is a 1-form ω at (m, P) with the property: $\omega(t) = 0$ for all $t \in G_p(M)_{(m,P)}$ such that $\pi_* t \in P$, where $\pi = \pi_1^G = \pi[G_p(M) \to M]$. A* lift form on $G_p(M)$ *is a 1-form on $G_p(M)$ which is a lift at each point of $G_p(M)$.*

We note that the set of 1-forms at (m, P) which are lift forms at (m, P) is the annihilator of the subspace $\pi_{1*}^{G-1}(P)$ of $G_p(M)$ hence *an element of $G_p(M)_{(m,P)}$ projects* (under π_{1*}^G) *into P if and only if it is annihilated by all lift forms at* (m, P). As a consequence we obtain

LEMMA (3.1). *Let $(m, P, Q) \in G_p^2(M)$ and $\pi = \pi_1^G$. Then $\pi_* Q = P$ if and only if $(m, P, Q) \in G_p^2(M)^0$ and all lift forms at (m, P) vanish on Q.*

Proof. If $\pi_* Q = P$ then clearly $(m, P, Q) \in G_p(M)^0$, and all lift forms at (m, P) vanish on Q, by the definition of a lift form. If all lift forms vanish on Q then the above remark shows each element of Q maps into P and then, because $(m, P, Q) \in G_p^2(M)^0$ we have $\pi_* Q = P$.

We call the above forms 'lift forms' because of the following property: if A' is any map of an open $0 \subseteq R^p$ into $G_p(M)$ such that $\pi_1^G \circ A' = A$ is non-singular, then

$A' = A^{[1]}$ if and only if $\omega \circ A'_* = 0$ for all lift forms ω.

Proof. First suppose $A' = A^{[1]}$, hence $A'(q) = (q, A_*(0_q))$ for $q \in 0$. Let ω be any lift form and we wish to show that ω vanishes on $A'_*(0)$. So let $t \in A'_*(0_q)$, thus t is a tangent vector at $(q, A_*(0_q))$ (using $A' = A^{[1]}$). Then $\pi_1^G \circ A' = A$ implies $\pi_{1*}^G A'_* = A_*$, hence $\pi_{1*}^G t \in A_*(0_q)$, so $\omega(t) = 0$, by definition of a lift form. Now suppose that $\omega \circ A'_* = 0$ for all lift forms ω. Then $A'_*(0_q) \subseteq$ the annihilator of all lift forms, hence $A'_*(0_q) = P'$ is a p-dimensional subspace of the tangent space at $A'(q) = (m, P)$ and projects to P under π_{1*}^G, i.e. $\pi_{1*}^G A'_* = A'$, so $(\pi_1^G \circ A')_* = A'$, i.e. $A^{[1]} = A'$.

We shall denote the set of integrable points of $G_p^z(M)$ by $IG_p^z(M)$. One aim of this section is to prove

THEOREM (3.1). *A point $(m, P, Q) \in G_p^2(M)$ is integrable if and only if $(m, P, Q) \in G_p^2(M)^0$ and every C^∞ lift form ω on $G_p(M)$ has the property that ω and $d\omega$ vanish on Q, i.e. that $\omega(t) = 0$ and $d\omega(s, t) = 0$ for all $s, t \in Q$.*

We also wish to prove:

THEOREM (3.2). *$IG_p^2(M)$ is a submanifold of $G_p^2(M)$ and $IS_p^2(M)$ is, in a natural way, a bundle over $IG_p^2(M)$.*

The proofs will depend on co-ordinate expressions for the lift forms, co-ordinate conditions describing $IG_p^2(M)$, etc. so we begin by obtaining co-ordinate expressions for the lift forms. In the following $\{x_a\}$ will be a co-ordinate system of M and y_i^0, y_r^j the associated co-ordinate system of $G_p(M)$, with $1 \le i \le p, 0 \le j \le p, p+1 \le r \le d$. Let (m, P) be any point in the domain of these co-ordinates, t any tangent vector to $G_p(M)$ at (m, P), so

$$t = \sum_i c_i \frac{\partial}{\partial y_i^0}(m, P) + \sum_r c_r \frac{\partial}{\partial y_r^0}(m, P) + \sum_{i,r} c_{i,r} \frac{\partial}{\partial y_r^i}(m, P).$$

Then $\pi_{1*}^G t \in P$ if and only if $\pi_{1*}^G t$ is a linear combination of the

$$e_i = \frac{\partial}{\partial x_i}(m) + \sum_r y_r^i(m, P) \frac{\partial}{\partial x_r}(m).$$

Since

$$\pi_{1*}^G \frac{\partial}{\partial y_a^0}(m, P) = \frac{\partial}{\partial x_a}(m)$$

this shows that $\pi_1^G \in P$ if and only if

$$\sum_{i=1}^p c_i y_r^i(m, P) = c_r \qquad \text{for all } r.$$

We define ω_r, a 1-form of M, by

$$\omega_r = \sum_{i=1}^p y_r^i dy_i^0 - dy_r^0, \qquad p+1 \le r \le d.$$

Then

$$\omega_r(t) = \sum_{i=1}^p y_r^i(m, P) c_i - c_r$$

hence $\pi_{1*}^G t \in P$ if and only if all $\omega_r(t) = 0$. Hence on the domain of this co-ordinate system, *ω is a lift form if and only if*

$$\omega = \sum_{r=p+1}^d f_r \omega_r = \sum_r f_r\left(\sum_i y_r^i dy_i^0 - dy_r^0\right)$$

for some functions f_r. This shows that C^∞ lift forms exist and that if ω is a lift form at a point of $G_p(M)$ then it can be extended to a C^∞ lift form on the whole of $G_p(M)$.

LEMMA (3.2). *Let $\{x_a\}$ be any co-ordinate system of M and $\{y_i^0, y_r^{j,k}\}$ the associated co-ordinate system of $G_p^2(M)$ $(1 \le i \le p, 0 \le j, k \le p, p+1 \le r \le d)$. For points (m, P, Q) in the domain of this co-ordinate system each of the following conditions is equivalent*:

(a) $y_r^{(0,j)}(m, P, Q) = y_r^{(j,0)}(m, P, Q)$;

(b) $\pi_{1*}^G Q = P$;

(c) all lift forms at (m, P) vanish on Q.

Proof. The equivalence of (b) and (c) follows from Lemma (3.1) and the fact, from §2, that all points in the domain of a special co-ordinate system of $G_p^2(M)$ are in $G_p^2(M)^0$.

Now we prove the equivalence of (a) and (b). We shall use also the co-ordinate system $\{y_i^0, y_r^j\}$ of $G_p(M)$ obtained from $\{x_a\}$; and we shall use the projections ρ_m, $\rho_{(m,P)}$ defined in §2 from these co-ordinate systems. Throughout the calculations below we write π for π_1^G.

From (2.2) and (2.3) we have

$$\text{(i)} \quad \begin{cases} \pi_* \dfrac{\partial}{\partial y_a^0}(m, P) = \dfrac{\partial}{\partial x_a}(m) \\[2ex] \pi_* \dfrac{\partial}{\partial y_r^i}(m, P) = 0 \end{cases}$$

(ii) $\pi_* \circ \rho_{(m,P)} = \rho_m \circ \pi_*$.

We let e_i be the element of P that projects to $\frac{\partial}{\partial x_i}(m)$ under ρ_m and f_i be the element of Q that projects to $\frac{\partial}{\partial y_i^0}(m, P)$ under $\rho_{(m,P)}$ so,

(iii) $e_i = \frac{\partial}{\partial x_i}(m) + \sum_r y_r^i(m, P) \frac{\partial}{\partial x_r}(m)$

(iv) $f_i = \frac{\partial}{\partial y_i^0}(m, P) + \sum_{r,j} y_r^{j,i}(m, P, Q) \frac{\partial}{\partial y_r^j}(m, P).$

Now we show

(v) $\pi_* Q = P$ if and only if $\pi_* f_i = e_i$ for all i.

Proof of (*v*). If all $\pi_* f_i = e_i$ it is clear that $\pi_* Q = P$. Now suppose $\pi_* Q = P$. Then by (i), (iv) and (ii),

$$\rho_m(\pi_* f_i) = (\pi_* \circ \rho_{(m,P)}) f_i = \pi_* \left(\frac{\partial}{\partial y_i^0}(m, P) \right) = \frac{\partial}{\partial x_i}(m).$$

Since ρ_m is non-singular on $P, f_i \in Q$, and $\pi_* Q = P$, this implies $\pi_{1*}^G f_i = e_i$, proving (v).

Now we finish the proof that (a) is equivalent to (b). From (i) and (iv) we have

$$\pi_{1*}^G f_i = \frac{\partial}{\partial x_i}(m) = \sum_r y_r^{0,i}(m, P, Q) \frac{\partial}{\partial x_r}(m)$$

Comparing with (iii) we see that $\pi_{1*}^G f_i = e_i$ (for all i) if and only if $y_r^{0,i}(m, P, Q) = y_r^i(m, P)$ (for all i, r) and since $y_r^i(m, P) = y_r^{i,0}(m, P, Q)$ this shows that $y_r^{i,0}(m, P, Q) = y_r^{0,i}(m, P, Q)$ (for all i, r) if and only if $\pi_{1*}^G f_i = e_i$ (for all i). Using (v) this gives the equivalence of (a) and (b), and hence proves the lemma.

LEMMA (3.3). *Let* $\{x_a\}$ *be any co-ordinate system of* M *and* $\{y_i^0, y_r^{j,k}\}$ *the associated co-ordinate system* $G_p^2(M)$ *the following conditions are equivalent*:

(a) *for all* C^∞ *lift forms* ω *on* $G_p(M)$, *both* ω *and* $d\omega$ *vanish on* Q, *i.e.* $\omega(s) = 0$ *and* $d\omega(s, t) = 0$ *for all* $s, t \in Q$;

(b) $y_r^{j,k}(m, P, Q) = y_r^{k,j}(m, P, Q)$ *for all* j, k, r $(0 \le j, k \le p, p + 1 \le r \le d)$.

Proof. Because all C^∞ lift forms on $G_p(M)$ are, locally, linear combinations with C^∞ coefficients of the

$$\omega_r = \sum_i y_r^i dy_i^0 - dy_r^0 \qquad (p + 1 \le r \le d)$$

we see that (a) is equivalent to

(a′) each ω_r and $d\omega_r$ vanishes on Q.

