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Vertex-neighbor-integrity of magnifiers, expanders, and hypercubes

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Abstract

A set of vertices S is subverted from a graph G by removing the closed neighborhood $N[S]$ from G. We denote the survival subgraph of the vertex subversion strategy S by G/S. The *vertex-neighbor-integrity* of G is defined to be $\mathrm{VNI}(G) = \min_{S \subseteq V(G)}\{|S| + \omega(G/S)\}$, where $\omega(H)$ is the order of the largest connected component in the graph H. The graph parameter VNI was introduced by Cozzens and Wu [3] to measure the vulnerability of a spy network. Cozzens and Wu showed that the VNI of paths, cycles, trees and powers of paths on n vertices are all on the order of $\sqrt{n}$. Here we prove that the VNI of any member of a family of magnifier graphs is linear in the order of the graph. We also find upper and lower bounds on the VNI of hypercubes. Finally, we show that the decision problem corresponding to computing the vertex-neighbor-integrity of a graph is NP-complete. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

In 1996, Cozzens and Wu [2,3] introduced a new graph parameter called the ‘vertex-neighbor-integrity’. They motivate the use of this parameter by viewing a graph as a model of a spy network where the vertices represent agents and the edges represent lines of communication. If a spy is discovered, the espionage agency can no longer trust any of the spies with whom she was in direct communication. With this model in mind, we determine the robustness of the network by examining the effect of removing a vertex (or set of vertices) and all of its neighbors from the graph. Our definitions follow Cozzens and Wu [2,3].

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Let $G=(V,E)$ be a graph and u be a vertex in G. The *open neighborhood* of u is $N(u)=\{v\in V(G)|\{u,v\}\in E(G)\}$; the *closed neighborhood* of u is $N[u]=\{u\}\cup N(u)$. We define analogously for any $S\subseteq V(G)$ the open neighborhood $N(S)=\bigcup_{u\in S}N(u)$ and the closed neighborhood $N[S]=\bigcup_{u\in S}N[u]$. A vertex $u\in V(G)$ is *subverted* by removing the closed neighborhood $N[u]$ from G. For a set of vertices $S\subseteq V(G)$, the *vertex subversion strategy* S is applied by subverting each of the vertices of S from G. Define the *survival subgraph* G/S to be the subgraph left after the subversion strategy S is applied to G, i.e., $G/S=G\setminus N[S]$. We define the *order* of G to be $|V(G)|$.

Definition (*Cozzens and Wu* [2,3]). The *vertex-neighbor-integrity* (VNI) of a graph G is defined as

$$\mathrm{VNI}(G)=\min_{S\subseteq V(G)}\{|S|+\omega(G/S)\},$$

where $\omega(H)$ is the order of the largest connected component in the graph H.

For a subversion strategy $S\subseteq V(G)$ we define the *resistance* of G to S by

$$R_G(S)=|S|+\omega(G/S).$$

Thus we have $\mathrm{VNI}(G)=\min_{S\subseteq V(G)}\{R_G(S)\}$.

Note that any simple graph of order n that has a vertex of degree $n-1$ has vertex-neighbor-integrity of 1. In particular, complete graphs, complete multipartite graphs of the form $K_{1,n_2,\dots,n_k}$, wheels, fans, and windmills have vertex-neighbor-integrity 1. Graphs such as double fans $P_n\vee\bar{K}_2$, double cones $C_n\vee\bar{K}_2$, and $K_{n_1,n_2,\dots,n_k}$, where $n_i>1$ for all i have vertex-neighbor-integrity 2.

Cozzens and Wu proved the following results about the vertex-neighbor-integrity of trees, cycles, and powers of cycles.

