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Fixed subrings of Noetherian graded regular rings

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Rings of invariants can have nice homological properties even if they do not have finite global dimension. Watanabe's Theorem [W] gives conditions when the fixed subring of a commutative ring under the action of a finite group is a Gorenstein ring. The Gorenstein condition was extended to noncommutative rings by a condition explored by Idun Reiten in the 1970s, called k-Gorenstein in [FGR]. This condition, also known as the Auslander–Gorenstein condition, has proved to be a very useful one, and now has been generalized further in the notion of an Auslander dualizing complex (see e.g. [YZ]). Artin and Schelter defined another Gorenstein condition for connected graded rings (see [AS]); this condition is now called the Artin–Schelter Gorenstein condition.

Noncommutative versions of Watanabe's Theorem, giving conditions when a fixed ring satisfies a Gorenstein condition, were proved by Jørgensen and Zhang [JoZ], and extended by Jing and Zhang [JZ2]. These conditions involve the "homological determinant" of the automorphisms in the group, as defined in [JoZ]. In this paper we apply these results to down–up algebras, their extensions, and certain generalized Weyl algebras, and thus expand the class of algebras for which the homological determinant can be easily computed. The fact that rings of invariants in some cases give algebras that are Artin–Schelter regular was one motivation for this paper.

The homological determinant defines a group homomorphism from the group of graded automorphisms of a connected graded K-algebra A to the multiplicative group of the field K^*; its name is due to the fact that when $A = K[x_1, \dots, x_n]$ is a commutative poly-

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nomial ring each graded automorphism g is associated to an element of $GL_n(K)$, and the homological determinant of g is the usual (matrix) determinant of this linear map.

Let g be an automorphism of an algebra A over a field K, let V be the K-space spanned by a set of generators of A as an algebra, and assume that $g|_V$ is a linear automorphism. For many classes of regular algebras, including the Weyl algebras, universal enveloping algebras of finite-dimensional Lie algebras, the 4-dimensional Sklyanin algebras, and the nth quantum Weyl algebra $A_n(q, p_{i,j})$ (cf. [GZ, Section 2.3]) the homological determinant of a filtered (with respect to the filtration induced by V) automorphism g is the usual determinant of the linear map $g|_V$, and for a graded automorphism of the exterior algebra $\Gamma(V)$ the homological determinant is $(\det g|_V)^{-1}$ (see [JZ2]). Jing and Zhang gave examples to show that the homological determinant is not always either $(\det g|_V)$ or $(\det g|_V)^{-1}$ [JZ2, Examples 2.8 and 2.9]. We show that for graded down–up algebras using the usual generating set $V = \{u, d\}$ the homological determinant of g is $(\det g|_V)^2$. The homological determinants of certain automorphisms of generalized Weyl algebras are also computed.

First we recall the basic definitions. A Noetherian connected graded K-algebra A is called Artin–Schelter Gorenstein (AS–Gorenstein) if A has finite right and left injective dimensions d and if there is an integer ℓ such that the graded Ext-group $\underline{\mathrm{Ext}}$ has the property that $\underline{\mathrm{Ext}}^i_A(K, A) = \underline{\mathrm{Ext}}^i_{A^\circ}(K, A) = 0$ for $i \neq d$ and $\underline{\mathrm{Ext}}^d_A(K, A) = \underline{\mathrm{Ext}}^d_{A^\circ}(K, A) = K(\ell)$, where $K(\ell)$ is the ℓth degree shift of the trivial module K. An AS–Gorenstein ring is called AS-regular if it has finite global dimension.

Let f be a K-linear homomorphism from an A-module M to an A-module N, and let g be a graded automorphism of A. We say that f is g-linear if $f(ma) = f(m)g(a)$ for all $m \in M$ and all $a \in A$. The map f is g-linear if and only if f is an A-module homomorphism from M to the g-twisted module N^g. If g is a graded automorphism of A then g is g-linear. Let A_k be the vector space of degree k elements of A, $A_{\geqslant n} = \bigoplus_{k \geqslant n} A_k$ and $\mathfrak{m} = A_{\geqslant 1}$ be the graded maximal ideal of A, and let $H^*_{\mathfrak{m}}$ denote the local cohomology functors, so

$$H^i_{\mathfrak{m}}(M) = \varinjlim_{n} \underline{\mathrm{Ext}}^i_A(A/A_{\geqslant n}, M).$$

A g-linear map can be extended to an injective (or projective) resolution, and the local cohomology functor $H^*_{\mathfrak{m}}$ can be applied to g-linear maps. If A is a graded AS–Gorenstein ring with injective dimension d then by [JoZ, Sections 2.2 and 2.3] $g : A \to A$ induces a g-linear map $H^d_{\mathfrak{m}}(g) : A'(\ell) \to A'(\ell)$, where ℓ is the integer in the definition of AS–Gorenstein and $'$ is the graded vector space dual. Moreover, $H^d_{\mathfrak{m}}(g) = c(g^{-1})'$ for some constant c. The constant c^{-1} is called the homological determinant of g, and we denote this fact by $\operatorname{hdet} g = c^{-1}$.

The trace of g on A (see [JZ1]) is defined to be

$$\mathrm{Tr}_A(g, t) = \sum_{n \geqslant 0} \mathrm{tr}(g|_{A_n}) t^n,$$

where tr is the usual trace of the linear map g on A_n. As an example, the Hilbert series of A is the trace of the identity map. It follows from [JoZ, Lemma 2.6 and Theorem 4.2] that when A is an AS-regular algebra then the hdet g can be computed from the trace of g: since $\mathrm{Tr}_A(g,t)$ is a rational function in t it can be written as a Laurent series in t^{-1}, and we can write

$$\mathrm{Tr}_A(g,t) = (-1)^d(\operatorname{hdet} g)^{-1}t^{-\ell} + \text{lower degree terms},$$

where d and ℓ are as in the definition of AS-regular.

For a (not necessarily graded) ring A the grade of an A-module M is defined to be $j(M) = \min\{i\colon \mathrm{Ext}^i_A(M,A) \neq 0\}$ or ∞ if no such i exists. We say that A satisfies the Auslander–Gorenstein condition if A has finite left- and right-injective dimension, and for every Noetherian A-module M, all $i \geqslant 0$, and all submodules $N \subseteq \mathrm{Ext}^i_A(M,A)$ the relation $j(N) \geqslant i$ holds. The ring A is called Auslander-regular if it satisfies the Auslander–Gorenstein condition and has finite global dimension.