For $1 \le j \le p$ let f_j be the element of Q such that $\rho_{(m,P)} f_j = \frac{\partial}{\partial y_j^0}(m, P)$, so

(c) $d\omega_r = 0$ on Q if and only if $d\omega_r(f_j, f_k) = 0$ for all j, k, r.

Clearly

$$d\omega_r = \sum_{i=1}^{p} dy_r^i dy_i^0$$

hence

$$\begin{aligned} d\omega_r(f_j, f_k) &= \sum_i dy_r^i(f_j)dy_i^0(f_k) - \sum_i dy_r^i(f_k)dy_i^0(f_j) \\ &= \sum_i y_r^{i,j}(m, P, Q)\delta_{ik} - \sum_i y_r^{i,k}(m, P, Q)\delta_{ij} \\ &= y_r^{k,j}(m, P, Q) - y_r^{j,k}(m, P, Q). \end{aligned}$$

Combining Lemma (3.2) with (a′), (c) and this calculation we see that this lemma holds.

THEOREM (3.1). *If* $(m, P, Q) \in G_p^2(M)$ *then the following conditions are equivalent*:

(1) (m, P, Q) is integrable;

(2) $(m, P, Q) \in G_p^2(M)^0$ and *for every* C^∞ *lift form* ω *on* $G_p(M)$, *both* ω *and* $d\omega$ vanish on Q;

(3) if $(m, P, Q) \in G_p^2(M)^0$ and $\{y_i^0, y_r^{j,k}\}$ is a special co-ordinate system at (m, P, Q) then $y_r^{j,k}(m, P, Q) = y_r^{k,j}(m, P, Q)$ for all r, j, k $(0 \le j, k \le p, p+1 \le r \le d)$.

(4) There exists an $(m, e) \in IS_p^2(M)$ such that $\pi_2(m, e) = (m, P, Q)$.

Proof. By Lemma (3.3), (2) and (3) are equivalent, so it will be sufficient to show that (e) implies (4), (4) implies (1) and (1) implies (2).

Proof that (3) *implies* (4): Applying (2.4), (2.5), (2.7) one has, in the notation of §2, if $1 \le i, j \le p$,

$$\begin{aligned} & y_r^{i,0} \circ \pi_2 = \sum v_l^{i,0} x_r^{l,0} \\ (3.1) \qquad & y_r^{0,i} \circ \pi_2 = \sum v_l^{0,i} x_r^{0,l} \\ & y_r^{i,j} \circ \pi_2 = -\sum v_l^{0,j} v_m^{n,0} x_n^{u,l} v_u^{i,0} x_r^{m,0} + \sum v_l^{0,j} v_m^{i,0} x_r^{m,l} \end{aligned}$$

(This is the explicity expression, for the case $z = 2$, of (2.6)). The summation indices in the above are l, m, n, u and satisfy $1 \le l, m, n, u \le p$. We know by Lemma (2.2) that for any choice of numbers (c_i^{jk}) $(1 \le i \le p, 0 \le j, k \le p)$ such that the $p \times p$ matrices $(c_i^{j,0})$ and $(c_i^{0,j})$ are non-singular, there exists a unique $(m, e) \in S_p^2(M)$ such that $\pi_2(m, e) = (m, P, Q)$ and $x_i^{j,k}(m, e) = c_i^{j,k}$. We now consider any fixed set of such numbers $c_i^{j,k}$ with the additional property: $c_i^{j,k} = c_i^{k,j}$ (for all i, j, k). (For our purposes it would suffice to make a particular choice, e.g. $c_i^{j,0} = c_j^{i,0} = \delta_{ij}$ and all other $c_i^{j,k} = 0$.) Using that $y_r^{j,k}(m, P, Q) = y_r^{k,j}(m, P, Q)$ and (3.1) it then follows easily that the (m, e) obtained with this choice of the $c_i^{j,k}$ satisfies

$$x_a^{j,k}(m, e) = x_a^{k,j}(m, e)$$

for all a, j, k. Hence, by Theorem (1.1), $(m, e) \in IS_p^2(M)$, so (4) holds.

Proof that (4) *implies* (1). If A is a map of an open $0 \subseteq R^p \to M$ which is non-singular and with $A^2(q) = m, e)$ then, because $A^{[2]} = \pi_2 \circ A^2$ we have $A^{[2]}(q) = (m, P, Q)$, proving (1).

Proof that (1) implies (2): Let A be a non-singular map of 0 (open, in R^p) into M with $A^{[2]}(q) = (m, P, Q)$. Let ω be any C^∞ lift form on $G_p(M)$. Then, by Lemma (3.1), $\omega \circ A_*^{[1]} = 0$ hence $d\omega \circ A_*^{[1]} = d(\omega \circ A_*^{[1]}) = 0$. In particular, applied at $A^{[1]}(q) = (m, P)$, this says ω and $d\omega$ vanish on Q. Hence the theorem is proved.

Remark. It is clear from the proof that it is equivalent to state (3) for all co-ordinate systems or for a single one.

LEMMA (3.4). *$IG_p^z(M)$ is a submanifold of $G_p^z(M)^0$ of dimension $p + (d-p)(p_z)$. If $\{x_a\}$ is any co-ordinate system of M and $\{y_i^0, y_r^\alpha\}$ the associated co-ordinate system of $G_p^z(M)$, with domain $Q^{[z]}$, then*

$$Q^{[z]} \cap IG_p^z(M) = [(m, P) \in Q^{[z]} | y_r^\alpha(m, P) = y_r^{\pi\alpha}(m, P) \quad \text{for all permutations } \pi \text{ of } \{1, \dots, z\}].$$

Proof. The first statement is immediate from the second. We prove the second by induction. For $z = 0$ or 1 it is immediate; for $z = 2$ it was proved in Theorem (3.1). We now assume it for $z - 1 \geq 2$ and prove it for z. By the reduction theorem of §2 we know that $(m, P_1, \dots, P_z) \in G_p^z(M)$ is integrable if and only if $(m, P_1, \dots, P_{z-1}) \in IG_p^{z-1}(M)$ and $(m, P_1, \dots, P_z)$ is integrable over $G_p^{z-2}(M)$. For points in $Q^{[z]}$ the first of these is characterized by $y_r^{(\alpha',0)} = y_r^{(\pi'\alpha',0)}$ (using our induction assumption) for all $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$ and permutations π' of $\{1, \dots, z-1\}$. And the second is characterized by $y_r^{(\alpha'',j,k)} = y_r^{(\alpha'',k,j)}$ for all $\alpha'' = (\alpha_1, \dots, \alpha_{z-2})$ and all j, k. Together these give the desired condition for z, thus proving the lemma.

DEFINITION. *We define a 1-form θ of $G_p^z(M)$ to be a* lift form *of $G_p^z(M)$ if and only if it can be expressed either as $\theta = \omega \circ (\pi_z^G \circ \dots \circ \pi_{w+1}^G)_*$ where is a lift form of $G_p^w(M)$, considered as $G_p(G_p^{w-1}(M))$, (for some w with $1 \leq w \leq z-1$), or as a lift form of $G_p^z(M)$ over $G_p^{z-1}(M)$.*

COROLLARY. *The point $(m, P_1, \dots, P_z) \in G_p^z(M)$ is integrable if and only if θ and $d\theta$ vanish on P_z for all lift forms θ of $G_p^{z-1}(M)$.*

Proof. Immediate from Lemma (3.3) and (3.4).

For the proof of Lemma (2.2) we defined a certain open subset $0(z)$ of Euclidean space of dimension $p(p+1)^z - p$. We now consider the subset $0_I(z)$ of $0(z)$ consisting of all those $(c_i^\alpha) \in 0(z)$ such that $c_i^\alpha = c_i^{\pi\alpha}$ for all permutations π of the integers 1 through z. So $0_I(z)$ is naturally diffeomorphic to the space $Lp \times R^t$ where L_p is, as above, the group of non-singular $p \times p$ matrices, and $t = p(p_z - 1) - p^2 = p(p_z - p - 1)$. The following lemma contains the essential part of the proof that $IS_p^z(M)$ is a bundle over $IG_p^z(M)$ with fibre $L_p \times R^t$. We omit, as before, the proof that the strip maps of the type given by the following are properly related (it is easy and not necessary for us). Again we assume a co-ordinate system $\{x_a\}$ given for M and use, in the following lemma, its associated co-ordinate systems for $G_p^z(M)$ and $S_p^z(M)$, and with our usual notation for those. We let, as before, $Q^{[z]}$ and Q^z be the domains of those co-ordinate systems.

LEMMA (3.5). If $A(z)$ is the map of Lemma (2.2) then the restriction of $A(z)$ to $(Q^{[z]} \cap IG_p^z(M) \times O_I(z))$ is a diffeomorphism of the space onto $\pi_z^{-1}(Q^{[z]}) \cap IS_p^z(M)$.

Proof. The proof consists in performing an induction, essentially the same as in the proof of Lemma (2.2), to show that $A(z)(m, e) \in IS_p^z(M)$ if and only if $(c_i^\alpha) \in 0_I(z)$. We omit the details because of the close similarity with the previous proof.

From Lemma (3.4) we see how to obtain, for each co-ordinate system $\{x_a\}$ of M, a co-ordinate system $\{y_i^0, y_r^\lambda\}$ of $IG_p^z(M)$, where the ranges of these indices are: $1 \leq i \leq p$, $p+1 \leq r \leq d$, $\lambda = (\lambda_1, \dots, \lambda_p)$, $0 \leq \lambda_i \leq z$, $\sum \lambda_i \leq z$. We define y_r^λ to be y_r^α where α is any superscript such that, for all i, the number of α_w which equal i is λ_i. These co-ordinate systems will be used often.

§4. HIGHER ORDER GRASSMANN BUNDLES AND THE KURANISHI DIFFEOMORPHISM THEOREM

We now wish to define the analogs of $S_p(M)$ and $G_p(M)$ for higher order contact and shall denote those for zth order contact by ${}^zS_p(M)$ and ${}^zG_p(M)$. We shall then establish a natural diffeomorphism between ${}^zG_p(M)$ and $IG_p^z(M)$; from our point of view this diffeomorphism carries the information that a higher order system of partial differential equations is equivalent to a system of first order partial differential equations. It also provides an essential structure on ${}^zG_p(M)$.