Theorem 1 (Cozzens and Wu [3]). *Let P_n be the path on n vertices. Then we have*

$$\mathrm{VNI}(P_n)=\begin{cases}\lceil 2\sqrt{n+3}\rceil-4 & \text{if } n\geqslant 2,\\ 1 & \text{if } n=1.\end{cases}$$

Theorem 2 (Cozzens and Wu [3]). *The path P_n has the maximum vertex-neighbor-integrity among all trees of order $n\geqslant 1$.*

Theorem 3 (Cozzens and Wu [2]). *Let C_n be the n-cycle, where $n\geqslant 3$. Then*

$$\mathrm{VNI}(C_n)=\begin{cases}\lceil 2\sqrt{n}\rceil-3 & \text{if } n>4,\\ 2 & \text{if } n=4,\\ 1 & \text{if } n=3.\end{cases}$$

The kth power of a simple graph G is the graph G^k with vertex set $V(G)$ and edge set $\{\{u,v\}$: $d_G(u,v)\leqslant k\}$.

Theorem 4 (Cozzens and Wu [2]). *Let C_n^k be the kth power of the n-cycle, where $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor n/2 \rfloor$. Then*

$$\mathrm{VNI}(C_n^k) = \begin{cases} \lceil 2\sqrt{n} \rceil - (2k+1) & \text{if } 1 \leqslant k < \frac{\sqrt{n}-1}{2}, \\ \left\lceil \frac{n}{2k+1} \right\rceil & \text{if } \frac{\sqrt{n}-1}{2} \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor. \end{cases}$$

In particular, C_n has the highest VNI *among all powers of C_n.*

We observe that the maximum VNI for powers of C_n and for trees on n vertices is on the order of $\sqrt{n}$. In this paper we present several examples of infinite families of graphs whose VNI grows faster than the square root of the number of vertices. In particular, we show that if G belongs to a family of magnifier graphs, then VNI(G) is on the order of $\alpha|V(G)|$, where α depends on the parameters of the family. We also find upper and lower bounds on the VNI of hypercubes by relating them to magnifier graphs. We conclude by showing that the decision problem related to finding the VNI of a graph is NP-complete. We begin with some elementary results about vertex-neighbor-integrity.

2. Elementary bounds on vertex-neighbor-integrity

Since the vertex-neighbor-integrity of a graph G is the minimum value of $R_G(S)$ over all subversion strategies S, we see that each $R_G(S)$ is an upper bound on VNI(G). If S is any dominating set in $V(G)$, (i.e., $N[S]=V(G)$), then $R_G(S)=|S|$, since $\omega(G/S)=0$. We obtain the following upper bound on the vertex-neighbor-integrity of any connected graph.

Fact 1. *For any connected graph G of order n,* $\mathrm{VNI}(G) \leqslant n/2$.

Fact 1 follows from the observation that any connected graph has a dominating set of order at most half the order of the graph.

We can also determine bounds for Cartesian products of graphs.

Fact 2. *For any graphs G* and *H,* $\mathrm{VNI}(G \times H) \geqslant \max\{\mathrm{VNI}(G), \mathrm{VNI}(H)\}$.

Fact 3. *For any graph G,* $\mathrm{VNI}(G \times P_n) \leqslant n\mathrm{VNI}(G)$.

Fact 2 follows from the observation that if $S \subseteq V(G \times H)$ is such that $\mathrm{VNI}(G \times H) = R_{G\times H}(S)$, then we can project S onto either G or H to get subversion strategies S_G or S_H for G or H, respectively, with $R_G(S_G), R_H(S_H) \leqslant R_{G\times H}(S) = \mathrm{VNI}(G \times H)$. Fact 3 follows from the observation that any strategy $S \subseteq V(G)$ can be extended in the natural way to a strategy $S' \subseteq V(G \times P_n)$ such that $R_{G\times P_n}(S') = nR_G(S)$.

3. Vertex-neighbor-integrity of magnifier graphs and expander graphs

Adding edges to a graph could increase or decrease its VNI. For a graph G and any subversion strategy S, additional edges can make $N[S]$ larger, which would tend to make $R_G(S)$ smaller. However, additional edges could also connect together two different connected components of G/S, which would instead increase $R_G(S)$. Thus, to obtain a graph with high VNI, we would look for highly connected graphs with low vertex degree.

One such type of graphs is magnifier graphs.

Definition (*see West* [9]). An (n,k,ε)-*magnifier* is an n-vertex graph G, such that the maximum vertex degree $\Delta(G)\leqslant k$ and that $|N(A)\cap(V(G)\setminus A)|\geqslant\varepsilon|A|$ for every $A\subset V(G)$ with $|A|\leqslant n/2$.