Throughout we will use [JoZ, Theorem 3.3] that guarantees that if G is a finite group of graded automorphisms acting on an AS–Gorenstein (Auslander–Gorenstein) ring A with $|G| \neq 0 \in K$, and if the homological determinant of g satisfies $\operatorname{hdet} g = 1$ for all $g \in G$, then the fixed subring A^G is AS–Gorenstein (Auslander–Gorenstein). Furthermore, by [JoZ, Theorems 1.2(5) and 3.5] if A is a filtered algebra, G is a group of automorphisms preserving the filtration, and the associated graded ring $\mathrm{gr}(A)$ is a Noetherian Auslander–Gorenstein ring, then the fixed subring A^G satisfies the Auslander–Gorenstein condition.

In Section 1 we consider the case when A is a down–up algebra, hence an Auslander-regular (and AS-regular when graded) algebra of dimension 3; we determine the filtered automorphisms g of A that are of the form $g(u) = wu + yd + r_1$ and $g(d) = xu + zd + r_2$ for elements $r_i, w, x, y, z \in K$. We use results of Jing and Zhang, and of Jørgensen and Zhang, to give conditions when the fixed ring will satisfy a Gorenstein condition. In Section 2 we consider the fixed rings R^G when R is a particular regular extension of a down–up algebra, as considered by Benkart and Roby [BR], Bauwens [Bau], and Cassidy [C1,C2]. In Section 3 we consider a more general class of rings, generalized Weyl algebras $R(\sigma,h)$. A down–up algebra is a generalized Weyl algebra over the commutative polynomial ring $K[x,y]$, and its defining homomorphism σ is a linear map σ. We show that there is a generalized Weyl algebra over $K[x,y]$ with a linear map σ and linear polynomial h in $R = K[x,y]$ where $R(\sigma,h)$ is not isomorphic to a down–up algebra. We consider fixed rings of some generalized Weyl algebras $R(\sigma,h)$ under certain groups of automorphisms that preserve the generalized Weyl algebra structure of $R(\sigma,h)$.

1. Down–up algebras

Let K be a field, and fix parameters $\alpha,\beta,\gamma \in K$. In [B] and [BR] Benkart and Roby considered algebras $A(\alpha,\beta,\gamma)$ that they called down–up algebras. The algebra $A = A(\alpha,\beta,\gamma)$ is defined by generators u and d and relations:

$$d^2u = \alpha dud + \beta ud^2 + \gamma d \quad \text{and} \quad du^2 = \alpha udu + \beta u^2d + \gamma u.$$

The algebra A has a Poincare–Birkhoff–Witt type basis over K of the form $u^i(du)^jd^k$. It follows from results in [KMP,KK] that $A(\alpha,\beta,\gamma)$ is a Noetherian ring if and only if it is Auslander-regular if and only if $\beta \neq 0$. When $\gamma = 0$ the ring $A(\alpha, \beta, 0)$ is a connected graded algebra with degree$(u) =$ degree$(d) = 1$; we will refer to a down–up algebra with $\gamma = 0$ as a graded down–up algebra. The algebra $A(\alpha, \beta, 0)$ is of type S_1 in the classification of regular algebras of dimension 3 [AS]. The representation theory of graded down–up algebras was studied in [BW], and the enveloping algebra of the three-dimensional Heisenberg Lie algebra $A(2, -1, 0)$ is one example. As another example, consider the quantized enveloping algebra $U_q(\mathfrak{sl}_3)$, which has generators $E_i, F_i, K^{\pm 1}, i = 1, 2$, and defining relations as in e.g. [Ja]. Then the subalgebra $U^+(\mathfrak{sl}_3)$ generated by E_1, E_2 is the down–up algebra $A([2], -1, 0)$, where for a scalar $q \neq 0 \in K$ we define $[n] = (q^n - q^{-n})/(q - q^{-1})$. The algebra $A([2], -1, 0)$ was called H_q in [KS1], and the algebra H_q' of [KS1] is the down–up algebra $A(2q, -q^2, 0)$. In [KMP] it is shown that when an arbitrary down–up algebra $A(\alpha, \beta, \gamma)$ is filtered in the generators u and d, the associated graded algebra is the graded down–up algebra $A(\alpha, \beta, 0)$. For $\gamma \neq 0$, the down–up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$. Some examples of down–up algebras with $\gamma \neq 0$ include $A(2, -1, \gamma)$, which is isomorphic to the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, and $A(0, 1, \gamma)$, which is isomorphic to the enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1, 2)$.

We fix the notation we will use throughout this section. Let A be a down–up algebra. If A is graded and g is a graded automorphism of A, we can restrict g to the graded vector space $V = Ku \oplus Kd$; we will represent this linear map by

$$g|_V = \begin{bmatrix} w & x \\ y & z \end{bmatrix},$$

where $U = g(u) = wu + yd$ and $D = g(d) = xu + zd$. For any down–up algebra A an invertible linear map g as defined above will extend to a graded (filtered) automorphism of A if and only if U and D satisfy the same relations as u and d. Expressing U and D in terms of u and d we see that the equation

$$D^2U = \alpha DUD + \beta UD^2 + \gamma D$$

holds if and only if the coefficients of each of the different monomials in u and d in the PBW-basis of A are equal. This leads to the following equations ("the D^2U equations"):

- u^3 coefficient: $(1 - \alpha - \beta)wx^2 = 0$,
- d^3 coefficient: $(1 - \alpha - \beta)yz^2 = 0$,
- u^2d coefficient: $(1 - \beta^2)x^2y = \alpha(1 + \beta)wxz$,
- udu coefficient: $(1 + \alpha - \alpha^2 - \beta)wxz = \alpha(1 + \beta)x^2y$,
- ud^2 coefficient: $(1 - \alpha - \alpha\beta - \beta^2)xyz = 0$,
- dud coefficient: $(1 - \beta - \alpha\beta - \alpha^2)xyz = 0$,
- u coefficient: $\gamma x((1 - \alpha)wz - \beta xy - 1) = 0$,
- d coefficient: $\gamma z(wz - (\alpha + \beta)xy - 1) = 0$.