We first discuss ${}^zS_p(M)$. The elements of ${}^zS_p(M)$ will be certain bases of the spaces of zth order tangent vectors at the points of M and we now explain just which bases. If A is a non-singular map of the open $0 \subseteq R^p$ into M with $A(q) = m$ then A_* maps the bases $\dfrac{\partial}{\partial u_\alpha}(q)$ $(|\alpha| \leq z)$ of z the space of zth order tangent vectors to R^p at q onto a base of a zth order tangent p-plane to M at m and ${}^zS_p(M)$ is to consist of all such bases at all points of M. We would like to characterize such bases intrinsically, without reference to such an A. We shall do this in the following way, which does eliminate A but depends upon $IS_p^z(M)$, which is also not intrinsic to ${}^zS_p(M)$. In fact our definition of ${}^zS_p(M)$ will make it only trivially different from $IS_p^z(M)$ and for this reason ${}^zS_p(M)$ does not seem of great interest.

Each $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) \in S_p^z(M)$ gives, by §1, a family $\{e_\alpha\}$ of tangent vectors (of zth order) to M at m. This set will be linearly dependent and if $(m, e) \in IS_p^z(M)$ then various members of this set will be equal. More precisely, if $(m, e) \in IS_p^z(M)$ then e_α will equal e_β if and only if β is a permutation of α. For such families $\{e_\alpha\}$ we now make a change of notation to eliminate this redundancy, i.e. we shall change to a system of subscripts in which different subscripts will indicate different (zth order) tangent vectors. We shall always use the letter λ to denote a p-tuple of integers $(\lambda_1, \dots, \lambda_p)$ where each λ_i satisfies $0 \leq \lambda_i \leq z$. Then to each α as above we assign $\lambda = \lambda(\alpha)$ where $\lambda_i =$ the number of w for which $\alpha_w = i$. Clearly, if $(m, e) \in IS_p^z(M)$ then $e_\alpha = e_\beta$ if and only if $\lambda(\alpha) = \lambda(\beta)$ so we now associate to the sequence $\{e_\alpha\}$ the sequence $\{e_\lambda\}$ where the e_λ are defined by $e_\lambda = e_\alpha$ if and only if $\lambda = \lambda(\alpha)$. Clearly $\{e_\lambda\}$ is a base of the zth order tangent space to M at m. We define ${}^zS_p(M)$ to be the set of all bases of zth order tangent spaces to M at all points of M, i.e. ${}^zS_p(M)$ is the set of all $(m, \{e_\lambda\})$ where $\{e_\lambda\}$ is obtained from an $\{e_\alpha\}$ obtained from an $(m, e) \in IS_z^p(M)$ as above.

We now put the differentiable structure on ${}^zS_p(M)$. Recall our map H of §1, from $S_p^z(M)$ to families $(m, \{e_\alpha\})$. Then we define a map K of $IS_p^z(M)$ to ${}^zS_p(M)$ by $K(m, e) =$ the $(m, \{e_\lambda\})$ associated with $H(m, e) = (m, \{e_\alpha\})$ by the preceding paragraph. Because H is $1:1$ it follows that K is $1:1$. We put on ${}^zS_p(M)$ that differentiable structure carried over via K from our differentiable structure on $IS_z^p(M)$. This completes our definition of ${}^zS_p(M)$.

Now we discuss the higher order Grassmann bundles ${}^zG_p(M)$. The elements of ${}^zG_p(M)$ are to be of the form $(m, {}^zP)$ where ${}^zP$ is the 'right kind' of zth order contact space at m. We could define the 'right kind' of a ${}^zP$ to be one of the form A_*V, where V is all tangent vectors of order $\leq z$ at a point of R^p, and A is a non-singular map carrying that point to m; we could also define it to be the span of the e_λ belonging to a point $(m, \{e_\lambda\})$ of ${}^zS_p(M)$. We prefer, however, to define it through the local ring of M, because that is more intrinsic. Our definition will thus be in the spirit of the Chevalley definition of a tangent vector at m. Let R_m be the local ring of the manifold M at m (i.e. the C^∞ functions at m, or the ring of germs of C^∞ functions at m, or an equivalent). We shall define the 'right kind' of ${}^zP$ to be the space of linear functions on R_m which annihilate the 'right kind' of ideal in R_m. We now motivate the definition of the 'right kind' of an ideal in R_m.

If N is any p-dimensional submanifold through $m \in M$ it gives rise to a certain ideal I_N in R_m, namely $I_N =$ all germs of C^∞ functions on M which vanish on N. We would like first an algebraic characterization of those ideals I in R_m which are of the form I_N for some such N. One such characterization (though non-algebraic) is: there exists a set of generators $f_{p+1}, \dots, f_d$ of I_N whose differentials are linearly independent at m. This definition can be rendered more algebraic to the extent of being phrased purely in terms of the local ring R_m, in the following way.

One can first give the definition of a germ of a C^∞ vector field and a C^∞ differential form at m, in terms of R_m, i.e. defining $V_m =$ the algebra of derivations of R_m, and $D_m =$ the space of R_m – linear maps of ∇_m into R_m. It is easily proved that V_m is naturally isomorphic to the germs of C^∞ vector fields at m and D_m to the germs of C^∞ differential 1-forms at m. If $f \in R_m$ we also define $df \in D_m$ by $df(X) = Xf$ for all $X \in V_m$; thus df is an R_m linear map of V_m into R_m. Then the statement made above, that the differentials of $f_{p+1}, \dots, f_d$ are linearly independent at m translates to: for each choice of real numbers $c_{p+1}, \dots, c_d$, not all zero, $d(\sum c_r f_r)$ does not map V_m into I_m, where I_m is the maximal ideal of R_m. This gives a rather weak characterization of those ideals I in R_m which are of the form I_N for some p-dimensional submanifold N through m, but we shall use it and shall call such ideals p-ideals.

DEFINITION. *A* p-ideal *in R_m is an ideal I for which there exists a set of d-p generators $f_{p+1}, \dots, f_d$ with the property that for each choice of real numbers $c_{p+1}, \dots, c_d$, not all zero, $d(\sum c_r f_r)$ does not map V_m into I_m.*

It is trivial that every p-ideal is an I_N, where N is a p-dimensional submanifold through. m. Now we note that the local ring of such an N at m, which we shall denote by $R_m(N)$, can be constructed from I_N without reference to N itself, i.e. we have

(a) $R_m/I_N \approx R_m(N)$.

We have this because the restriction homormorphism $J: R_m \to R_m(N)$, is onto and has kernel I_N. Also, under the isomorphism (a),

(b) $I_m/I_N \approx I_m(N)$

where $I_m(N)$ is the maximal ideal in $R_m(N)$. The space of kth order differentials of N at m is, by definition, $I_m(N)/I_m(N)^{k+1}$. This is isomorphic to $I_m/(I_m^{k+1} + I_N)$ because we have the natural homomorphisms

$$I_m \to I_m(N) \to (I_m/I_N)/(I_m/I_N)^{k+1}.$$

The composite here is onto and its kernel is $I_m^{k+1} + I_N$, hence

$$I_m(N)/I_m(N)^{k+1} \approx I_m/(I_m^{k+1} + I_N).$$

Thus the zth order tangent space to N at m is isomorphic to the space of linear (over R) functions l from R_m to R which have the properties: (1) $l(f) = 0$ if f is constant (i.e. f is the germ of a function constant on a neighbourhood of m), (2) $l(f) = 0$ if $f \in I_m^{k+1} + I_N$.

DEFINITION. *A* zth order p-ideal *at m is any ideal in R_m of the form $I + I_m^{k+1}$ where I is any p-ideal in R_m. A* zth order p-space *at m is any dual space of an $I_m/(I + I_m^{z+1})$ where I is any p-ideal in R_m. We shall sometimes denote the dual space of $I_m/(I + I_m^{z+1})$ by $(I_m/(I + I_m^{z+1}))^*$.*

A linear function on $I_m/(I + I_m^{z+1})$ can naturally be identified with a linear function on I_m which vanishes on $I + I_m^{z+1}$. We can then extend this function uniquely to a linear function on R_m which vanishes on C, where C is the germs of functions constant on a neighbourhood of m. Thus a zth order p-space at m is essentially a subspace of the linear functions from R_m to R which is the annihilator of some $C + I + I_m^{z+1}$, where I is any p-ideal at m. We shall usually write ${}^zP$ or ${}^zQ$ for a zth order space. We note that such a ${}^zP$ is a linear space over R of dimension $(d-p)(p_z - 1)$. We may have $I + I_m^{z+1} = J + I_m^{z+1}$ with I and J distinct p-ideals, however, in this case each $f \in I$ has the form $g + h$ where $g \in J$ and $h \in I_m^{z+1}$, so I and J have the same elements if we neglect higher orders than the zth.

We define ${}^zG_p(M)$ to be the set of all $(m, {}^zP)$ where m is any point of M and ${}^zP$ is any zth order p-space at m. We have projection mappings ${}^z\pi^G : {}^zG_p(M) \to {}^{z-1}G_p(M)$, defined by

$${}^z\pi^G(m, {}^zP) = (m, \text{span of all tangent vectors of order} \leq z - 1 \text{ at } m \text{ which lie in } {}^zP).$$

One must show for this that span of this set of tangent vectors is a $(z - 1)$st order p-space but this is easy because if ${}^zP$ is the annihilator of $C + I + I_m^{z+1}$ then this set is the annihilator of $C + I + I_m^z$.

Let $\{z_a\}$ be any co-ordinate system of M with domain Q and we shall define a co-ordinate system for ${}^zG_p(M)$ consisting of functions $\{w_i^0, w_r^\lambda\}$ with i, r, λ satisfying $1 \leq i \leq p$, $p + 1 \leq r \leq d$, $\lambda = (\lambda_1, \dots, \lambda_p)$, $0 \leq \lambda_i \leq z$. We also define $|\lambda| = (\sum_i \lambda_i$ and $\lambda! = \lambda_1! \dots \lambda_p!$. Let N be the submanifold of M consisting of the slice, defined through this co-ordinate system by

$$N = [m \in M | x_{p+1}(m) = \dots = x_d(m) = 0]$$

and τ be the projection of Q into N defined by : if m has co-ordinates $(c_1, \dots, c_d)$ then $\tau(m)$ has co-ordinates $(c_1, \dots, c_p, 0, \dots, 0)$. Let ${}^zQ$ be the set of all zth order p-spaces $(m, {}^zP)$ such that $m \in Q$ and ${}^zP$ has the form

$${}^zP = (I_m/(I_m^{z+1} + I))^*$$

where I is a p-ideal having a set of generators of the form

$$f_{p+1} = x_{p+1} - g_{p+1} \circ \tau, \dots, f_d = x_d - g_d \circ \tau$$

where the $g_{p+1}, \dots, g_d$ are C^∞ functions on N. We then define the functions w_i^0, w_r^λ $|\lambda| \leq z$ by,

$$w_i^0(m, {}^zP) = x_i(m)$$

$$w_r^\lambda(m, {}^zP) = \frac{\partial g_r}{\partial x_\lambda}(m), \; |\lambda| \leq z.$$

Although neither I nor the g_r are uniquely determined by ${}^zP$ (even when the x_a are given) it is trivial that these derivatives of order $\leq z$ are independent of the possible choices of I and the g_r. Said in other words, if ${}^zP = (I_m/(I + I_m^{z+1}))^*$ and if $I = I_N$, then, for $(m, {}^zP) \in {}^zQ$, N' is locally the graph of certain functions g_r on the slice N, and we define the co-ordinates of ${}^zP$ to be the derivatives of these g_r of order $\leq z$, in all co-ordinate directions of N. Clearly the dimension of ${}^zG_p(M)$ is $p + (d - p)(p)_z$.

