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Identities for Classical Group Characters of Nearly Rectangular Shape

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We derive several identities that feature irreducible characters of the general linear, the symplectic, the orthogonal, and the special orthogonal groups. All of the identities feature characters that are indexed by shapes that are "nearly" rectangular, by which we mean that the shapes are rectangles except for one row or column that might be shorter than the others. As applications we prove new results in plane partitions and tableaux enumeration, including new refinements of the Bender–Knuth and MacMahon (ex-)conjectures. © 1998 Academic Press

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1. INTRODUCTION

We prove identities that set into relation irreducible characters of classical Lie groups. All of them feature a classical group character indexed by a "nearly" rectangular shape. What is remarkable about these identities is that they have simple explicit forms (as opposed to many, however more general, identities in the literature), and that many of them exhibit multiplicity-free expansions, typically featuring sums of characters indexed by shapes that have a fixed number of rows or columns of odd length.

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There are three types of identities that we consider.

First, in Theorem 1, we express (irreducible) characters of the general linear group $GL(N)$ (these characters are also known as *Schur functions*) of "nearly" rectangular shape in terms of (irreducible) characters of the symplectic group $Sp(N)$, and in terms of (irreducible) characters of the orthogonal group $O(N)$. As may be expected, for the proofs of these identities we utilize Littlewood's [25] branching rules for restricting representations of the general linear groups to representations of the symplectic or orthogonal groups.

Second, in Theorem 2, we express (irreducible) characters of the symplectic group $Sp(N)$ (where $N = 2n$) and (irreducible) characters of the special orthogonal group $SO(N)$ (and its spin covering group) of "nearly" rectangular shape in terms of (irreducible) characters of the general linear group $GL(N)$. For the proofs of these identities we use the tableaux descriptions for symplectic characters developed by DeConcini and Procesi [3, 4], and for special orthogonal characters as developed by Lakshmibai, Musili, and Seshadri [22, 23, 21].

Finally, in Theorem 3, we express the product of two (irreducible) characters of the symplectic group $Sp(2n)$, respectively, of the special orthogonal group $SO(N)$ (and its spin covering group), of "nearly" rectangular shape in terms of (irreducible) characters of the same type. For the proofs of these identities, we rely on Littelmann's extension [24] of the Littlewood–Richardson rule for all classical group characters, which also uses the tableaux by DeConcini, Procesi, Lakshmibai, Musili, and Seshadri. In fact, as a first "approximation," we convert Littelmann's rule in the special case of the product of an (irreducible) symplectic or special orthogonal character of *rectangular shape* by an *arbitrary* (irreducible) character of the same type to a simpler and more explicit form, the coefficients in the expansion being expressed in terms of modified Littlewood–Richardson coefficients. The resulting formulas are given in Proposition 1 in Section 6.

We want to emphasize that it is particularly the tableaux descriptions by DeConcini, Procesi, Lakshmibai, Musili, and Seshadri that turn out to be very useful and effective here. The more classical rules (cf. [12, 13, 16]) involving Littlewood–Richardson coefficients, which would also apply to the second and third problem types that we consider, are nice in theory but useless in practice, since it seems to be impossible to keep track of the cancellations that are caused by the application of modification rules.

As will become apparent, the reason that classical group characters indexed by "nearly" rectangular shapes allow particularly nice identities is because Littlewood–Richardson fillings and the above-mentioned tableaux behave in a special way for "nearly" rectangular shapes. This fact was previously observed by Proctor and Stanley [28, 29, 31] for rectangular shapes. Aside from these three papers, the inspiration for the current paper comes from [20]. There it was discovered that symplectic characters of "nearly" rectangular shape have a nice explicit expansion in terms of general linear characters. This gave us the idea of exploring the other identities there may be for classical group characters of "nearly" rectangular shapes.

Finally, we remark that Okada's paper [27] may be considered as a precursor to this paper, as it contains the specializations to rectangular shapes of all of our identities. Okada, however, uses a completely different approach (namely, he uses the minor summation formula of Ishikawa and Wakayama [9]). It must be pointed out, on the other hand, that there is one type of identity that Okada addresses, but I am not able to address. This is restrictions of representations of $G(N)$ (G can be GL, Sp, or SO) to a representation of $G(l) \times G(N-l)$. Although computational evidence exists that the respective identities of Okada can be extended from rectangular shapes to "nearly" rectangular shapes as well, so far I have not been able to prove anything because of the lack of an efficient combinatorial rule. (Again, the rules in [12, 16] are apparently useless.)

Our paper is organized as follows. In Section 2 we recall the definitions of irreducible general linear, symplectic, orthogonal, and special orthogonal characters. Then, in Section 3, we state our results. These are subsequently proved in Sections 4, 5, and 6. Section 7 contains applications of our identities to plane partition theory, including new refinements of the Bender–Knuth and MacMahon (ex-)conjectures. Finally, in the Appendix we provide the necessary background information; in particular, we describe all of the different types of tableaux that we use, Littlewood–Richardson fillings, the Littlewood–Richardson rule, Littelmann's extension of it to other classical groups, and Littlewood's branching rules for restricting representations of the general linear groups to representations of the symplectic or orthogonal groups.

2. GENERAL LINEAR CHARACTERS (SCHUR FUNCTIONS), AND SYMPLECTIC, ORTHOGONAL, AND SPECIAL ORTHOGONAL CHARACTERS

Here we recall the classical character formulas. We refer the reader to [5; 6, Chap. 24; 13; 15; 33, Appendix; 39] for surveys and more detailed background information concerning classical group characters.

In what follows, $\mathbf{x} = (x_1, x_2, \dots)$ will always be an infinite sequence of indeterminates. We call a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ a *partition* if all of the λ_i's are *integers*, and a *half-partition* if all of the λ_i's are *half-integers*, by which we mean numbers of the form

$k+1/2$, where k is an integer. For (ordinary) partitions we adopt the convention that partitions that differ only by trailing zeros are considered to be the same. The components $\lambda_1, \lambda_2, \dots$ of λ are also called the *parts* of λ. We call a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{r-1} \ge |\lambda_r|$ an *r-orthogonal partition* if all of the λ_i's are *integers*, and an *r-orthogonal half-partition* if all of the λ_i's are *half-integers*.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, we define the (formal) *general linear character*, also called a *Schur function*, $s(\lambda; \mathbf{x})$ by (see [26, I, (3.4)])

$$s(\lambda; \mathbf{x}) = \det_{1 \le i, j \le r} \big(h_{\lambda_i - i + j}(\mathbf{x})\big), \tag{2.1}$$

where $h_m(\mathbf{x}) = \sum_{1 \le i_1 \le \cdots \le i_m} x_{i_1} \cdots x_{i_m}$ denotes the *complete homogeneous symmetric function* of degree m. If $r \le n$,

$$s(\lambda; x_1, x_2, \dots, x_n, 0, 0, \dots)$$

is the irreducible character for $GL(n, \mathbb{C})$ indexed by λ (see, e,g., [6, (24.10)]). Following King [13], we write

$$s_n(\lambda; \mathbf{x}) := s(\lambda; x_1, x_2, \dots, x_n, 0, 0, \dots). \tag{2.2}$$

Two further similar notations for Schur functions that we use are

$$s_{2n}(\lambda; \mathbf{x}^{\pm 1}) := s\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots\big) \tag{2.3}$$

and

$$s_{2n+1}(\lambda; \mathbf{x}^{\pm 1}) := s\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots\big). \tag{2.4}$$

Again, let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition. Following Koike and Terada [15, Def. 2.1.1], we define the (formal) *symplectic character* $sp(\lambda; \mathbf{x})$ by

$$sp(\lambda; \mathbf{x}) = \det_{1 \le i, j \le r} \big(h_{\lambda_i - i + 1}(\mathbf{x}) \;\vdots\; h_{\lambda_i - i + j}(\mathbf{x}) + h_{\lambda_i - i - j + 2}(\mathbf{x})\big). \tag{2.5}$$

Here, the notation of the determinant means that the first expression gives the entries of the first column and the second the entries for the remaining columns, $j \ge 2$. If $r \le n$,

$$sp\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots\big)$$

is the irreducible character for $Sp(2n, \mathbb{C})$ indexed by λ (see [6, Prop. 24.22]). We write

$$sp_{2n}(\lambda; \mathbf{x}^{\pm 1}) := sp\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots\big). \tag{2.6}$$

If $r \le n+1$,

$$sp\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 1, 0, \dots\big)$$

is a character for Proctor's odd symplectic group $\mathrm{SSp}(2n+1,\mathbb{C})$ (see [30, Prop.3.1, with $x_N = 1$]). We write

$$sp_{2n+1}(\lambda; \mathbf{x}^{\pm 1}) := sp\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots\big). \quad (2.7)$$

Following Koike and Terada [15, Def. 2.1.1], we define the (formal) *orthogonal character* $o(\lambda; \mathbf{x})$ by

$$o(\lambda; \mathbf{x}) = \det_{1\le i,j\le r}\big(h_{\lambda_i - i + j}(\mathbf{x}) - h_{\lambda_i - i - j}(\mathbf{x})\big). \quad (2.8)$$

If $r \le n$,

$$o\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots\big)$$

is the irreducible character for $\mathrm{O}(2n,\mathbb{C})$ indexed by λ (see [6, Ex. 24.46]), for which we write

$$o_{2n}(\lambda; \mathbf{x}^{\pm 1}) := o\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots\big). \quad (2.9)$$

Similarly, if $r \le n$,

$$o\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots\big)$$

is the irreducible character for $\mathrm{O}(2n+1,\mathbb{C})$ indexed by λ (see [6, Ex. 24.46]), for which we write

$$o_{2n+1}(\lambda; \mathbf{x}^{\pm 1}) := o\big(\lambda; x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots\big). \quad (2.10)$$

The character $o_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$ is also the irreducible character for $\mathrm{SO}(2n+1,\mathbb{C})$ indexed by λ. Therefore we shall sometimes write $so_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$ for $o_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$. However, for the spin covering group of $\mathrm{SO}(2n+1,\mathbb{C})$ there are also irreducible characters indexed by half-partitions. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a half-partition; then the irreducible character $so_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$ is given by the Weyl formula (see [6, (24.28)]),

$$so_{2n+1}(\lambda; \mathbf{x}^{\pm 1}) = \frac{\det_{1\le i,j\le n}\Big(x_j^{\lambda_i + n - i + 1/2} - x_j^{-(\lambda_i + n - i + 1/2)}\Big)}{\det_{1\le i,j\le n}\Big(x_j^{n - i + 1/2} - x_j^{-(n - i + 1/2)}\Big)}. \quad (2.11)$$

In fact (see again [6, (24.28)]), if λ is an ordinary partition, then $so_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$ is given by the same formula.

The situation is even more delicate for $\mathrm{SO}(2n,\mathbb{C})$. The irreducible characters for $\mathrm{SO}(2n,\mathbb{C})$ and its spin covering group are indexed by n-orthogonal partitions or half-partitions. Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be an n-orthogonal partition or half-partition, i.e., $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_{n-1}\ge|\lambda_n|$, where the λ_i's are integers, respectively half-integers. Then the irreducible special orthogonal character $so_{2n}(\lambda;\mathbf{x}^{\pm1})$ is given by the Weyl formula (see [6, (24.40)]),

$$so_{2n}(\lambda;\mathbf{x}^{\pm1})\\ =\frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i}+x_j^{-(\lambda_i+n-i)}\right)+\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i}-x_j^{-(\lambda_i+n-i)}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i}+x_j^{-(n-i)}\right)}. \tag{2.12}$$

The irreducible character for $\mathrm{O}(2n,\mathbb{C})$ indexed by the partition λ, $o_{2n}(\lambda,\mathbf{x}^{\pm1})$ equals the irreducible character for $\mathrm{SO}(2n,\mathbb{C})$ indexed by λ, $so_{2n}(\lambda;\mathbf{x}^{\pm1})$, if $\lambda_n=0$, but splits into two irreducible characters for $\mathrm{SO}(2n,\mathbb{C})$, if $\lambda_n\ne0$, one indexed by $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$, the other indexed by λ^-, which by definition is $(\lambda_1,\lambda_2,\dots,\lambda_{n-1},-\lambda_n)$ (see [6, first paragraph on p. 411]).

The characters $s(\lambda;\mathbf{x}),sp(\lambda;\mathbf{x}),o(\lambda;\mathbf{x})$ are also called *universal characters*, meaning that by specializing one obtains the actual (general linear, symplectic, orthogonal, respectively) characters for any dimension of the corresponding group. There is no such thing in the even special orthogonal case and for odd special orthogonal characters indexed by half-partitions. Of course, it has to be mentioned that there is not only a Weyl formula for $so_N(\lambda;\mathbf{x}^{\pm1})$, but also for $s_n(\lambda;\mathbf{x})$ (see [6, p. 403, (A.4); 26, I, (3.1)]) and $sp_{2n}(\lambda;\mathbf{x})$ (see [6, (24.18)]); the latter reads

$$sp_{2n}(\lambda;\mathbf{x}^{\pm1})=\frac{\det\limits_{1\le i,j\le n}\left(x_j^{\lambda_i+n-i+1}-x_j^{-(\lambda_i+n-i+1)}\right)}{\det\limits_{1\le i,j\le n}\left(x_j^{n-i+1}-x_j^{-(n-i+1)}\right)}. \tag{2.13}$$

Actually, the Weyl formulas for all of these characters allow a uniform statement (see [6, Theorem 24.2; 39, Theorem 4.1]).

3. CHARACTER IDENTITIES

In this section we collect our identities for classical group characters of "nearly" rectangular shapes. By "nearly" rectangular shapes we mean partitions of the form $(c,c,\dots,c,c-p)$ or $(c,\dots,c,c-1,\dots,c-1)$, i.e., partitions whose Ferrers diagram (see Section A1 of the Appendix) is a rectangle, except that one row or column might be shorter. We write $(c^{r-1}, c-p)$ for the first partition (given that the partition has r components) and $(c^{r-p}, (c-1)^p)$ for the second (given that the partition has r components, p of which are equal to $c-1$).

The first five identities express general linear characters (Schur functions) in terms of symplectic, respectively, orthogonal characters. Most of the "partition terminology" that we use here and henceforth is explained in Section A1 of the Appendix. Other terminology concerns the parity of rows or columns: When we say "an odd (even) row" we mean "a row of odd (even) length." We use the same convention with columns. For convenience, we denote the number of odd rows of some (ordinary or skew) shape σ by oddrows(σ) and the number of odd columns of σ by oddcols(σ), and the same with "even" instead of "odd."

THEOREM 1. *Let r, c, p be nonnegative integers. For $c \ge p$ there holds*

$$s\big((c^{r-1}, c-p); \mathbf{x}\big) = \sum_{\substack{\nu \subseteq (c^r) \\ \operatorname{oddcols}((c^r)/\nu) = p}} sp(\nu; \mathbf{x}), \tag{3.1}$$

and for $r \ge p$ there holds

$$s\big((c^{r-p}, (c-1)^p); \mathbf{x}\big) = \sum_{\substack{\nu \subseteq (c^r) \\ \operatorname{oddrows}((c^r)/\nu) = p}} o(\nu; \mathbf{x}). \tag{3.2}$$

In particular, for $r \le N$ and $c \ge p$ there hold

$$s_N\big((c^{r-1}, c-p); \mathbf{x}^{\pm 1}\big) = \begin{cases} \displaystyle\sum_{\substack{\nu \subseteq (c^r) \\ \operatorname{oddcols}((c^r)/\nu) = p}} sp_N(\nu; \mathbf{x}^{\pm 1}) & r \le \lceil N/2 \rceil \\ \displaystyle\sum_{\substack{\nu \subseteq (c^{N-r+1}) \\ \operatorname{oddcols}((c^{N-r+1})/\nu) = c-p}} sp_N(\nu; \mathbf{x}^{\pm 1}) & r > \lceil N/2 \rceil, \end{cases} \tag{3.3}$$

and

$$s_{2n+1}\big((c^r); \mathbf{x}^{\pm 1}\big) = \begin{cases} \displaystyle\sum_{\nu \subseteq (c^r)} sp_{2n}(\nu; \mathbf{x}^{\pm 1}) & r \le n \\ \displaystyle\sum_{\nu \subseteq (c^{2n+1-r})} sp_{2n}(\nu; \mathbf{x}^{\pm 1}) & r > n. \end{cases} \tag{3.4}$$

For $p \le r \le \lfloor N/2 \rfloor$ there holds

$$s_N\big((c^{r-p},(c-1)^p);\mathbf{x}^{\pm 1}\big) = \sum_{\substack{\nu \subseteq (c^r)\\ \mathrm{oddrows}((c^r)/\nu)=p}} o_N(\nu;\mathbf{x}^{\pm 1}). \quad (3.5)$$

Remark 1. The identities (3.3)–(3.5) have obvious interpretations as branching rules for the restriction of representation modules of $\mathrm{GL}(N,\mathbb{C})$ to $\mathrm{Sp}(N,\mathbb{C})$, $\mathrm{Sp}(N-1,\mathbb{C})$, respectively, $\mathrm{O}(N,\mathbb{C})$.

The case $N=2n$, $r\le n$ of (3.3) is implicit in Proctor's paper [28, Proof of Lemma 4, Claim on p. 558]. In the same paper, there appear, explicitly, the case $N=2n$, $r\le n$, $p=0$ of (3.3) [28, Lemma 4, equation for $A_{2n-1}(m\omega_r)$] and the case $r\le n$ of (3.4) [28, Lemma 4, equation for $A_{2n}(m\omega_r)$]. The cases $N=2n$, $r\le n$, $p=0$ of (3.3)–(3.5) appear in [27, Theorem 2.6].

The next four identities express symplectic and special orthogonal characters in terms of general linear characters (Schur functions).

THEOREM 2. *Let n, c, p be nonnegative integers. For $n \ge p$ there holds*

$$sp_{2n}\big((c^{n-p},(c-1)^p);\mathbf{x}^{\pm 1}\big) = (x_1x_2\cdots x_n)^{-c}\cdot \sum_{\substack{\nu\subseteq((2c)^n)\\ \mathrm{oddrows}(\nu)=p}} s_n(\nu;\mathbf{x}). \quad (3.6)$$

If c is a nonnegative integer or half-integer and p is a nonnegative integer, $2c\ge p$, then there holds

$$so_{2n}\big((c^{n-1},c-p);\mathbf{x}^{\pm 1}\big) = (x_1x_2\cdots x_n)^{-c}\cdot \sum_{\substack{\nu\subseteq((2c)^n)\\ \mathrm{oddcols}(((2c)^n)/\nu)=p}} s_n(\nu;\mathbf{x}). \quad (3.7)$$

Next, if c is a nonnegative integer or half-integer and p is a nonnegative integer, $n\ge p$, then there holds

$$\begin{aligned} so_{2n+1}&\big((c^{n-p},(c-1)^p);\mathbf{x}^{\pm 1}\big)\\ &= (x_1x_2\cdots x_n)^{-c}\cdot \sum_{\nu\subseteq((2c)^n)} a_{n,p}(\nu)\cdot s_n(\nu;\mathbf{x}), \end{aligned} \quad (3.8)$$

where

$$a_{n,p}(\nu) = \begin{array}{l}\textit{number of vertical strips of length } p \textit{ on the rim of } \nu\\ \textit{avoiding the } (2c)\textit{th column}.\end{array} \quad (3.9)$$

Finally, if c is a nonnegative integer or half-integer and p is a nonnegative integer, $c \ge p$, then there holds

$$so_{2n+1}\big((c^{n-1}, c-p); \mathbf{x}^{\pm 1}\big) = (x_1 x_2 \cdots x_n)^{-c} \cdot \sum_{\nu \subseteq ((2c)^n)} b_{n,p}(\nu) \cdot s_n(\nu; \mathbf{x}), \tag{3.10}$$

where

$$b_{n,p}(\nu) = \begin{array}{l}\textit{number of horizontal strips of length } p \textit{ on the rim of } \nu \\ \textit{such that the ith cell of the strip (counted from left to} \\ \textit{right) comes before the } (2c-2p+2i)\textit{th column.}\end{array} \tag{3.11}$$

Remark 2. The identities (3.6)–(3.8) and (3.10) have obvious interpretations as decomposition formulas for representation modules of $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SO}(2n, \mathbb{C})$, or $\mathrm{SO}(2n+1, \mathbb{C})$ as representations of the subgroup $\mathrm{GL}(n, \mathbb{C})$. For the interested reader we add that, without much additional effort, it is also possible to derive decomposition formulas for $sp_{2n}((c^{n-1}, c-p); \mathbf{x}^{\pm 1})$ and $so_{2n}((c^{n-p}, (c-1)^p); \mathbf{x}^{\pm 1})$ that are in the spirit of (3.10), respectively, (3.8), by slightly modifying the arguments in the proofs of (3.10) and (3.8).

Formula (3.6) appeared for the first time in [20], although it is implicit already in [7, Theorem 2.6; 18, (2.2)]. The case $p = 0$ of (3.6) has already appeared in a number of papers [31, Theorem 3; 37, Theorem 4.1; 38, Cor. 7.4.(b); 27, Theorem 2.3.(2)].

