Internal waves in an unbounded non-Boussinesq flow

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A B S T R A C T
Weakly nonlinear internal waves in an unbounded non-Boussinesq flow with uniform stratification are treated with a Laurent-type expansion. The expansion eliminates the problem encountered with a traditional expansion in wave amplitude where higher harmonics grow exponentially faster with higher order. The results show that the second-order wave correction to the linear estimate of the wave speed of internal waves in an unbounded layer is always negative, meaning that higher amplitude waves travel slower.

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1. Introduction

The linear theory of internal waves in a continuously stratified non-Boussinesq flow with constant stratification shows that monochromatic waves have an exponentially growing wave amplitude [1,2]:

\[ \psi = Ae^{\beta z} \sin kx \sin mz, \quad (1) \]

where \( \psi \) is the streamfunction, \((x, z)\) are the horizontal and vertical distances, respectively, \(k\) and \(m\) are horizontal and vertical wavenumbers, respectively, \(A\) is the wave amplitude, and \(\beta\) is the density scale height, which is constant for this present case of uniform stratification. Numerical simulations [3,4] of horizontally periodic waves created at a lower boundary have demonstrated that this exponential increase does indeed exist. These simulations also show that wave breaking will occur when the wave-generated mean flow (or other mean flow) becomes strong enough to exceed the horizontal phase speed of the wave, thereby forming a critical layer.

When nonlinear effects are included for small amplitude waves using a straight-forward power series in wave amplitude, the exponential growth is magnified such that the higher harmonics grow at a faster rate [1,2]. This dominance of higher harmonics is not evident in the numerical simulations, and is the result of an expansion that is not uniformly valid [5].

I recently considered [5] internal waves in a non-Boussinesq flow with the expansion

\[ \delta = \sum_{p=1}^{\infty} \left( \frac{e^{\alpha z}}{1 + e^{\alpha z}} \right)^p \phi_p(x, z), \quad (2) \]

where \( \delta \) is the vertical displacement of a streamline (related to the streamfunction), \(p\) is an integer, and \( \epsilon \ll 1 \). The exponent \( \alpha \) was chosen to be \( \frac{\beta}{2} \), so that the linear solution with (2) matched the traditional linear solution. The expansion in (2) does not suffer the same fate as the traditional expansion, as each base function is bounded. The fluid layer in [5] was bounded on top and bottom by rigid surfaces, as with previous work by Thorpe [1] and Yih [2]. For the nonlinear case, only the first few terms of this expansion could be determined, due to the complexity of the equations.

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The results found with this new expansion show that the second-order correction to the wave speed is positive or negative, depending on wave parameters. This appears to contrast the previous results of Yih [2], who showed that the wave speed correction is negative, however the new results actually agree with Yih’s results for his limited parameter region ($k \approx 1$). The negative value of the wave speed correction (originally found by Yih [2]) is surprising, and means that larger amplitude waves travel slower than small amplitude waves, opposite the case for most wave systems, such as free-surface waves.

An increase in wave speed with increasing wave amplitude is generally known as ‘amplitude dispersion’. Amplitude dispersion is important when considering the formation of solitary waves or cnoidal waves. Early work by Long [6] and Benjamin [7] treated such long waves assuming amplitude dispersion, but they did not actually determine the wave speed behavior (see the detailed discussion by Yih [2]). Yih [2] was the first to show that the correction to the wave speed for finite-amplitude effects is negative, and my recent work [5] showed that it may sometimes be positive. A recent review of the theory of solitary waves in a variety of environmental flows is provided by Grimshaw [8].

Here I treat waves of permanent form in an unbounded region, a problem that could not be treated meaningfully with the traditional expansion.