Similarly the equation

$$DU^2 = \alpha UDU + \beta U^2D + \gamma U$$

holds if and only if the following equations ("the DU^2 equations") hold:

- u^3 coefficient: $(1-\alpha-\beta)w^2x=0$,
- d^3 coefficient: $(1-\alpha-\beta)y^2z=0$,
- u^2d coefficient: $(1-\alpha-\alpha\beta-\beta^2)wxy=0$,
- udu coefficient: $(1-\beta-\alpha\beta-\alpha^2)wxy=0$,
- ud^2 coefficient: $(1-\beta^2)xy^2=\alpha(1+\beta)wyz$,
- dud coefficient: $(1+\alpha-\alpha^2-\beta)wyz=\alpha(1+\beta)xy^2$,
- u coefficient: $\gamma w(wz-(\alpha+\beta)xy-1)=0$,
- d coefficient: $\gamma y((1-\alpha)wz-\beta xy-1)=0$.

We use these relations to determine the graded (filtered) automorphisms of down–up algebras that are of the form above; the results are described in the following proposition (which is also true if A is not Noetherian). Notice that the down–up algebras that have non-diagonal graded automorphisms include some of the examples of down–up algebras mentioned above.

Proposition 1.1. *All of the automorphisms of down–up algebras $A=A(\alpha,\beta,\gamma)$ given by $g(u)=wu+yd$ and $g(d)=xu+zd$ are described below.*

1. *The diagonal map $g|_V=\left[\begin{smallmatrix} w & 0\\ 0 & z\end{smallmatrix}\right]$ with $wz\neq 0$ is an automorphism of any $A(\alpha,\beta,0)$. When $\gamma\neq 0$ the diagonal map $g|_V=\left[\begin{smallmatrix} w & 0\\ 0 & w^{-1}\end{smallmatrix}\right]$ is an automorphism of any $A(\alpha,\beta,\gamma)$.*
2. *The map $g|_V=\left[\begin{smallmatrix} 0 & x\\ y & 0\end{smallmatrix}\right]$ with $xy\neq 0$ is an automorphism of $A(0,1,0)$ or $A(\alpha,-1,0)$ for any $\alpha\in K$. When $\gamma\neq 0$ the algebra $A(0,1,\gamma)$ has the automorphism $g|_V=\left[\begin{smallmatrix} 0 & x\\ -x^{-1} & 0\end{smallmatrix}\right]$, and for any $\alpha\in K$ the algebra $A(\alpha,-1,\gamma)$ has the automorphism $g|_V=\left[\begin{smallmatrix} 0 & x\\ x^{-1} & 0\end{smallmatrix}\right]$.*
3. *An arbitrary invertible linear map $g|_V=\left[\begin{smallmatrix} w & x\\ y & z\end{smallmatrix}\right]$ is an automorphism of $A(0,1,0)$ or $A(2,-1,0)$. When $\gamma\neq 0$ and $\det(g|_V)=1$ then g is an automorphism of $A(0,1,\gamma)$.*

Proof. It is clear that the diagonal automorphism will always preserve the relations when $\gamma=0$. When $\gamma\neq 0$ the u coefficient from the DU^2 equations (and also the d coefficient from the D^2U equations) give $wz=1$, and then all equations are satisfied.

Next consider the case when $w=z=0$ and $xy\neq 0$. Equating u^2d coefficients in the D^2U equations gives $\beta^2=1$, and equating coefficients of udu gives $\alpha(1+\beta)=0$; hence we have described the two classes of algebras in case 2. Furthermore, one can check that under these necessary conditions both the D^2U equations and the DU^2 equations hold in the case $\gamma=0$, so that these possible maps are algebra automorphisms. When $\gamma\neq 0$ the u coefficient from the D^2U equations (or the d coefficient from the DU^2 equations) gives $-\beta xy=1$, and the stated automorphisms follow.

If $g|_V$ is not as described in cases 1 or 2 then $g|_V$ has at least 3 nonzero entries since it is nonsingular; hence either $wx\neq 0$ or $yz\neq 0$. Equating the coefficient of u^3 (when $wx\neq 0$) or d^3 (when $yz\neq 0$) in the D^2U equations gives $\alpha+\beta=1$. If $x=0$ then $wyz\neq 0$ and the coefficient of ud^2 in the DU^2 equations gives $0=wyz(\alpha+\alpha\beta)$, so $\alpha(1+\beta)=0$, and either $\alpha = 0$ (and hence $\beta = 1$) or $\beta = -1$ and $\alpha = 2$. Hence we may assume that $x \neq 0$. Equating the coefficient of udu in the D^2U equations gives

$$x^2 y\alpha(1+\beta) = wxz\big(1+\alpha-\alpha^2-\beta\big),$$

and the coefficient of u^2d gives

$$x^2 y\big(1-\beta^2\big) = wxz(1+\beta)\alpha.$$

Dividing by x, subtracting equations, and using the relation $\alpha = 1-\beta$ gives

$$(wz-xy)\alpha(1+\beta) = xy\big(1-\beta^2\big) - wz\big(1+\alpha-\alpha^2-\beta\big),$$

so that

$$\begin{aligned} 0 &= xy\alpha(1+\beta) - wz\big(2\alpha-\alpha^2\big) = xy\alpha(1+\beta) - wz\alpha(2-\alpha) \\ &= xy\alpha(1+\beta) - wz\alpha(1+\beta) = -(wz-xy)\alpha(1+\beta). \end{aligned}$$

Since $wz - xy \neq 0$, we again get $\alpha(1+\beta) = 0$ and the cases described in case 3. Furthermore, one can check that in these cases when $\gamma = 0$ the maps satisfy both the D^2U and the DU^2 equations, so that these maps are algebra automorphisms. When $\gamma \neq 0$ and $wx \neq 0$ we get $(1-\alpha)wz - \beta xy - 1 = 0$ from the u coefficient of the D^2U equations. Either y or z must also be non-zero, and hence from the other linear coefficients we get $wz - (\alpha+\beta)xy - 1) = 0$. Subtracting these two equations we get $\alpha(wz - xy) = 0$, and hence $\alpha = 0$, which means $\beta = 1$ from above. Then the linear term coefficient conditions all are satisfied if $\det g = 1$. The case $yz \neq 0$ is similar. $\square$

1.2. More generally one can consider linear filtered automorphisms g of $A = A(\alpha,\beta,\gamma)$ that are of the form $g(u) = wu + yd + r_1$ and $g(d) = xu + zd + r_2$ for $r_i \in K$. Using an analysis similar to the above, one can show that either $r_i = 0$ for $i = 1, 2$, or $A = A(2,-1,0)$, and then for arbitrary $r_i \in K$ and arbitrary elements $w, x, y, z \in K$ that satisfy $wz - xy \neq 0$ the linear map defined by $g(u) = wu + yd + r_1$ and $g(d) = xu + zd + r_2$ give automorphisms of $A(2,-1,0)$, the enveloping algebra of the Heisenberg Lie algebra. There are automorphisms of this form that have finite order: e.g. $g(u) = d + 1$ and $g(d) = u - 1$ has order two.