The case $p = 0$ of (3.8) and (3.10), which is

$$so_{2n+1}\big((c^n); \mathbf{x}^{\pm 1}\big) = (x_1 x_2 \cdots x_n)^{-c} \cdot \sum_{\nu \subseteq ((2c)^n)} s_n(\nu; \mathbf{x}), \tag{3.12}$$

appeared also in a number of papers [31, Theorem 3; 38, Cor. 7.4.(a); 27, Theorem 2.3.(1)].

Finally, the cases $p = 0$ and $p = c$ of (3.7) have previously appeared in [1; 27, Theorem 2.3.(3)].

Our final identities of this section display the decomposition of the product of a rectangularly shaped and a "nearly" rectangularly shaped (general linear, symplectic, respectively special orthogonal) character. As will become apparent from the proofs, these identities follow rather easily from decomposition formulas for the product of a rectangularly shaped symplectic, respectively, special orthogonal, character and an *arbitrarily* shaped symplectic, respectively, special orthogonal, character, which we obtain in Proposition 1 in Section 6 from Littelmann's decomposition formula [24].

THEOREM 3. *Let n, c, d, p be nonnegative integers.*
First let $n \ge p$. If $c \le d$, then there holds

$$sp_{2n}\big((c^n);\mathbf{x}^{\pm 1}\big)\cdot sp_{2n}\big(((d+1)^p, d^{n-p});\mathbf{x}^{\pm 1}\big) = \sum_{\substack{((d-c)^n)\subseteq\nu\subseteq((c+d+1)^n)\\ \operatorname{oddrows}(\nu/((d-c)^n))=p}} sp_{2n}(\nu;\mathbf{x}^{\pm 1}), \tag{3.13}$$

and if $c \ge d$,

$$sp_{2n}\big((c^n);\mathbf{x}^{\pm 1}\big)\cdot sp_{2n}\big((d^{n-p},(d-1)^p);\mathbf{x}^{\pm 1}\big) = \sum_{\substack{((c-d)^n)\subseteq\nu\subseteq((c+d)^n)\\ \operatorname{oddrows}(\nu/((c-d)^n))=p}} sp_{2n}(\nu;\mathbf{x}^{\pm 1}). \tag{3.14}$$

Next, if c, d are nonnegative integers or half-integers and p is an integer with $p \le 2d$, then there hold

$$so_{2n}\big((c^n);\mathbf{x}^{\pm 1}\big)\cdot so_{2n}\big((d^{n-1}, d-p);\mathbf{x}^{\pm 1}\big) = \sum_{\substack{(|c-d|^{n-1},c-d)\subseteq\nu\subseteq((c+d)^n)\\ \operatorname{oddcols}(((c+d)^n)/\nu)=p}} so_{2n}(\nu;\mathbf{x}^{\pm 1}), \tag{3.15}$$

with the understanding that ν ranges over n-orthogonal partitions if $c+d$ is an integer and over n-orthogonal half-partitions if $c+d$ is a half-integer, and

$$so_{2n}\big((c^{n-1},-c);\mathbf{x}^{\pm 1}\big)\cdot so_{2n}\big((d^{n-1}, d-p);\mathbf{x}^{\pm 1}\big) = \sum_{\substack{(|c-d|^{n-1},c-d)\subseteq\nu\subseteq((c+d)^n)\\ \operatorname{evencols}(((c+d)^n)/\nu)=p}} so_{2n}\big((\nu_1,\dots,\nu_{n-1},-\nu_n);\mathbf{x}^{\pm 1}\big) \tag{3.16}$$

with the same understanding.
For c, d nonnegative integers or half-integers and for $n > p$ there holds

$$so_{2n+1}\big((c^n);\mathbf{x}^{\pm 1}\big)\cdot so_{2n+1}\big((d^{n-p},(d-1)^p);\mathbf{x}^{\pm 1}\big) = \sum_{(|c-d|^{n-p},(\max\{c-d,d-c-1\})^p)\subseteq\nu\subseteq((c+d)^n)} c_{n,p}(\nu)\cdot so_{2n+1}(\nu;\mathbf{x}^{\pm 1}), \tag{3.17}$$

where, with $m_l(\nu)$ denoting the multiplicity of l in ν,

$$c_{n,p}(\nu) = \text{number of vertical strips of length } p - m_{d-c-1}(\nu) \text{ that can be added to } \nu \text{ to obtain another (half-) partition, avoiding the } (d-c)\text{th, the } (c-d+1)\text{th, and the } (c+d+1)\text{th columns of } \nu. \tag{3.18}$$

Again, the sum in (3.17) *is understood to range over partitions if* $c + d$ *is an integer and over half-partitions if* $c + d$ *is a half-integer.*

Finally, for c, d *nonnegative integers or half-integers and for* $d \geq p$ *there holds*

$$so_{2n+1}\big((c^n);\mathbf{x}^{\pm 1}\big) \cdot so_{2n+1}\big((d^{n-1}, d-p);\mathbf{x}^{\pm 1}\big) = \sum_{(|c-d|^{n-1}, \max\{c-d, d-c-p\}) \subseteq \nu \subseteq ((c+d)^n)} d_{n,p}(\nu) \cdot so_{2n+1}(\nu;\mathbf{x}^{\pm 1}), \tag{3.19}$$

where $d_{n,p}(\nu)$ *is the number of all horizontal strips* σ *that can be added to* ν *to obtain another* (*half-*)*partition, avoiding the* $(c + d + 1)$*th column and the nth row of* ν, *and which satisfy the following inequalities*: *If* σ_i *denotes the number of cells of the strip* σ *in the ith row,* $i = 1, 2, \dots, n - 1$, *and if* $|\sigma|$ *denotes the total number of cells in* σ, *then*

$$\big||\sigma| - c + d - p\big| \leq \nu_n, \qquad \nu_n + p - \nu_{n-1} \leq |\sigma| \leq p, \quad \text{and}$$
$$2\sigma_1 + \cdots + 2\sigma_{i-1} + \sigma_i + \nu_i \geq c - d + 2p \quad \text{for } i = 1, 2, \dots, n-1. \tag{3.20}$$

Again, the sum in (3.19) *is understood to range over partitions if* $c + d$ *is an integer and over half-partitions if* $c + d$ *is a half-integer.*

Remark 3. All of these formulas, that is, (3.13)–(3.20) and (6.1)–(6.5), have obvious interpretations as decomposition formulas for the tensor product of two representation modules of $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SO}(2n, \mathbb{C})$, or $\mathrm{SO}(2n + 1, \mathbb{C})$. For the interested reader we add that, without much additional effort, it is also possible to derive decomposition formulas for $sp_{2n}((c^n); \mathbf{x}^{\pm 1}) \cdot sp_{2n}((d^{n-1}, d - p); \mathbf{x}^{\pm 1})$ and $so_{2n}((c^n); \mathbf{x}^{\pm 1}) \cdot so_{2n}((d^{n-p}, (d-1)^p); \mathbf{x}^{\pm 1})$, which are in the spirit of (3.19), respectively, (3.17), by slightly modifying the arguments in the proofs of (3.19) and (3.17).

Formulas (3.13)–(3.20) generalize the symplectic and special orthogonal decompositions of Okada [27, Theorem 2.5], who proved the $p = 0$ special cases. Okada also proves a decomposition formula for the product of two rectangularly shaped general linear characters (Schur functions), thus generalizing Stanley's results [36, Lemma 3.3]. More generally, Carini [2, Sect. 3.3, p. 105ff] derived decomposition formulas for the product of two "nearly" rectangularly shaped general linear characters.

4. PROOF OF THEOREM 1

Here we use decomposition rules of Littlewood [25] (see Section A7 of the Appendix). All of the notions that appear in this section, like Littlewood–Richardson filling (LR-filling), Littlewood–Richardson condition (LR-condition), content, etc., are also explained in the Appendix, mainly in Section A6.

Proof of (3.1). Implicitly, this was already proved in [28, Proof of Lemma 4, Claim on p. 558]. However, since there is no explicit statement in [28] and since one would have to translate things appropriately, we include a detailed proof of (3.1) here.

According to (A.13) we have

$$s\big((c^{r-1}, c-p);\mathbf{x}\big) = \sum_{\nu} sp(\nu;\mathbf{x}) \sum_{\mu,\,\mu'\text{ even}} \mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}.$$

Hence we have to show that

$$\sum_{\mu,\,\mu'\text{ even}} \mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)} = \begin{cases} 1 & \operatorname{oddcols}\big((c^r)/\nu\big) = p \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

To see this, suppose that μ is a fixed partition whose Ferrers diagram has only even columns. Consider a LR-filling of shape $(c^{r-1}, c-p)/\mu$. By the LR-condition, there is no choice for the entries in the first $r-1$ rows; i.e., all of the rows except for the rth row are uniquely determined in a way that is exemplified in Fig. 1. There is only freedom in the rth row.

It should be observed that the content of the uniquely determined part of the LR-filling equals $\tilde{\mu} := (c-\mu_{r-1}, c-\mu_{r-2}, \dots, c-\mu_1)$. In particular, this implies that all of the columns in $(c^r)/\tilde{\mu}$ have odd length, except for columns $c-\mu_{r-1}+1, c-\mu_{r-1}+2, \dots, c$ (which would have length r), in the case where r is even.

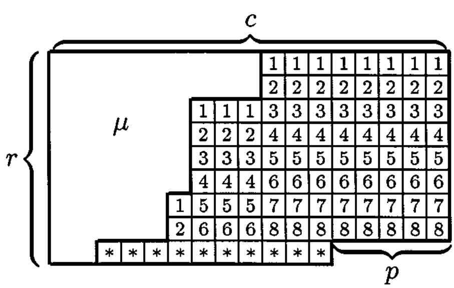

FIGURE 1

Suppose that ν is the content of the *complete* LR-filling. Then the LR-condition is equivalent to saying that $\nu/\tilde{\mu}$ is a horizontal strip (see Appendix A1 for the definition of a horizontal strip). Moreover, this horizontal strip is of length $c-p-\mu_r$, the length of the rth row of the LR-filling. Hence, there are $c-(c-p-\mu_r)=p+\mu_r$ odd columns in $(c^r)/\nu$ if c is odd, and there are $c-\mu_{r-1}-(c-p-\mu_r)=p+\mu_r-\mu_{r-1}$ odd columns in $(c^r)/\nu$ if r is even. The latter is due to the fact that 1 cannot be an entry in the rth row of the LR-filling if r is even; hence the horizontal strip has to avoid the first row. Actually, the number of odd columns is p in both cases, since in case r odd we must have $\mu_r=0$ because the Ferrers diagram of μ has only even columns, and in case r even we have $\mu_{r-1}=\mu_r$ for the same reason. Thus we have shown that, given a fixed μ whose Ferrers diagram has only even columns, then $\mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}\neq 0$ only if ν is a partition such that the number of odd columns in $(c^r)/\nu$ equals p. In particular, this establishes the "otherwise" part of (4.1).

Conversely, given a partition $\nu\subseteq(c^r)$ such that the number of odd columns in $(c^r)/\nu$ equals p, we claim that there is exactly one partition μ whose Ferrers diagram has only even columns and such that $\mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}\neq 0$. And, more precisely, we have $\mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}=1$, which means that there is exactly one LR-filling of shape $(c^{r-1},c-p)/\mu$ and content ν. Altogether, this would establish (4.1).

The claim is established by going through the preceding paragraphs, backwards. Suppose that $\nu\subseteq(c^r)$ is a partition with $\mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}\neq 0$, for some μ whose Ferrers diagram has only even columns. Then $\tilde{\mu}$, defined as above as $(c-\mu_{r-1},c-\mu_{r-2},\dots,c-\mu_1)$, is a partition in which (1) all the columns have length $\equiv r \bmod 2$, and (2) which is contained in ν and differs from ν by a horizontal strip of length $c-p-\mu_r$. It is straightforward to see that (1) and (2) determine $\tilde{\mu}$ uniquely. Hence, $\mu_1,\mu_2,\dots,\mu_{r-1}$ are uniquely determined, thus also μ_r, since the Ferrers diagram of μ has to contain only even columns. To be precise, the latter condition implies that μ_r equals μ_{r-1} if r is even, and 0 if r is odd.

Let μ be this uniquely determined partition. To show that $\mathrm{LR}_{\mu,\nu}^{(c^{r-1},c-p)}=1$ it remains to see that there is exactly one LR-filling of shape $(c^{r-1},c-p)/\mu$ with content ν. In fact, as we already observed, the entries of the first $r-1$ rows of a LR-filling of shape $(c^{r-1},c-p)/\mu$ are uniquely determined. The content of this partial filling of these first $r-1$ rows is $\tilde{\mu}$. Since ν, which should be the content of the complete LR-filling, differs from $\tilde{\mu}$ by a horizontal strip, also the entries of the r-th row are uniquely determined. To be precise, the length of the i-th row of $\nu/\tilde{\mu}$ gives the multiplicity of i in the r-th row of the LR-filling.

All together, this establishes (4.1), and hence (3.1), as desired. ∎

Proof of (3.2). According to (A.14) we have

$$s\big((c^{r-p},(c-1)^p);\mathbf{x}\big) = \sum_{\nu} o(\nu;\mathbf{x}) \sum_{\mu,\,\mu\text{ even}} \mathrm{LR}_{\mu,\nu}^{(c^{r-p},(c-1)^p)}.$$

Hence we have to show that

$$\sum_{\mu,\,\mu\text{ even}} \mathrm{LR}_{\mu,\nu}^{(c^{r-p},(c-1)^p)} = \begin{cases} 1 & \text{oddrows}\big((c^r)/\nu\big)=p\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

We could prove this again directly, in a style similar to that of the proof of (4.1). However, once (4.1) is already known, the companion (4.2) follows straightforwardly from the well-known identity (see [8]) $\mathrm{LR}^{\lambda}_{\mu,\nu} = \mathrm{LR}^{\lambda'}_{\mu',\nu'}$. ∎

For the proofs of (3.3) and (3.4) we utilize the following auxiliary result.

LEMMA 1. *Let N be a positive integer. Then, for any partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_N)$ we have*

$$s_N\big((\lambda_1,\lambda_2,\dots,\lambda_N);\mathbf{x}^{\pm1}\big) = s_N\big((\lambda_1-\lambda_N,\lambda_1-\lambda_{N-1},\dots,\lambda_1-\lambda_2,0);\mathbf{x}^{\pm1}\big). \tag{4.3}$$

Proof. This identity follows upon little manipulation from the dual Jacobi–Trudi identity (the Nägelsbach–Kostka identity, see [26, I, (3.5)]) for Schur functions,

$$s(\lambda;\mathbf{x}) = \det_{1\le i,j\le\lambda_1}\big(e_{\lambda'_i-i+j}(\mathbf{x})\big), \tag{4.4}$$

where $e_m(\mathbf{x}) = \sum_{1\le i_1<\dots<i_m} x_{i_1}\cdots x_{i_m}$ denotes the *elementary symmetric function* of degree m. For, by (4.4), and because of

$$e_m\big(x_1,x_1^{-1},\dots,x_n,x_n^{-1}\big) = e_{2n-m}\big(x_1,x_1^{-1},\dots,x_n,x_n^{-1}\big)$$

and a similar identity for $e_m(x_1,x_1^{-1},\dots,x_n,x_n^{-1},1)$, we have (in the following calculation $e_m(\mathbf{x}^{\pm1})$ is short for $e_m(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$, respectively, for $e_m(x_1,x_1^{-1},\dots,x_n,x_n^{-1},1)$, depending on whether N is even or odd)

$$\begin{aligned} s_N\big((\lambda_1,\lambda_2,\dots,\lambda_N);\mathbf{x}^{\pm1}\big) &= \det_{1\le i,j\le\lambda_1}\big(e_{\lambda'_i-i+j}(\mathbf{x}^{\pm1})\big)\\ &= \det_{1\le i,j\le\lambda_1}\big(e_{N-\lambda'_i+i-j}(\mathbf{x}^{\pm1})\big)\\ &= \det_{1\le i,j\le\lambda_1}\big(e_{N-\lambda'_{\lambda_1+1-i}+(\lambda_1+1-i)-(\lambda_1+1-j)}(\mathbf{x}^{\pm1})\big)\\ &= s_N\big((\lambda_1-\lambda_N,\lambda_1-\lambda_{N-1},\dots,\lambda_1-\lambda_2,0);\mathbf{x}^{\pm1}\big). \end{aligned}$$

Proof of (3.3). In (3.1) we specialize $\mathbf{x}$ to $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots)$ if $N = 2n$, and to $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots)$ if $N = 2n + 1$, which leads to

$$s_N\big((c^{r-1}, c-p); \mathbf{x}^{\pm 1}\big) = \sum_{\substack{\nu \subseteq (c^r) \\ \operatorname{oddcols}((c^r)/\nu) = p}} sp_N(\nu; \mathbf{x}^{\pm 1}). \tag{4.5}$$

This gives (3.3) for $r \le \lceil N/2 \rceil$ immediately. In the case where $r > \lceil N/2 \rceil$, on the right-hand side of (4.5) the partition ν could have more than $\lceil N/2 \rceil$ parts, in which case one would have to apply modification rules for the corresponding symplectic characters $sp_N(\nu; \mathbf{x})$ (see, e.g., [10; 39, Theorem 5.4]). However, we circumvent this difficulty by appealing to Lemma 1. In fact, by (4.3) and (4.5) we have for $r > \lceil N/2 \rceil$,

$$\begin{aligned} s_N\big((c^{r-1}, c-p); \mathbf{x}^{\pm 1}\big) &= s_N\big((c^{N-r}, p); \mathbf{x}^{\pm 1}\big) \\ &= \sum_{\substack{\nu \subseteq (c^{N-r+1}) \\ \operatorname{oddcols}((c^{N-r+1})/\nu) = c-p}} sp_N(\nu; \mathbf{x}^{\pm 1}), \end{aligned}$$

which is (3.3) for $r > \lceil N/2 \rceil$. ∎

Proof of (3.4). In (3.1) we set $N = 2n + 1$, $p = 0$, and substitute $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots)$ for $\mathbf{x}$ to obtain

$$s_{2n+1}\big((c^r); \mathbf{x}^{\pm 1}\big) = \sum_{\substack{\rho \subseteq (c^r) \\ \operatorname{oddcols}((c^r)/\rho) = 0}} sp_{2n+1}(\rho; \mathbf{x}^{\pm 1}). \tag{4.6}$$

Now first let $r \le n$. The odd symplectic characters on the right-hand side of (4.6) are known to decompose in terms of even symplectic characters as (see [30, Cor. 8.1; 32, Lemma 9.1, with $z = 1$ and $b = 0$])

$$sp_{2n+1}(\rho; \mathbf{x}^{\pm 1}) = \sum_{\substack{\nu \subseteq \rho \\ \rho/\nu \text{ a horizontal strip}}} sp_{2n}(\nu; \mathbf{x}^{\pm 1}). \tag{4.7}$$

Combining this with (4.6) yields (3.4) for $r \le n$, as for any subpartition ν of the rectangle (c^r) there is exactly one way of adding a horizontal strip to ν such that a partition ρ with $\operatorname{oddcols}((c^r)/\rho) = 0$ is obtained.

The case $r > n$ of (3.4) then follows from the $r \le n$ case by use of (4.3) with $N = 2n + 1$. ∎

Proof of (3.5). Here, in (3.2) we specialize $\mathbf{x}$ to $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 0, 0, \dots)$ if $N = 2n$, and to $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1, 0, 0, \dots)$ if $N = 2n + 1$. ∎

5. PROOF OF THEOREM 2

In this section we combine the tableaux description of symplectic characters due to DeConcini and Procesi [3, 4] (see Section A3 of the Appendix) and tableaux descriptions of special orthogonal characters due to Lakshmibai, Musili, and Seshadri [22, 23, 21] (see Sections A4 and A5 of the Appendix) with Robinson−Schensted−Knuth-type algorithms. We remark that while there are restriction rules that would apply here (they involve ordinary Littlewood−Richardson coefficients; see [12, (4.20)−(4.22); 16, Theorem 2.1 with $k = n$, Theorem A1]), these do not appear to be very helpful for our purposes. This is because they involve modification rules for characters, these cause alternating signs, and these in turn cause a lot of cancellations, and all of this is simply not tractable for the applications that we have in mind.

Proof of (3.6). By (A.4) we know that the left-hand side of (3.6) equals

$$\sum_{\substack{S \text{ a } (2n)\text{-symplectic tableau} \\ \text{of shape } (c^{n-p},(c-1)^p)}} (\mathbf{x}^{\pm 1})^S. \tag{5.1}$$

On the other hand, by (A.2) we have that the right-hand side of (3.6) equals

$$(x_1x_2 \cdots x_n)^{-c} \cdot \sum_{\substack{T \text{ an } n\text{-tableau} \\ \text{of shape } \nu\subseteq((2c)^n) \\ \text{with oddrows}(\nu)=p}} \mathbf{x}^T. \tag{5.2}$$

Comparing (5.1) and (5.2), we see that (3.6) will be proved if we can find a bijection, Φ say, between $(2n)$-symplectic tableaux S of shape $(c^{n-p},(c-1)^p)$ and n-tableaux T with at most $2c$ columns and exactly p odd rows such that

$$(\mathbf{x}^{\pm 1})^S = (x_1x_2 \cdots x_n)^{-c} \cdot \mathbf{x}^T, \qquad \text{if } T = \Phi(S). \tag{5.3}$$

The bijection that we are going to construct proceeds in two steps. In the first step we map $(2n)$-symplectic tableaux of shape $(c^{n-p},(c-1)^p)$ to certain pairs (see the paragraph including (5.4)) by analyzing how these symplectic tableaux look. In the second step we map these pairs to the above-described n-tableaux by a Robinson−Schensted−Knuth-type correspondence.