We have a natural projection map ${}^z\pi : {}^zS_p(M) \to {}^zG_p(M)$, defined by

$$ {}^z\pi(m, \{e_\lambda\}) = (m, sp\{e_\lambda\}). $$

It is clear that this span is a zth order p-space for these e_λ come from an integrable point (m, e) in $S_p^z(M)$ and if A is a non-singular map of $Q \subseteq R^p$ into M with $A^z(q) = (m, e)$ then ${}^zP = A_*$ (the space of tangent vectors of order $\leq z$ at q). It is also clear, in the same way, that ${}^z\pi$ maps ${}^zS_p(M)$ onto ${}^zG_p(M)$.

THEOREM (4.1). (Kinanishi factoring theorem). *There exists a unique* $1 : 1$ *map* L *of* $IG_p^z(M)$ *onto* ${}^zG_p(M)$ *such that*

$$ L \circ \pi_z = {}^z\pi \circ K. $$

This L *is a diffeomorphism and* $L \circ \pi_z^G = {}^z\pi^G \circ L$ *(the* L *on the left side being that associated with* $z - 1$).

Proof. To prove the first statement we shall prove, if (m, e) and (m, e^*) are any points of $IS_p^z(M)$, that

(a) $({}^z\pi \circ K)(m, e) = ({}^z\pi \circ K)(m, e^*)$ if and only if $\pi_z(m, e) = \pi_z(m, e^*)$.

This clearly gives the existence of L and the fact that L is $1 : 1$. Because π_z, K, ${}^z\pi$ are onto it is then trivial that L is onto. Hence to prove the first statement of the theorem it is sufficient to prove (a).

For this we shall need the following fact: if $\{x_a\}$ is any co-ordinate system of M and $\{x_a^\alpha\}$ and $\{y_i^0, y_r^\alpha\}$ the associated co-ordinate systems of $S_p^z(M)$ and $G_p^z(M)$ then, for all α,

$$ \text{(i)} \qquad \frac{\partial}{\partial x_i^0}(y_r^\alpha \circ \pi_z) = 0 \qquad \text{if} \qquad 1 \leq i \leq p,\ p + 1 \leq r \leq d. $$

We prove this by induction on z. For $z = 0$ it is trivial. If true for $z - 1$ then (2.6) shows it is true for z, hence (i) is proved.

Because $({}^z\pi \circ K)(m, e) = (m, {}^zP)$ where ${}^zP$ is the span of the e_α obtained from (m, e) we see that (a) is equivalent to

(a′) $\mathrm{sp}\{e_\alpha\} = \mathrm{sp}\{e_{\alpha *}\}$ if and only if $\pi_z(m, e) = \pi_z(m, e^*)$.

We now choose, using Theorem (1.1), a co-ordinate system x_a at m such that

$$ \text{(ii)} \qquad e_\alpha^* = \frac{\partial}{\partial x_\alpha}(m) \qquad \text{for all } \alpha $$

Because the sets $\mathrm{sp}\{e_\alpha\}$ and $\mathrm{sp}\{e_{\alpha *}\}$ have the same dimension we see that (a′) is equivalent to

(a″) each e_α is a linear combination of the $\dfrac{\partial}{\partial x_\beta}(m)$ if and only if $\pi_z(m, e) = \pi_z(m, e^*)$.

Since every e_α is expressible as in (1.2) we see that (a″) is equivalent to

(a‴) $e_\alpha x_\beta = 0$ if one or more β_w is greater than p if and only if $\pi_z(m, e) = \pi(m, e^*)$.

Then, because the set $\{e_\alpha\}$ contains all the $e_{E\alpha}$, in the notation of §1, we have, still assuming that (ii) holds, (a″) is equivalent to

(a^{iv}) $e_\alpha x_r = 0$ for all α and r $(p+1 \leq r \leq d)$ if and only if $\pi_z(m, e) = \pi_z(m, e^*)$.

Proof of (a^{iv}): We induct on z. For $z = 1$ (a^{iv}) is the statement: if $e_1, \ldots, e_p$ are linearly independent vectors in M_m then $e_i x_r = 0$ for all r if and only if $\mathrm{sp}\{e_i\} = \mathrm{sp}\left\{\frac{\partial}{\partial x_i}(m)\right\}$. This is trivially true. So we now assume (a^{iv}) for $z-1$ and prove it for z.

Clearly either of the conditions, $\pi_z(m, e) = \pi_z(m, e^*)$ or $({}^z\pi \circ K)(m, e) = ({}^z\pi \circ K)(m, e^*)$ implies (m, e) and (m, e^*) are both in the domain of the co-ordinate system $\{x_a^\alpha\}$ so we only need consider such points. Clearly from (ii), all $x_r^\alpha(m, e^*) = 0$. Then (2.6) shows (by virtue of condition g) of Lemma (D) that all $y_r^\alpha(\pi_z(m, e^*)) = 0$.

If all $e_\alpha x_r = 0$, i.e. all $x_r^\alpha(m, e) = 0$ then (2.6) shows (again using g)) that $y_r^\alpha(\pi_z(m, e)) = 0$, hence $\pi_z(m, e) = \pi_z(m, e^*)$. On the other hand, if $\pi_z(m, e) = \pi_z(m, e^*)$ then all $y_r^\alpha(\pi_z(m, e)) = 0$ and an easy induction on $|\alpha|$ (z being fixed), using (2.6), and condition (e) of Lemma D, gives that all $x_r^\alpha(m, e) = 0$, i.e. all $e_\alpha x_r = 0$. This proves (a^{iv}) and hence (a).

The remainder of Theorem (4.1) is now easily proved for one proves easily that $w_i^0 \circ L = y_i^0$ and $w_r^\lambda \circ L = y_r^\alpha$ if $\lambda = \lambda(\alpha)$ (in the notation of the beginning of this section) and if these co-ordinate systems are defined, as previously, from the same x_a of M. This shows L is a diffeomorphism. The remaining relation is trivial. Hence Theorem (4.1) is proved.

§5. PARTIAL DIFFERENTIAL EQUATIONS AND THEIR CHARACTERISTICS

DEFINITION. *Let M be a d-dimensional manifold, p any integer satisfying* $1 \leq p < d$, *and z any integer* ≤ 1. *A* system of zth order partial differential equations for a p-dimensional submanifold of M *is a subset E of* ${}^zG_p(M)$.

DEFINITION. *A* solution *of a system E, as above, is a p-dimensional submanifold N of M whose lift* ${}^{[z]}N$, *which is a submanifold of* ${}^zG_p(M)$, *lies in E.* Here ${}^{[z]}N$ *is the natural lift of N into* ${}^zG_p(M)$, *i.e. it is the submanifold of* ${}^zG_p(M)$ *consisting of all* $(n, {}^zP)$ *where* $n \in N$ *and* ${}^zP$ *is the zth order tangent space to N at n.*

Usually the system E is given as the set of common zeros of a family $\{F_\gamma\}$ of functions defined on a part of ${}^zG_p(M)$ and one then says that E *is defined by* $\{F_\gamma\}$. In fact, one usually defines the system to be the family $\{F_\gamma\}$ and then considers two systems $\{F_\gamma\}$ and $\{G_\gamma\}$ to be equivalent if they define the same E. Since the notion of a solution depends only on E and since, if E is given, we can always find a family that defines it (e.g. by choosing all functions that vanish on E) we have defined the system to be just E. However most theorems in this subject depend on properties of the family of functions which vanish on E.

We point out the relation of our notions to the classical notions. Suppose we are given a system of zth order partial differential equations in the classical sense, i.e. a family of expressions

$$(*) \qquad f_\lambda\left(u_1, \ldots, u_p, g_1, \ldots, g_q, \ldots, \frac{\partial^\lambda g_r}{\partial u_\lambda}, \ldots\right) = 0$$

where λ runs through some set (usually finite and most often the integers from 1 through q), $\lambda = (\lambda_1, \dots, \lambda_p)$, $\sum_i \lambda_i \leq z$, and

$$\frac{\partial^\lambda}{\partial u_\lambda} = \frac{\partial^{\lambda_1}}{\partial u_1^{\lambda_1}} \cdots \frac{\partial^{\lambda_p}}{\partial u_p^{\lambda_p}}$$

One calls the u_i 'independent variables', calls the g_j 'unknown functions' and defines a 'solution' to be a set of functions g_j for which (*) holds.

We translate this to a system in our sense. Let $M = R^{p+q}$ and $u_1, \dots, u_{p+q}$ be the usual co-ordinate functions on R^{p+q}. This co-ordinate system for R^{p+q} gives us, as in §4, a co-ordinate system $\{y_i^0, y_r^\lambda\}$ for ${}^zG_p(M)$ (in §4 these were denoted by $\{w_i^0, w_r^\lambda\}$. Then the above functions f_γ give, via this co-ordinate system, functions F_γ of ${}^zG_p(M)$, defined by

$$F_\lambda = f_\lambda(y_1^0, \dots, y_p^0, y_{p+1}^0, \dots, y_{p+q}^0, \dots, y_r, \dots).$$

(The domain of the F_γ will be that part of the domain of the co-ordinate system $\{y_i^0, y_r^\lambda\}$ which maps, under the homomorphism defining this co-ordinate system, to points in the domain of the f_γ.) We define the system E to be that defined by these F_γ. It is easy to see that any solution (in our sense) N of this E will lie in R^{p+q} as the graph of a map from part of R^p to R^q and thus will give q functions $g_1, \dots, g_q$ of p real variables, and these g_j will be a solution (in the classical sense) of (*). In fact, if $g_1, \dots, g_q$ are any C^∞ functions defined on an open subset 0 of R^p then (*) is just an analytic expression of the fact that the graph N of the map from 0 to R^q defined by these g_j has its lift ${}^{[z]}N$ in the E defined above, via the F_γ, from the f_γ.