2. Weakly nonlinear theory

2.1. Physical model

The flow is assumed to be incompressible, inviscid, and two-dimensional. A coordinate system moving with the horizontal wave speed is chosen, making the flow steady. For steady flow the governing equations can be reduced to Long’s equation [9]:

$$\nabla^2 \delta + \frac{1}{2} \frac{dq}{dz_0} \left[ 2 \frac{\partial \delta}{\partial z} - \left( \nabla \delta \right)^2 \right] + \frac{g \beta}{u_0^2} \delta = 0, \tag{3}$$

where $\delta(x, z)$ is the displacement of a streamline, defined by

$$\delta = z - z_0. \tag{4}$$

$g$ is the gravitational constant, and

$$\beta = -\frac{1}{\rho_0} \frac{d \rho_0}{dz_0}, \tag{5}$$

$$q = \rho_0 u_0^2. \tag{6}$$

The quantity, $\rho_0$, is the background density, defined to be the spatially averaged part of the total density field. The quantity, $u_0$, is the background horizontal velocity. This background velocity is not required to be merely the horizontally averaged velocity field, but is required to reduce to the linear wave speed when the wave amplitude becomes infinitesimal. The distance, $z_0$, is the vertical position of streamlines in the background state. The fluid velocity at any position is related to $\delta$ by

$$u = u_0 \left( 1 - \frac{\partial \delta}{\partial z} \right), \tag{7}$$

$$w = u_0 \left( \frac{\partial \delta}{\partial x} \right), \tag{8}$$

where $u, w$ are the velocities in the horizontal and vertical directions, respectively. The Brunt–Vaisala frequency is defined by

$$g \beta = -\frac{g}{\rho_0} \frac{d \rho_0}{dz_0}. \tag{9}$$

The results below are not restricted to Long’s equation, as the same results can be obtained using the primitive equations directly. Long’s equation is merely convenient for the present work.

The boundary conditions on the sides of the domain are that $\delta$ is periodic. For the top and bottom boundaries, the linear wave solution shows that the waves cannot be truly periodic in the vertical, due to the exponentially decaying density. Instead the condition on the top and bottom is that wave radiation must be upwards, and $\delta$ must be bounded.

The system is made dimensionless using a length scale, $\lambda$, which is the horizontal wavelength, a velocity scale, $U = \sqrt{g \lambda}$, and a constant value of density, $\rho_{00}$:

$$\hat{\delta} = \frac{\delta}{\lambda}, \quad \hat{x} = \frac{x}{\lambda}, \quad \hat{z} = \frac{z}{\lambda}, \quad \hat{\rho} = \frac{\rho}{\rho_{00}}, \quad \hat{q} = \frac{q}{\sqrt{g \lambda}}. \tag{10}$$
Long’s equation becomes
\[ \hat{\nabla}^2 \hat{\delta} + \frac{1}{2} \frac{\partial q}{\partial z_0} \left[ 2 \frac{\partial \hat{\delta}}{\partial z} - (\hat{\nabla} \hat{\delta})^2 \right] + \frac{\hat{\beta}}{\hat{u}_0^2} \hat{\delta} = 0, \]  \tag{11}

where
\[ \hat{\beta} = \frac{\lambda}{\rho_0} \frac{d \rho_0}{d z_0}, \]  \tag{12}
\[ \hat{u}_0 = \frac{u_0}{\sqrt{g \lambda}}. \]  \tag{13}

Dropping the circumflex, Long’s equation becomes
\[ \nabla^2 \delta + \frac{1}{2} \frac{\partial q}{\partial z_0} \left[ 2 \frac{\partial \delta}{\partial z} - (\nabla \delta)^2 \right] + \frac{\beta}{u_0^2} \delta = 0, \]  \tag{14}

where all quantities are now dimensionless [including \( \beta \) and \( u_0 \), which are now defined by (12) and (13) rather than their previous definitions].