Example 1.3. Consider the graded automorphism $g|_V = \left[\begin{smallmatrix} 0 & a \\ a^{-1} & 0 \end{smallmatrix}\right]$ of $A(0,1,0)$ or $A(2,-1,0)$ (Proposition 1.1(2)). As a linear map g is diagonalizable with eigenvectors $T_1 = u + a^{-1}d$ and $T_2 = u - a^{-1}d$ that generate $A(0,1,0)$ (respectively $A(2,-1,0)$) as an algebra and satisfy $g(T_1) = T_1$ and $g(T_2) = -T_2$. Furthermore, by Proposition 1.1(3) we know that the change of basis matrix $h|_V = \left[\begin{smallmatrix} 1 & 1 \\ a^{-1} & -a^{-1} \end{smallmatrix}\right]$ is also an automorphism of $A(0,1,0)$ (respectively $A(2,-1,0)$) and the proof of Proposition 1.1 showed that the elements T_1 and T_2 satisfy the same relations as u and d in the algebra $A(0,1,0)$ (respectively $A(2,-1,0)$), so that the set of monomials $T_1^i(T_2T_1)^jT_2^k$ for $i, j, k \geqslant 0$ form a PBW-type basis of $A(0,1,0)$ (respectively $A(2,-1,0)$). Hence the subalgebra of $A(0,1,0)$ (respectively $A(2,-1,0)$) invariant under $G=\langle g\rangle$ has K-basis $T_1^i(T_2T_1)^jT_2^k$ with $j+k$ even, and hence is generated as a K-algebra by T_1, $(T_2T_1)T_2$, and T_2^2.

Example 1.4. For $\gamma\neq 0$ consider the filtered automorphism $g|_V=\left[\begin{smallmatrix}0 & a\\ -a^{-1} & 0\end{smallmatrix}\right]$ of $A(0,1,\gamma) = U(\mathfrak{osp}(1,2))$ (Proposition 1.1(2)). As a linear map g is diagonalizable with eigenvectors $T_1=ia2^{-1}u-2^{-1}d$ and $T_2=u-ia^{-1}d$ satisfying $g(T_1)=iT_1$ and $g(T_2)=-iT_2$ and with change of basis matrix

$$h|_V=\begin{bmatrix} ia2^{-1} & 1\\ -2^{-1} & -ia^{-1}\end{bmatrix},$$

which is also an automorphism of $A(0,1,\gamma)$ by Proposition 1.1, case 3, since $\det h=1$. Furthermore the elements T_1 and T_2 satisfy the same relations as u and d, so that the monomials $T_1^i(T_2T_1)^jT_2^k$ form a K-basis of $A(0,1,\gamma)$. Hence a K-basis of the subalgebra of A invariant under $G=\langle g\rangle$ is monomials of the form $T_1^i(T_2T_1)^jT_2^k$ with $i+3k\equiv 0 \bmod 4$. Notice that the form of the fixed ring A^G under the cyclic group $G=\langle g\rangle$ of order 4 is independent of a for any $a\neq 0$, and so the A^G are all isomorphic for any a.

Theorem 1.5. *Let g be a graded automorphism of a Noetherian graded down–up algebra. If the matrix of $g|_V$ with respect to the basis $\{u,d\}$ of V is $\left[\begin{smallmatrix}w & x\\ y & z\end{smallmatrix}\right]$, then*

$$\operatorname{Tr}(g,t)=\frac{1}{(1-\lambda t)}\frac{1}{(1-\mu t)}\frac{1}{(1-\lambda\mu t^2)}=\frac{1}{p(t)}$$

where λ and μ are the (not necessarily distinct) eigenvalues of $g|_V$, and

$$\begin{aligned}p(t)&=1-(w+z)t+(w+z)(wz-xy)t^3-(wz-xy)^2t^4\\ &=\bigl(1-\operatorname{tr}(g|_V)t+\det(g|_V)t^2\bigr)\bigl(1-\det(g|_V)t^2\bigr).\end{aligned}$$

The homological determinant hdet g *is* $\det^2(g|_V)=\lambda^2\mu^2$.

Proof. We use a minimal free resolution of K to compute the $\operatorname{Tr}(g,t)$ and hence the homological determinant hdet g.

The Noetherian graded down–up algebra $A(\alpha,\beta,0)$ is regular of global dimension 3 and generated by two degree 1 elements. By [AS], the trivial module K has a minimal free resolution

$$0\to A(-4)\to A^{\oplus 2}(-3)\to A^{\oplus 2}(-1)\to A\to K\to 0.$$

By [JZ1, Theorem 3.1], $\operatorname{Tr}_A(g,t)=1/p(t)$ where $p(t)$ is a polynomial of degree 4. We can find $p(t)=1+a_1t+a_2t^2+a_3t^3+a_4t^4$ from the first 5 terms of the trace function $\operatorname{Tr}(g,t)=1+b_1t+b_2t^2+b_3t^3+b_4t^4+\cdots$ using the standard method of undetermined coefficients to get:

$$a_1 = -b_1, \qquad a_2 = -b_2 + b_1^2, \qquad a_3 = -b_3 + 2b_1b_2 - b_1^3,$$
$$a_4 = -b_4 + 2b_1b_3 + b_2^2 - 3b_1^2b_2 + b_1^4.$$