The Kuranishi factoring theorem shows every system of zth order partial differential equations is 'equivalent' to some system of first-order partial differential equations. For this we first note that $IG_p^z(M)$ is 'contained' in $G_p(IG_p^{z-1}(M))$ sense that if $(m, P_1, \dots, P_z) \in IG_p^z(M)$ then P_z is tangent to the submanifold $IG_p^{z-1}(M)$ of $G_p^{z-1}(M)$ (in its definition P_z is only given tangent to $G_p^{z-1}(M)$). For if A is any non-singular map (of an open $Q \subseteq R^p$ into M) with $A^{[z]}(q) = (m, P_1, \dots, P_z)$ then $A^{[z-1]}(Q) \subseteq IG_p^{z-1}(M)$, hence $P_z = (A^{[z-1]})_*(R_q^p)$ is tangent to $IG_p^{z-1}(M)$. Now let E be any subset of ${}^zG_p(M)$ and E_1 the corresponding subset of $IG_p^z(M)$ under our diffeomorphism theorem. Then, by the above remarks, $E_1 \subseteq G_p(M')$ where $M' = IG_p^{z-1}(M)$. If N is any solution of E then $N^{[z-1]}$ will be a solution of E_1 and if N_1 is any solution of E_1 then $\pi_1^G \circ \dots \circ \pi_{z-1}^G(N_1)$ will be a solution of E, by the consistency part of our diffeomorphism theorem. Thus there is a natural 1 : 1 correspondence between solutions of the given system E and solutions of the associated first order system E_1. In the most usual way of establishing such an equivalence between a zth order system and a first order system one gets not a 1 : 1 correspondence between solutions in general but only a 1 : 1 correspondence between solutions of Cauchy problems for the two. That is because one usually uses, instead of our process just described, a process which passes from E to the first prolongation of E_1. In that usual process one gets a quasi-linear system which our E_1 will not, in general be; by prolonging any system one gets a quasi-linear system so if that feature is desired we can obtain it by prolongation, but losing the strict equivalence between solutions of the systems. We also remark that the classical procedure introduces certain new independent variables which, in our procedure, appear as the co-ordinate systems of $IG^z_p(M)$ derived from co-ordinate systems of M.

We now wish to define the *characteristics* of a system E of zth order partial differential equations. We define this for $E \subseteq {}^zG_p(M)$ in terms of the corresponding (under our basic diffeomorphism) $E_1 \subseteq IG^z_p(M)$. The notion of a characteristic will depend on an $(m, P_1, \dots, P_z) \in E_1$ and a P'_z where P'_z is a $(p-1)$-dimensional subspace of P_z. Essentially, the definition says the following. The collection $(m, P_1, \dots, P_z)$ and P'_z is non-characteristic if and only if for each 'appropriate' $(m^*, P^*_1, \dots, P_{z-1})$ and $P^{*\prime}_z$, which are close to $(m, P_1, \dots, P_{z-1})$ and P'_z, there exists a unique P^*_z containing $P^{*\prime}_z$ such that $(m^*, P^*_1, \dots, P^*_z) \in E_1$; and P^*_z is a differentiable function of $(m^*, P^*_1, \dots, P^*_{z-1})$ and $P^{*\prime}_z$. By 'appropriate' we mean here: $P^{*\prime}_{z-1} \subseteq$ some p-dimensional $\tilde{P}_z$ for which $(m^*, P^*_1, \dots, P^*_{z-1}, P_z) \in IG^z_p(M)$. To express this differentiability carefully and for the proof that this notion of characteristic coincides with the classical notion we introduce the following bundles.

$$I'G^z_p(M) = [(m, P_1, \dots, P_{z-1}, P'_z) \mid \text{there exists an } (m, P_1, \dots, P_z) \in IG^z_p(M) \textit{ such that } P'_z \text{ is a } (p-1)\text{-dimensional subspace of } P_z]$$

$$I^+G^z_p(M) = [(m, P_1, \dots, P_{z-1}, P'_z, P_z) \mid (m, P_1, \dots, P_{z-1}, P_z) \in IG^z_p(M) \text{ and } P'_z \text{ is a } (p-1)\text{-dimensional subspace of } P_z].$$

We also define

$$\pi' : I'G^z_p(M) \to IG^{z-1}_p(M) : (m, P_1, \dots, P_{z-1}, P'_z) \to (m, P_1, \dots, P_{z-1})$$

$$\pi^+ : I^+G^z_p(M) \to IG^z_p(M) : (m, P_1, \dots, P'_z, P_z) \to (m, P_1, \dots, P_{z-1}, P_z)$$

$$\pi^0 : I^+G^z_p(M) \to I'G^z_p(M) : (m, P_1, \dots, P'_z, P_z) \to (m, P_1, \dots, P'_z).$$

We now put the differentiable structure on $I^+G^z_p(M)$ and $I'G^z_p(M)$ under which these become bundles with the above maps as projections of the bundles.

Let $\{x_a\}$ be any co-ordinate system of M and we shall consider now three associated co-ordinate systems: (1) the usual associated co-ordinate system $\{y^0_i, y^{\alpha'}_r\}$ of $G^{z-1}_p(M)$, (2) the usual associated co-ordinate system $\{y^0_i, y^\alpha_r\}$ of $G^z_p(M)$, (3) the co-ordinate system of $G_{p-1}(G^{z-1}_p(M))$ associated in the usual way with the co-ordinate system $\{y^0_i, y^{\alpha'}_r\}$ of $G^{z-1}_p(M)$; we denote the functions in this co-ordinate system by $z^0_l, z^0_p, z^{(\alpha',0)}_r, z^{0,l}_p, z^{(\alpha',l)}_r$, where $1 \le l \le p-1, p+1 \le r \le d$, $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$, $0 \le \alpha_w \le p$. The explicit definition of these co-ordinate functions is:

$$z^0_a = y^0_a \circ \pi$$

$$z^{(\alpha',0)}_r = y^{\alpha'}_r \circ \pi \qquad \pi = \pi[G_{p-1}(G^{z-1}_p(M)) \to G^{z-1}_p(M)]$$

$$f'_l = \frac{\partial}{\partial y^0_l}(m, P_1, \dots, P_{z-1}) + z^{0,l}_p(m, P_1, \dots, P_{z-1}, P'_z)\frac{\partial}{\partial y^0_p}(m, P_1, \dots, P_{z-1})$$
$$+ \textstyle\sum_{r,\alpha'} z^{(\alpha',l)}_r(m, P_1, \dots, P_{z-1}, P'_z)\dfrac{\partial}{\partial y^{\alpha'}_r}(m, P_1, \dots, P_{z-1})$$

where, as usual, $f'_1, \dots, f'_{p-1}$ are the elements of P'_z that project to $\dfrac{\partial}{\partial y^0_1}, \dots, \dfrac{\partial}{\partial y^0_{p-1}}$ at

$(m, P_1, \dots, P_{z-1})$ under the projection ρ' defined as usual from the base $\frac{\partial}{\partial y_i^0}, \frac{\partial}{\partial y_r^{\alpha'}}$ of the tangent space to $G_p^{z-1}(M)$ at $(m, P_1, \dots, P_{z-1})$, and projecting onto the span of the first $p-1$ of the $\frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1})$. We denote the domains of our three co-ordinate systems by $Q^{[z-1]}$, $Q^{[z]}$, and Q'.

If $(m, P_1, \dots, P_{z-1}, P_z', P_z) \in I^+G_p^z(M)$ we say the co-ordinate system $\{x_a\}$ of M is *suitable* for $(m, P_1, \dots, P_{z-1}, P_z', P_z)$ if and only if $(m, P_1, \dots, P_{z-1}, P_z') = \pi^0(M, P_1, \dots, P_{z-1}, P_z', P_z)$ is in Q' and $(m, P_1, \dots, P_{z-1}, P_z) = \pi^+(m, P_1, \dots, P_{z-1}, P_z', P_z)$ is in $Q^{[z]}$, i.e. if and only if $(m, P_1, \dots, P_{z-1}, P_z', P_z) \in (\pi^0)^{-1}(Q') \cap (\pi^+)^{-1}(Q^{[z]})$. The existence of a suitable co-ordinate system for any $(m, P^+) = (m, P_1, \dots, P_{z-1}, P_z', P_z) \in I^+G_p^z(M)$ is proved as follows. Let $(m, P) = \pi^+(m, P^+) = (m, P_1, \dots, P_z)$. Choose $(m, e) = (m, e_1^1, \dots, e_p^1, \dots, e_1^z, \dots, e_p^z) \in IS_p^z(M)$ such that $\pi_z(m, e) = (m, P)$. Let $f_1, \dots, f_p$ be a base for P_z such that $f_1, \dots, f_{p-1}$ is a base for P_z'. Then $f_j = \sum a_{ij}\pi_{z-1*}e_i^z$. Define $(m, f) = (m, f_1^1, \dots, f_p^1, \dots, f_1^z, \dots, f_p^z)$ by $f_j^w = \sum a_{ij}e_i^w$. Clearly $(m, f) \in IS_p^z(M)$. Choose a co-ordinate system $\{x_a\}$ of M such that $f_\alpha = \frac{\partial}{\partial x_\alpha}(m)$ if $|\alpha| \le z$, where, as usual, $(m, \{f_\alpha\}) = H(m, f)$. It is then trivial that this $\{x_a\}$ is suitable for (m, P^+) and has furthermore the property: the $\frac{\partial}{\partial y_i^0}$ and $\frac{\partial}{\partial y_l^0}$ with $1 \le i \le p$ and $1 \le l \le p-1$ span P_z', P_z.

LEMMA (5.1). *If* $(m, P_1, \dots, P_{z-1}, P_z') \in Q'$, $(m, P_1, \dots, P_z) \in Q^{[z]}$, *and* $P_z' \subseteq P_z$ *then*

$$
\begin{aligned}
z_r^{(\alpha',l)}(m, P_1, \dots, P_{z-1}, P_z') &= y_r^{(\alpha',l)}(m, P_1, \dots, P_z) \\
&\quad + z_p^{0,l}(m, P_1, \dots, P_{z-1}, P_z')y_r^{(\alpha',p)}(m, P_1, \dots, P_z)
\end{aligned}
\tag{5.1}
$$

for all $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$, $0 \le \alpha_w \le p$, $1 \le l \le p-1, p+1 \le r \le d$.