First step. Let S be a $(2n)$-symplectic tableaux of shape $(c^{n-p},(c-1)^p)$. By the definition of $(2n)$-symplectic tableaux in Section A3 of the Appendix, S is a $(2n)$-tableau of shape $((2c)^{n-p},(2c-2)^p)$ such that columns

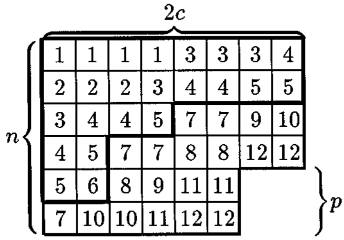

FIGURE 2

$2c-1, 2c$, columns $2c-3, 2c-2$, etc., form $(2n)$-symplectic admissible pairs. An example with $n=6$, $c=4$, $p=2$ is displayed in Fig. 2.

We claim that $(2n)$-symplectic tableaux S of shape $(c^{n-p}, (c-1)^p)$ are in bijection with pairs $(\overline{T}, \{e_1, e_2, \dots, e_p\})$, by a bijection Φ_1 say, where $\overline{T}$ is an n-tableau whose shape has only even rows and is contained in $((2c)^{n-p}, (2c-2)^p)$, and where $\{e_1, e_2, \dots, e_p\}$ is a set of numbers satisfying

$$1 \le e_1 < e_2 < \cdots < e_p \le n,$$
$$\text{and} \quad e_l \notin [i(m), j(m)] \text{ for } 1 \le l \le p, \quad 1 \le m \le s, \quad (5.4)$$

given that

$$\begin{matrix} i(1) & j(1) \\ \vdots & \vdots \\ i(s) & j(s) \end{matrix}$$

are the $(2c-1)$th and $(2c)$th columns of $\overline{T}$, such that

$$(\mathbf{x}^{\pm 1})^S = (x_1 x_2 \cdots x_n)^{-c} \cdot x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}},$$
$$\text{if } \left(\overline{T}, \{e_1, e_2, \dots, e_p\}\right) = \Phi_1(S). \quad (5.5)$$

The construction of the bijection Φ_1 is based on an analysis of the symplectic tableaux under consideration. By Observation 2 in Section A3 of the Appendix, both columns in a $(2n)$-symplectic admissible pair have the same number of entries $\le n$. Hence, the entries $\le n$ in S form an n-tableau, $\overline{T}$ say, with only even rows. Of course, the shape of $\overline{T}$ is contained in $((2c)^{n-p}, (2c-2)^p)$. In Fig. 2 we have marked the area that is covered by entries $\le n$ by a bold line. The resulting tableau is displayed in the left half of Fig. 3.

$$\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&3&3&3&4\\\hline 2&2&2&3&4&4&5&5\\\hline 3&4&4&5\\\cline{1-4} 4&5\\\cline{1-2} 5&6\\\cline{1-2}\end{array}\quad,\quad\{2,6\}\right)$$

FIGURE 3

The next observation is that, given $\overline{T}$, we can recover S almost completely; only the $(2c-1)$th and $(2c)$th columns cannot be necessarily recovered. For, all of the columns of S except columns $2c-1$ and $2c$ have length n. Thus, by Observation 3 in Section A3 of the Appendix, if the entries $\le n$ in a column of length n are $\{i(1), i(2), \dots, i(s)\}$, then the entries $> n$ are $\{n+1, n+2, \dots, 2n\} \setminus \{2n+1-i(1), 2n+1-i(2), \dots, 2n+1-i(s)\}$. Only for recovering columns $2c-1$ and $2c$ of S from $\overline{T}$ do we need more information than just the $(2c-1)$th and $(2c)$th columns of $\overline{T}$.

Let the $(2c-1)$th and $(2c)$th column of S be

$$\begin{array}{cc} i(1) & j(1)\\ \vdots & \vdots\\ i(s) & j(s)\end{array}\Bigg\}\text{entries } \le n$$
$$\begin{array}{cc} i(s+1) & j(s+1)\\ \vdots & \vdots\\ i(n-p) & j(n-p)\end{array}\Bigg\}\text{entries } > n \tag{5.6}$$

Observe that by definition of a $(2n)$-symplectic admissible pair (see Definition 1 in the Appendix) we have

$$\begin{aligned}\{i(1), \dots, i(s), 2n+1-i(s+1), \dots, 2n+1-i(n-p)\}\\ = \{j(1), \dots, j(s), 2n+1-j(s+1), \dots, 2n+1-j(n-p)\}.\end{aligned} \tag{5.7}$$

Let $\{e_1, e_2, \dots, e_p\}$ be the complement of this set in $\{1, 2, \dots, n\}$. In other words, $e_1, e_2, \dots, e_p$ are the numbers e between 1 and n with the property that neither e nor its "conjugate" $2n+1-e$ occurs in the $(2c-1)$th or $(2c)$th columns of S. Without loss of generality we may assume $e_1 < e_2 < \cdots < e_p$. In our running example (recall $p=2$) we have $\{e_1, e_2\} = \{2, 6\}$.

Obviously, because of (5.7) and the definition of $\overline{T}$ and $\{e_1, e_2, \dots, e_p\}$, all of the information about the $(2c-1)$th and the $(2c)$th columns (displayed in (5.6)) is contained in the $(2c-1)$th and $(2c)$th columns of $\tilde{T}$ (the entries $\le n$ in (5.6)) and $\{e_1, e_2, \dots, e_p\}$. Moreover, because of the definition of a $(2n)$-symplectic admissible pair, we have $e_l \notin [i(m), j(m)]$ for all l and m. And conversely, if we have

$$\begin{matrix} i(1) & j(1) \\ \vdots & \vdots \\ i(s) & j(s) \end{matrix}, \qquad \{e_1, e_2, \dots, e_p\}$$

such that (5.4) is satisfied, then (5.6) with

$$\begin{aligned}\{i(s+1), \dots, i(n-p)\} \\ &:= \{n+1, \dots, 2n\} \setminus \big\{2n+1-i(1), \dots, 2n+1-i(s), \\ &\qquad 2n+1-e_1, \dots, 2n+1-e_p\big\} \\ \{j(s+1), \dots, j(n-p)\} \\ &:= \{n+1, \dots, 2n\} \setminus \big\{2n+1-j(1), \dots, 2n+1-j(s), \\ &\qquad 2n+1-e_1, \dots, 2n+1-e_p\big\}\end{aligned}$$

will be a $(2n)$-symplectic admissible pair.

Hence, we have defined the desired bijection Φ_1. It is easy to check that the weight property (5.5) holds under this correspondence.

Our running example in Fig. 2 is mapped under Φ_1 to the pair in Fig. 3.

Second step. In the second step we construct a bijection Φ_2 between pairs $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ satisfying the above conditions including (5.4), and n-tableaux T with at most $2c$ columns and exactly p odd rows, such that

$$\mathbf{x}^T = x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}}. \tag{5.8}$$

Let $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ be such a pair. We insert $e_p, e_{p-1}, \dots, e_1$, in this order, into $\overline{T}$, according to the following procedure. Let $\overline{T}_0 := \overline{T}$. Suppose that, by inserting $e_p, e_{p-1}, \dots, e_{p-l+1}$ we already formed $\overline{T}_l$. Next we insert e_{p-l} into $\overline{T}_l$ in the following way. Choose the first row (from top to bottom) of $\overline{T}_l$ such that e_{p-l} is less than the entry in the $(2c-1)$th column in that row of $\overline{T}_l$. If there is no such row, choose the first row that does not have an entry in the $(2c-1)$th column of that row. Then, starting with that row of $\overline{T}_l$, ROW-INSERT e_{p-l} into $\overline{T}_l$, i.e. (cf. [14, p. 712; 19, pp. 87–88]), find the leftmost entry in that row that is larger than e_{p-l}, bump it, and replace it by e_{p-l}; if there is none, then place e_{p-l} at the end of that row. If an entry was bumped, then repeat this same procedure with the bumped entry and the next row, etc. Thus one obtains $\overline{T}_{l+1}$. Finally, set $T = \Phi_2((\overline{T}, \{e_1, e_2, \dots, e_p\})) := \overline{T}_p$. Our running example from Fig. 3 is mapped under Φ_2 to the tableau in Fig. 4.

Since $\overline{T}$ was an n-tableau with only even rows, and since later "insertion paths" are (weakly) to the left of previous ones, T is an n-tableau with exactly p odd rows. Trivially, (5.8) is satisfied.

To show that Φ_2 is a bijection, we have to construct the inverse mapping. Take an n-tableau T with at most $2c$ columns and exactly p odd rows. Choose the last row (from top to bottom) of T that has odd length. Now, starting with that row, perform a slightly modified ROW-DELETE (cf. [14, p. 713; 19, pp. 88]). Namely, remove the last entry, x say, from that row and find the rightmost entry, y say, in the previous row that is less than x, replace y by x, and repeat this same procedure with y and the row before the row that contained y, etc., until no row is left to be considered or until an entry in the $(2c)$th column would be replaced. In the latter case (this is the modification), do not replace the entry in the $(2c)$th column, but stop the procedure. Thus we obtain an n-tableau T_1 with $p-1$ odd rows and an entry, e_1 say, that was replaced in the last step of the procedure. This procedure is repeated with T_1, thus obtaining T_2 and e_2, etc. In the end we obtain T_p, which is an n-tableau with only even rows, and in the course of our algorithm we obtained the elements $e_1, e_2, \dots, e_p$. By standard properties of ROW-INSERT and ROW-DELETE (cf. [34]), it is not difficult to see that this algorithm exactly reverses Φ_2, step by step. In particular, the start of ROW-INSERT in a row that is possibly different from the first and the modified ending of ROW-DELETE complement each other exactly.

The composition $\Phi_2 \circ \Phi_1$ is by definition the desired bijection between $(2n)$-symplectic tableaux S of shape $(c^{n-p}, (c-1)^p)$ and n-tableaux T with at most $2c$ columns and exactly p odd rows. From (5.5) and (5.8) the weight property (5.3) follows immediately. This completes the proof of (3.6). ▮

Proof of (3.7). Using (A.6) and (A.2) in (3.7), we see that (3.7) will be proved if we can find a bijection, Ψ say, between $(2n)$-orthogonal tableaux

1	1	1	1	2	3	3	4
2	2	2	3	3	4	5	5
3	4	4	4	6			
4	5	5					
5	6						

FIGURE 4

S of shape $(c^{n-1}, c-p)$ and n-tableaux T with at most $2c$ columns and exactly p columns with parity different from n (by which we mean that the *lengths* of the columns have parity different from n) such that

$$(\mathbf{x}^{\pm 1})^S = (x_1 x_2 \cdots x_n)^{-c} \cdot \mathbf{x}^T, \qquad \text{if } T = \Psi(S). \tag{5.9}$$

Again, the bijection that we are going to construct proceeds in two steps. In the first step we map $(2n)$-orthogonal tableaux of shape $(c^{n-1}, c-p)$ to certain pairs (see the paragraph including (5.10)) by analyzing how these orthogonal tableaux look. In the second step we map these pairs to the above-described n-tableaux by Robinson–Schensted–Knuth insertion.

First step. Let S be a $(2n)$-orthogonal tableaux of shape $(c^{n-1}, c-p)$. By Observation 1 in Section A5 of the Appendix, S consists of a pair (S_3, S_2) of $(2n)$-tableaux, S_3 being of shape $((2c-p)^n)$ and each column of which containing an even number of entries $> n$, S_2 being of shape (p^n), each column of which containing an odd number of entries $> n$, and all of the entries in the first row of S_2 being at most n, such that the concatenation $S_3 \cup S_2'$ is a $(2n)$-tableau, where S_2' is the tableau arising from S_2 by replacing the topmost element, e_i say, in column i of S_2 by its "conjugate" $2n+1-e_i$, for all $i = 1, 2, \dots, p$, and by rearranging the columns in increasing order. An example with $n=6$, $c=7/2$, $p=2$ is displayed in the left half of Fig. 5. The right half shows the concatenation $S_3 \cup S_2'$ (note that $e_1 = 1$, $e_2 = 2$).

We claim that $(2n)$-orthogonal tableaux S of shape $(c^{n-1}, c-p)$ are in bijection with pairs $(\overline{T}, \{e_1, e_2, \dots, e_p\})$, by a bijection Ψ_1 say, where $\overline{T}$ is an n-tableau whose shape has only columns with the same parity as n and is contained in $((2c)^{n-1}, 2c-p)$, and where $\{e_1, e_2, \dots, e_p\}$ is a set of numbers satisfying

$$1 \le e_1 \le e_2 \le \cdots \le e_p \le n, \tag{5.10}$$

e_i has less than the topmost element of the $(2c-p+i)$th column of $\overline{T}$, such that

$$(\mathbf{x}^{\pm 1})^S = (x_1 x_2 \cdots x_n)^{-c} \cdot x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}},$$
$$\text{if } \left(\overline{T}, \{e_1, e_2, \dots, e_p\}\right) = \Psi_1(S). \tag{5.11}$$

The construction of the bijection Ψ_1 is based on an analysis of the orthogonal tableaux under consideration. By definition, all of the columns in $S_3 \cup S_2'$ contain an even number of entries $> n$. Hence, the entries $\le n$ in $S_3 \cup S_2'$ form an n-tableau, $\overline{T}$ say, with all columns having the same parity as n. Of course, the shape of $\overline{T}$ is contained in $((2c)^{n-1}, 2c-p)$. In the right half of Fig. 5 we have marked the area that is covered by

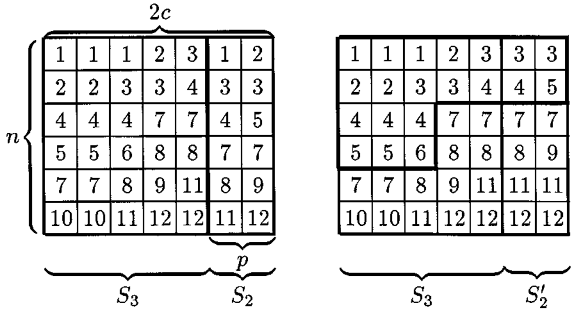

FIGURE 5

entries $\le n$ by a bold line. The resulting tableau is displayed in the left half of Fig. 6.

Now, let $\Psi_1(S)$ be defined by $(\overline{T}, \{e_1, e_2, \dots, e_p\})$, where, as before, e_i is the topmost element of the ith column of S_2. Our running example in Fig. 5 is mapped under Ψ_1 to the pair in Fig. 6. It is obvious that (5.10) and (5.11) hold under this mapping. Besides, it is trivial to recover S from $(\overline{T}, \{e_1, e_2, \dots, e_p\})$. Hence, Ψ_1 is a bijection, as desired.

Second step. In the second step we construct a bijection Ψ_2 between pairs $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ satisfying the above conditions including (5.10), and n-tableaux T with at most $2c$ columns and exactly p columns of parity different from n, such that

$$\mathbf{x}^T = x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}}. \tag{5.12}$$

Let $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ be such a pair. We ROW-INSERT $e_1, e_2, \dots, e_p$, in this order, into $\overline{T}$. (The reader should observe the differences in the algorithm in the second step of the proof of (3.6): here we use *genuine* ROW-INSERT, without any modification, as opposed to the proof of (3.6). Furthermore, the order in which the elements $e_1, e_2, \dots, e_p$ are inserted is exactly the reversed order in the proof of (3.6).) I.e. (cf. [14, p. 712; 19, pp. 87–88]), let $\overline{T}_0 := \overline{T}$. Suppose that, by inserting $e_1, e_2, \dots, e_l$ we have already formed $\overline{T}_l$. Next we insert e_{l+1} into $\overline{T}_l$. Namely, we find the

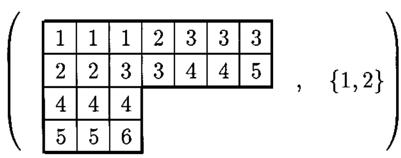

FIGURE 6

leftmost entry in the first row of $\overline{T}_l$ that is larger than e_{l+1}, bump it, and replace it by e_{l+1}; if there is none, we place e_{l+1} at the end of that row. If an entry was bumped, then we repeat this same procedure with the bumped entry and the next row, etc. Thus one obtains the tableau $\overline{T}_{l+1}$. Finally, we set $T = \Psi_2((\overline{T}, \{e_1, e_2, \dots, e_p\})) := \overline{T}_p$. Our running example from Fig. 6 is mapped under Ψ_2 to the tableau in Fig. 7.

Since $\overline{T}$ was an n-tableau with all columns having the same parity as n, and since later "insertion paths" are strictly to the right of previous ones, T is an n-tableau with exactly p columns of parity different from n. Moreover, since e_i is less than the topmost element of the $(2c - p + i)$th column of $\overline{T}$, the insertion process will at no stage produce more than $2c$ columns. Hence, T has at most $2c$ columns. Trivially, (5.12) is satisfied.

To show that Ψ_2 is a bijection, we have to construct the inverse mapping. Take an n-tableau T with at most $2c$ columns and exactly p columns of parity different from n. Choose the rightmost column of T that has parity different from n. Suppose that the bottommost entry in this column is in row I. Now, starting with that row, perform ROW-DELETE, i.e. (cf. [14, p. 713; 19, pp. 88]), remove the last entry, x_I say, from the Ith row and find the rightmost entry, x_{I-1} say, in the $(I-1)$th row that is less than x_I, replace x_{I-1} by x_I, and repeat this same procedure with x_{I-1} and the $(I-2)$th row, etc., until no row is left to be considered. Thus we obtain an n-tableau T_1 with $p-1$ columns of parity different from n and an entry, e_p say, that was replaced in the last step of the procedure. This procedure is repeated with T_1, thus obtaining T_2 and e_{p-1}, etc. In the end we obtain T_p, which is an n-tableau with all columns having the same parity as n, and in the course of our algorithm we obtained the elements $e_p, e_{p-1}, \dots, e_1$. Again, by standard properties of ROW-INSERT and ROW-DELETE (cf. [34]), it is not difficult to see that this algorithm exactly reverses Ψ_2, step by step.

The composition $\Psi_2 \circ \Psi_1$ is by definition the desired bijection between $(2n)$-orthogonal tableaux S of shape $(c^{n-1}, c-p)$ and n-tableaux T with at most $2c$ columns and exactly p columns of parity different from n. From (5.11) and (5.12) the weight property (5.9) follows immediately. This completes the proof of (3.7). ∎

1	1	1	1	2	3	3
2	2	2	3	3	4	5
3	4	4	4			
4	5	6				
5						

FIGURE 7

Proof of (3.8). Here we have to deal with $(2n+1)$-orthogonal tableaux. Since these are not too far from $(2n)$-symplectic tableaux, the arguments here are very similar to those in the proof of (3.6). In fact, the basic steps are the same; only the details differ. So sometimes we shall be sketchy and provide details only if necessary.

Using (A.5) and (A.2) in (3.8), we see that (3.8) will be proved if we can find a bijection, Θ say, between $(2n+1)$-orthogonal tableaux S of shape $(c^{n-p},(c-1)^p)$ and pairs (T,σ), where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a vertical strip of length p on the rim of ν that avoids the $(2c)$th column, such that

$$(\mathbf{x}^{\pm 1})^S = (x_1x_2\cdots x_n)^{-c}\cdot \mathbf{x}^T, \qquad \text{if } (T,\sigma)=\Theta(S) \text{ for some } \sigma. \tag{5.13}$$

Again, we proceed in two steps.

First step. Let S be a $(2n+1)$-orthogonal tableau of shape $(c^{n-p},(c-1)^p)$. By the definition of $(2n+1)$-orthogonal tableaux in Section A4 of the Appendix, S is a $(2n)$-tableau of shape $((2c)^{n-p},(2c-2)^p)$ such that columns $2c-1$, $2c$, columns $2c-3$, $2c-2$, etc., form $(2n+1)$-orthogonal admissible pairs. An example with $n=6$, $c=4$, $p=3$ is displayed in Fig. 8.