For ease of notation, we will let ∂A denote the *boundary* of A defined by $\partial A = N(A)\cap(V(G)\setminus A) = N[A]\setminus A$ for any $A\subseteq V(G)$, so we have $N[A]\setminus\partial A = A$.

Note that any connected graph G is a magnifier with $k=\Delta(G)$ and $\varepsilon\geqslant 2/n$. We refer to the constant ε as the *vertex magnification* of the graph. Magnifier graphs have the property that for any sufficiently small subset A of the vertex set, we can guarantee that A has a relatively large number of neighbors. By using this property, we can determine a lower bound on $\omega(G/S)$ for certain subversion strategies S. We show that for any (n,k,ε)-magnifier G, $\mathrm{VNI}(G)\geqslant n\varepsilon/2k$. First, we need several lemmas.

In general, for magnifier graphs, $\varepsilon\leqslant 1$, so we assume that $\varepsilon < k/(1+k)$.

Lemma 1. *Let G be an (n,k,ε)-magnifier. For any subsets $A,S\subseteq V(G)$ such that $k|S|/\varepsilon < |A| < n/2$, we have $|\partial S| < |\partial A|$.*

Proof. First of all, we have $k|S| < |A|\varepsilon$. Since G is a magnifier, we know by definition that $|\partial A| > \varepsilon|A|$ since $|A| < n/2$. Since $\Delta(G)\leqslant k$, we have $|\partial S|\leqslant k|S|$. Therefore, $|\partial S| < |\partial A|$. □

This lemma guarantees that for certain subsets $A\subset V(G/S)$, A has more neighbors than the subversion strategy S. We now show that for specific strategies S we can find such a subset A.

Lemma 2. *Let G be an (n,k,ε)-magnifier. If S is a subversion strategy of G with $|S| < n\varepsilon/2k$, then there exists a set $A\subseteq V(G/S)$ such that $|\partial A| > |\partial S|$ and $|A| < n/2$.*

Proof. By Lemma 1, to prove that such a subset A exists, we need only show that at least $k|S|/\varepsilon$ vertices remain after we subvert the strategy S, that is, that $|G| - |N[S]| > k|S|/\varepsilon$. For magnifier graphs, since $\varepsilon < k/(1+k)$, we get

$$|S| < \frac{n\varepsilon}{2k} < \frac{n}{1+k+k/\varepsilon}.$$

This gives us

$$n > |S|\left(1 + k + \frac{k}{\varepsilon}\right) = |S|(1 + k) + \frac{|S|k}{\varepsilon}.$$

Since $N[S] = \partial S \cup S$, we have $|N[S]| \leqslant k|S| + |S| = |S|(1 + k)$, so we obtain

$$n > \frac{k|S|}{\varepsilon} + |N[S]|.$$

This gives the desired result. □

We achieve the desired lower bound on the VNI of a magnifier graph by exploiting the vertex magnification property of the graph. Namely, we show that for any vertex subset of the survival subgraph of a predetermined size, we can 'grow' the set by adding its neighbors which were not subverted to find a new larger subset of the vertices in G/S.

Lemma 3 (Growing lemma). *Let G be an (n,k,ε)-magnifier graph, and let S be a subversion strategy of G such that $|S| < n\varepsilon/2k$. If a subset $A \subseteq V(G/S)$ is such that $k|S|/\varepsilon < |A| < n/2$, then there exists a subset $B \subseteq V(G/S)$ such that $A \subset B$, $|B| \geqslant n/2$, and the subgraph of G induced by B, $G[B]$, has at most as many connected components as $G[A]$.*

Proof. Let A be such a subset of $V(G/S)$. We define a chain of subsets: let $C_1 = A$, and let $C_{i+1} = N[C_i] \setminus \partial S$. We see that

$$A = C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$$

and

$$\begin{aligned} C_{i+1} \setminus C_i &= N[C_i] \setminus \partial S \setminus C_i \\ &= \partial C_i \setminus \partial S. \end{aligned}$$