We proceed to calculate the $b_i = \operatorname{tr}(g|_{A_i})$, $i = 1, \ldots, 4$. Clearly $b_1 = w + z$, and $b_2 = (w+z)^2$, so that $a_1 = -(w+z)$ and $a_2 = 0$ as claimed. It can be checked that

$$\begin{aligned} b_3 &= w^3 + 2w^2z + 2wz^2 + z^3 + xy(w+z)(\alpha+\beta) \\ &= (w+z)^3 - wz(w+z) + xy(w+z)(\alpha+\beta) \end{aligned}$$

and

$$\begin{aligned} b_4 = w^4 + 2w^3z + 3w^2z^2 + 2wz^3 + z^4 + \bigl(2\beta + \alpha^2 + \alpha + \alpha\beta\bigr)xyw^2 \\ + (3\alpha + 2\beta + \alpha\beta)xywz + \beta^2x^2y^2 + \bigl(2\beta + \alpha^2 + \alpha + \alpha\beta\bigr)xyz^2. \end{aligned}$$

Hence in the diagonal case (case 1), $b_3 = (w+z)^3 - wz(w+z)$ and $b_4 = w^4 + 2w^3z + 3w^2z^2 + 2wz^3 + z^4 = (w+z)^4 - 2wz(w+z)^2 + z^2w^2$, and the $p(t)$ is as given. In the second case $w = z = 0$ and $\beta^2 = 1$, so $b_3 = 0$ and $b_4 = x^2y^2$ and the trace function is as given. In the third case we get $b_3 = w^3 + 2w^2z + 2wz^2 + z^3 + xy(w+z)$, and $b_4 = w^4 + 2w^3z + 3w^2z^2 + 2wz^3 + z^4 + 2xyw^2 + 2xywz + x^2y^2 + 2xyz^2$, and a laborious calculation shows that the trace function is as given. However, in the third case the trace function can be calculated more easily by noting that by Proposition 1.1, case 3, any linear map on V gives an automorphism of A, so if we change basis on V, the new set of basis elements $\{U, D\}$ will satisfy the same relations as u and d. Without loss of generality we can assume that K is algebraically closed and that $g|_V$ is triangular with respect to a set of generators U, D satisfying the original relations. We have a monomial basis $\{U^i(DU)^jD^k\}$, with $i + 2j + k = n$, for A_n the terms of degree n. Since the relations of A preserve degree in U and D in each term, and since applying g gives either the same terms, or monomials with U factors replaced by D factors, then after all monomials are put in standard form we have

$$g\bigl(U^i(DU)^jD^k\bigr) = \bigl(\lambda^i(\lambda\mu)^j\mu^k\bigr)\bigl(U^i(DU)^jD^k\bigr) + \text{possibly other terms}.$$

Hence, just as in the case of commutative polynomial rings (see [JZ1, Proposition 1.1]), we can easily compute the $\operatorname{tr}(g|A_n)$ and get

$$\operatorname{Tr}(g,t) = \sum_{n \geqslant 0}\Biggl(\sum_{i+2j+k=n} \lambda^i(\lambda\mu)^j\mu^k\Biggr)t^n = \frac{1}{(1-\lambda t)}\frac{1}{(1-\mu t)}\frac{1}{(1-\lambda\mu t^2)}.$$

The computation of the leading term of the Laurent series in t^{-1} follows easily, giving the value of the homological determinant equal to $\det^2(g|_V)$. □

1.6. Jing and Zhang [JZ2] extended their results to the filtered case. If g is an automorphism of a filtered ring A that preserves the filtration on A, then g induces a homomorphism $\bar{g}$ on the associated graded ring $\operatorname{Gr} A$. The homological determinant of g is defined to be $\operatorname{hdet}_{\operatorname{Gr} A} \bar{g}$. If G is a finite group of automorphisms, $|G| \neq 0$ in K, and if $\operatorname{hdet} g = 1$ for all $g \in G$, then it follows from [JZ2, Theorem 3.5] that when $\operatorname{Gr} A$ is AS–Gorenstein (respectively Auslander–Gorenstein) then $\operatorname{Gr}(A^G)$ is AS–Gorenstein (respectively Auslander–Gorenstein). It follows from [Bj, Theorem 4.1] that A^G satisfies the Auslander–Gorenstein condition. Notice in Proposition 1.1 that all the automorphisms g of down–up algebras with $\gamma \neq 0$ have $\operatorname{hdet} g = 1$, and the groups G of Examples 1.3 and 1.4 both satisfy the condition of the Corollary below.

Corollary 1.7. *Let G be a finite group of graded automorphisms of $A(\alpha, \beta, 0)$ for $\beta \neq 0$, with (not necessarily distinct) eigenvalues λ, μ satisfying $\lambda^2\mu^2 = 1$ and $|G| \neq 0 \in K$, then the ring of invariants A^G satisfies the Artin–Schelter and Auslander–Gorenstein conditions. If G is a finite group of automorphisms of $A(\alpha, \beta, \gamma)$ with $\beta, \gamma \neq 0$ having the form above, then the ring of invariants A^G satisfies the Auslander–Gorenstein condition.*

The ring of invariants need not have finite global dimension. In fact, in Example 3.5 we show that for any down–up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0$, and any root of unity λ, the diagonal map $\Theta_\lambda : A \to A$ with $\Theta_\lambda(d) = \lambda d$, $\Theta_\lambda(u) = \lambda^{-1} u$ is an automorphism with fixed ring having infinite global dimension, but satisfying the Gorenstein condition.

Note that the down–up algebra $A = A(2, -1, \gamma)$ is the enveloping algebra of a three-dimensional Lie algebra L with basis u, d, and $[u, d]$, where $[u, [u, d]] = \gamma u$ and $[[u, d], d] = \gamma d$. Under the filtration on A in u, d, a filtered automorphism g is of the form:

$$g(u) = a_{1,1}u + a_{1,2}d + b_1, \qquad g(d) = a_{2,1}u + a_{2,2}d + b_2.$$

Let M be the matrix $M = (a_{i,j})$. It is not difficult to check that $g([u, d]) = (\det M)[u, d]$, and the homological determinant $\operatorname{hdet}(g) = (\det M)^2$, which is the same as determinant of the matrix of $g|L$ (cf. [JZ2, Lemma 6.1]).