Proof. We let $f_1', \dots, f_{p-1}'$ be as above and $f_1, \dots, f_p$ be the elements of P_z such that

$$f_i = \frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1}) + \textstyle\sum_{r,\alpha'} y_r^{(\alpha',i)}(m, P_1, \dots, P_{z-1})\frac{\partial}{\partial y_r^{\alpha'}}(m, P_1, \dots, P_{z-1}).$$

Because $P_z' \subseteq P_z$ there exist numbers c_{il} such that

$$f_l' = \textstyle\sum_{i=1} c_{il}f_i$$

and because of the way the $\frac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1})$ appear in the co-ordinate expressions for the f' and f_i we see that the c_{il} are given by

$$c_{il} = \delta_{il} + z_p^{0,l}(m, P_1, \dots, P_{z-1}, P_z')$$

hence

$$f_l' = f_l + z_p^{0,l}(m, P_1, \dots, P_{z-1}, P_z')f_p.$$

Writing the co-ordinate expressions for each side and equating coefficients gives (5.1).

LEMMA (5.2). *If* $(m, P_1, \dots, P_{z-1}, P_z') \in Q'$ *then* $(m, P_1, \dots, P_{z-1}, P_z') \in I'G_p^z(M)$ *if and only if the following all hold at* $(m, P_1, \dots, P_{z-1}, P_z')$:

$$
(5.2)\quad
\begin{aligned}
&\text{(a)}\quad z_r^{(\alpha',0)} = z_r^{(\beta',0)} \textit{ and } z_r^{(\alpha',l)} = z_r^{(\beta',l)} \textit{ whenever } \alpha' \textit{ is a permutation of } \beta',\\
&\text{(b)}\quad z_r^{(\alpha'',0,l)} = z_r^{(\alpha'',l,0)} + z_p^{0,l} z_r^{(\alpha'',p,0)}\\
&\text{(c)}\quad z_r^{(\alpha'',n,l)} = z_r^{(\alpha'',l,n)} + z_p^{0,l} z_r^{(\alpha'',p,n)} - z_p^{0,n} z_r^{(\alpha'',p,l)}
\end{aligned}
$$

for all $l, n, r, \alpha'', \alpha', \beta'$, *with* $1 \le l$, $n \le p-1$, $p+1 \le r \le d$, $\alpha'' = (\alpha''_1, \dots, \alpha''_{z-2})$, $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$, $\beta' = (\beta_1, \dots, \beta_{z-1})$, *and* $0 \le \alpha''_w, \alpha_w, \beta_w \le p$.

Proof. First suppose $(m, P_1, \dots, P_{z-1}, P'_z) \in I'G_p^z(M)$, and let $P'_z \subseteq P_z$ with $(m, P_1, \dots, P_z) \in IG_p^z(M)$. We have $z_r^{(\alpha',0)} = y_r^{\alpha'} \circ \pi$ and since $y_r^{\alpha'} = y_r^{\beta'}$ at $(m, P_1, \dots, P_{z-1})$ whenever β' is a permutation of α' (since $(m, P_1, \dots, P_{z-1}) \in IG_p^{z-1}(M)$) we have $z_r^{(\alpha',0)} = z_r^{(\beta',0)}$ for such α', β'. To prove $z_r^{(\alpha',l)} = z_r^{(\beta',l)}$ for such α', β' recall that P_z is tangent to $IG_p^{z-1}(M)$ hence the above f', since they are in P_z, are tangent to $IG_p^{z-1}(M)$. Since $y_r^{\alpha'} = y_r^{\beta'}$ on $IG_p^{z-1}(M)$ (if β' is a permutation of α') it then follows that $f'_l y_r^{\alpha'} = f'_l y_r^{\beta'}$ and then from the co-ordinate expressions for the f'_l we see that $z_r^{(\alpha',l)} = z_r^{(\beta',l)}$, proving (a).

Now we prove (b) and (c) by showing

(*) (b) plus (c) is equivalent to the statement that for every C^∞ lift from ω of $G_p^{z-1}(M) = G_p(G_p^{z-2}(M))$, both ω and $d\omega$ vanish on P'_z.

Proof of (*). Since every such C^∞ lift form is, by section 3, locally expressible as a linear combination of the

$$\omega_r^{\alpha''} = \textstyle\sum_{i=1}^p y_r^{(\alpha'',i)} dy_i^0 - dy_r^{(\alpha'',0)}$$

$(p+1 \le r \le d$, $\alpha'' = (\alpha''_1, \dots, \alpha''_{z-2})$, $0 \le \alpha''_w \le p)$ it will be sufficient to show (b) plus (c) is equivalent to

(d) all $\omega_r^{\alpha''}$ and $d\omega_r^{\alpha''}$ vanish on P'_z.

Let $f'_1, \dots, f'_{p-1}$ be as above so (d) is equivalent to

(e) $\omega_r^{\alpha''}(f'_l) = d\omega_r^{\alpha''}(f'_l), f'_n = 0$ for all α'', r, l, n.

We now obtain co-ordinate expressions for these $\omega_r^{\alpha''}(f')$ and $d\omega_r^{\alpha''}(f', f'_n)$. Here we will write y_σ^τ for $y_\sigma^\tau(m, P_1, \dots, P_{z-1})$ and z_σ^τ for $z_\sigma^\tau(m, P_1, \dots, P_{z-1}, P'_z)$. We have

$$
\begin{aligned}
\omega_r^{\alpha''}(f') &= \textstyle\sum_{i=1}^p y_r^{(\alpha'',i)} dy_i^0(f'_l) - dy_r^{(\alpha'',0)}(f'_l)\\
&= \textstyle\sum_{i=1}^{p-1} y_r^{(\alpha'',i)} \delta_{li} + y_r^{(\alpha'',p)} z_p^{0,l} - z_r^{(\alpha'',0,l)}\\
&= y_r^{(\alpha'',l)} + y_r^{(\alpha'',p)} z_p^{0,l} - z_r^{(\alpha'',0,l)}\\
&= z_r^{(\alpha'',l,0)} + z_r^{(\alpha'',p,0)} z_p^{0,l} - z_r^{(\alpha'',0,l)}\\
d\omega_r^{\alpha''}(f'_l, f'_n) &= (\textstyle\sum_{i=1}^p dy_r^{(\alpha'',i)} dy_i^0)(f'_l, f'_n)\\
&= \textstyle\sum_{i=1}^p dy_r^{(\alpha'',i)}(f'_l) dy_i^0(f'_n) - \sum_{i=1}^p dy_r^{(\alpha'',i)}(f'_n) dy_i^0(f'_l)\\
&= \textstyle\sum_{i=1}^{p-1} z_r^{(\alpha'',i,l)} \delta_{in} + z_r^{(\alpha'',p,l)} z_p^{0,n} - \sum_{i=1}^{p-1} z_r^{(\alpha'',i,n)} \delta_{il} - z_r^{(\alpha'',p,h)} z_p^{0,l}\\
&= z_r^{(\alpha'',n,l)} + z_r^{(\alpha'',p,l)} z_p^{0,n} - z_r^{(\alpha'',l,n)} - z_r^{(\alpha'',p,n)} z_p^{0,l}
\end{aligned}
$$

From these formulae it is clear that (d) is equivalent to (b) plus (c), hence (*) is proved. Thus we have proved that if $(m, P_1, \dots, P_{z-1}, P'_z) \in I'G^z_p(M)$ then (a), (b), (c) hold.

Now suppose that (a), (b), (c) hold. Because $z^{(\alpha',0)}_r = z^{(\beta',0)}_r$ whenever β' is a permutation of α' we see that $y^{\alpha'}_r = y^{\beta'}_r$ for such α', β', hence $(m, P_1, \dots, P_{z-1}) \in IG^{z-1}_p(M)$. Using (*) we see that for all C^∞ lift forms ω of $G_p(G^{z-2}_p(M))$ both ω and $d\omega$ vanish on P'_z. Since the dimension of a maximal subspace on which such a set of functions vanishes is independent of the subspace we conclude that P'_z is contained in some P_z, of dimension p, on which all such ω and $d\omega$ vanish. Then $(m, P_1, \dots, P_{z-1}, P_z) \in IG^z_p(M)$, by our reduction theorem, hence $(m, P_1, \dots, P_{z-1}, P'_z, P_z) \in I'G^z_p(M)$.

We remark that a part of (a), namely that $z^{(\alpha',l)}_r = z^{(\beta',l)}_r$ if β' is a permutation of α', has not been used here. That is because it is dependent on the other conditions—this being essentially a consequence of our reduction theorem.

Now we shall put the differentiable structures on $I'G^z_p(M)$ and $I^+G^z_p(M)$. Let (m, P^+) be any point of $I^+G^z_p(M)$, $(m, P') = \pi^0(m, P^+)$, and $(m, P) = \pi^+(m, P^+)$. Let $\{x_a\}$ be any co-ordinate system of M suitable for (m, P^+). Let $\{y^0_i, y^\alpha_r\}$ and $\{z^0_i, z^{(\alpha',0)}_r, z^{0,l}_p, z^{(\alpha',l)}_r\}$ be the associated co-ordinate systems of $G^z_p(M)$ and $G_{p-1}(G^{z-1}_p(M)$ described above, with domains $Q^{[z]}$ and Q'. We define $Q^+ = (\pi^0)^{-1}(Q') \cap (\pi^+)^{-1}(Q^{[z]})$. Then we have on Q^+:

$$
\begin{aligned}
& z^0_i \circ \pi^0 = y^0_i \circ \pi^+ \\
(5.3) \qquad & z^{(\alpha',0)}_r \circ \pi^0 = y^{(\alpha',0)}_r \circ \pi^+ \\
& z^{(\alpha',l)}_r \circ \pi^0 = y^{(\alpha',l)}_r \circ \pi^+ + (z^{0,l}_p \circ \pi^0)(y^{(\alpha',p)}_r \circ \pi^+)
\end{aligned}
$$

for $1 \le i \le p$, $1 \le l \le p-1$, $p+1 \le r \le d$, $\alpha' = (\alpha_1, \dots, \alpha_{z-1})$, $0 \le \alpha_w \le p$. The first two lines here are trivial and the third is just (5.1) in another notation.