By Lemma 1, when $|C_i| < n/2$, we have $|\partial C_i| > |\partial S|$ since $A \subseteq C_i$ implies that $|C_i| > k|S|/\varepsilon$. This gives us $C_i \neq C_{i+1}$ when $|C_i| < n/2$. Let B be C_i for the smallest i such that $|C_i| \geqslant n/2$. B exists since $C_i \neq C_{i+1}$ for $|C_i| < n/2$. By construction, $A \subset B$ and $|B| \geqslant n/2$. Since we 'grow' B from A by appending at each step only neighbors of elements from the previous step, $G[B]$ has at most as many connected components as $G[A]$. □

We refer to a subset $B \subseteq V(G/S)$ whose existence is guaranteed by the above lemma as a subset which we can *grow* from A.

Theorem 5. *Let G be an (n,k,ε)-magnifier, with $\varepsilon < k/(k+1)$. Then*

$$\mathrm{VNI}(G) \geqslant \frac{n\varepsilon}{2k}.$$

Proof. If $S \subseteq V(G)$ is a subversion strategy such that $|S| \geqslant n\varepsilon/2k$, then $R_G(S) = |S| + \omega(G/S) \geqslant n\varepsilon/2k$ as desired.

Assume $|S| < n\varepsilon/2k$. By the proof of Lemma 2, there exists a subset $A \subseteq V(G/S)$ such that $k|S|/\varepsilon < |A| < n/2$. This subset satisfies the hypothesis of the Growing Lemma. Of all such subsets $A \subset V(G/S)$ satisfying the hypothesis of the Growing Lemma, consider one for which $G[A]$ has the fewest connected components. Use the Growing Lemma to grow A to $B \subseteq V(G/S)$ with $|B| \geqslant n/2$ and the number of connected components of $G[B]$ at most the number of connected components of $G[A]$. Since $G[A]$ had the minimum number of connected components for any set of vertices A such that $k|S|/\varepsilon < |A| < n/2$, removing connected components from $G[B]$ must not result in a subgraph $G[C]$ for $C \subseteq V(G/S)$ with $k|S|/\varepsilon < |C| < n/2$. Thus, some connected component of $G[B]$ must have at least $n/2 - k|S|/\varepsilon$ vertices. In particular, this tells us that the order of the largest connected component in G/S is $\omega(G/S) \geqslant n/2 - k|S|/\varepsilon$. So, for $|S| < n\varepsilon/2k$, we have the following:

$$\begin{aligned} R_G(S) &\geqslant |S| + \frac{n}{2} - \frac{k|S|}{\varepsilon} \\ &= -|S|\left(\frac{k}{\varepsilon} - 1\right) + \frac{n}{2} \\ &> \left(-\frac{n\varepsilon}{2k}\right)\left(\frac{k}{\varepsilon} - 1\right) + \frac{n}{2} \\ &= \frac{n\varepsilon}{2k}. \end{aligned}$$

We know that $R_G(S) \geqslant n\varepsilon/2k$ for all subversion strategies S, and thus $\mathrm{VNI}(G) \geqslant n\varepsilon/2k$ for any (n, k, ε)-magnifier G. □

Infinite families of magnifier graphs exist with fixed constants k and ε. This follows from the existence of infinite families of a special kind of magnifier graph, expander graphs. An (n, k, c)-*expander* is an n-vertex graph G such that $\Delta(G) \leqslant k$ and $|\partial A| \geqslant c(1 - |A|/n)|A|$ for any subset $A \subseteq V(G)$ (see [6]). Expander graphs (where G is k-regular) have many applications in computer science, especially communication networks. Every n-vertex graph G is an (n, k, c)-expander for $k = \Delta(G)$ and some value of c (though c may be difficult to calculate). In particular, if G is an (n, k, c)-expander, then G is an $(n, k, c/2)$-magnifier. The concept becomes useful when we consider an infinite family of graphs with fixed k and c while n tends to infinity. Using a counting argument, one can prove the existence of such infinite families of expanders (and hence, of magnifiers) [6], but the explicit construction of expander families is very difficult. In 1973, Margulis constructed an explicit example of such a family [7].

As we saw in Fact 1, for any connected graph of order n, $\mathrm{VNI}(G) \leqslant n/2$. This gives us upper and lower bounds on the VNI of any expander graph which are linear in the order of the graph.