1.8. Kraft and Small [KS] have called an algebra A an FCR-algebra if every finite-dimensional A-module is completely reducible (i.e. a direct sum of simple A-modules), and if the intersection of the annihilators of all the finite-dimensional simple A-modules is zero. The down–up algebras that are FCR-algebras are known (see e.g. [KS2, Theorem 1.2 and Proposition 1.4]) and include $A(-2, 1, \gamma) \cong U(\mathfrak{sl}_2))$ and $A(0, 1, \gamma) \cong U(\mathfrak{osp}(2, 1))$ for $\gamma \neq 0$ (see e.g. [KS2, Theorem 2.12]). It follows from [KS, Proposition 1] that if R is a Noetherian FCR-algebra over K and G is a finite group of automorphisms of R, where $|G|$ and the characteristic of K are relatively prime, then the ring of invariants $A = R^G$ is also an FCR-algebra. Hence one can consider invariants of finite groups of the automorphisms described in Proposition 1.1 for FCR down–up algebras. The case Proposition 1.1(1) was considered by Jordan and Wells in [JW]. Proposition 1.1(2) applies to $U(\mathfrak{sl}_2)$, and Proposition 1.1 shows that any finite subgroup G of $SL(K, 2)$ acts on $U(\mathfrak{osp}(1, 2))$, and hence it follows that $(U(\mathfrak{osp}(1, 2)))^G$ is also an FCR-algebra.

2. Dimension four extensions of down–up algebras

M. Artin, J. Tate, and M. Van den Bergh [ATV] have given a complete characterization of connected graded Artin–Schelter regular algebras of dimension three, generated in degree one; this work has been extended by D. Stephenson [S1,S2] for algebras generated in other degrees. Dimension four is less well understood. Work initiated by L. Le Bruyn, S.P. Smith, and M. Van den Bergh [LSV] on producing AS-algebras of dimension 4 as central extensions of AS-algebras of dimension three has been generalized by T. Cassidy [C1]. Cassidy considers extensions H by a normal element z where $A = H/\langle z\rangle$ is an AS-regular algebra of global dimension 3. Cassidy's results give sufficient conditions for such an extension H to be AS-regular of global dimension 4 when degree $z = 1$ (in this case z is central, and by [C1, Remark 3.13] any normal extension by an element of degree 1 is a Zhang-twist of a central extension) [C1, Theorem 1.2], degree $z = 2$ [C1, Theorem 1.4], degree $z = 3$ [C1, Theorem 3.10], and degree $z > 3$ [C1, Theorem 3.8].

Cassidy's results can be used to produce connected graded Artin–Schelter regular algebras H of dimension 4 generated by u, d, z, where z is a normalizing element of H and $H/\langle z\rangle \cong A(\alpha, \beta, 0) = A$. Specifically consider the algebras $H = H(\alpha, \beta, \gamma)$ satisfying:

$$\begin{aligned} d^2u - \alpha dud - \beta ud^2 - \gamma dz^2 &= 0,\\ du^2 - \alpha udu - \beta u^2d - \gamma uz^2 &= 0,\\ uz = zu, \quad \text{and} \quad dz &= zd. \end{aligned} \tag{1}$$

These algebras have been considered by Benkart and Roby [BR, open problem (f)], Bauwens [Bau] (see also [CM, Section 6.2]), and Cassidy [C2]. When $\beta \neq 0$ then H is an AS-regular algebra of global dimension 4 (see e.g. [C2, Proposition 3.2]). Note that by [C2, Lemma 4.2] H has a basis of elements of the form $u^i(du)^j d^k z^\ell$.

Let g be a graded automorphism of H that when restricted to the span of $\{u, d, z\}$ is given by the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

that is,

$$\begin{aligned} U = g(u) &= a_{11}u + a_{21}d + a_{31}z,\\ D = g(d) &= a_{12}u + a_{22}d + a_{32}z, \quad \text{and}\\ Z = g(z) &= a_{13}u + a_{23}d + a_{33}z. \end{aligned}$$

We will show that $a_{13} = a_{23} = a_{31} = a_{32} = 0$. Since z is central, so is $g(z)$ and $g(z)u = ug(u)$. Hence $a_{13}u^2 + a_{23}du + a_{33}zu = a_{13}u^2 + a_{23}ud + a_{33}zu$, and thus $a_{23}(du - ud) = 0$, which implies that $a_{23} = 0$. In a similar manner, commuting with d yields that $a_{13} = 0$, and hence $a_{33} \neq 0$.

Since g is an automorphism, U, D and Z must satisfy the relations

$$D^2U - \alpha DUD - \beta UD^2 - \gamma DZ^2 = 0, \tag{2}$$

$$DU^2 - \alpha UDU - \beta U^2D - \gamma UZ^2 = 0. \tag{3}$$

Substituting and calculating the coefficient of z^3 (in the original basis) yields the equations

$$a_{32}^2 a_{31}(1-\alpha-\beta) - \gamma a_{32} a_{33}^2 = 0$$
$$a_{31}^2 a_{32}(1-\alpha-\beta) - \gamma a_{31} a_{33}^2 = 0.$$

If either $a_{31} \neq 0$ or $a_{32} \neq 0$ we have that $\gamma = a_{31}a_{32}(1-\alpha-\beta)/a_{33}^2$. If $\alpha+\beta = 1$ then since $\gamma \neq 0$ we have $a_{32} = a_{31} = 0$. Thus we may assume that $\alpha+\beta \neq 1$.

Since g leaves $\langle z\rangle$ invariant, g induces an automorphism of $H/\langle z\rangle$. This is isomorphic to the down–up algebra $A(\alpha,\beta,0)$. Except for the case that $\beta = -1$, Proposition 1.1 gives that g must be given by a matrix of the form

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and $U = g(u) = a_{11}u + a_{31}z$, $D = g(d) = a_{22}d + a_{32}z$, and $Z = g(z) = a_{33}z$. Substituting in Eq. (2) and calculating the coefficient of d^2z, we obtain the equation

$$a_{22}^2 a_{31} - \alpha a_{22}^2 a_{31} - \beta a_{22}^2 a_{31} = 0.$$

Since $a_{22} \neq 0$ and $(1-\alpha-\beta) \neq 0$, we have that $a_{31} = 0$. Calculating the coefficient of uz^2 we obtain the equation

$$a_{32}^2 a_{11} - \alpha a_{32}^2 a_{11} - \beta a_{32}^2 a_{11} = 0.$$

This implies that $a_{32} = 0$.