We claim that $(2n+1)$-orthogonal tableaux S of shape $(c^{n-p},(c-1)^p)$ are in bijection with pairs $(\overline{T},\{e_1,e_2,\dots,e_p\})$, by a bijection Θ_1 say, where $\overline{T}$ is an n-tableau contained in $((2c)^{n-p},(2c-2)^p)$, and where $\{e_1,e_2,\dots,e_p\}$ is a set of numbers satisfying

$$\begin{gathered} 1\le e_1<e_2<\cdots<e_p\le n,\\ e_l\notin [i(m),j(m)] \qquad \text{for } 1\le l\le p,\quad 1\le m\le s, \\ \text{and}\quad e_l\notin [i(s+1),n] \qquad \text{for } 1\le l\le p, \end{gathered} \tag{5.14}$$

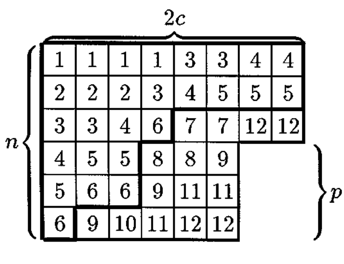

FIGURE 8

given that

$$\begin{matrix} i(1) & j(1) \\ \vdots & \vdots \\ i(s) & j(s) \\ i(s+1) & \\ \vdots & \\ i(t) & \end{matrix}$$

are the $(2c-1)$th and $(2c)$th columns of $\overline{T}$, such that

$$(\mathbf{x}^{\pm 1})^{S} = (x_1 x_2 \cdots x_n)^{-c} \cdot x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}},$$
$$\text{if } \left(\overline{T}, \{e_1, e_2, \dots, e_p\}\right) = \Theta_1(S). \quad (5.15)$$

The construction of the bijection Θ_1 is based on an analysis of the orthogonal tableaux under consideration. Clearly, the entries $\leq n$ in S form an n-tableau, $\overline{T}$ say, whose shape is contained in $((2c)^{n-p}, (2c-2)^{p})$. Note that in contrast to the symplectic case, here $\overline{T}$ is not necessarily a tableau with only even rows. In Fig. 8 we have marked the area that is covered by entries $\leq n$ by a bold line. The resulting tableau is displayed in the left half of Fig. 9.

Now, let the $(2c-1)$th and $(2c)$th columns of S be

$$n \geq \left\{ \begin{matrix} i(1) & j(1) \\ \vdots & \vdots \\ \vdots & j(s) \\ i(t) & j(s+1) \\ i(t+1) & \vdots \\ \end{matrix} \right. \quad \begin{matrix} \leq n \\ \\ > n \end{matrix} \qquad (5.16)$$

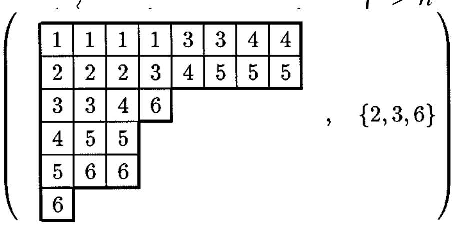

FIGURE 9

As in the symplectic case, we define $\{e_1, e_2, \dots, e_p\}$ to be the set of numbers e between 1 and n with the property that neither e nor its "conjugate" $2n + 1 - e$ occurs in the $(2c - 1)$th or $(2c)$th columns of S. Again, without loss of generality we may assume $e_1 < e_2 < \cdots < e_p$. In our running example (recall $p = 3$) we have $\{e_1, e_2, e_3\} = \{2, 3, 6\}$. From the definition of a $(2n + 1)$-orthogonal admissible pair (see Definition 2 in the Appendix) it follows that the numbers $e_1, e_2, \dots, e_p$ satisfy (5.14). Note that (5.14) differs from the "symplectic analogue" (5.4) by the additional condition in the third line.

We define $\Theta_1(S)$ to be $(\overline{T}, \{e_1, e_2, \dots, e_p\})$. Our running example in Fig. 8 is mapped under Θ_1 to the pair in Fig. 9.

As in the symplectic case, it can be shown that S can be uniquely recovered from $(\overline{T}, \{e_1, e_2, \dots, e_p\})$. Furthermore, it is easy to check that the weight property (5.15) holds under this correspondence.

Second step. In the second step we construct a bijection Θ_2 between pairs $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ satisfying (5.14) as before, and pairs (T, σ), where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a vertical strip of length p on the rim of ν that avoids the $(2c)$th column, such that

$$\mathbf{x}^T = x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}}. \tag{5.17}$$

To obtain T from such a pair $(\overline{T}, \{e_1, e_2, \dots, e_p\})$ we use the mapping Φ_2 from the second step of the proof of (3.6). On the other hand, σ is defined to be the vertical strip by which the shapes of T and $\overline{T}$ differ. Thus, our running example from Fig. 9 is mapped under Θ_2 to the pair in Fig. 10. There, the vertical strip σ is visualized by bold lines embedded in the shape of T.

By definition of Θ_2 (recall that an element is never inserted into the $(2c)$th column of some $\overline{T}_i$), σ is a vertical strip on the rim of the shape of T that avoids the $(2c)$th column. It is obvious that (5.17) holds.

The composition $\Theta_2 \circ \Theta_1$ is by definition the desired bijection between $(2n + 1)$-orthogonal tableaux S of shape $(c^{n-p}, (c - 1)^p)$ and pairs (T, σ),

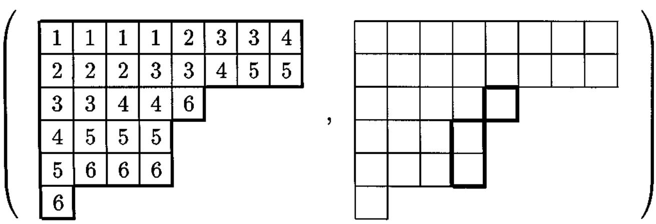

FIGURE 10

where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a vertical strip of length p on the rim of ν that avoids the $(2c)$th column. From (5.15) and (5.17) the weight property (5.13) follows immediately. This completes the proof of (3.8). ∎

Proof of (3.10). Here we proceed very similarly to the preceding proof of (3.8).

Using (A.5) and (A.2) in (3.10), we see that (3.10) will be proved if we can find a bijection, Ω say, between $(2n+1)$-orthogonal tableaux S of shape $(c^{n-1}, c-p)$ and pairs (T, σ), where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a horizontal strip of length p on the rim of ν such that the ith cell of the strip comes before the $(2c-2p+2i)$th column, such that

$$(\mathbf{x}^{\pm 1})^S = (x_1 x_2 \cdots x_n)^{-c} \cdot \mathbf{x}^T, \qquad \text{if } (T, \sigma) = \Omega(S) \text{ for some } \sigma. \quad (5.18)$$

Again, we proceed in two steps.

First step. Let S be a $(2n+1)$-orthogonal tableau of shape $(c^{n-1}, c-p)$. By the definition of $(2n+1)$-orthogonal tableaux in Section A4 of the Appendix, S is a $(2n)$-tableau of shape $((2c)^{n-1}, 2c-2p)$ such that columns $2c-1, 2c$, columns $2c-3, 2c-2$, etc., form $(2n+1)$-orthogonal admissible pairs. An example with $n=6$, $c=7/2$, $p=3$ is displayed in Fig. 11.

We claim that $(2n+1)$-orthogonal tableaux S of shape $(c^{n-1}, c-p)$ are in bijection with pairs $(\overline{T}, (e_1, e_2, \dots, e_p))$, by a bijection Ω_1 say, where $\overline{T}$ is an n-tableau contained in $((2c)^{n-1}, 2c-2p)$, and where $(e_1, e_2, \dots, e_p)$ is a vector of numbers (we definitely mean *vector* here, i.e., the order of the numbers is important) satisfying

$$\begin{aligned} 1 \le e_1, e_2, \dots, e_p \le n,& \\ e_l \notin [i_l(m), j_l(m)] \qquad &\text{for } 1 \le l \le p, \quad 1 \le m \le s_l, \qquad (5.19) \\ e_l \notin [i_l(s_l+1), n] \qquad &\text{for } 1 \le l \le p, \end{aligned}$$

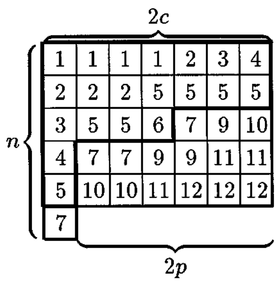

FIGURE 11

and for all m,

$$\left|\{1,2,\dots,m\}\setminus\{e_l,j_l(1),\dots,j_l(s_l)\}\right|$$
$$\le\left|\{1,2,\dots,m\}\setminus\{e_{l+1},i_{l+1}(1),\dots,i_{l+1}(t_{l+1})\}\right|, \qquad (5.20)$$

given that

$$\begin{matrix} i_l(1) & j_l(1)\\ \vdots & \vdots\\ i_l(s_l) & j_l(s_l)\\ i_l(s_l+1) & \\ \vdots & \\ i_l(t_l) & \end{matrix}$$

are the $(2c-2p+2l-1)$th and $(2c-2p+2l)$th columns of $\overline{T}$, such that

$$(\mathbf{x}^{\pm 1})^S=(x_1x_2\cdots x_n)^{-c}\cdot x_{e_1}\cdots x_{e_p}\cdot \mathbf{x}^{\overline{T}},$$
$$\text{if } \left(\overline{T},\{e_1,e_2,\dots,e_p\}\right)=\Omega_1(S). \qquad (5.21)$$

Again, the construction of the bijection Ω_1 is based on an analysis of the orthogonal tableaux under consideration. Clearly, the entries $\le n$ in S form an n-tableau, $\overline{T}$ say, whose shape is contained in $((2c)^{n-1},2c-2p)$. In Fig. 11 we have marked the area that is covered by entries $\le n$ by a bold line. The resulting tableau is displayed in the left half of Fig. 12.

If we apply the paragraph containing (5.16) with $p=1$, then we see that, with e_l being the number that together with its conjugate $2n+1-e_l$ does not appear in the $(2c-2p+2l-1)$th or $(2c-2p+2l)$th columns of S, $l=1,2,\dots,p$, the map Ω_1 defined by

$$S\to\left(\overline{T},(e_1,e_2,\dots,e_p)\right)$$

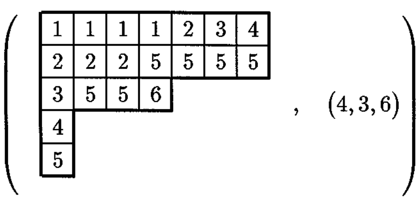

FIGURE 12

defines the desired bijection. In particular, the fact that the entries $>n$ in S also form a (skew) tableau is reflected by condition (5.20). For, the entries $>n$ from the $(2c-2p+2l)$th column of S are

$$\{n+1, n+2, \dots, 2n\} \setminus \{2n+1-e_l, 2n+1-j_l(1), \dots, 2n+1-j_l(s_l)\},$$

and the entries $>n$ from the $(2c-2p+2l+1)$th column of S are

$$\{n+1, n+2, \dots, 2n\} \setminus \{2n+1-e_{l+1}, 2n+1-i_{l+1}(1), \dots, 2n+1-i_{l+1}(t_{l+1})\}.$$

That all entries $>n$ from the $(2c-2p+2l)$th column of S are less than or equal to their right neighbors from the $(2c-2p+2l+1)$th column of S, is exactly equivalent to requiring

$$\begin{aligned}&\big|\{2n, 2n-1, \dots, 2n-m\} \setminus \\ &\{2n+1-e_l, 2n+1-j_l(1), \dots, 2n+1-j_l(s_l)\}\big| \\ &\quad \le \big|\{2n, 2n-1, \dots, 2n-m\} \\ &\qquad \setminus \{2n+1-e_{l+1}, 2n+1-i_{l+1}(1), \dots, 2n+1-i_{l+1}(t_{l+1})\}\big|\end{aligned}$$

for all m, which is clearly equivalent to (5.20).

Again, it is easy to check that the weight property (5.21) holds under this correspondence. Our running example in Fig. 11 is mapped under Ω_1 to the pair in Fig. 12.

Second step. In the second step we construct a bijection Ω_2 between pairs $(\overline{T}, (e_1, e_2, \dots, e_p))$ satisfying (5.19) as before, and pairs (T, σ), where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a horizontal strip of length p on the rim of ν such that the ith cell of the strip comes before the $(2c-2p+2i)$th column, such that

$$\mathbf{x}^T = x_{e_1} \cdots x_{e_p} \cdot \mathbf{x}^{\overline{T}}. \tag{5.22}$$

Let $(\overline{T}, (e_1, e_2, \dots, e_p))$ be such a pair. We insert $e_1, e_2, \dots, e_p$, in this order, into $\overline{T}$, according to a procedure that is very similar to Φ_2, or Θ_2, which were used in the proofs of (3.6), respectively, (3.8). That we have to modify these procedures is due to the fact that e_l satisfies a condition, namely (5.19), that depends on the $(2c-2p+2l-1)$th and $(2c-2p+2l)$th columns of $\overline{T}$, and not just on the $(2c-1)$th and $(2c)$th columns, as was the case in (5.4) or (5.14). Let $\overline{T}_0 := \overline{T}$. Suppose that, by inserting $e_1, e_2, \dots, e_l$ we have already formed $\overline{T}_l$. Next we insert e_{l+1} into $\overline{T}_l$ in the following way. Choose the first row (from top to bottom) of $\overline{T}$ such that e_{l+1} is less than the entry in the $(2c-2p+2l-1)$th column in that row of $\overline{T}_l$. If there is no such row, choose the first row that does not have an entry in the $(2c-2p+2l-1)$th column of that row. Then, starting with that row of $\overline{T}$, ROW-INSERT e_{l+1} into $\overline{T}$; see the definition of Φ_2 in the second step of the proof of (3.6). Thus one obtains the tableau $\overline{T}_{l+1}$. Finally, set $T = \Omega_2((\overline{T},(e_1,e_2,\dots,e_p))) := \overline{T}_p$. On the other hand, σ is defined to be the vertical strip (it is indeed a vertical strip, as will be shown in a moment) by which the shapes of T and $\overline{T}$ differ. Thus, our running example from Fig. 12 is mapped under Ω_2 to the pair in Fig. 13. There, the vertical strip σ is visualized by bold lines embedded in the shape of T.

It is obvious that (5.22) holds under this correspondence. Moreover, it is immediate from the definition of σ as a result of the above insertion procedure that the ith cell of σ comes before the $(2c-2p+2i)$th column. However, it is not so immediate that σ is indeed a vertical strip. This will be established next. We will be done if we are able to show that later "insertion paths" are strictly to the right of previous ones. It suffices to consider two successive insertion paths.

Let the columns $2c-2p+2l-1$, $2c-2p+2l$, $2c-2p+2l+1$, $2c-2p+2l+2$ be given by the four columns in Fig. 14 (ignore for the moment "$e_l \to$ " and "$e_{l+1} \to$ ").

Suppose that the insertion of e_l would start in row u, as is symbolized by $e_l \to i_l(u)$ in Fig. 14, and that the subsequent insertion of e_{l+1} would start in row v, as symbolized by $e_l \to i_{l+1}(v)$ in Fig. 14. This would mean that we have

$$j_l(u-1) < e_l < i_l(u) \tag{5.23}$$

and

$$j_{l+1}(v-1) < e_{l+1} < i_{l+1}(v). \tag{5.24}$$

During the insertion of e_{l+1}, first the element e_{l+1} bumps $i_{l+1}(v)$ (which is an element of the $(2c-2p+2l+1)$th column of $\overline{T}$) or an element to the left of $i_{l+1}(v)$ in the same row. Let us further suppose that until row

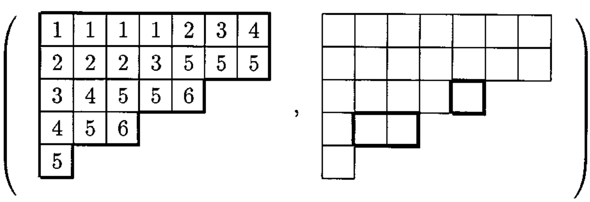

FIGURE 13

$$\begin{array}{cccccc}
 & i_l(1) & j_l(1) & & i_{l+1}(1) & j_{l+1}(1) \\
 & \vdots & \vdots & & \vdots & \vdots \\
 & \vdots & \vdots & e_{l+1} \to & i_{l+1}(v) & j_{l+1}(v) \\
 & \vdots & \vdots & & \vdots & \vdots \\
e_l \to & i_l(u) & j_l(u) & & i_{l+1}(s_{l+1}) & j_{l+1}(s_{l+1}) \\
 & \vdots & \vdots & & \vdots & \\
 & \vdots & \vdots & & i_{l+1}(t_{l+1}) & \\
 & i_l(s_l) & j_l(s_l) & & & \\
 & i_l(s_l+1) & & & & \\
 & \vdots & & & & \\
 & i_l(t_l) & & & &
\end{array}$$

FIGURE 14

$w-1$, $w \geq v$, elements of the $(2c-2p+2l+1)$th column are bumped, i.e., $i_{l+1}(v)$ bumps $i_{l+1}(v+1), \dots$; finally, $i_{l+1}(w-2)$ bumps $i_{l+1}(w-1)$. This would mean

$$j_l(v+1) \leq i_{l+1}(v), \dots, j_l(w-1) \leq i_{l+1}(w-2). \tag{5.25}$$

We suppose that then in row w the insertion path jumps to the left of the $(2c-2p+2l+1)$th column (note that the case $w=v$ covers the case that e_{l+1} bumps an element to the left of $i_{l+1}(v)$), which means that

$$i_{l+1}(w-1) < j_l(w), \qquad \text{or } e_{l+1} < j_l(v) \text{ in case } w=v. \tag{5.26}$$

We do not care what happens afterward.

We claim that $u \leq w$ and that $e_l \leq i_{l+1}(w-1)$, respectively, $e_l \leq e_{l+1}$ in case $w=v$. This would imply that in the uth row the insertion path caused by e_{l+1} is strictly to the right of the insertion path caused by e_l and therefore has to stay strictly to the right from there on, by an elementary property of ROW-INSERT. And this is what we want to show.

To prove the claim we consider (5.20) with $m = i_{l+1}(w-1)$, respectively, $m = e_{l+1}$ in case $w=v$. For this choice of m, the right-hand side of (5.20) equals $i_{l+1}(w-1) - w$, since we have

$$i_{l+1}(1) < i_{l+1}(2) < \cdots < i_{l+1}(w-1),$$

and by (5.24),

$$e_{l+1} < i_{l+1}(v) < \cdots < i_{l+1}(w-1),$$

respectively, equals $e_{l+1}-v$ in case $w=v$, since by (5.24) we have

$$i_{l+1}(1) < \cdots < i_{l+1}(v-1) \le j_{l+1}(v-1) < e_{l+1} < i_{l+1}(v).$$

Hence, the left-hand side of (5.20) is bounded above by $i_{l+1}(w-1)-w$, respectively, $e_{l+1}-v$ in case $w=v$.

On the other hand, the left-hand side of (5.20) is at least $i_{l+1}(w-1)-w$, respectively, $e_{l+1}-v$ in case $w=v$, and equal to the lower bound only if $e_l \le i_{l+1}(w-1)$, respectively, $e_l \le e_{l+1}$ in case $w=v$. For, by (5.25) and (5.26) we have

$$j_l(1) < \cdots < j_l(w-1) \le i_{l+1}(w-2) < i_{l+1}(w-1) < j_l(w),$$

and in case $w=v$ we have, by (5.24) and (5.26),

$$j_l(1) < \cdots < j_l(v-1) \le j_{l+1}(v-1) < e_{l+1} < j_l(v).$$

So the lower bound can only be reached if $e_l \le i_{l+1}(w-1)$, respectively, $e_l \le e_{l+1}$ in case $w=v$, which is what we wanted to show.

Summarizing, we have shown that indeed $e_l \le i_{l+1}(w-1)$, respectively, $e_l \le e_{l+1}$ in case $w=v$. Combining this with (5.23) and (5.26), we obtain the inequality chain

$$j_l(u-1) < e_l \le i_{l+1}(w-1) \text{ (respectively, } e_{l+1}) < j_l(w);$$

hence $j_l(u-1) < j_l(w)$. Since columns are strictly increasing, this immediately implies $u \le w$, as desired.

To show that Ψ_2 is a bijection, we have to construct the inverse mapping. Experienced with three other similar proofs, this is rather straightforward. Let (T,σ) be a pair, where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a horizontal strip of length p on the rim of ν such that the ith cell of the strip comes before the $(2c-2p+2i)$th column. For $l=1,2,\dots,p$ start a ROW-DELETE with the entry of T that is located in the cell corresponding to the lth cell of σ (counted from the right), but stop before an entry in the $(2c-2l+2)$th column would be bumped. This procedure reverses the algorithm Ω_2, step by step.

The composition $\Omega_2 \circ \Omega_1$ is by definition the desired bijection between $(2n+1)$-orthogonal tableaux S of shape $(c^{n-1}, c-p)$ and pairs (T,σ), where T is an n-tableau whose shape is contained in $((2c)^n)$, and where σ is a horizontal strip of length p on the rim of ν such that the ith cell of the strip comes before the $(2c-2i+2)$th column. From (5.21) and (5.22) the weight property (5.18) follows immediately. This completes the proof of (3.10). ∎

6. PROOF OF THEOREM 3

In this section we use Littelmann's extension [24] of the Littlewood–Richardson rule to symplectic and special orthogonal characters. This extension is described in Section A6 of the Appendix. We remark that while there are other rules for the decomposition of the product of two symplectic or orthogonal characters involving ordinary Littlewood–Richardson coefficients (the *Newell–Littlewood rules*; see [13, Theorem 4.1; 15, Cor. 2.5.3/Prop. 2.5.2; 39, Theorem 5.3]), these do not appear to be very helpful for our purposes. Again, this is because they involve modification rules for characters, these cause alternating signs, and these in turn cause a lot of cancellations, and all of this is simply not tractable for the applications that we have in mind.

Before we move on to the proofs of (3.13)–(3.20), in Proposition 1 we supply decomposition formulas for the product of a rectangularly shaped symplectic, respectively, special orthogonal, character and an *arbitrarily* shaped character of the same type. All of these expansions involve slightly modified Littlewood–Richardson coefficients, which even reduce to ordinary Littlewood–Richardson coefficients in a number of cases; see Remark 4. But there are no alternating signs here, and hence there is no cancellation. The formulas (3.13)–(3.20) then follow rather easily from (6.1)–(6.5).