Corollary 1. *For any family of magnifiers with constants k and ε, the vertex-neighbor-integrities of the graphs in the family are linear in the order of the graph.*

4. Vertex-neighbor-integrity of hypercubes

While hypercubes are not a family of magnifier graphs, they do have good connectivity and symmetry. It seems reasonable to hope that they might have high vertex-neighbor-integrity. Let Q_n denote the n-dimensional hypercube. From [1], we get the following connection between the eigenvalues of the adjacency matrix of a regular graph and its magnifier properties.

Theorem 6 (Alon and Milman [1]). *If G is a k-regular n-vertex graph with second-larg-est eigenvalue* λ, *then G is an* (n,k,ε)*-magnifier, where* $\varepsilon \geqslant (2k-2\lambda)/(3k-2\lambda)$.

It is known that the eigenvalues of the hypercube Q_n are $n-2j$ with multiplicity $\binom{n}{j}$ (see [8]). Thus the second largest eigenvalue of Q_n is $n-2$, and Q_n is an $(2^n, n, \varepsilon)$-magnifier, with $\varepsilon \geqslant 4/(n+4)$. Combining Theorems 5 and 6 we get the following corollary.

Corollary 2. *Let* Q_n *be the n-dimensional hypercube, then*

$$\mathrm{VNI}(Q_n) \geqslant \frac{2^{n+1}}{n^2+4n}.$$

We know that the order of any dominating set for a graph, in particular, the order of the smallest dominating set, the *domination number*, is an upper bound for its vertex-neighbor-integrity. The domination number of hypercubes is an unsolved problem, and finding the domination number of a graph has been shown to be an NP-complete problem. From Jha and Slutzki [5] we get the following bounds on the domination number of Q_n.

Theorem 7 (Jha and Slutzki [5]). *Let* $\mathrm{dom}(Q_n)$ *denote the domination number of* Q_n. *Then*

$$\frac{2^n}{n+1} \leqslant \mathrm{dom}(Q_n) \leqslant \frac{2^n}{2^{\lfloor \log_2(n+1) \rfloor}}.$$

Note that the upper bound is the least power of 2 that is at least $2^n/(n+1)$, thus for n of the form 2^k-1, the bounds coincide.

Corollary 3.

$$\mathrm{VNI}(Q_n) \leqslant \frac{2^n}{2^{\lfloor \log_2(n+1) \rfloor}}.$$

5. Complexity results

Consider the decision problem

Vertex-Neighbor-Integrity

Instance: Graph $G=(V,E)$, positive integer $K \leqslant |V(G)|$.

Question: Is $\mathrm{VNI}(G) \leqslant K$, i.e., does there exist a subset $S \subseteq V(G)$ such that $R_G(S) = |S| + \omega(G/S) \leqslant K$?

We know that finding the size of the smallest dominating set in any graph is an upper bound for the vertex-neighbor-integrity of the graph. In this section we will show that Vertex-Neighbor-Integrity is NP-complete by reducing the following well-known NP-complete problem to a special case of Vertex-Neighbor-Integrity.

Dominating Set (*see Garey and Johnson* [4]).

Instance: Graph $G=(V,E)$, positive integer $K \leqslant |V(G)|$.

Question: Is there a dominating set of size K or less for G, i.e., does there exist a subset $V' \subseteq V(G)$ with $|V'| \leqslant K$ such that for all $u \in V(G) \setminus V'$ there is a $v \in V'$ for which $\{u,v\} \in E(G)$?

For our NP-completeness proof we use a construction involving the strong direct product. The *strong direct product* of the graphs G and H is defined to be the graph $G \cdot H$ with vertices (g,h) for all $g \in V(G)$ and $h \in V(H)$ and with an edge between (g_1,h_1) and (g_2,h_2) if $g_1 = g_2$ or $\{g_1,g_2\} \in E(G)$, and $h_1 = h_2$ or $\{h_1,h_2\} \in E(H)$.