2.2. The expansion

Define \( Q \):
\[ Q = \frac{e^{\alpha z}}{1 + e^{\alpha z}}. \]  \tag{15}

Expand \( \delta \) in a power series:
\[ \delta = Q \phi_1 + Q^2 \phi_2 + \cdots, \]  \tag{16}
where \( \phi_j \) will be subsequently determined. Note that the analysis is aided by the relation
\[ \frac{d}{dz} (Q)^p = \alpha p (Q)^{p-1} - (Q)^{p+1}. \]  \tag{17}

As in my previous work [5], the wave speed and part of the wave-generated mean flow are merged together as \( u_0 \), and then this is expanded in the same manner:
\[ u_0 = c_0 \left( 1 + Q c_1 + Q^2 c_2 + \cdots \right), \]  \tag{18}
where the \( c_j \)'s are constants. The constant, \( c_0 \), will be equal to the linear wave speed, and the remaining \( c_j \)'s will be chosen to obtain a uniformly valid solution.

The upstream density profile must be adjusted so that the average of the total density in the presence of waves results in the desired profile. This correction is included at this early stage by expanding \( \beta \):
\[ \beta = \beta_0 \left( 1 + Q \beta_1 + Q^2 \beta_2 + \cdots \right), \]  \tag{19}
where the \( \beta_j \)'s are also constants.

2.3. First order

The first-order equation consists of all coefficients of \( Q^1 \) in (14). With the choice
\[ \alpha = \frac{\beta_0}{2}, \]  \tag{20}
Eq. (14) becomes
\[ \nabla^2 \phi_1 + \left[ \frac{\beta_0}{c_0^2} - \frac{\beta_0^2}{4} \right] \phi_1 = 0. \]  \tag{21}

The linear solution for monochromatic internal waves with upward group velocity is chosen to be
\[ \phi_1 = A \sin(kx + mz). \]  \tag{22}
where the wave amplitude, $A$, is assumed to be $\bigcirc(1)$. The resulting dispersion relation is

$$c_0^2 = \frac{\beta_0}{k^2 + m^2 + \frac{\beta_0^2}{4}}. \quad (23)$$

Note that since the length scale is chosen to be the horizontal wavelength, then the value of $k$ can only have a value of $2\pi$. However, the symbol $k$ will be retained for clarity, and then set to $2\pi$ at a later time. The vertical wavenumber, $m$, may have any real value.

### 2.4. Second order

The second-order equation is

$$\nabla^2 \phi_2 + \beta_0 \phi_{2x} + \frac{\beta_0}{c_0^2} \phi_2 = \beta_0 \phi_{1x} + \frac{\beta_0^2}{4} \phi_1 - \frac{1}{2} \beta_0 \left[ \phi_{1x}^2 + \left( \phi_{1x} + \frac{\beta_0}{2} \phi_1 \right)^2 \right] - c_1 \left[ \beta_0 \phi_{1x} + \frac{\beta_0^2}{2} \phi_1 - 2 \frac{\beta_0}{c_0} \phi_1 \right] + \beta_1 \left[ \beta_0 \phi_{1x} + \frac{\beta_0^2}{2} \phi_1 - \frac{\beta_0}{c_0} \phi_1 \right]. \quad (24)$$

The value for $c_1$ and $\beta_1$ have been found to be zero, in agreement with my previous work and the work of Yih [2]. Some details of this correction are given later. The final expression for $\phi_2$ is

$$\phi_2 = -\frac{1}{2} \beta_0 A^2 \left[ \frac{1}{2} + 4 \frac{PD_2 - 16B_2^2}{N} \cos 2(kx + mz) - 4B_3 \frac{P + 16D_2}{N} \sin 2(kx + mz) \right] \quad (25)$$

where $G_0$, $B_1$, $B_3$, $M$, $P$, and $N$ are given in the Appendix, and are similar to my previous definitions [5].