PROPOSITION 1. *For any nonnegative integer c and any partition λ with at most n parts, there holds*

$$sp_{2n}((c^n);\mathbf{x}^{\pm 1})\cdot sp_{2n}(\lambda;\mathbf{x}^{\pm 1}) = \sum_{\nu\subseteq\lambda+(c^n)} sp_{2n}(\nu;\mathbf{x}^{\pm 1}) \sum_{\substack{\mu\subseteq((2c)^n)\\ \mu \text{ even}}} \overline{LR}_{\lambda,\mu}^{\nu+(c^n)}(c) \tag{6.1}$$

("μ *even*" *means that all of the rows of* μ *are even*), *where* $\overline{LR}_{\lambda,\mu}^{\nu+(c^n)}(c)$ *is the number of LR-fillings F of shape* $(\nu+(c^n))/\lambda$ *with content* μ *and with the additional property that*

> *If there is an entry e in the nth row of F, in column j say* (*see Section A1 in the Appendix for how columns are counted*), *then F must contain at least* $2c-2j+1$ *other entries e to the right of column* j. (6.2)

Next, for any nonnegative integer or half-integer c and any partition or half-partition λ *with at most n parts, there holds*

$$\begin{aligned} &so_{2n+1}((c^n);\mathbf{x}^{\pm 1})\cdot so_{2n+1}(\lambda;\mathbf{x}^{\pm 1})\\ &\quad= \sum_{\nu\subseteq\lambda+(c^n)} so_{2n+1}(\nu;\mathbf{x}^{\pm 1}) \sum_{\mu\subseteq((2c)^n)} \overline{LR}_{\lambda,\mu}^{\nu+(c^n)}(c), \end{aligned} \tag{6.3}$$

with the understanding that ν *ranges over partitions if* $\lambda + (c^n)$ *is a partition and over half-partitions if* $\lambda + (c^n)$ *is a half-partition, and where* $\overline{LR}_{\lambda,\mu}^{\nu+(c^n)}(c)$ *is defined as before*. (*Again, see Section A1 in the Appendix for how columns are counted*; *in particular, in condition* (6.2) *the column index j ranges over the integers if* λ *is a partition and over half-integers if* λ *is a half-partition.*)

Finally, for any nonnegative integer or half-integer c and any $(2n)$*-orthogonal partition or half-partition* λ *with at most n parts, there holds*

$$so_{2n}\big((c^n);\mathbf{x}^{\pm1}\big)\cdot so_{2n}(\lambda;\mathbf{x}^{\pm1}) = \sum_{\nu\subseteq\lambda+(c^n)} so_{2n}(\nu;\mathbf{x}^{\pm1})\left(\sum_{\substack{\mu\subseteq((2c)^n)\\ \operatorname{oddcols}(((2c)^n)/\mu)=0}} \widetilde{LR}_{\lambda,\mu}^{\nu+(c^n)}(c)\right), \quad (6.4)$$

where $\widetilde{LR}_{\lambda,\mu}^{\nu+(c^n)}(c)$ *is the number of LR-fillings F of shape* $(\nu+(c^n))/\lambda$ *with content* μ *and with the additional property that for* $l=1,2,\dots,2c$ *there holds*

> *If the subfilling that arises from F by deleting the rightmost* $2c-l$ *entries 1, the rightmost* $2c-l$ *entries* $2,\dots,$ *the rightmost* $2c-l$ *entries n* (*if there are fewer than* $2c-l$ *entries of some size, then delete all of these*) *has shape* $\nu(l)/\lambda$, *then* $\nu(l)_{n-1}+\nu(l)_n\ge l$. (6.5)

Again, the sum in (6.4) *is understood to range over n-orthogonal partitions if* $\lambda+(c^n)$ *is an n-orthogonal partition and over n-orthogonal half-partitions if* $\lambda+(c^n)$ *is an n-orthogonal half-partition.*

Remark 4. It should be noted that in case $\lambda_n\ge c$ the condition (6.2) is void, so that the coefficients $\overline{\mathrm{LR}}_{\lambda,\mu}^{\nu+(c^n)}(c)$ that appear in (6.1) and (6.3) reduce to the *ordinary* Littlewood–Richardson coefficients $\mathrm{LR}_{\lambda,\mu}^{\nu+(c^n)}$. Likewise, if $\lambda_{n-1}+\lambda_n\ge 2c$ the condition (6.5) is void, so that the coefficients $\widetilde{\mathrm{LR}}_{\lambda,\mu}^{\nu+(c^n)}(c)$ that appear in (6.4) reduce to the *ordinary* Littlewood–Richardson coefficients $\mathrm{LR}_{\lambda,\mu}^{\nu+(c^n)}$.

Proof of Proposition 1. For convenience, we start with the proof of (6.3).

Proof of (6.3). By (A.10) with $\chi_n(\cdot)=so_{2n+1}(\,\cdot\,;\mathbf{x}^{\pm1})$ we know that

$$so_{2n+1}\big((c^n);\mathbf{x}^{\pm1}\big)\cdot so_{2n+1}(\lambda;\mathbf{x}^{\pm1}) = \sum_T so_{2n+1}\big(\lambda+\operatorname{con}(T);\mathbf{x}^{\pm1}\big), \quad (6.6)$$

where the sum is over all $(2n+1)$-orthogonal tableaux T of shape (c^n) such that for all $l=1,2,\dots,2c$ the vector $\nu(l):=\lambda+\operatorname{con}(T(l))$ is in the Weyl chamber (A.8) of type B, i.e., it satisfies

$$\nu(l)_1\ge\nu(l)_2\ge\cdots\ge\nu(l)_n\ge 0. \quad (6.7)$$

The content $\operatorname{con}(T)$ of T is defined after (A.3).

By comparing (6.6) with (6.3) we see that (6.3) will be proved once we construct a bijection, Υ say, between $(2n+1)$-orthogonal tableaux T of shape (c^n) and content ρ that satisfy (6.7) for $l = 1, 2, \dots, 2c$ and LR-fillings F of shape $(\lambda + \rho + (c^n))/\lambda$ that satisfy property (6.2).

The bijection Υ is defined as follows. Let λ be fixed and let T be a $(2n+1)$-orthogonal tableau of shape (c^n) with content ρ. By Observation 2 in Section A4 of the Appendix, $(2n+1)$-orthogonal tableaux of shape (c^n) are nothing but $(2n)$-tableaux of shape $((2c)^n)$, where each column contains one of e or $2n+1-e$ for all $e = 1, 2, \dots, n$. An example with $n = 5$ and $c = 5/2$ is displayed in the left half of Fig. 15. It satisfies the required property that $\lambda + \mathrm{con}(T(l))$ is in the Weyl chamber of type B, $l = 1, 2, \dots, 2c$ for $\lambda = (4, 4, 3, 1, 1)$, as is easily checked.

To obtain the image of T under Υ, we construct a sequence $F_0, F_1, \dots, F_{2c}$ of fillings by reading T columnwise, from right to left. The desired filling F will then be defined to be the last filling, F_{2c}. Define F_0 to be the only filling of the shape λ/λ (which is, of course, the empty filling). Suppose that we already constructed F_l. To obtain F_{l+1}, we add for $i = 1, 2, \dots, n$ an entry e to row i of F_l if i is an entry in the $(2c-l)$th column and the eth row of T. As already announced, we define F to be F_{2c}. Thus, with $\lambda = (4, 4, 3, 1, 1)$, our tableau in the left half of Fig. 15 is mapped by Υ to the filling in the right half of Fig. 15.

It is straightforward from this construction that the mapping Υ can be reversed, step by step. So we shall be done if we show that, given that T is mapped to F by Υ, T is a $(2n+1)$-orthogonal tableau of shape (c^n) with content ρ if and only if F is a LR-filling of shape $(\lambda + \rho + (c^n))/\lambda$ satisfying (6.2). We provide the details only for the forward implication. Since the arguments for the backward implication are similar, the reader will have no difficulty filling in the respective details.

Let T be a $(2n+1)$-orthogonal tableau of shape (c^n) with content ρ that is mapped by Υ to the filling F. What we have to show is that F is an n-tableau, i.e., that entries are weakly increasing along rows and strictly increasing along columns, that the LR-condition holds, that the shape of F is $(\lambda + \mathrm{con}(T) + (c^n))/\lambda$, and that F satisfies (6.2). The reader is advised

1	1	1	4	6
2	3	4	6	7
3	4	5	8	8
5	5	8	9	9
7	9	9	10	10

$\longleftrightarrow$

				1	1	1
				2		
			2	3		
	1	2	3			
	3	4	4			

FIGURE 15

to keep the example of Fig. 15 in mind. It will help to follow the subsequent arguments.

For the first statement, let e and f be entries in the ith row of F, e being the left neighbor of f. Then, by construction of Υ, e was caused by some entry i in the eth row and c_eth column, say, of T, while f was caused by some entry i in the fth row and c_fth column, say, of T. Since f is to the right of e, f was added "later"; hence $c_f < c_e$. Now, since T is a tableau, the entry i in column c_f cannot be higher than the entry i in column c_e. Therefore we have $e \le f$. This holds for any left-right neighbors in F, so rows are weakly increasing, as desired. To see that columns of F are strictly increasing, we consider entries e and f in the same column of F, e being the top neighbor of f. Let e be located in the ith row of F (and, thus, f be located in the $(i+1)$th row of F). Then, by construction of Υ, e was caused by some entry i in the eth row and c_eth column, say, of T, while f was caused by some entry $(i+1)$ in the fth row and c_fth column, say, of T. It is easily seen by induction that for all l the shape of the partial filling F_l is given by $(\lambda + \mathrm{con}(T(l)) + ((l/2)^n))/\lambda$. In particular, because of (6.7), this implies that the "outer shape" of F_l is always "well behaved" in the sense that lower rows always terminate earlier than higher rows. Therefore the entry f of F was added "later" than the entry e, which means that the column c_f of T must be weakly to the left of column c_e. We already know that there is an entry i in column c_e and an entry $(i+1)$ in column c_f. Since column c_f is located weakly to the left of column c_e, the entry $(i+1)$ must be in a lower row than the entry i. As the entry $(i+1)$ is located in the fth row and the entry i is located in the eth row, this means nothing more than $e < f$. This holds for any top-bottom neighbors in F, so columns are strictly increasing, as desired.

Now we turn to the LR-condition. We have to show that, while reading the entries of F row-wise from top to bottom, and in each row from right to left, at any stage we have

$$\text{number of 1's} \ge \text{number of 2's} \ge \text{number of 3's} \ge \cdots .$$

Now, by construction of Υ, each entry in F corresponds to some entry in T. Thus, to the above described reading of the entries of F there corresponds the following reading of the entries of T: First read the entries 1 in T, from left to right, then the entries 2, from left to right, then the entries 3, etc. The LR-condition is equivalent to saying that at any stage during this reading of T the number of entries read from the first row is greater than or equal to the number of entries read from the second row, which in turn is greater than or equal to the number of entries read from the third row, etc. But this is obviously true because T is a tableau.

With regard to the shape, we have already noticed that for all l the shape of F_l is given by $(\lambda + \mathrm{con}(T(l)) + ((l/2)^n))/\lambda$. Hence $F = F_{2c}$ has shape $(\lambda + \mathrm{con}(T) + (c^n))/\lambda$.

Finally, we want to show that F satisfies (6.2). Let e be a fixed entry in the nth row and jth column of F. Write again $\nu(l) = \lambda + \mathrm{con}(T(l))$. By assumption, $\nu(l)$ lies in the Weyl chamber of type B; see (6.7). In particular, we have $\nu(l)_n \geq 0$. Now we have already observed that

$$\big(\lambda + \mathrm{con}(T(l)) + ((l/2)^n)\big)/\lambda = \big(\nu(l) + ((l/2)^n)\big)/\lambda$$

is the shape of F_l. Hence, the condition $\nu(l)_n \geq 0$ is equivalent to saying that F_l contains an entry in the nth row that is located in the $(l/2)$th (respectively, $(l+1)/2$th) column (depending on whether λ is a partition or half-partition). (In passing, we note that the $l = 2c$ case of this fact implies that $F = F_{2c}$ contains an entry in the cth (respectively, $(c + 1/2)$th) column. Hence the shape of F can indeed by written in the form $(\nu + (c^n))/\lambda$, where ν is a partition or half-partition.) On the other hand, F_l is the subfilling of F that arises by deleting the rightmost $2c - l$ entries 1, the rightmost $2c - l$ entries $2, \dots, 2c - l$ *entries* e, etc., from F. Now, suppose that the fixed entry e in the nth row and jth column is also contained in F_l, i.e., $j \leq (l+1)/2$. Then there must be at least $2c - l$ entries e in F to the right of the jth column. This last property holds for all l with $j \leq (l+1)/2$. Hence there must be at least $2c - (2j+1)$ entries e in F to the right of the jth column. This is exactly what we wanted to show.

Thus, the proof of (6.3) is complete.

Proof of (6.1). By (A.10) with $\chi_n(\cdot) = sp_{2n}(\,\cdot\,; \mathbf{x}^{\pm 1})$ we know that

$$sp_{2n}\big((c^n); \mathbf{x}^{\pm 1}\big) \cdot sp_{2n}(\lambda; \mathbf{x}^{\pm 1}) = \sum_T sp_{2n}\big(\lambda + \mathrm{con}(T); \mathbf{x}^{\pm 1}\big), \quad (6.8)$$

where the sum is over all $(2n)$-symplectic tableaux T of shape (c^n) such that for all $l = 1, 2, \dots, 2c$ the vector $\nu(l) := \lambda + \mathrm{con}(T(l))$ is in the Weyl chamber (A.8) of type C, i.e., it satisfies (6.7). (Recall that the Weyl chambers of types B and C are the same.) By comparing (6.8) with (6.1) we see that (6.1) will be proved once we construct a bijection between $(2n)$-symplectic tableaux T of shape (c^n) with content ρ and LR-fillings F of shape $(\lambda + \rho + (c^n))/\lambda$ with even content that satisfy property (6.2).

As bijection we can take the mapping Υ from the preceding proof of (6.3). We only have to observe (see Observation 3 in Section A3 of the Appendix) that $(2n)$-symplectic tableaux T of shape (c^n) are the same as $(2n+1)$-orthogonal tableaux of shape (c^n), with the additional property that the entries $\leq n$ form a subtableau with only even rows. Suppose that

T is mapped by Υ to F. Then the length of the ith row of this subtableau of T is the same as the number of occurrences of i in the filling F. In other words, the shape of the subtableau equals the content of the corresponding filling F. Since the shape of the subtableau is even, the filling must have even content, as desired. The final observation is that since (6.7) holds here, too, the filling must again satisfy (6.2).

Proof of (6.4). By (A.10) with $\chi_n(\cdot) = so_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$ we know that

$$so_{2n}\big((c^n);\mathbf{x}^{\pm 1}\big)\cdot so_{2n}(\lambda;\mathbf{x}^{\pm 1}) = \sum_{T} so_{2n}\big(\lambda + \operatorname{con}(T);\mathbf{x}^{\pm 1}\big), \quad (6.9)$$

where the sum is over all $(2n)$-orthogonal tableaux T of shape (c^n) such that for all $l = 1, 2, \dots, 2c$ the vector $\nu(l) := \lambda + \operatorname{con}(T(l))$ is in the Weyl chamber (A.9) of type D. By comparing (6.9) with (6.4) we see that (6.4) will be proved once we construct a bijection between $(2n)$-orthogonal tableaux T of shape (c^n) with content ρ and LR-fillings F of shape $(\lambda + \rho + (c^n))/\lambda$ and content μ, where all of the columns of μ have the same parity as n and where property (6.5) is satisfied.

Again, we can take the mapping Υ from the proof of (6.3) as the bijection. Here, this is because of the observation (see Observation 1 in Section A5 of the Appendix, with $\lambda_{n-1} = \lambda_n = c$) that $(2n)$-orthogonal tableaux T of shape (c^n) are the same as $(2n+1)$-orthogonal tableaux of shape (c^n), with the additional property that the entries $\le n$ form a subtableau whose shape has only columns of the same parity as n. Again, since the shape of the subtableau equals the content of the corresponding filling, the content μ of the filling must have columns of the same parity as n throughout, as desired. This time we have to impose (6.5) (instead of (6.2)), since $\nu(l) = \lambda + \operatorname{con}(T(6))$ has to be in the Weyl chamber of type D (and not of type B or C).

This completes the proof of Proposition 1. ▮

Now we are in a position to prove (3.13)–(3.20).

Proof of (3.13). We apply (6.1) with $\lambda = ((d+1)^p, d^{n-p})$. Because of the assumption $c \le d$, Remark 4 applies, which says that the coefficients $\overline{\mathrm{LR}}_{\lambda,\mu}^{\nu+(c^n)}(c)$ reduce to ordinary Littlewood–Richardson coefficients $\mathrm{LR}_{\lambda,\mu}^{\nu+(c^n)}$. Then, obviously, (3.13) is equivalent to the claim

$$\sum_{\substack{\mu\subseteq((2c)^n)\\ \mu\ \text{even}}} \mathrm{LR}^{\nu+(c^n)}_{((d+1)^p,\,d^{n-p}),\,\mu} = \begin{cases} 1 & \big((d-c)^n\big)\subseteq\nu\subseteq\big((c+d+1)^n\big)\\ & \text{and } \operatorname{oddrows}\big(\nu/\big((d-c)^n\big)\big)=p\\ 0 & \text{otherwise.}\end{cases} \quad (6.10)$$

Let F be a LR-filling of shape $(\nu + (c^n))/((d+1)^p, d^{n-p})$ with even content. Because of the LR-condition, almost all of the entries of F are uniquely determined. To be precise, except for the entries in column $d+1$, all of the entries in the ith row have to equal i throughout, $i = 1, 2, \dots, n$, as exemplified in Fig. 16.

However, the entries in the $(d+1)$th column of F are also uniquely determined because the content of F should be even. Namely, i occurs in the $(d+1)$th column if and only if the number of the other entries i, which is $\nu_i + c - (d+1)$, is odd. In our example in Fig. 16, the entries in the $(d+1)$th column would have to be $1, 2, 5$. Now, suppose that F contains exactly q entries in the $(d+1)$th column, $q \le n-p$, of course. Equivalently, $\nu = (\nu_1, \dots, \nu_{p+q}, d-c, \dots, d-c)$. Then there are exactly q quantities $\nu_i + c - (d+1)$, $i \le p+q$, that are odd, plus $n-p-q$ quantities $\nu_i + c - (d+1) = -1$ for $i = p+q+1, p+q+2, \dots, n$. Therefore, the number of odd rows in $\nu/((d+1-c)^n)$ is exactly $q + (n-p-q) = n-p$, or equivalently, the number of odd rows in $\nu/((d-c)^n)$ is exactly p.

Summarizing, we have shown that the left-hand side in (6.10) is different from zero only if $((d-c)^n) \subseteq \nu \subseteq ((d+c+1)^n)$ and $\operatorname{oddrows}(\nu/((d-c)^n)) = p$, the inclusions being trivial constraints. In addition, we have also seen that there is exactly one LR-filling under those conditions. Thus, (6.10) and thus also (3.13) are established. ▮

Proof of (3.14). Here we apply (6.1) with $\lambda = (d^{n-p}, (d-1)^p)$. Now the assumption is $c \ge d$, so Remark 4 does not apply. We have to show

$$\sum_{\substack{\mu \subseteq ((2c)^n) \\ \mu \text{ even}}} \overline{\mathrm{LR}}^{\nu + (c^n)}_{(d^{n-p}, (d-1)^p), \mu}(c) = \begin{cases} 1 & ((c-d)^n) \subseteq \nu \subseteq ((c+d)^n) \\ & \text{and } \operatorname{oddrows}\big(\nu/((c-d)^n)\big) = p \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

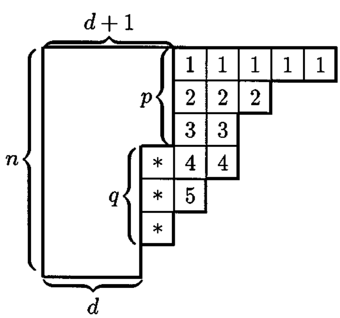

FIGURE 16

From (6.10) it follows directly that the left-hand side in (6.11) can only be nonzero if $((d-c-1)^n) \subseteq \nu \subseteq ((c+d)^n)$ and if $\operatorname{oddrows}(\nu/((d-1-c)^n)) = n-p$. Note that the latter condition is equivalent to $\operatorname{oddrows}(\nu/((c-d)^n)) = p$ (if this makes sense, i.e., if $((c-d)^n) \subseteq \nu$). It also follows from (6.10) that if the left-hand side of (6.11) is nonzero, then it can only be 1. So what remains to be seen is that it is nonzero if and only if in addition $((c-d)^n) \subseteq \nu$, or equivalently, $c-d \le \nu_n$.