Theorem 8. *Vertex-Neighbor-Integrity is NP-complete.*

Proof. It is easy to see that Vertex-Neighbor-Integrity is in NP, since a nondeterministic algorithm need only guess a subset $S \subseteq V(G)$ and check in polynomial time that $R_G(S) \leqslant K$.

For any n-vertex graph G we construct an n^2-vertex graph $G^* := G \cdot K_n$, where K_n denotes the complete graph on n vertices. This construction can obviously be completed in polynomial time. We will show that answering Dominating Set for the graph G is equivalent to answering Vertex-Neighbor-Integrity with $K < n$ for G^*.

Consider the structure of G^* for any graph G. For each vertex v in G, the graph G^* has $n = |V(G)|$ corresponding copies of v. Two vertices are connected in G^* if and only if they are either copies of the same vertex v in G or if they are copies of vertices which are connected in G. Subverting any vertex v^* in G^* which is a copy of v in G removes all the vertices in G^* which are also copies of v in G or which are copies of neighbors of v in G. Thus, we see that having two vertices in a subversion strategy $S \subseteq V(G^*)$ which are copies of the same vertex in G is redundant because subverting one of them removes exactly the same set of vertices as subverting both does. Therefore, to answer Vertex-Neighbor-Integrity for G^* we need only look at strategies with no more than one copy of each vertex from G. Note that each strategy

$S' \subseteq V(G)$ induces such a strategy $S \subseteq V(G^*)$, and moreover, $R_{G^*}(S) = |S'| + n \cdot \omega(G/S')$. If we consider Vertex-Neighbor-Integrity for G^* and $K < n$ then we are searching for an S' such that $\omega(G/S') = 0$ and $|S'| \leqslant K$. This is equivalent to answering Dominating Set for the graph G.

Thus Dominating Set is a special case of Vertex-Neighbor-Integrity because every instance of Dominating Set can be transformed in polynomial time into an instance of Vertex-Neighbor-Integrity. The NP-completeness of Vertex-Neighbor-Integrity follows by this restriction to Dominating Set. □

6. Discussion and open questions

Cozzens and Wu reasoned that 'for a connected representing graph [of a spy network] the more edges it has, the more jeopardy the spy network is in' [2]. Hence they presented a criterion as follows:

Criterion $(*)$. A connected graph G is said to satisfy criterion $(*)$ if for any connected spanning subgraph H of G, then $\mathrm{VNI}(H) \geqslant \mathrm{VNI}(G)$. [2]

Their criterion ignores the fact that adding edges to a graph may make the network more robust by making it harder to disconnect into small connected components. The espionage agency would reasonably want to add lines of communication to their existing spy network to maximize its robustness. We present an alternate criterion for a graph to model an optimal spy network.

Criterion $(**)$.

A connected graph G is said to satisfy criterion $(**)$ if for any supergraph H such that $V(H) = V(G)$ and $E(H) \supseteq E(G)$, then $\mathrm{VNI}(H) \leqslant \mathrm{VNI}(G)$

Not all graphs satisfy this criterion.

Example 1. The path P_{13} does not satisfy the criterion $(**)$ since by adding the edge to obtain the cycle C_{13} we can increase the VNI because by Theorems 1 and 2 $\mathrm{VNI}(C_{13}) = 5$ while $\mathrm{VNI}(P_{13}) = 4$.

Example 2. The cycle C_{2^n} for n sufficiently large does not satisfy the criterion $(**)$ because C_{2^n} is contained in the hypercube Q_n since Q_n always has a Hamiltonian cycle, and $\mathrm{VNI}(Q_n) \geqslant (2^{n+1})/(n^2 + 4n)$ grows faster than $\mathrm{VNI}(C_{2^n}) = \lceil 2^{(n+2)/2} \rceil - 3$.

This leaves open the question:

Which graphs satisfy the criterion $(**)$ for the model of an optimal spy network?

We can also show, by recursively looking at survival subgraphs after subverting one vertex, that for any connected n-vertex graph G with $n \leqslant 15$, the vertex-neighbor-integrity $\mathrm{VNI}(G) \leqslant \lceil n/3 \rceil$.

Conjecture. For any connected n-vertex graph G, $\mathrm{VNI}(G) \leqslant \lceil n/3 \rceil$.

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