We shall now choose functions which will (by definition) make up a co-ordinate system for $I'G^z_p(M)$. Their domain will be $Q'' = Q' \cap I'G^z_p(M)$ and the functions will be obtained by choosing an 'independent set on Q''' from among the z^0_i, $z^{(\alpha',0)}_r$, $z^{0,l}_p$, $z^{(\alpha',l)}_r$, and restricting them to Q''; the others among these will then be determined on Q'' by (5.2). First however we make a change of notation, similar to that made in section 4 in passing from the e_α to the e_λ. This time however we must be more careful since (5.2) does not give invariance under all permutations of the superscripts.

Recall as before that for each $\alpha = (\alpha_1, \dots, \alpha_z)$ we have defined $\lambda(\alpha) = (\lambda_1, \dots, \lambda_p)$ by: $\lambda_i =$ the number of w for which $\alpha_w = i$. Recall we have also defined, for such α and λ, $|\alpha| =$ the number of w such that $\alpha_w \ne 0$, $|\lambda| = \sum_i \lambda_i$, so $|\alpha| = |\lambda(\alpha)|$. We now consider all $\lambda = (\lambda_1, \dots, \lambda_p)$ such that $|\lambda| \le z$ and $\lambda_p < z$. Each such λ is of the form $\lambda(\alpha)$ for some α satisfying both: (i) $\alpha_z < p$, (ii) $\alpha_z \le \min(\alpha_1, \dots, \alpha_{z-1})$. And by (5.2) (a), if $\lambda = \lambda(\alpha) = \lambda(\beta)$ with α and β both satisfying (i) and (ii) then $z^\alpha_r = z^\beta_r$ on $I'G^z_p(M)$. Hence we may now define z^λ_r for all $\lambda = (\lambda_1, \dots, \lambda_p)$ such that $|\lambda| \le z$ and $\lambda_p < z$, by

$$z^\lambda_r = z^\alpha_r \,|\, Q'' \text{ if } \lambda = \lambda(\alpha),\ \alpha_z < p,\ \alpha_z \le \min(\alpha_1, \dots, \alpha_{z-1})$$

where $1 \le i \le p$, $1 \le l \le p-1$, $p+1 \le r \le d$, $\lambda = (\lambda_1, \dots, \lambda_p)$, and $|\lambda| \le z$, $\lambda_p < z$. We now define the functions z^0_i, $z^{0,l}_p$, z^λ_r for such values of the indices by taking z^λ_r as above and taking z_i^0 and $z_p^{0,l}$ to be the restrictions of our previous z_i^0 and $z_p^{0,l}$ to Q''. (This introduces some inconsequential ambiguity about the domains of z_i^0 and $z_p^{0,l}$.) It is easily proved that the $\{z_i^0, z_p^{0,l}, z_r^\lambda\}$ provide a $1:1$ map of Q'' onto an open set in Euclidean space of dimension $p + (p-1) + (d-p)[(p)_z - 1]$. Furthermore, by facts proved in section 3, the range of this co-ordinate system is the product of the range of $\{x_a\}$ by an entire Euclidean space. By this process we get for each co-ordinate system $\{x_a\}$ of M which is suitable for any point of $I^+G_p^z(M)$ such a map to a Euclidean space, and the collection of all these makes $I'G_p^z(M)$ into a manifold since it is easily checked that any two are C^∞ related. This defines our differentiable structure on $I'G_p^z(M)$, making it a manifold of dimension $3p - d - 1 + (d-p)(p)_z$.

Now we put the differentiable structure on $I^+G_p^z(M)$. Starting again with our co-ordinate system $\{x_a\}$ of M we shall define what will be, by definition, a co-ordinate system of $I^+G_p^z(M)$, with domain Q^+. We shall use here the associated co-ordinate system $\{y_i^0, y_r^\lambda\}$ of $IG_p^z(M)$ obtained from the $\{y_i^0, y_r^\alpha\}$ associated with $\{x_a\}$.

We define the functions which shall constitute this co-ordinate system by

$$\begin{aligned} w_i^0 &= y_i^0 \circ \pi^+ = z_i^0 \circ \pi^0 \\ w_r^\lambda &= y_r^\lambda \circ \pi^+ \\ w_p^{0,l} &= z_p^{0,l} \circ \pi^0 \end{aligned} \tag{5.4}$$

these being defined on Q^+ for $1 \le i \le p$, $1 \le l \le p-1$, $p+1 \le r \le d$, $\lambda = (\lambda_1, \dots, \lambda_p)$, where the λ_i are integers ≥ 0 and $|\lambda| = \sum_i \lambda_i \le z$. Again, by section 3, the range of this co-ordinate system will be the range of $\{x_a\}$ times a full Euclidean space. One proves easily that this set of functions provides a $1:1$ map to a Euclidean space and that the set of all such maps are C^∞ related thus defining our differentiable structure on $I^+G_p^z(M)$ and making it into a manifold of dimension $2p - 1 + (d-p)(p)_z$. From (5.3) we have, on Q^+,

$$\begin{aligned} z_r^\lambda \circ \pi^0 &= w_r^\lambda \qquad \text{if } |\lambda| < z \\ z_r^\lambda \circ \pi^0 &= w_r^\lambda + w_p^{0,l} w_r^{\lambda - \delta_l} + \delta_p \qquad \text{if } |\lambda| = z \end{aligned} \tag{5.5}$$

where $\lambda_p < z$ and $l = \min[i | \lambda_i \neq 0]$. Here δ_i = the ith canonical unit vector in R^p: $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$.

From (5.4) and (5.5) we see that if we fix $(m, P_1, \dots, P_{z-1}, P_z')$ and vary P_z containing P_z' then this variation is described, locally, by the co-ordinates $w_r^{z\delta}p (p+1 \le r \le d)$. For if $(m, P_1, \dots, P_{z-1}, P_z')$ is fixed then all the w_i^0, $w_p^{0,l}$, and w_r^λ with $|\lambda| < z$ are fixed and so are the z_r^λ with $|\lambda| = z$ but $\lambda_p < z$. The last line in (5.5) then shows that varying P_z containing P_z' is equivalent to varying the $w_r^{z\delta}p$, including the fact that any choice of $w_r^{z\delta}p$ gives a (unique) P_z containing P_z'. Thus the dimension of the fibre of $I^+G_p^z(M)$ as a bundle over $I'G_p^z(M)$ is $d-p$. On the other hand, if we fix $(m, P_1, \dots, P_z)$ and vary $P_z' \subseteq P_z$ then this variation is described, locally, by the co-ordinates $w_p^{0,l} (1 \le l \le p-1)$. So the dimension of the fibre of $I^+G_p^z(M)$ as a bundle over $IG_p^z(M)$ is $p-1$.

DEFINITION. *Let E by a system of* zth *order partial differential equations, i.e. a subset of* ${}^zG_p(M)$. *Let E_1 be the corresponding subset of $IG_p^z(M)$ (corresponding that is under our* *diffeomorphism of Theorem (4.1), and let* $E_1^+ = (\pi^+)^{-1}(E_1)$. *A point* $(m, P^+) = (m, P_1, \dots, P_{z-1}, P'_z, P_z) \in I^+G_p^z(M)$ *is said to be* non-characteristic *for* E *if and only if both*:

(1) $(m. P) = \pi^+(m, P^+)$ *is in* E_1, *i.e.* $(m, P^+) \in E_1^+$,

(2) *in some neighbourhood (in* $I^+G_p^z(M)$) *of* (m, P^+), E_1^+ *is locally a cross-section over* $I'G_p^z(M)$, *i.e. there exists a neighbourhood* $\tilde{Q}'$ *of* $(m, P') = \pi^0(m, P^+)$ *and a neighbourhood* $\tilde{Q}$ *of* (m, P^+) *and a map* χ *of* $\tilde{Q}'$ *into* $I^+G_p^z(M)$ *such that*

(a) $\chi \circ \pi^0 = $ *identity, on* $\tilde{Q}'$;

(b) $\chi(m, P') = (m, P^+)$;

(c) $E_1^+ \cap \tilde{Q} = \chi(\tilde{Q}')$;

(d) $\tilde{Q}' = \pi^0\tilde{Q}$.

As remarked above, this says (m, P^+) is non-characteristic if and only if each $(m^*, P^{*\prime})$ near (m, P') determines a unique (m^*, P^*) satisfying the partial differential equation (i.e. belonging to E_1) and (m^*, P^*) is a differentiable function of $(m^*, P^{*\prime})$. This definition could be expressed directly in terms of ${}^zG_p(M)$ (without referring to $IG_p^z(M)$ or our basic diffeomorphism) in the following way. Consider a fixed $(m, {}^zP) \in E$. Consider an $(m, {}^{z-1}P) \in {}^{z-1}G_p(M)$ with ${}^{z-1}P \subseteq {}^zP$ and an $(m, {}^zP') \in {}^zG_{p-1}(M)$ with ${}^zP' \subseteq {}^zP$; so ${}^{z-1}P + {}^zP' \subseteq {}^zP$. Then the triple $(m, {}^{z-1}P + {}^zP', {}^zP)$ is non-characteristic for E if and only if each $(m^*, {}^{z-1}P^* + {}^zP^{*\prime})$ near to $(m, {}^{z-1}P + {}^zP')$ determines a unique $(m^*, {}^zP^*)$ with ${}^{z-1}P^* + {}^zP^{*\prime} \subseteq {}^zP^*$ and $(m^*, {}^zP^*) \in E$, and if the map thereby defined is differentiable. To express this carefully one would need auxiliary bundles corresponding to $I'G_p^z(M)$ and $I^+G_p^z(M)$ so this procedure is just a translation of the other.

We shall show that our definition of non-characteristic is equivalent to a classical one. The classical one says the system is non-characteristic at a point if one can 'solve' the system locally for all zth order derivatives with respect to one of the independent variables, i.e. the system is 'equivalent' (in the sense, say, of having the same solutions as) to one of the form:

$$\frac{\partial^z g_1}{\partial u_p^z} = f_1\left(u_1, \dots, u_p, g_1, \dots, g_q, \dots, \frac{\partial^\lambda g_j}{\partial u_\lambda}, \dots\right)$$

$$\frac{\partial^z g_q}{\partial u_p^z} = f_q\left(u_1, \dots, u_p, g_1, \dots, g_q, \dots, \frac{\partial^\lambda g_j}{\partial u_\lambda}, \dots\right)$$

where the λ on the right side satisfy $|\lambda| \leq z$ and $\lambda_p < z$. In case $q = 1$ (the most interesting case) this is the only classical definition but for $q > 1$ there is at least one alternative which is more general, namely, one supposes one can solve for the highest derivatives with respect to u_p without assuming those highest derivatives are all of the same order. We do not consider this more general version. The following lemma expresses the equivalence of our notion with the above classical notion.