We begin with the forward implication. Suppose that there is a LR-filling F of shape $(\nu+(c^n))/(d^{n-p},(d-1)^p)$ with even content satisfying (6.2). If $c=d$ there is nothing to show. So let $c>d$. Then, by considering the shape of F, we see that there must be an entry in the nth row and $(d+1)$th column of F. (If there is no entry in the $(d+1)$th column of F, then $\nu_n + c \le d$. So, $\nu_n \le d-c < 0$, which is impossible, since ν has to be a partition.) Clearly, this entry equals n, since it is located in the last row of a column of length n. Now, condition (6.2) applied to this entry says that there must be $2c-2(d+1)+1 = 2c-2d-1$ more entries n to the right of this column. All of them necessarily have to be in the nth row of F; hence $\nu_n + c \ge (d+1)+(2c-2d-1)$, or equivalently, $\nu_n \ge c-d$, as desired.

For the backward implication, assume $\nu_n \ge c-d$. We have to establish the existence of a LR-filling of shape $(\nu+(c^n))/(d^{n-p},(d-1)^p)$ with even content satisfying (6.2). This LR-filling can only be the uniquely determined filling that was described in the proof of (3.13). In fact, the arguments of the first paragraph of this proof and those in the proof of (3.13) actually show that this uniquely determined LR-filling satisfies all of the required properties, except for, possibly, (6.2) for the entry in the dth column.

We now verify (6.2) for this entry, e say, by distinguishing between two cases. First let $e \ne n$. Condition (6.2) would require that there are $2c-2d+1$ more entries e to the right. All of these entries necessarily have to be located in the eth row. Hence, condition (6.2) would require $\nu_e + c - d \ge 2c-2d+1$. Now, because of $\nu_n \ge c-d$, we have $\nu_e + c - d \ge \nu_n + c - d \ge 2c-2d$. So the total number of e's, which is $\nu_e + c - d + 1$, is at least $2c-2d+1$. Since the content of the filling is even, this number must be even. So it is actually at least $2c-2d+2$. Hence, $\nu_e + c - d + 1 \ge 2c-2d+2$, or equivalently, $\nu_e + c - d \ge 2c-2d+1$, as required.

Now let $e=n$. Condition (6.2) would require that there are $2c-2d+1$ more entries n to the right. Now, the total number of n's (all of them are located in the nth row) equals $\nu_n + c - d + 1$. Because of $\nu_n \ge c-d$ this number is at least $2c-2d+1$. Again, since the content of the filling is even, this number must be even. So it is actually at least $2c-2d+2$. Hence, there are at least $2c-2d+1$ more n's to the right of the n in the dth column, as required.

This completes the proof of (3.14). ∎

Proof of (3.15). We apply (6.4) with $\lambda = (d^{n-1}, d-p)$. Thus, we have to show that

$$\sum_{\substack{\mu \subseteq ((2c)^n) \\ \operatorname{oddcols}(((2c)^n)/\mu)=0}} \widetilde{\mathrm{LR}}^{\nu+(c^n)}_{(d^{n-1},\, d-p),\ \mu}(c)$$

$$= \begin{cases} 1 & \left(|c-d|^{n-1}, c-d\right) \subseteq \nu \subseteq \left((c+d)^n\right) \\ & \text{and } \operatorname{oddcols}\left(\left((c+d)^n\right)/\nu\right) = p \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Let F be a LR-filling of shape $(\nu + (c^n))/(d^{n-1}, d-p)$ with content μ, where $\operatorname{oddcols}(((2c)^n)/\mu) = 0$. For later use we note right here that the left-hand side of (6.12) can only be nonzero if

$$\nu \subseteq \left((c+d)^n\right). \tag{6.13}$$

Similarly here, because of the LR-condition, the entries in the first $n-1$ rows are uniquely determined. To be precise, all of the entries in the ith row have to equal i, $i = 1, 2, \dots, n-1$, as is exemplified in Fig. 17.

However, the entries in the nth row of F are also uniquely determined because the content μ of F should satisfy $\operatorname{oddcols}(((2c)^n)/\mu) = 0$. For convenience, let $\tilde{F}$ denote the subfilling of F consisting of the first $n-1$ rows of F. Namely, there have to be as many entries i in the nth row of F as there are columns in $\tilde{F}$ of length $i-1$ with parity different from n, $i = 1, 2, \dots, n$. (In passing, we note that therefore all of the entries in the nth row have the same parity as n.) In our examples in Fig. 17, the entries in the nth row would have to be $2, 2, 4, 6$ and $2, 2, 4, 6, 6, 6$, respectively. Now, suppose that F contains exactly q entries in the nth row. Then there are exactly q columns in $\tilde{F}$ with parity different from n. If $q \le p$, then the number of columns of F whose parity is different from n equals the aforementioned q columns plus the $p-q$ empty columns $d-p+q+1, d-p+q+2, \dots, d$; see the left filling in Fig. 17. If $q \ge p$, then the number of columns of F whose parity is different from n equals the

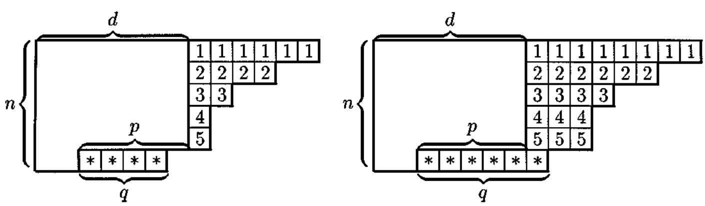

FIGURE 17

aforementioned q columns minus the $q-p$ columns $d+1, d+2, \dots, d+q-p$ of length n; see the right filling in Fig. 17. Hence, in both cases the number of columns of F whose parity is different from n equals p, or equivalently, $\operatorname{oddcols}(((c+d)^n)/\nu) = p$. (Recall that by (6.13) the shape ν is indeed contained in $((c+d)^n)$.)

Summarizing, we have shown that the left-hand side of (6.12) is different from zero only if $\nu \subseteq ((c+d)^n)$ and $\operatorname{oddcols}(((c+d)^n)/\nu) = p$. We have also shown that if the left-hand side is nonzero, it can only be 1 since there is at most one LR-filling. So, what remains is to show that it is nonzero if and only if in addition $(|c-d|^{n-1}, c-d) \subseteq \nu$.

Again, we begin with the forward implication. Suppose that there is a LR-filling F of shape $(\nu + (c^n))/(d^{n-1}, d-p)$ with $\operatorname{oddcols}(((2c)^n)/\mu) = 0$, where μ is the content of F, satisfying (6.5). Because of $(d^{n-1}, d-p) \subseteq \nu + (c^n)$, we have $((d-c)^{n-1}, d-c-p) \subseteq \nu$. Hence, we will be done if we can show $c - d \le \nu_n$.

We have already noted that the first $n-1$ rows of F are uniquely determined; in particular, the $(n-1)$th row of F contains only entries $n-1$. Now we apply condition (6.5) with $l = c + d - \nu_{n-1}$. (Note that in (6.5) this choice of l has the effect of removing $2c - l = \nu_{n-1} + c - d$ entries of each size. In particular, all of the entries from the $(n-1)$th row are removed.) Since the content μ of F satisfies $\operatorname{oddcols}(((2c)^n)/\mu) = 0$, the number of n's must equal the number of $(n-1)$'s in F. Therefore the number of n's, all of which have to be in the nth row, is at least as large as the number of $(n-1)$'s in the $(n-1)$th row of F, the latter being $\nu_{n-1} + c - d$. Therefore condition (6.5) with $l = c + d - \nu_{n-1}$ reads

$$\big(c + \nu_{n-1} - (\nu_{n-1} + c - d)\big) + \big(c + \nu_n - (\nu_{n-1} + c - d)\big) \\ \ge c + d - \nu_{n-1}.$$

Simplifying, we obtain $\nu_n \ge c - d$, which is what we wanted.

For the backward implication, assume $\nu_n \ge c - d$. Recall that there is a uniquely determined LR-filling of shape $(\nu + (c^n))/(d^{n-1}, d-p)$ with content μ, where $\operatorname{oddcols}(((2c)^n)/\mu) = 0$. What has to be shown is that it also satisfies (6.5). A careful reading of the previous paragraph reveals that $\nu_n \ge c - d$ is actually *equivalent* to condition (6.5) with $l = c + d - \nu_{n-1}$. Now, a moment's thought will convince the reader that in our particular situation (the $(n-1)$th row consists of $(n-1)$'s throughout; $l = c + d - \nu_{n-1}$ in (6.5) therefore empties the complete $(n-1)$th row), condition (6.5) holds for *all* l if and only if it holds for the *particular choice* $l = c + d - \nu_{n-1}$. Thus, (6.5) is established.

This completes the proof of (3.15). ∎

Proof of (3.16). Equation (3.16) is an easy corollary of Eq. (3.15) with p replaced by $2d - p$. This is due to the observation that the substitution $x_n \to 1/x_n$ in a character $so_{2n}(\lambda; \mathbf{x}^{\pm 1})$ has the effect

$$so_{2n}\big((\lambda_1, \dots, \lambda_{n-1}, \lambda_n); \mathbf{x}^{\pm 1}\big)\big|_{x_n \to 1/x_n} = so_{2n}\big((\lambda_1, \dots, \lambda_{n-1}, -\lambda_n); \mathbf{x}^{\pm 1}\big),$$

which follows easily from the definition (2.12). ∎

Proof of (3.17). We apply (6.3) with $\lambda = (d^{n-p}, (d-1)^p)$. Note that the only difference between (6.3) and (6.1) is that in (6.1) the partition μ is required to be even. Hence, we can use those arguments from the proofs of (3.13) (here one has to replace d by $d-1$ and p by $n-p$) and (3.14) that do not rely on this requirement.

Again, we have to consider LR-fillings F of shape $(\nu + (c^n))/(d^{n-p}, (d-1)^p)$ that satisfy (6.2), but with *arbitrary* content. So, again all entries to the right of the dth column are uniquely determined, and in the same way, see Fig. 16. But the entries in the dth column are *not* unique now. In fact, there is almost complete freedom; the constraints are that the entries along the columns have to be strictly increasing, i.e., each entry of a particular size can occur only once, that the total content μ of F has to be a partition of course, and that (6.2) must be satisfied. It is easy to see that the first and second constraints are equivalent to $(\mu + (d^n))/(\nu + (c^n))$ being a vertical strip of length $p - m_{d-c-1}(\nu)$ avoiding the dth column and, because of $\mu \subseteq ((2c)^n)$, the $(d + 2c + 1)$th column. Note that the latter "avoidance" conditions give the first and third "avoidance" condition in (3.18) when columns are counted with respect to ν, i.e., when everything is shifted back by c.

The inclusion $(|c-d|^{n-p}, (\max\{d-c-1, c-d\})^p) \subseteq \nu$ follows from the trivial inclusion $((d-c)^{n-p}, (d-c-1)^p) \subseteq \nu$, and from the fact that $c - d \le \nu_n$ if $c \ge d$, which is shown in the same way as in the proof of (3.14).

Finally, we claim that if $c \ge d$ the vertical strip $(\mu + (d^n))/(\nu + (c^n))$ has to avoid the $(2c - d + 1)$th column. Note that this "avoidance" condition gives the second "avoidance" condition in (3.18) when columns are counted with respect to ν, i.e., when everything is shifted back by c. The former "avoidance" condition comes from considering condition (6.2) for the entry in the nth row and dth column of the filling F. See the analogous considerations at the end of the proof of (3.14). The difference here is that the content μ can be arbitrary. Hence the argument in the proof of (3.14) that the number of e's or n's is even does not apply here. We leave the details to the reader.

This finishes the proof of (3.17). ∎

Sketch of Proof of (3.19). As in the preceding proof of (3.17) we apply (6.3), this time with $\lambda = (d^{n-1}, d-p)$. The arguments are very similar here. We have to consider LR-fillings F of shape $(\nu + (c^n))/(d^{n-1}, d-p)$ with *arbitrary* content. As in the proof of (3.15), all of the entries in the first $n-1$ rows of such a filling are uniquely determined, and in the same way, see Fig. 17. But the entries in the nth row are not. They can be chosen arbitrarily as long as the LR-condition and (6.2) are satisfied. The LR-condition implies that, with μ denoting the content of the filling, $(\mu_1 + d, \dots, \mu_{n-1} + d)/(\nu_1 + c, \dots, \nu_{n-1} + c)$ is a *horizontal* strip, σ say. Of course, the length of σ, which equals the number of entries $\le n-1$ in the nth row of the LR-filling, is at most $\nu_n + c - d + p$, the length of the nth row of the filling. Note that this is one part of the first inequality in (3.20). At the same time, the length of σ is at most p. For there cannot be more than p entries $\le n-1$ in the nth row, since the entry in the $(d+1)$th column (which is a column of length n) and the nth row has to be n. Note that this proves the second part of the second inequality in (3.20). Moreover, the LR-condition applied to entries $n-1$ and n implies the first part of the second inequality in (3.20). Finally, condition (6.2) implies the third inequality in (3.20) and, when applied to the first entry n in the nth row, that the length of σ is at least $c - d + p - \nu_n$, the latter being the missing part of the first inequality in (3.20). We leave it to the reader to check this in detail.

The inclusion $(|c-d|^{n-1}, \max\{c-d, d-c-p\}) \subseteq \nu$ follows on the one hand from the trivial inclusion $((d-c)^{n-1}, d-c-p) \subseteq \nu$, and on the other hand from $c - d \le \nu_n$, which is derived from the first and second inequalities in (3.20) as follows: $\nu_n \ge -|\sigma| + c - d + p \ge c - d$. ∎

7. APPLICATIONS TO PLANE PARTITIONS AND TABLEAUX ENUMERATION

We now apply results from Section 3 to derive some enumeration results for plane partitions of trapezoidal shape and for tableaux. Recall [28] that the *trapezoidal shape* $(N, N-2, \dots, N-2r+2)$ is an array of cells with r rows, each row indented by one cell to the right with respect to the previous row, and $N - 2i + 2$ cells in row i. A plane partition of shape $(N, N-2, \dots, N-2r+2)$ is a filling of the trapezoidal shape $(N, N-2, \dots, N-2r+2)$ with *nonnegative* integers (note that we allow 0 as an entry) such that entries along rows and columns are weakly decreasing.

We begin with an application of (3.3). The second statement in the theorem below, (7.2), is a result of Proctor [28, Corollary on p. 554].

THEOREM 4. *The number of plane partitions of trapezoidal shape* $(N, N-2, \dots, N-2r+2)$ *with entries between* 0 *and* c *and where the entries on the main diagonal form a partition with exactly* p *columns of parity different from* r *(equivalently,* $\sum_{i=1}^{r}(-1)^{r-i+1}a_{ii} = p$ *if* r *is even, respectively* $\sum_{i=2}^{r}(-1)^{r-i+1}a_{ii} = p$ *if* r *is odd, with* a_{ii} *denoting the first entry in row* i*), equals*

$$\binom{c}{p}\frac{\binom{p+r-1}{p}}{\binom{N-r+c}{p}}\prod_{\substack{1\le i\le r\\ 1\le j\le c}}\frac{N-i+j}{i+j-1}. \tag{7.1}$$

In particular, the number of plane partitions of trapezoidal shape $(N, N-2, \dots, N-2r+2)$ *with entries between* 0 *and* c *equals*

$$\prod_{\substack{1\le i\le r\\ 1\le j\le c}}\frac{N+1-i+j}{i+j-1}. \tag{7.2}$$

Proof. We use (3.3) with $x_i = 1$, $i = 1, 2, \dots, \lfloor N/2 \rfloor$. For this choice of x_i's, the Schur functions reduce to a number (which is the dimension of the corresponding irreducible representation of $GL(N, \mathbb{C})$) that has a nice closed form (see [6, Ex. A.30.(ii); 26, I, Ex. 4 on p. 45; 39, Theorem 4.4]), and is therefore easily computed. Thus, the left-hand side of (3.3) turns into (7.1). On the right-hand side of (3.3), we have a certain sum of symplectic characters evaluated at $x_i = 1$, $i = 1, 2, \dots, \lfloor N/2 \rfloor$. Now, besides the $(2n)$-symplectic tableaux of DeConcini and Procesi in Section A3 of the Appendix, there are other symplectic tableaux. Namely, the (even) symplectic characters $sp_{2n}(\lambda; \mathbf{x}^{\pm 1})$ can also be described by King's [11] symplectic tableaux of shape λ (see also [33, Theorem 4.2; 39, Theorem 2.3]), and the (odd) symplectic characters $sp_{2n+1}(\lambda; \mathbf{x}^{\pm 1})$ can also be described by Proctor's [30] odd symplectic tableaux of shape λ (see also [33, Theorem 4.2 with $z = 1$]). There is a uniform definition. Let $N = 2n$ or $N = 2n + 1$. Then a King/Proctor symplectic tableau of shape λ is an N-tableau of shape λ such that the entries in the ith row are at least $2i - 1$ for all i. Thus, the right-hand side of (3.3) can be interpreted as the number of King/Proctor symplectic tableaux with entries $\le N$ and of some shape ν, where ν is contained in (c^r) and $\operatorname{oddcols}((c^r)/\nu) = p$. These tableaux are now translated into plane partitions of trapezoidal shape as described in [31, bottom of p. 295]. Namely, given such a King/Proctor symplectic tableau, replace each entry e by $2n + 1 - e$, then interpret each row of the resulting array as a partition and replace it by its conjugate partition. Next, shift the ith row by $(i - 1)$ cells to the right, $i = 1, 2, \dots,$ to obtain a plane partition of "shifted" shape that is contained in the trapezoidal shape $(N, N-2, \dots, N-2r+2)$. Finally, place a zero in each cell of the trapezoidal shape that is not yet filled. It is easy to see that during this transformation the lengths of the rows of a King/Proctor symplectic tableau become the entries on the main diagonal of the resulting plane partition of trapezoidal shape. This establishes the first assertion of Theorem 4.

The number in (7.2) is obtained by summing the numbers in (7.1) over all p, by means of the Vandermonde sum (cf. [35, (1.7.7)]). This completes the proof of Theorem 4. ∎

The next two theorems give applications of Theorem 2. For the proof of these theorems we need a few determinant evaluations that are listed in Lemma 2. We remark that the evaluations (7.3), (7.4), (7.5) are basically the Weyl denominator factorizations of types C, B, D, respectively (cf. [6, Lemma 24.3, Ex. A.52, Ex. A.62, Ex. A.66]).

LEMMA 2. *The following identities hold true*:

$$\det_{1\le i,j\le n}\left(x_i^j - x_i^{-j}\right) = (x_1\cdots x_n)^{-n}\prod_{1\le i<j\le n}\big((x_i-x_j)(1-x_ix_j)\big)\prod_{i=1}^n\left(x_i^2-1\right), \tag{7.3}$$

$$\det_{1\le i,j\le n}\left(x_i^{j-1/2} - x_i^{-(j-1/2)}\right) = (x_1\cdots x_n)^{-n+1/2}\prod_{1\le i<j\le n}\big((x_i-x_j)(1-x_ix_j)\big)\prod_{i=1}^n(x_i-1), \tag{7.4}$$

$$\det_{1\le i,j\le n}\left(x_i^{j-1} + x_i^{-(j-1)}\right) = 2\cdot(x_1\cdots x_n)^{-n+1}\prod_{1\le i<j\le n}\big((x_i-x_j)(1-x_ix_j)\big), \tag{7.5}$$

$$\det_{1\le i,j\le n}\left(x_i^{j} + x_i^{-j}\right) = (x_1\cdots x_n)^{-n}\prod_{1\le i<j\le n}\big((x_i-x_j)(1-x_ix_j)\big)\sum_{k=0}^n e_k(x_1,\dots,x_n)^2, \tag{7.6}$$

$$\det_{1\le i,j\le n}\left(x_i^{j-1/2}+x_i^{-(j-1/2)}\right)$$
$$=(x_1\cdots x_n)^{-n+1/2}\prod_{1\le i<j\le n}\left((x_i-x_j)(1-x_ix_j)\right)\prod_{i=1}^{n}(x_i+1). \tag{7.7}$$

Proof. Identities (7.3)–(7.5) and (7.7) are readily proved by the standard argument that proves Vandermonde-type determinant evaluations.