### 2.5. Third order

The third-order equation is

$$\nabla^2 \phi_3 + 2 \beta_0 \phi_{3x} + \left[ \frac{\beta_0}{c_0^2} + 3 \frac{\beta_0^2}{4} \right] \phi_3 = 2 \beta_0 \phi_{2x} + 6 \frac{\beta_0^2}{4} \phi_2 - \frac{\beta_0^2}{2} \phi_1 - \beta_0 \left[ \phi_{1x} \phi_{2x} + \left( \phi_{1x} + \frac{\beta_0}{2} \phi_1 \right) \phi_{2x} + \left( \phi_{1x} + \frac{\beta_0}{2} \phi_1 \right) \left( \phi_{2x} + \frac{\beta_0}{2} \phi_2 - \frac{\beta_0}{2} \phi_1 \right) \right] - 2c_2 \left[ \beta_0 \phi_{1x} + \frac{\beta_0^2}{2} \phi_1 - \frac{\beta_0}{c_0} \phi_1 \right] + \beta_2 \left[ \beta_0 \phi_{1x} + \frac{\beta_0^2}{2} \phi_1 - \frac{\beta_0}{c_0} \phi_1 \right]. \quad (26)$$

This equation will be used to determine $c_2$; no attempt will be made to determine $\phi_3$ in its entirety.

Before finding $c_2$, it is necessary to determine $\beta_2$ so that the background density profile matches the density profile in the presence of waves. The definition of $\delta$ in (4) and the above solution to second order are used to obtain

$$z = z_0 + Q \phi_1 + Q^2 \phi_2. \quad (27)$$

This is a nonlinear algebraic equation for the shape of a streamline, $z = \eta(x, z_0)$, for a chosen value of $z_0$. A streamline in this scenario is also a line of constant density, and the relationship between $\rho$ and $z$ upstream could be used to eliminate $z_0$, resulting in $z = \eta(x, \rho)$. The inversion of this relationship is the density in the disturbed field, $\rho(x, z)$.

Eq. (27) is inverted using the method of successive approximations, producing

$$z = z_0 + QA \left[ \sin kx \cos mz_0 + \cos kx \sin mz_0 \right] + Q^2 A^2 \left[ \frac{\beta_0}{2} \left[ \cos^2 kx + \sin^2 kx \right] \right.$$  

$$+ \left[ \frac{1}{2} m \left( \cos^2 kx - \sin^2 kx \right) \right] \sin 2mz_0$$  

$$+ \left[ -\frac{1}{2} \frac{\beta_0}{2} \left( \cos^2 kx - \sin^2 kx \right) \right] \cos 2mz_0 \left. \right] + Q^2 \phi_2, \quad (28)$$

where $\phi_2$ is given by (25). This final form in (28) is achieved after only two iterations. Averaging (28) over one horizontal wavelength causes all of the second-order terms to be zero, resulting in

$$\overline{\eta} = z_0 + \bigcirc(Q^3). \quad (29)$$
The correction to $\beta$ is now found using
\[
\frac{1}{\rho_0} \frac{d\rho_0}{dz_0} = \frac{1}{\rho} \frac{d\rho}{d\eta} \frac{d\eta}{dz_0} = -\beta_0 \frac{d\eta}{dz_0},
\]
which may be further evaluated by taking a derivative of (29). As the second-order terms are zero in (30),
\[
\beta_2 = 0.
\]
This result is surprising, as $\beta_2$ plays an important role when top and bottom boundaries are present.

To determine $c_2$, the right-hand side of (26) is chosen to be orthogonal to $\phi_1$, which is achieved by multiplying by
\[
e^{i\omega t} \sin(\kappa x + mz),
\]
integrating over one period in both the horizontal and vertical directions, and setting to zero. The result is
\[
c_2 = -\frac{\beta_0}{4} A^2 \left( 4 \left( k^2 + m^2 + \frac{\beta_0^2}{8} \right) \left( \frac{PD_2 - 16B_0^2}{D_2N} \right) - m\beta_0 \left( \frac{PB_3 + 16B_3D_2}{D_2N} \right) - \frac{\beta_0^3}{4} \frac{1}{16D_2} \right).
\]

3. Results

The parameter, $\beta_0$, in geophysical situations has a small value, suggesting an approximation to (33) for small $\beta_0$, as was treated by Yih [2]. For small $\beta_0$, (33) reduces to
\[
c_2 = -A^2 \frac{\beta_0^2}{3}.
\]
The primary result here is that this value is always negative, meaning that the nonlinear waves move slower as the amplitude increases. Note also that $c_2$ for small $\beta_0$ is independent of the wavenumbers, $k$ and $m$.