In the final case with $\beta = -1$, $H/\langle z\rangle$ is the down–up algebra $A(\alpha,-1,0)$ with $\alpha \neq 2$. This admits automorphisms g having matrix

$$\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

that is, $U = g(u) = a_{21}d + a_{31}z$, $D = g(d) = a_{12}u + a_{32}z$, and $Z = g(z) = a_{33}z$. Substituting in Eq. (2) and calculating the coefficient of dz^2 gives the equation

$$a_{32}^2 a_{21} - \alpha a_{32}^2 a_{21} + a_{32}^2 a_{21} = 0.$$

Since $a_{21} \neq 0$ and $\alpha \neq 2$ we have $a_{32} = 0$. In a similar manner calculating the coefficient of u^2z yields that $a_{31} = 0$.

Hence a graded automorphism g of H has $g(z) = \lambda z$ for $0 \neq \lambda \in K$, and g induces an automorphism of $A = H/\langle z\rangle$, and so by [JZ2, Proposition 2.4] $\operatorname{hdet}_H g = (\operatorname{hdet}_A g)\lambda = (\det g|V)^2\lambda$. We will show that this computation can be used to produce examples of rings of invariants of dimension 4 that satisfy the Gorenstein property. It is a straightforward exercise to find the graded automorphisms of H of the type above.

Proposition 2.1. *Let $H = H(\alpha, \beta, \gamma)$ be the homogenization of $A = A(\alpha, \beta, \gamma)$ defined by the relations* (1) *above with $\beta \neq 0$ and $\gamma \neq 0$. The graded automorphisms of H are as listed below; in each case the conditions that guarantee g has finite order and that* $\operatorname{hdet} g = 1$ *are given.*

1. *For any α and β the maps $g(u) = ru$, $g(d) = wd$, and $g(z) = \lambda z$ for $r, w, \lambda \in K$ give graded automorphisms of H if and only if $\lambda^2 = rw$. The element g is of finite order with* $\operatorname{hdet} g = 1$ *if and only if $\lambda^5 = 1$ and r, w are roots of unity.*
2. *When $\beta = -1$ the maps $g(u) = td$, $g(d) = su$ and $g(z) = \lambda z$ for $s, t, \lambda \in K$ give graded automorphisms of H if and only if $\lambda^2 = st$. The element g is of finite order with* $\operatorname{hdet} g = 1$ *if and only if $\lambda^5 = 1$ and s and t are roots of unity.*
3. *When $\alpha = 0, \beta = 1$ the linear map $g(u) = ru + td$, $g(d) = su + wd$, and $g(z) = \lambda z$ for $r, t, s, w, \lambda \in K$ gives a graded automorphism of H if and only if $\lambda^2 = \det g|_V = rw - st$. An element g of finite order has* $\operatorname{hdet} g = 1$ *if and only if $\lambda^5 = 1$.*

Example 2.2. Next consider [C1, Example 5.1], which gives a normal AS-regular extension H of the down–up algebra $A = A(2, -1, 0)$, the enveloping algebra of the three-dimensional Heisenberg Lie algebra. Let $f_1 = d^2u - \alpha dud - \beta ud^2$ and $f_2 = du^2 - \alpha udu - \beta u^2d$. The connected graded algebra H is generated by $x = u$, $y = d$, z, where z has degree 2, and the relations in H are:

$$f_1 + (x + y)z = 0, \qquad f_2 + (x + y)z = 0, \qquad xz = zy, \quad \text{and} \quad yz = zx.$$

In Cassidy's notation we have

$$M = \begin{bmatrix} y^2 & xy - 2yx \\ yx - 2xy & x^2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let g be a graded automorphism of H of the form $g(x) = a_{1,1}x + a_{2,1}y$, $g(y) = a_{1,2}x + a_{2,2}y$, and $g(z) = \lambda z$. Since $xz = zy$ we have

$$g(x)\lambda z = \lambda z g(y),$$
$$(a_{1,1}x + a_{2,1}y)z = z(a_{1,2}x + a_{2,2}y),$$
$$a_{1,1}xz + a_{2,1}yz = a_{1,2}yz + a_{2,2}xz.$$

The condition that $yz = zx$ adds no new restriction. Hence $g|_{K\langle x,y\rangle}$ is the linear map with coordinate matrix Σ of the form:

$$\Sigma = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Since $g(x) = ax + by$ and $g(y) = bx + ay$, one can check that in $K\langle x, y\rangle$

$$g(f_1) = \bigl(a^2 - b^2\bigr)(af_1 - bf_2) \quad \text{and} \quad g(f_2) = \bigl(a^2 - b^2\bigr)(-bf_1 + af_2)$$

and hence g applied to the relations of H gives the following equations in $K\langle x, y, z\rangle$:

$$g\bigl(f_1 + (x + y)z\bigr) = \bigl(a^2 - b^2\bigr)(af_1 - bf_2) + (a + b)\lambda(x + y)z,$$
$$g\bigl(f_2 + (x + y)z\bigr) = \bigl(a^2 - b^2\bigr)(-bf_1 + af_2) + (a + b)\lambda(x + y)z,$$

which must be linear combinations of $f_1 + (x + y)z$ and $f_2 + (x + y)z$. This forces $\lambda = (a-b)^2$. Hence g gives a graded automorphism of H when $a \neq b$. Furthermore, the inverse of such a g is an automorphism of this form. The hdet $\Sigma = (a+b)^2(a-b)^2$, and hdet $g = (a-b)^4(a+b)^2$. Next we find the finite order automorphisms of this form, where a and b are chosen so hdet $g = 1$. Since g has finite order, λ and hence $a - b$ must be a root of unity, and since Σ has finite order, det $\Sigma = (a - b)(a + b)$ and hence $a + b$ must be roots of unity. Hence let $\omega = a - b$ be any primitive root of unity; then $a + b = \pm\omega^{-2}$. This leads to two parameterized families of automorphisms. First those with $a = (\omega + \omega^{-2})/2$, $b = (\omega^{-2} - \omega)/2$, and $\lambda = \omega^2$; second those with $a = (\omega - \omega^{-2})/2$, $b = (-\omega^{-2} - \omega)/2$, and $\lambda = \omega^2$. The eigenvalues of Σ are $a - b = \omega$ and $a + b = \pm\omega^{-2}$ and hence any matrix of this form has finite order. Hence there are a countable number of matrices of finite order of this form with hdet $= 1$. Matrices of the form of Σ commute, so any finite group of such automorphisms must be an Abelian group. As one example the Klein-4 group acts on H as the matrices

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

with $\lambda = 1$, and each element of the group has hdet $g = 1$. The fixed rings of H under these groups are rings satisfying the AS–Gorenstein condition.