For (7.6) there is a little work to do. First, by reversing the order of columns, and adding some factors, we rewrite the determinant in (7.6) as

$$(-1)^{\binom{n}{2}}\frac{\det_{1\le i,j\le n}\left(x_i^{n+1-j}+x_i^{-(n+1-j)}\right)}{\det_{1\le i,j\le n}\left(x_i^{n-j}+x_i^{-(n-j)}\right)}\det_{1\le i,j\le n}\left(x_i^{n-j}+x_i^{-(n-j)}\right). \tag{7.8}$$

Next we observe that because of (2.12) the quotient of determinants is one-half of the orthogonal character $o_{2n}((1,1,\dots,1);\mathbf{x}^{\pm 1})$ (recall that $o_{2n}(\lambda;\mathbf{x}^{\pm 1})$ is the sum of $so_{2n}(\lambda;\mathbf{x}^{\pm 1})$ and $so_{2n}(\lambda^-;\mathbf{x}^{\pm 1})$ if $\lambda_n\neq 0$). In addition, we reverse the order of columns in the single determinant in (7.8) to obtain for (7.8),

$$\tfrac{1}{2}o_{2n}\left((1,1,\dots,1);\mathbf{x}^{\pm 1}\right)\det_{1\le i,j\le n}\left(x_i^{j-1}+x_i^{-(j-1)}\right). \tag{7.9}$$

By the "orthogonal Jacobi–Trudi identity" [6, Cor. 24.45; 15, Theorem 2.3.3, (6)], the orthogonal character in (7.9) is nothing but the elementary symmetric function $e_n(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$. Moreover, the determinant in (7.9) can be evaluated by (7.5). Thus, the expression (7.9) becomes

$$e_n\left(x_1,x_1^{-1},\dots,x_n,x_n^{-1}\right)\cdot(x_1\cdots x_n)^{-n+1}\prod_{1\le i<j\le n}\left((x_i-x_j)(1-x_ix_j)\right). \tag{7.10}$$

The elementary symmetric function in (7.10) can be transformed as follows:

$$e_n\left(x_1,x_1^{-1},\dots,x_n,x_n^{-1}\right)=\sum_{k=0}^{n}e_k(x_1,\dots,x_n)\cdot e_{n-k}\left(x_1^{-1},\dots,x_n^{-1}\right)$$
$$=(x_1\cdots x_n)^{-1}\sum_{k=0}^{n}e_k(x_1,\dots,x_n)^2.$$

Plugging this into (7.10) completes the proof of (7.6). ∎

As a first application of Theorem 2 we give new proofs of two theorems of the author [19, Theorems 21 and 11] by means of (3.6). These are refinements of the Bender–Knuth and MacMahon (ex-)conjectures; see [19, Sections 3.3, 4.3] for more information and references. To be able to formulate the Theorem, we have to introduce a few q-notations. We write $[\alpha]_q := 1 - q^\alpha$, $[n]_q! := [1]_q[2]_q \cdots [n]_q$, $[0]_q! := 1$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q!} & k \geq 0 \\ 0 & k < 0. \end{cases}$$

The base q in $[\alpha]_q$, $[n]_q!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ will in most cases be omitted. Only if the base is different from q will it be stated explicitly.

THEOREM 5. *Let $n(T)$ denote the sum of all of the entries of a tableau T. The generating function $\sum q^{n(T)}$ for tableaux T with p odd rows, with at most c columns, and with entries between 1 and n, is given by*

$$q^{\binom{p+1}{2}} \frac{[2r]}{[2r+p]} \begin{bmatrix} n \\ p \end{bmatrix} \frac{\begin{bmatrix} n+2r \\ n \end{bmatrix}}{\begin{bmatrix} n+2r+p \\ n \end{bmatrix}} \prod_{1 \leq i \leq j \leq n} \frac{[2r+i+j]}{[i+j]} \qquad \text{if } c = 2r \tag{7.11}$$

and

$$q^{\binom{p+1}{2}} \begin{bmatrix} n \\ p \end{bmatrix} \prod_{1 \leq i \leq j \leq n} \frac{[2r+i+j]}{[i+j]} \qquad \text{if } c = 2r+1. \tag{7.12}$$

The generating function $\sum q^{n(T)}$ for tableaux T with p odd rows, with at most c columns, and with only odd entries that lie between 1 and $2n-1$, is given by

$$q^{p^2} \frac{[2r+2p]_{q^2}[r]_{q^2}}{[2r+p]_{q^2}[r+p]_{q^2}} \begin{bmatrix} n \\ p \end{bmatrix}_{q^2} \frac{\begin{bmatrix} n+2r \\ n \end{bmatrix}_{q^2}}{\begin{bmatrix} n+2r+p \\ n \end{bmatrix}_{q^2}} \times \prod_{i=1}^{n} \frac{[r+i]_{q^2}}{[i]_{q^2}} \prod_{1 \leq i < j \leq n} \frac{[2r+i+j]_{q^2}}{[i+j]_{q^2}} \qquad \text{if } c = 2r \tag{7.13}$$

and

$$q^{p^2}\begin{bmatrix} n \\ p \end{bmatrix}_{q^2} \prod_{i=1}^{n} \frac{[r+i]_{q^2}}{[i]_{q^2}} \prod_{1\le i<j\le n} \frac{[2r+i+j]_{q^2}}{[i+j]_{q^2}} \qquad \text{if } c = 2r+1. \quad (7.14)$$

Proof. To prove (7.11), replace x_i by q^i, $i = 1, 2, \dots, n$, in (3.6). By (2.13), the left-hand side of (3.6) can be written as a quotient of two determinants, which for this choice of x_i's factor by means of (7.3). On the other hand, by (A.2) the right-hand side of (3.6) with $x_i = q^i$, $i = 1, 2, \dots, n$, clearly equals the generating function for the tableaux in the first assertion of Theorem 5.

The expression (7.12) is obtained without much effort from the $p = 0$ case of (7.11) as described in [19, Proof of (4.3.2b)].

The proof of (7.13) proceeds in the same way as the proof of (7.11). To point out the differences, instead of substituting q^i now substitute q^{2i-1} for x_i, $i = 1, 2, \dots, n$, and use (7.4) instead of (7.3). The expression (7.14) is obtained without much effort from the $p = 0$ case of (7.13) as described in [19, Proof of (3.3.3b)]. ∎

Our second application of Theorem 2 leads to different refinements of the Bender–Knuth and MacMahon (ex-)conjectures that are new. What we do is use (3.7) to obtain a "column-analogue" of Theorem 5. Now, the tableaux generating functions in question do not factor completely in terms of cyclotomic polynomials, but it is still possible to write the results in a reasonably compact form.

THEOREM 6. *Let $n(T)$ denote the sum of all of the entries of a tableau T. The generating function $\sum q^{n(T)}$ for tableaux T with at most c columns, p of which are odd, and with entries between* 1 *and n, is given by*

$$q^p \frac{\begin{bmatrix} p+n-1 \\ p \end{bmatrix}\begin{bmatrix} c-p+n-1 \\ c-p \end{bmatrix}}{2\begin{bmatrix} c+n-1 \\ c \end{bmatrix}} \prod_{1\le i<j\le n} \frac{[c+i+j-2]}{[i+j]}$$

$$\times\left(\left(\sum_{k=0}^{n} q^{k(k+c-1)} \left(q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{-p} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) \right)^2 \right.$$

$$\left. + (1 - q^{c-2p}) \prod_{i=1}^{n-1} (1 - q^{c+2i}) \right). \qquad (7.15)$$

The generating function $\sum q^{n(T)}$ for tableaux T with at most c columns, p of which are odd, and with only odd entries that lie between 1 *and* $2n-1$, *is* *given by*

$$q^p \frac{\begin{bmatrix} p+n-1 \\ p \end{bmatrix}_{q^2} \begin{bmatrix} c-p+n-1 \\ c-p \end{bmatrix}_{q^2}}{\begin{bmatrix} c+n-1 \\ c \end{bmatrix}_{q^2}} \times \prod_{i=1}^{n} \frac{1}{1+q^{2(i-1)}} \prod_{1\le i<j\le n} \frac{[c+i+j-2]_{q^2}}{[i+j-2]_{q^2}} \times \left((1+q^{c-2p}) \prod_{i=1}^{n-1} (1+q^{c+2i}) + (1-q^{c-2p}) \prod_{i=1}^{n-1} (1-q^{c+2i}) \right). \tag{7.16}$$

Proof. Since it is simpler, we start with the proof of (7.16). Replace c by $c/2$ and substitute q^{2i-1} for x_i in (3.7), $i = 1, 2, \dots, n$. By (2.12), the left-hand side of (3.7) can be written as a quotient of a sum of two determinants by another determinant. The determinants, with this choice of x_i's, factor by means of (7.4) and (7.7), respectively. After simplification we arrive at the expression

$$q^p \frac{\begin{bmatrix} p+n-1 \\ p \end{bmatrix}_{q^2} \begin{bmatrix} c-p+n-1 \\ c-p \end{bmatrix}_{q^2}}{\begin{bmatrix} c+n-1 \\ c \end{bmatrix}_{q^2}} \times \prod_{i=1}^{n} \frac{1}{1+q^{2(i-1)}} \prod_{1\le i<j\le n} \frac{[c+i+j-2]_{q^2}}{[i+j-2]_{q^2}} \times \left((1+q^{c-2p}) \prod_{i=1}^{n-1} (1+q^{c+2i}) + (-1)^n (1-q^{c-2p}) \prod_{i=1}^{n-1} (1-q^{c+2i}) \right). \tag{7.17}$$

Note that the only difference from (7.16) is the term $(-1)^n$ in the last line of (7.17).

On the other hand, by (A.2) the right-hand side of (3.7) with c replaced by $c/2$ and with $x_i = q^{2i-1}$, $i = 1, 2, \dots, n$, equals the generating function for tableaux with at most c columns, p of which have parity different from n, and with only odd entries that lie between 1 and $2n - 1$. Now we distinguish between n even or odd. If n is even, then the right-hand side of

(3.7) with these replacements equals the generating function for the tableaux in the second assertion of Theorem 6, and the expressions (7.17) and (7.16) agree. If n is odd, then the right-hand side of (3.7), with the above replacements and with p replaced by $c-p$, equals the generating function for the tableaux in the second assertion of Theorem 6, and, as a few manipulations show, the expressions (7.17), with p replaced by $c-p$, and (7.16) agree.

For the proof of (7.15) we proceed in the same manner, but things are more complicated here. Now we replace c by $c/2$ and substitute q^i for x_i, $i=1,2,\dots,n$, in (3.7). Again, by (2.12), the left-hand side of (3.7) can be written as a quotient of a sum of two determinants by another determinant. The determinant in the denominator, with this choice of x_i's, can be evaluated by means of (7.5). The second determinant in the numerator, with this choice of x_i's, can be evaluated by means of (7.3). The first determinant in the numerator is evaluated by means of (7.6). After simplification we arrive at the expression

$$q^p \frac{\begin{bmatrix} p+n-1 \\ p \end{bmatrix}\begin{bmatrix} c-p+n-1 \\ c-p \end{bmatrix}}{2\begin{bmatrix} c+n-1 \\ c \end{bmatrix}} \prod_{1\le i<j\le n} \frac{[c+i+j-2]}{[i+j]}$$
$$\times\left(\sum_{k=0}^{n} q^{k(k+c-1)}\left(q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}+q^{-p}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right)^2\right.$$
$$\left.+(-1)^n(1-q^{c-2p})\prod_{i=1}^{n-1}(1-q^{c+2i})\right). \qquad (7.18)$$

Again, note that the only difference from (7.15) is the term $(-1)^n$ in the last line of (7.18).

On the other hand, by (A.2) the right-hand side of (3.7) with c replaced by $c/2$ and with $x_i=q^i$, $i=1,2,\dots,n$, equals the generating function for tableaux with at most c columns, p of which have parity different from n, and with entries between 1 and n. Again, we distinguish between n even or odd. If n is even, then the right-hand side of (3.7) with these replacements equals the generating function for the tableaux in the first assertion of Theorem 6, and the expressions (7.18) and (7.15) agree. If n is odd, then the right-hand side of (3.7), with the above replacements and with p replaced by $c-p$, equals the generating function for the tableaux in the first assertion of Theorem 6, and, as a few manipulations show, the expressions (7.18), with p replaced by $c-p$, and (7.15) agree. In particular, in the last step we use the fact that

$$\sum_{k=0}^{n} q^{k(k+c-1)}\left(q^k\begin{bmatrix} n-1\\ k\end{bmatrix}+q^{-c+p}\begin{bmatrix} n-1\\ k-1\end{bmatrix}\right)^2$$
$$=q^{-c+2p}\sum_{k=0}^{n} q^{k(k+c-1)}\left(q^k\begin{bmatrix} n-1\\ k\end{bmatrix}+q^{-p}\begin{bmatrix} n-1\\ k-1\end{bmatrix}\right)^2,$$

which is verified by expanding the squares.

This finishes the proof of Theorem 6. ∎

Since the results in Theorem 6 are completely combinatorial in nature, it would of course be desirable to find a combinatorial proof of Theorem 6. The papers [19, 20] (see also [18]) contain combinatorial proofs for the assertions in Theorem 5, in particular avoiding representation theory. But these are already quite difficult. The fact that the expressions in (7.15) and (7.16) are more complicated than those in (7.11)–(7.14) supports the suspicion that it will be even much harder to find combinatorial proofs for the assertions of Theorem 6.

We conclude by providing an application of Theorem 3, (3.13), (3.14).

THEOREM 7. *The number of plane partitions of trapezoidal shape* $(2n, 2n-2, \dots, 2)$ *with entries between* 0 *and* N, *where the entries on the main diagonal are at least* M *and exactly* p *of them have parity different from* N, *is*

$$\binom{n}{p}\frac{\binom{N-M+n}{n+1}}{\binom{N-M+n+p}{n+1}}\prod_{1\le i\le j\le n}\frac{(N+M+i+j)(N-M+i+j)}{(i+j)^2}$$
$$\text{if } N+M \text{ is even} \qquad (7.19)$$

and

$$\binom{n}{p}\frac{\binom{N+M+n+1}{n+1}}{\binom{N+M+1+n+p}{n+1}}$$
$$\times\prod_{1\le i\le j\le n}\frac{(N+M+1+i+j)(N-M-1+i+j)}{(i+j)^2}$$
$$\text{if } N+M \text{ is odd}. \qquad (7.20)$$

In particular, the number of plane partitions of trapezoidal shape $(2n, 2n-2, \dots, 2)$ *with entries between* 0 *and* N, *where the entries on the main diagonal are at least* M, *equals*

$$2^n \prod_{1 \le i \le j \le n} \frac{(N+M+i+j)(N-M-1+i+j)}{(i+j)^2}. \qquad (7.21)$$

Proof. We set $c = (N+M)/2$, $d = (N-M)/2$, $x_i = 1$, $i = 1, 2, \dots, n$, in (3.14), and we set $c = (N-M-1)/2$, $d = (N+M-1)/2$, $x_i = 1$, $i = 1, 2, \dots, n$, and replace p by $n-p$ in (3.13). For this choice of x_i's, a symplectic character reduces to a number (which is the dimension of the corresponding irreducible representation of $\mathrm{Sp}(2n, \mathbb{C})$) that has a nice closed form (see [6, Ex. 24.20; 39, Theorem 4.5.(1)]) and is therefore easily computed. Thus, the left-hand sides of (3.14) and (3.13) turn into (7.19) and (7.20), respectively. On the right-hand sides of (3.14) and (3.13), we have certain sums of symplectic characters evaluated at $x_i = 1$, $i = 1, 2, \dots, n$. In the same way as in the proof of Theorem 4, these sums can be interpreted as the cardinalities of certain sets of plane partitions of trapezoidal shape. These are exactly the plane partitions in the first assertion of Theorem 7; they are counted by the specialized right-hand side of (3.14) if $N+M$ is even, and by the specialized right-hand side of (3.13) if $N+M$ is odd. This proves the first assertion of Theorem 7.

To obtain (7.21), we sum the respective expressions in (7.19) and (7.20) by means of Kummer's very well-poised ${}_2F_1[-1]$ summation (cf. [35, (2.3.2.9), Appendix (III.5)]). Thus, Theorem 7 is proved. ∎

It would be easy to generalize the above theorem to trace generating functions of the type appearing in [31, Theorem 1; 17, Sect. 5] by replacing x_i by q^i or q^{2i-1} in (3.13) and (3.14). We omit the details here.

APPENDIX

A1. *Partitions and their Diagrams*

In Section 2 we already defined a *partition* to be a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of integers such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0$. A partition can be viewed geometrically, in terms of its *Ferrers diagram*. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ is an array of cells with r left-justified rows and λ_i cells in row i. Figure A.a shows the Ferrers diagram corresponding to $(4, 3, 3, 1)$. We identify partitions with their Ferrers diagram. For example, if we say "the first row of the partition λ," then we mean "the first row of the Ferrers diagram of the partition λ."

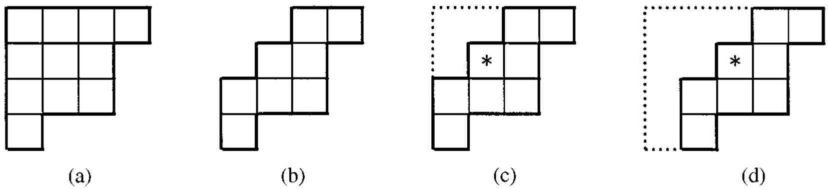

FIG. A. a. Ferrers diagram b. Skew Ferrers diagrams c. $(4, 3, 3, 1)/(2, 1)$. d. $(5, 4, 4, 2)/(3, 2, 1, 1)$.

The *conjugate* of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$, where λ'_j is the length of the jth column in the Ferrers diagram of λ.

Given two partitions, half-partitions, or orthogonal (half-)partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, we write $\lambda + \mu$ for the vector sum $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots)$. We write $\mu \subseteq \lambda$ if $\mu_i \le \lambda_i$ for all i. Given two partitions λ, μ with $\mu \subseteq \lambda$, the *skew Ferrers diagram* λ/μ consists of all cells that are contained in (the Ferrers diagram of) λ but not in (the Ferrers diagram of) μ. Figure A.b shows the skew Ferrers diagram $(4, 3, 3, 1)/(2, 1)$. Of course, it also shows the skew Ferrers diagram $(5, 4, 4, 2)/(3, 2, 1, 1)$. So, the notation for skew Ferrers diagrams is not unique. We might even allow vectors containing negative coordinates for denoting skew Ferrers diagrams, e.g., $(3, 2, 2, 0)/(1, 0, -1, -1)$ for the same skew Ferrers diagram in Fig. A.b, or vectors with half-integer coordinates, e.g., $(9/2, 7/2, 7/2, 3/2)/(5/2, 3/2, 1/2, 1/2)$ for the same skew Ferrers diagram. We do all this freely in the text, without further notice. However, when we count columns of skew Ferrers diagrams λ/μ, we do distinguish between the different notations for the same skew Ferrers diagram. Each column gets the number that it has as a column of λ. Thus, the first cell in the second row of the skew Ferrers diagram in Fig. A.b is located in the second column of $(4, 3, 3, 1)/(2, 1)$ (see Fig. A.c), in the third column of $(5, 4, 4, 2)/(3, 2, 1, 1)$ (see Fig. A.d), in the first column of $(3, 2, 2, 0)/(1, 0, -1, -1)$, in the $5/2$th column of $(9/2, 7/2, 7/2, 3/2)/(5/2, 3/2, 1/2, 1/2)$, etc. A *horizontal strip* is a skew Ferrers diagram with no more than one cell in each of its columns. A *vertical strip* is a skew Ferrers diagram with no more than one cell in each of its rows.

A2. *n-Tableaux* (*Ordinary Tableaux*)

Let λ, μ be partitions with $\mu \subseteq \lambda$. An *n-tableau of shape* λ (respectively, *of shape* λ/μ) is a filling of the cells of λ (respectively, λ/μ) with integers between 1 and n such that entries along rows are weakly increasing and entries along columns are strictly increasing. If we just say *tableau* instead of *n-tableau*, then we mean the same but without requiring that the entries are bounded above by n. Figure 4 shows a 6-tableau (7-tableau, ...) of shape $(8, 8, 5, 3, 2)$.

The *weight* $\mathbf{x}^T$ for an n-tableau T is defined by

$$\mathbf{x}^T := x_1^{\#(1\text{'s in } T)} x_2^{\#(2\text{'s in } T)} \cdots x_n^{\#(n\text{'s in } T)}. \tag{A.1}$$

The vector $(\#(1\text{'s in } T), \#(2\text{'s in } T), \dots, \#(n\text{'s in } T))$ of exponents in (A.1) is called the *content* of T and is denoted by $\operatorname{con}(T)$. For example, the weight and content of the tableau in Fig. 4 are $x_1^4 x_2^4 x_3^5 x_4^6 x_5^5 x_6^2$ and $(4, 4, 5, 6, 5, 2)$, respectively.

As is well known (see [26, (5.12) with $\mu = \mathbf{0}$; 34, Def. 4.4.1]), the irreducible general linear character (Schur function) $s_n(\lambda; \mathbf{x})$ equals the generating function for all n-tableaux of shape λ,

$$s_n(\lambda; \mathbf{x}) = \sum_{\substack{T \text{ an } n\text{-tableau} \\ \text{of shape } \lambda}} \mathbf{x}^T, \tag{A.2}$$

with $\mathbf{x}^T$ as defined in (A.1).

*A*3. $(2n)$-*Symplectic Tableaux*

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. A $(2n)$-*symplectic tableau of shape* λ is a $(2n)$-tableau of shape $2\lambda = (2\lambda_1, 2\lambda_2, \dots, 2\lambda_n)$ such that columns $1, 2$, columns $3, 4, \dots,$ columns $2\lambda_1 - 1, 2\lambda_1$ form $(2n)$-*symplectic admissible pairs*.

DEFINITION 1. A pair (C, D) of two columns of the length $k \le n$ is called a $(2n)$-*symplectic admissible pair* if the following conditions are satisfied:

(a) Entries in C and D are between 1 and $2n$ and in strictly increasing order.

(b) If e is in D then $2n + 1 - e$ is not in D. The same holds for C.

(c) C arises from D by a (possibly empty) sequence of operations O of the following type: The operation O to be described applies only to columns E and integers e_1, e_2 with $e_2 < e_1 \le n$, where $e_1 \in E$, $2n + 1 - e_2 \in E$, and for all t between e_2 and e_1 either t or $2n + 1 - t$ belongs to E. The operation O itself consists of forming the new column $O(E)$ out of E by replacing e_1 by e_2 and $2n + 1 - e_2$ by $2n + 1 - e_1$ and rearranging the new set of entries in strictly increasing order.