LEMMA (5.3). *Let* $E \subseteq {}^zG_p(M)$, E_1 *be the corresponding set in* $IG_p^z(M)$ *and* $E^+ = (\pi^+)^{-1}(E_1)$. *Let* $(m, P^+) \in E_1^+$. *Then* (m, P^+) *is non-characteristic for* E *if and only if* *there exists a suitable co-ordinate system $\{x_a\}$ for (m, P^+) such that if $\{y_i^0, y_r^\lambda\}$ is the associated co-ordinate system for $IG_p^z(M)$ (or equivalently, for ${}^zG_p(M)$) then E_1 (or E) is defined locally by the family of functions*:

$$y_{p+1}^{z\delta_p} - f_{p+1}(y_i^0, y_r^\lambda), \ldots, y_d^{z\delta_p} - f_d(y_i^0, y_r^\lambda)$$

where all λ occuring in these f_r satisfy $|\lambda| \le z$ and $\lambda_p < z$, and the f_r are C^∞ (on a neighbourhood of the point in R^e, with $e = p + (d-p)[(p)_z - 1]$, whose co-ordinates are the numbers $y_i^0(m, P)$, $y_r^\lambda(m, P)$, for $(m, P) = \pi^+(m, P^+)$).

Remark. When we say E is *locally defined* by such functions we mean there exists a neighbourhood Q of (m, P) such that these functions are defined on Q and the set of their common zeros in Q is $E \cap Q$.

Proof. The only thing to prove is that if (m, P^+) is non-characteristic then it is defined by such functions, for the converse is trivial. Let $(m, P^+) = (m, P_1, \ldots, P_{z-1}, P_z', P_z)$, $(m, P) = \pi^+(m, P^+) = (m, P_1, \ldots, P_{z-1}, P_z)$, and $(m, P') = \pi^0(m, P^+) = (m, P_1, \ldots, P_{z-1}, P_z')$. Choose a co-ordinate system $\{x_a\}$ of M which is suitable for (m, P^+) and with the further properties: for the associated $\{y_i^0, y_r^\lambda\}$, P_z is the span of the $\dfrac{\partial}{\partial y_i^0}(m, P_1, \ldots, P_{z-1})(1 \le i \le p)$ and P_z' is the span of the first $p-1$ of these. This is possible by the remarks preceeding Lemma (5.1). We shall show that for this co-ordinate system the given E is defined, locally, by functions of the required type. In the following we use the notation $\{y_i^0, y_r^\lambda\}$, $\{z_i^0, z_p^{0,l}, z_r^\lambda\}$, $\{w_i^0, w_p^{0,l}, w_r^\lambda\}$ for the co-ordinate systems defined previously from the x_a and also will denote their domains by $Q^{[z]}$, Q', Q^+ (thus deviating slightly from previous notation in which $Q^{[z]}$ and Q' were open sets in $G_p^z(M)$ and $G_{p-1}(G_p^{z-1}(M))$ instead of, as now, $IG_p^z(M)$ and $I'G_p^z(M)$). We also use ρ and ρ' for the projections of the tangent space to $G_p^{z-1}(M)$ at $(m, P_1, \ldots, P_{z-1})$ onto the span of the first p, and first $p-1$, of the $\dfrac{\partial}{\partial y_i^0}(m, P_1, \ldots, P_{z-1})$, defined from the base $\left\{\dfrac{\partial}{\partial y_i^0}(m, P_1, \ldots, P_{z-1}), \dfrac{\partial}{\partial y_r^\alpha}(m, P_1, \ldots, P_{z-1})\right\}$.

We note that

$$\pi^+ Q^+ = Q^{[z]}, \qquad \pi^0 Q^+ = Q'.$$

The first holds because if $(m, P_1, \ldots, P_{z-1}, P_z) \in Q^{[z]}$ then we can find a $(p-1)$-dimensional P_z' contained in P_z with ρ' non-singular on P_z', e.g. by choosing $P_z' =$ the span of those elements in P_z which project under ρ to the first $p-1$ of the $\dfrac{\partial}{\partial y_i^0}(m, P_1, \ldots, P_{z-1})$. The second holds because (5.4) and (5.5) show how to choose from the co-ordinates z_i^0, $z_p^{0,l}$, z_r^λ of (m, P') values of the co-ordinates w_i^0, $w_p^{0,l}$, w_r^λ that will define an (m, P^+) with $\pi^+(m, P^+) = (m, P')$ (using here that the range of these co-ordinates is the range of the x_a times a full Euclidean space).

We now define a local cross-section φ of $I^+G_p^z(M)$ over $IG_p^z(M)$ with domain $Q^{[z]}$, by

$$\varphi(m, P_1, \ldots, P_z) = (m, P_1, \ldots, P_{z-1}, P_z', P_z)$$

where $P'_z =$ the span of the elements of P_z that project under ρ to the first $p-1$ of the $\dfrac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1})$. Clearly,

$$(*) \qquad \begin{aligned} &\varphi(Q^{[z]}) \subseteq Q^+ \\ &\pi^+ \circ \varphi = \text{identity, on } Q^{[z]} \\ &w_p^{0,l} \circ \varphi = 0 \qquad \text{for } 1 \le l \le p-1 \end{aligned}$$

the last being proved by checking the co-ordinate expressions, given earlier, for those $f_i \in P_z$ such that $f_i = \dfrac{\partial}{\partial y_i^0}(m, P_1, \dots, P_{z-1})$.

Now let $\tilde{Q}$ and $\tilde{Q}'$ and χ be as in the definition of non-characteristic for (m, P^+) and we may assume $\tilde{Q}' \subseteq Q'$. Define

$$\tilde{Q}^{[z]} = \varphi^{-1}(\tilde{Q} \cap Q^+).$$

We define functions F_r on $\tilde{Q}^{[z]}$ by

$$F_r = w_r^{z\delta_p} \circ \varphi - w_r^{z\delta_p} \circ \chi \circ \pi^0 \circ \varphi$$

and we shall prove these F_r satisfy the lemma. For this it is sufficient to prove the following two statements:

(1) $F_r = y_r^{z\delta_p} - f_r(y_i^0, y_r^\lambda)$ where the f_r are C^∞ and the only λ occurring have $|\lambda| \le z$ and $\lambda_p < z$;

(2) if $(m^*, P^*) \in {}^{[}\tilde{Q}^{z]}$ then $(m^*, P^*) \in E_1$ if and only if all $F_r(m^*, P^*) = 0$.

Proof of (1). First we note

(a) $w_r^{z\delta_p} \circ \varphi = y_r^{z\delta_p} \circ \pi^+ \circ \varphi = y_r^{z\delta_p}$.

Because $w_r^{z\delta_p} \circ \chi$ is a C^∞ function on $\tilde{Q}'$ there exist C^∞ functions f_r on $\tilde{Q}'$ such that

$$w_r^{z\delta_p} \circ \chi = f_r(z_i^0, z_p^{0,l}, z_r^\lambda)$$

where the only λ occuring have $|\lambda| \le z$ and $\lambda_p < z$. Then using (5.4), (5.5) and (*), we have

(b) $$\begin{aligned} w_r^{z\delta_p} \circ \chi \circ \pi^0 \circ \varphi &= f_r(z_i^0 \circ \pi^0 \circ \varphi, z_p^{0,l} \circ \pi^0 \circ \varphi, z_r^\lambda \circ \pi^0 \circ \varphi) \\ &= f_r(w_i^0 \circ \varphi, w_p^{0,l} \circ \varphi, w_r^\lambda \circ \varphi + (w_p^{0,l} \circ \varphi)(w_r^{\lambda-\delta_l+\delta}p \circ \varphi)) \\ &= f_r(y_i^0 \circ \pi^+ \circ \varphi, 0, y_r^\lambda \circ \pi^+ \circ \varphi) \text{ (by (*))} \\ &= f_r(y_i^0, 0, y_r^\lambda). \end{aligned}$$

Together (a) and (b) prove (1).

Proof of (2). Let $(m^*, P^*) \in \tilde{Q}^{[z]}$ and first suppose $(m^*, P^*) \in E_1$. Then $\varphi(m^*, P^*) \in E_1^+ \wedge Q$, which implies $\varphi(m^*, P^*) = \chi(m^*, P'_1)$ for some (m^*, P'_1). Applying π^0 to both sides, $\pi^0 \circ \varphi(m^*, P^*) = (m^*, P'_1)$, hence $\chi \circ \pi^0 \circ \varphi(m^*, P^*) = \chi(m^*, P'_1) = \varphi(m^*, P^*)$, hence all co-ordinate functions and in particular the $w_r^{z\delta_p}$ satisfy $w_r^{z\delta_p}(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = w_r^{z\delta_p}(\varphi(m^*, P^*))$, i.e. $F_r(m^*, P^*) = 0$.

Now suppose $(m^*, P^*) \in \tilde{Q}^{[z]}$ and all $F_r(m^*, P^*) = 0$. This says, for all r,

$$w_r^{z\delta_p}(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = w_r^{z\delta_p}(\varphi(m^*, P^*)).$$

For any r and λ with $|\lambda| < z$ we have, by (5.5),

$$w_r^\lambda(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = z_r^\lambda(\pi^0 \circ \chi \circ \pi^0 \circ \varphi(m^*, P^*))$$
$$= z_r^\lambda(\pi^0 \circ \varphi(m^*, P^*)) = w_r^\lambda(\varphi(m^*, P^*)$$

and in the same way we have

$$w_i^0(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = w_i^0(\varphi(m^*, P^*))$$

and

$$w_p^{0,l}(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = w_p^{0,l}(\varphi(m^*, P^*)) = 0$$

hence, now with $|\lambda| = z$ but $\lambda_p < z$, we get from (5.5) that

$$w_r^\lambda(\chi \circ \pi^0 \circ \varphi(m^*, P^*)) = w_r^\lambda(\varphi(m^*, P^*)).$$

Because all their co-ordinates are the same we then have $\chi \circ \pi^0 \circ \varphi(m^*, P^*) = \varphi(m^*, P^*)$, proving $\varphi(m^*, P^*) \in \chi(Q') \in E_1$, hence $(m^*, P^*) = \pi^+ \circ \varphi(m^*, P^*) \in E_1$.

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