Large values of $\beta_0$ are treated with numerical evaluation of (33). A wide variety of parameter values indicates that $c_2$ is still always negative, although the magnitude of $c_2$ depends strongly on $\beta_0$. The value of $c_2$ also depends on the vertical wavenumber, $m$, but only weakly (the horizontal wavenumber is $2\pi$ by the previous rescaling). This dependence of $c_2$ on $\beta_0$ and $m$ can be traced to the characteristics of the second harmonic, represented by $\phi_2$.

As $\beta_2$ is zero, the only significant contribution to the value of $c_2$ is from the nonlinear terms in (26). This is quite different than the case in a finite layer with rigid boundaries on top and bottom. For a finite layer, the contribution from the terms associated with $\beta_2$ dominates the value of $c_2$. For the unbounded layer considered here, $\beta_2$ is zero and as a result the nonlinear terms are most important, and yet still the value of $c_2$ is negative.

An insight into this negative value is gained by a closer examination of the relative magnitudes of the terms in (33). The dominant term in (33) for small $\beta_0$ is the first term, which can be traced to the expression for $\phi_2$ given in (25): the dominant term is the coefficient of $\cos(kz + mz)$. Furthermore, it is true for small $\beta_0$ that the coefficient of $\cos(kz + mz)$ is much larger than the coefficient of $\sin(kz + mz)$. This means that the second harmonic given by (25) is $90^\circ$ out-of-phase with the primary wave given by (22).

A horizontal slice of $\delta$ is shown in Fig. A.1. The case in Fig. A.1 has unrealistically large values of $A$, $\beta_0$, and $\epsilon$ to exaggerate the wave shape for display. Fig. A.1 shows that the wave crest is flattened and the wave trough is sharpened by the addition of the second-order contribution. In contrast, free-surface waves, which have a positive value for the wave speed correction, have a second harmonic that is in-phase with the primary wave. These first two harmonics for Stokes’ waves act to magnify the height of the wave crest, and it is well known that Stokes’ free-surface waves have sharpened crests and broad troughs. The wave shape in Fig. A.1 suggests that the sharpened trough is responsible for the negative value of $c_2$. The velocity in any wave trough is slower than the wave speed, opposite that for the wave crest. Hence if the trough is exaggerated the overall effect is a slower moving wave.

The horizontal mean flow may be found by forming an expression for the horizontal component of velocity using (7) then substituting for $u_0$, $\delta$, $\phi_1$ and $\phi_2$ from (16), (18), (22), (25), and finally averaging over one horizontal wavelength. The result is
\[
\bar{u} = c_0 \left( 1 - \frac{1}{12} \beta_0^2 A^2 Q^2 \right).
\]

This is similar to the second-order correction for the wave speed, but weaker due to the appearance of $1/12$ in (34). Hence the reduction in wave speed due to the nonlinear effects is not compensated by a corresponding change in the mean flow. Overall, the nonlinear waves move slower than the infinitesimal waves.
Appendix. Constants

\[ G_0 = k^2 + m^2 + \frac{\beta_0^2}{4}, \]  
\[ B_1 = \frac{1}{8} G_0, \]  
\[ B_2 = \frac{1}{8} \left[ -k^2 + m^2 - \frac{\beta_0^2}{4} \right], \]  
\[ B_3 = \frac{1}{8} m \beta_0, \]  
\[ D_1 = \frac{1}{8} \left[ k^2 - m^2 - \frac{\beta_0^2}{4} \right], \]  
\[ D_2 = \frac{1}{8} \left[ -k^2 - m^2 + \frac{\beta_0^2}{4} \right], \]  
\[ M = (G_0 - 4m^2)^2 + 4m^2 \beta_0^2, \]  
\[ P = 4k^2 + 4m^2 - G_0, \]  
\[ N = P^2 + 4m^2 \beta_0^2. \]

References