For example, Fig. 2 displays a 12-symplectic tableau of shape $(4, 4, 4, 4, 3, 3)$. There, the next-to-last column arises from the last by one operation, as described in item (c) of Definition 1, with $e_1 = 4$, $e_2 = 3$.

Remark 5. Our description of $(2n)$-symplectic admissible pairs follows Lakshmibai [21]. The description in [24, Appendix A.2] is equivalent. Note that Littelmann's *rows* are our *columns*.

There are a few observations that are immediate from Definition 1.

OBSERVATION 1. $C \le D$, meaning that if the entries of C are $c_1, c_2, \dots, c_k$ (from top to bottom) and those of D are $d_1, d_2, \dots, d_k$ (from top to bottom), then $c_i \le d_i$ for all i.

OBSERVATION 2. If (C, D) is a $(2n)$-symplectic admissible pair, then the number of entries $\le n$ in C is the same as that in D.

OBSERVATION 3. If C and D are columns of length n that satisfy (a), (b) in Definition 1, $C \le D$, and have the same number of entries $\le n$, then (C, D) is a $(2n)$-symplectic admissible pair, i.e., condition (c) is satisfied automatically in this situation. This is due to the fact that a column of length n must contain either t or $2n+1-t$, for all $t = 1, 2, \dots, n$. Hence, the most obvious way to transform D into C is a legal sequence of operations according to (c): namely, apply the operation O described in (c) first with e_1 the topmost entry of C and e_2 the topmost entry of D (unless $e_1 = e_2$), then apply O with e_1 the next-to-the top entry of C and e_2 the next-to-the-top entry of D, etc.

The *weight* $(\mathbf{x}^{\pm 1})^S$ for a $(2n)$-symplectic tableau S is defined by

$$(\mathbf{x}^{\pm 1})^S := x_1^{(1/2)(\#(1\text{'s in } S) - \#((2n)\text{'s in } S))} x_2^{(1/2)(\#(2\text{'s in } S) - \#((2n-1)\text{'s in } S))} \times \cdots x_n^{(1/2)(\#(n\text{'s in } S) - \#((n+1)\text{'s in } S))}. \tag{A.3}$$

Again, the vector

$$\tfrac{1}{2}\big(\#(1\text{'s in } S) - \#((2n)\text{'s in } S), \#(2\text{'s in } S) - \#((2n-1)\text{'s in } S), \dots, \#(n\text{'s in } S) - \#((n+1)\text{'s in } S)\big)$$

of exponents in (A.3) is called the *content* of S and is denoted by $\operatorname{con}(S)$. For example, the weight and content of the tableau in Fig. 2 are $x_3 x_4^2 x_5 x_6^{-2}$ and $(0, 0, 1, 2, 1, -2)$, respectively.

It is a theorem (see [3, 4, 23]) that the irreducible symplectic character $sp_{2n}(\lambda; \mathbf{x}^{\pm 1})$ equals the generating function for all $(2n)$-symplectic tableaux of shape λ,

$$sp_{2n}(\lambda; \mathbf{x}^{\pm 1}) = \sum_{\substack{S \text{ a } (2n)\text{-symplectic tableau} \\ \text{of shape } \lambda}} (\mathbf{x}^{\pm 1})^S, \tag{A.4}$$

with $(\mathbf{x}^{\pm 1})^S$ as defined in (A.3).

*A*4. $(2n+1)$-*Orthogonal Tableaux*

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition or half-partition. A $(2n+1)$-*orthogonal tableau of shape* λ is a $(2n)$-tableau of shape $2\lambda = (2\lambda_1, 2\lambda_2, \dots, 2\lambda_n)$ such that columns $2\lambda_1 - 1, 2\lambda_1$, columns $2\lambda_1 - 3, 2\lambda_1 - 2, \dots,$ form $(2n+1)$-*orthogonal admissible pairs*.

DEFINITION 2. A pair (C, D) of two columns of the length $k \le n$ is called a $(2n+1)$-*orthogonal admissible pair* if the following conditions are satisfied:

(a) Entries in C and D are between 1 and $2n$ and in strictly increasing order.

(b) If e is in D then $2n+1-e$ is not in D. The same holds for C.

(c) C arises from D by a (possibly empty) sequence of operations that can be either operations of the type that are described in Definition 1 (c) or operations O of the following type: The operation O to be described applies only to columns E and an entry e of E, $e > n$, where for all t between $n+1$ and e, $n+1$ included, either t or $2n+1-t$ belongs to E. The operation O itself consists of forming the new column $O(E)$ out of E by replacing e by $2n+1-e$ and rearranging the new set of entries in strictly increasing order.

For example, Fig. 11 displays a 13-orthogonal tableau of shape $(7/2, 7/2, 7/2, 7/2, 7/2, 1/2)$. There, the 4th column arises from the 5th by one operation as in item (c) of Definition 2, with $e = 7$, and by one operation as in Definition 1 (c), with $e_1 = 2$, $e_2 = 1$.

Remark 6. Again, our description of $(2n+1)$-orthogonal admissible pairs follows Lakshmibai [21]. The description in [24, Appendix, $Spin_{2m+1}$-standard Young tableaux] is equivalent. Again, note that Littelmann's *rows* are our *columns*.

We make similar observations here, also immediate from Definition 2.

OBSERVATION 1. $C \le D$, meaning that if the entries of C are $c_1, c_2, \dots, c_k$ (from top to bottom) and those of D are $d_1, d_2, \dots, d_k$ (from top to bottom), then $c_i \le d_i$ for all i.

OBSERVATION 2. If C and D are columns of length n that satisfy (a), (b) in Definition 2, and $C \le D$, then (C, D) is a $(2n+1)$-orthogonal admissible pair, i.e., condition (c) is satisfied automatically in this situation.

The *weight* $(\mathbf{x}^{\pm 1})^S$ for a $(2n+1)$-orthogonal tableau S is again defined by (A.3). Also here, the vector of exponents in (A.3) is called the *content* of S and is denoted by $\mathrm{con}(S)$.

It is a theorem (see [23]) that the irreducible orthogonal character $so_{2n+1}(\lambda;\mathbf{x}^{\pm1})$ equals the generating function for all $(2n+1)$-orthogonal tableaux of shape λ,

$$so_{2n+1}(\lambda;\mathbf{x}^{\pm1}) = \sum_{\substack{S \text{ a } (2n+1)\text{-orthogonal tableau} \\ \text{of shape } \lambda}} (\mathbf{x}^{\pm1})^S, \tag{A.5}$$

with $(\mathbf{x}^{\pm1})^S$ as defined in (A.3).

*A*5. $(2n)$-*Orthogonal Tableaux*

The definition of $(2n)$-orthogonal tableaux is the most intricate one. We provide the full definition for the sake of completeness. However, in the current paper we actually need only a special case, the one that is discussed in Observation 1 below. The reader who is only interested in this paper's applications of $(2n)$-orthogonal tableaux can safely skip the full definition and move on directly to Observation 1.

We basically reproduce Littelmann's description [24, Appendix, A.3], with a little modification in the description of $(2n)$-orthogonal admissible pairs, where we follow [21]. Note again that Littelmann's *rows* are our *columns*.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be an n-orthogonal partition or half-partition. A $(2n)$-*orthogonal tableau of shape* λ is a triple (S_3, S_2, S_1) of $(2n)$-tableaux of respective shapes $((\lambda_{n-1}+\lambda_n)^n)$, $((\lambda_{n-1}-\lambda_n)^n)$, and $(2\lambda_1 - 2\lambda_{n-1}, 2\lambda_2 - 2\lambda_{n-1}, \dots, 2\lambda_{n-2} - 2\lambda_{n-1})$, subject to conditions (0)–(4) below. Note that when S_3, S_2, and S_1 are glued together, in that order, an array of shape $(2\lambda_1, 2\lambda_2, \dots, 2\lambda_{n-2}, 2\lambda_{n-1}, 2\lambda_{n-1})$ is obtained.

(0) If e is an element of a column of S_3, S_2, or S_1, then $2n+1-e$ is not an element of the column.

(1) Columns $1, 2$, columns $3, 4, \dots,$ columns $2\lambda_1 - 2\lambda_{n-1} - 1, 2\lambda_1 - 2\lambda_{n-1}$ of S_1 form $(2n)$-orthogonal admissible pairs (see Definition 3). In addition, for $i = 1, 2, \dots, \lambda_1 - \lambda_{n-1} - 1$, let the entries of the $(2i)$th column of S_1 be (from top to bottom) $k_1, k_2, \dots, k_s$, and let the entries of the $(2i+1)$th column be (again, from top to bottom) $l_1, l_2, \dots, l_t$, $s \le t$. Then for all sequences $1 \le j_1 < \cdots < j_q \le s$ with

$$n+1-q \le k_{j_1} < \cdots < k_{j_q} \le n+q$$

and

$$n+1-q \le l_{j_1} < \cdots < l_{j_q} \le n+q,$$

one has $k_{j_1} + \cdots + k_{j_q} \equiv l_{j_1} + \cdots + l_{j_q} \mod 2$. (Note that this condition is empty if neither n nor $n+1$ is an entry in one of the columns.)

(2) The number of entries $> n$ in each column of S_2 is odd. In addition, denote by R the column (of length n) that arises from the leftmost column of S_1 by adding all integers $e, n+1 \leq e \leq 2n$, that together with their "conjugate" $2n+1-e$ do not already appear in the leftmost column of S_1, arranging everything in strictly increasing order, and replacing the smallest added element, f say, by $2n+1-f$ in the case where the number of entries $> n$ would be odd otherwise. Denote by x the element of R that is closest to $n + 1/2$. To be precise, let $k_1, k_2, \dots, k_s$ be the entries (from top to bottom) of the leftmost column of S_1. Then R consists of the entries $k_1, \dots, k_s, l_1, \dots, l_{n-s-1}, x$ with the following properties:

> $2n \geq l_1 > \cdots > l_{n-s-1} > n, l_{n-s-1} > x, l_{n-s-1} > 2n+1-x, l_i \neq k_j$ and $x \neq k_j$ for all $1 \leq i \leq n-s-1, 1 \leq j \leq s$, and if $e \in R$, then $2n+1-e \notin R$. Furthermore, if the number of integers strictly greater than n in $R \setminus \{x\}$ is odd, then $x > n$; else $x \leq n$.

Let R' be the column with entries $(R \setminus \{x\}) \cup \{2n+1-x\}$ in increasing order. Then the concatenation $S_2 \cup R'$ forms a $(2n)$-tableau. By the concatenation $S_2 \cup R'$ we mean that R' is glued from the right to S_2 such that the topmost entries in each column are aligned in one row.

(3) The number of entries $> n$ in each column of S_3 is even. Denote by S_2' the tableau of shape $((\lambda_{n-1} - \lambda_n + 1)^n)$ obtained from S_2 as follows. The rightmost column of S_2' is R. Now assume that $1 \leq i \leq \lambda_{n-1} - \lambda_n$ and the $(\lambda_{n-1} - \lambda_n + 2 - i)$th column of S_2' has already been defined. The $(\lambda_{n-1} - \lambda_n + 1 - i)$th column of S_2' consists of the entries of the $(\lambda_{n-1} - \lambda_n + 1 - i)$th column of S_2 with one entry e replaced by $2n+1-e$, arranged in strictly increasing order. The entry e to be chosen is the smallest possible such that rows will be weakly increasing (so that S_2' indeed becomes a tableau). Then the concatenation $S_3 \cup S_2'$ is a $(2n)$-tableau. By the concatenation $S_3 \cup S_2'$ we mean that S_3 is glued to the left to S_2' such that the topmost entries in each column are aligned in one row.

DEFINITION 3. A pair (C, D) of two columns of length $k \leq n$ is called a $(2n)$-*orthogonal admissible pair* if the following conditions are satisfied:

(a) Entries in C and D are between 1 and $2n$ and in strictly increasing order.

(b) If e is in D then $2n+1-e$ is not in D. The same holds for C.

(c) C arises from D by a (possibly empty) sequence of operations that can be either operations of the type that are described in Definition 1 (c) or operations O of the following type: The operation O to be described applies only to columns E and entries e_1, e_2 of E, $n < e_1 < e_2$, where for all t between $n+1$ and e_2, $n+1$ included, either t or $2n+1-t$ belongs to E. The operation O itself consists of forming the new column $O(E)$ out of E by replacing e_1 by $2n+1-e_1$ and e_2 by $2n+1-e_2$, and rearranging the new set of entries in strictly increasing order.

OBSERVATION 1. $(2n)$-orthogonal tableaux of shape $(\lambda_{n-1}, \dots, \lambda_{n-1}, \lambda_n)$ are just pairs (S_3, S_2) of $(2n)$-tableaux of respective shapes $((\lambda_{n-1}+\lambda_n)^n)$ and $((\lambda_{n-1}-\lambda_n)^n)$ such that each column of S_3 or S_2 does not contain $2n+1-e$ if it contains e, such that the number of entries $> n$ in S_2 is odd and each entry in the first row of S_2 is $\le n$, such that the number of entries $> n$ in S_3 is even, and where the concatenation $S_3 \cup S_2'$ is a $(2n)$-tableau, with S_2' the tableau that arises from S_2 by replacing the topmost element, e_i say, in column i of S_2 by its "conjugate" $2n+1-e_i$, for all $i = 1, 2, \dots, \lambda_{n-1} - \lambda_n$, and by rearranging the columns in increasing order. For in the case of the above particular shape, the tableau S_1 is empty; hence R equals $\{n+1, n+2, \dots, 2n\}$ or $\{n, n+2, \dots, 2n\}$, depending on whether n is even or odd. Thus, R does not restrict S_2 in item (2) above, except that for odd n it forces all of the entries in the first row to be at most n. If n is even, then all of the entries in the first row have to be $\le n$ as well. For in each column the number of entries $> n$ is odd, with n being even. This implies that the number of entries $\le n$ has to be odd as well; in particular, it has to be at least 1. Finally, in item (3) above, when forming S_2' out of S_2, the smallest element in each column can always be replaced by its "conjugate."

The *weight* $(\mathbf{x}^{\pm 1})^S$ for a $(2n)$-orthogonal tableau S is again defined by (A.3). Also here, the vector of exponents in (A.3) is called the *content* of S and is denoted by $\mathrm{con}(S)$.

It is a theorem (see [23]) that the irreducible orthogonal character $so_{2n}(\lambda; \mathbf{x}^{\pm 1})$ equals the generating function for all $(2n)$-orthogonal tableaux of shape λ,

$$so_{2n}(\lambda; \mathbf{x}^{\pm 1}) = \sum_{\substack{S \text{ a } (2n)\text{-orthogonal tableau} \\ \text{of shape } \lambda}} (\mathbf{x}^{\pm 1})^S, \tag{A.6}$$

with $(\mathbf{x}^{\pm 1})^S$ as defined in (A.3).

A6. *Littelmann's Decomposition Rule and the Littlewood–Richardson Rule*

Littelmann's rule [24] for the decomposition of the product of two general linear, two symplectic, or two special orthogonal characters can be stated uniformly. (In fact, it is also valid for the simple, simply connected algebraic groups of type G_2 and E_6.) What is needed in the formulation of

Littelmann's theorem is the notion of a (*dominant*) *Weyl chamber* associated with each of the characters.

The (*dominant*) *Weyl chamber of type* A, which is associated with $s_n(\,\cdot\,;\mathbf{x})$, is the set of points (cf. [6, p. 215])

$$\{(x_1, x_2, \dots, x_n) \colon x_1 \geq x_2 \geq \cdots \geq x_n\}. \tag{A.7}$$

The (*dominant*) *Weyl chamber of type* C, which is associated with $sp_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$, is the set of points (cf. [6, p. 243, (16.5)])

$$\{(x_1, x_2, \dots, x_n) \colon x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}. \tag{A.8}$$

The (*dominant*) *Weyl chamber of type* B, which is associated with $so_{2n+1}(\,\cdot\,;\mathbf{x}^{\pm 1})$, is the same set (A.8) of points (cf. [6, p. 272]).

Finally, the (*dominant*) *Weyl chamber of type* D, which is associated with $so_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$, is the set of points (cf. [6, p. 272])

$$\{(x_1, x_2, \dots, x_n) \colon x_1 \geq x_2 \geq \cdots \geq |x_n|\}. \tag{A.9}$$

Furthermore, if T is a tableau, then by $T(l)$ we denote the tableau that consists of the *last* l columns of T.

Now we are in a position to state Littelmann's theorem.

THEOREM (Littelmann [24, Theorem (a), p. 346]). *Let* $\chi_n(\cdot)$ *be any of the characters* $s_n(\,\cdot\,;\mathbf{x})$, $sp_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$, $so_{2n+1}(\,\cdot\,;\mathbf{x}^{\pm 1})$, *or* $so_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$. *Then*

$$\chi_n(\lambda) \cdot \chi_n(\mu) = \sum_T \chi_n(\lambda + \operatorname{con}(T)), \tag{A.10}$$

where the sum is over all corresponding tableaux (*that is*, n-*tableaux in case* $\chi_n(\cdot) = s_n(\,\cdot\,;\mathbf{x})$, $(2n)$-*symplectic tableaux in case* $\chi_n(\cdot) = sp_{2n}(\,\cdot\,;\mathbf{x}^{\pm 1})$, *etc.*) *of shape* μ *such that* $\lambda + \operatorname{con}(T(l))$ *is in the Weyl chamber of the corresponding type for all* l.

In case $\chi_n(\cdot) = s_n(\,\cdot\,;\mathbf{x})$ this rule translates to the classical Littlewood–Richardson rule (cf. [26, I, Sect. 9; 34, (4.26) + Theorem 4.9.4]), which we want to describe next.

DEFINITION 4. A *Littlewood–Richardson filling (LR-filling) of shape* ν/λ *and content* μ is an (ordinary) tableau F of shape ν/λ and content μ where the *Littlewood–Richardson condition (LR-condition)* is satisfied. The latter condition is the following:

> Read the entries of F row-wise from top to bottom and in each row from right to left. Then at any stage during the reading the number of 1's is greater than or equal to the number of 2's, which in turn is greater than or equal to the number of 3's, etc.

Furthermore, define the *Littlewood–Richardson coefficient* $\mathrm{LR}^{\nu}_{\lambda,\mu}$ by

$$\mathrm{LR}^{\nu}_{\lambda,\mu} = \text{number of LR-fillings of shape } \nu/\lambda \text{ and content } \mu. \tag{A.11}$$

Then the *Littlewood–Richardson rule* reads as follows:

$$s_n(\lambda;\mathbf{x})\cdot s_n(\mu;\mathbf{x}) = \sum_{\nu} \mathrm{LR}^{\nu}_{\lambda,\mu} s_n(\nu;\mathbf{x}). \tag{A.12}$$

The translation from (A.10) with $\chi_n(\cdot) = s_n(\,\cdot\,;\mathbf{x})$ to (A.12) proceeds as follows. It is basically the same idea as the one we use in the proof of Proposition 1. What we do is construct, for any fixed λ, μ, ν, a bijection between n-tableaux T of shape μ, with $\nu = \lambda + \mathrm{con}(T)$, and where $\lambda + \mathrm{con}(T(l))$ is in the Weyl chamber (A.7) of type A for all l, and LR-fillings F of shape ν/λ and content μ. Clearly, this would establish the equivalence of (A.10) with $\chi_n(\cdot) = s_n(\,\cdot\,;\mathbf{x})$ and (A.12).

Given T as above, we construct a sequence $F_0, F_1, \dots, F_{\mu_1}$ of fillings by reading T column-wise, from right to left. The desired filling F will then be defined to be the last filling F_{μ_1}. Define F_0 to be the only filling of the shape λ/λ (which is, of course, the empty filling). Suppose that we have already constructed F_l. To obtain F_{l+1}, we add for $i = 1, 2, \dots, n$ an entry e to row i of F_l if i is an entry occurring in the lth last column and the eth row of T. As already announced, we define F to be F_{μ_1}.

A7. *Littlewood's branching rules*

Here we quote the branching rules for the restriction of Schur functions to symplectic or orthogonal characters, which we use in Section 4.

THEOREM (Littlewood). *There holds* (see [25, Appendix, p. 295; 13, (3.8b); 39, Theorem 3.13]

$$s(\lambda;\mathbf{x}) = \sum_{\nu} sp(\nu;\mathbf{x}) \sum_{\mu,\,\mu' \text{ even}} \mathrm{LR}^{\lambda}_{\mu,\nu} \tag{A.13}$$

(*"μ' even" means that all of the columns of μ are even*), *and* (see [25, p. 240, (II); 13, (3.8a); 39, Theorem 3.16])

$$s(\lambda;\mathbf{x}) = \sum_{\nu} o(\nu;\mathbf{x}) \sum_{\mu,\,\mu \text{ even}} \mathrm{LR}^{\lambda}_{\mu,\nu} \tag{A.14}$$

(“μ *even*” *means that all of the rows of* μ *are even*), *with* $\mathrm{LR}^{\lambda}_{\mu,\nu}$ *the Littlewood–Richardson coefficients as defined in* (A.11).

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