3. Generalized Weyl algebras

Let R be a ring, h an element in the center of R, and σ an automorphism of R. The ring $A = R(\sigma, h)$ generated over R by two generators X^+ and X^- subject to the relations $X^-X^+ = h$, $X^+X^- = \sigma(h)$, $X^+r = \sigma(r)X^+$, $X^-r = \sigma^{-1}(r)X^-$ is called a generalized Weyl algebra (or hyperbolic ring). These rings were studied by V.V. Bavula, D.A. Jordan, and A.L. Rosenberg (see, for example, [Ba1,J1,R]). In [KMP] it is shown that Noetherian down–up algebras $A(\alpha, \beta, \gamma)$, with $\beta \neq 0$, are generalized Weyl algebras $R(\sigma, h)$, where

$R = K[s,t]$, $h = s$, and σ is the automorphism given by $\sigma(s) = t$ and $\sigma(t) = \alpha t + \beta s + \gamma$ (namely take $X^+ \leftrightarrow d$, $X^- \leftrightarrow u$, $h = s \leftrightarrow ud$, and $t \leftrightarrow du$). The homogenization of A in Proposition 2.1 is a generalized Weyl algebra where $R = K[s,t,z]$, $h = s$, and σ is the automorphism given by $\sigma(s) = t$, $\sigma(t) = \alpha t + \beta s + \gamma z^2$, and $\sigma(z) = z$. Jordan has defined a closely related notion of ambiskew polynomial ring; over an algebraically closed field a Noetherian down–up algebra is also an ambiskew polynomial ring [J2]. We first note that the class of generalized Weyl algebras with $R = K[s,t]$, σ a linear map, and h a linear polynomial in R is larger than the class of down–up algebras.

Example 3.1. Let $R = K[s,t], \sigma(s) = r_1 s, \sigma(t) = r_2 t$ for $r_i \in K$ with $r_i \neq 1$ for $i = 1, 2$, and $h = s$. We will show that $A = R(\sigma, h)$ is not isomorphic to a down–up algebra by considering the largest commutative image A/I of A (as in [CM, Section 4.2]). First note that the relations in A are

$$X^-X^+ = s \quad \text{and} \quad X^+X^- = r_1 s,$$
$$X^- s = r_1^{-1} s X^- \quad \text{and} \quad X^- t = r_2^{-1} t X^-,$$
$$X^+ s = r_1 s X^+ \quad \text{and} \quad X^+ t = r_2 t X^+.$$

Hence $s \in I$. Since $X^- t - t X^- = (r^{-1} - 1) t X^-$ we have $t X^- \in I$ and $X^- t \in I$, and similarly $t X^+$ and $X^+ t \in I$. Note that $A/\langle t X^-, X^- t, t X^+, X^+ t, s\rangle$ is commutative. Then it follows that A/I is isomorphic to $B = K[a,b,c]/\langle ab, ac, bc\rangle$ where under the isomorphism $a \leftrightarrow t, b \leftrightarrow X^-$, and $c \leftrightarrow X^+$. We claim that B has exactly three minimal prime ideals $P_1 = \langle a, b\rangle$, $P_2 = \langle b, c\rangle$, and $P_3 = \langle a, c\rangle$; this follows since $B/P_1 \cong K[c]$, so it is prime (similarly for P_2 and P_3), and $P_1 P_2 P_3 = 0$ implies that these ideals P_i are all the minimal prime ideals of B. By [CM, Proposition 4.2] no Noetherian down–up algebra $A(\alpha, \beta, \gamma)$ has this property. The algebra A is an iterated Ore extension $K[t][X^+; \tau][X^-; \tau^{-1}]$, so is a graded AS-regular algebra of dimension 3.

More generally note that any generalized Weyl algebra $A = R(\sigma, h)$ with $R = K[s,t]$, $\sigma(s) = r_1 s, \sigma(t) = r_2 t$ and $h = a_0 s + a_1 t + a_2$ for $a_i, r_i \in K$ can be written as an iterated Ore extension of the form $A = K[t][X^+; \tau][X^-; \tau^{-1}, \delta]$, where $\tau(t) = r_2 t$, $\tau(X^+) = r_1 X^+$, $\delta(t) = 0$, and $\delta(X^+) = a_1(r_2 - r_1)t + (1 - r_1)a_2$; hence such a generalized Weyl algebra A has finite global dimension. It follows from [Ba2, Theorem 3.7] that since $\langle s, t\rangle$ is an ideal of height 2 invariant under σ that the global dimension of $A = R(\sigma, h)$ is 3.

In this section we consider certain automorphisms of generalized Weyl algebras. Note that if R is a connected graded ring, σ is a graded homomorphism, and if degree X^+ and degree X^- can be chosen so that degree X^++ degree X^- = degree (h), then $R(\sigma, h)$ is a graded ring. We first note that generalized Weyl algebras often satisfy the Auslander–Gorenstein conditions.

Proposition 3.2. *If R is Noetherian and satisfies the Auslander–Gorenstein conditions, then the generalized Weyl algebra $A = R(\sigma, h)$ satisfies the Auslander–Gorenstein conditions. If, in addition, A is a connected graded ring, then it satisfies the Artin–Schelter–Gorenstein conditions.*

Proof. As is shown in [Ba2, pp. 88–89] (or see [J1]), a generalized Weyl algebra $R(\sigma, h)$ is always a factor ring of an iterated skew polynomial extension (this iterated skew polynomial ring is also an ambiskew polynomial extension) of R in the following way. First form the polynomial ring $R[z]$, and consider the generalized Weyl algebra $R[z](\sigma, h+z)$, where σ is extended to $R[z]$ by taking $\sigma(z) = z$. It can be checked that $R[z](\sigma, x+z)$ is an iterated skew polynomial ring extension of R; $R[z](\sigma, h+z) \cong R[X^-; \sigma^{-1}][X^+; \sigma, \delta]$, where the automorphism σ is extended to $R[X^-; \sigma^{-1}]$ by $\sigma(X^-) = X^-$, and the σ-derivation δ is defined by $\delta(r) = 0$ for all $r \in R$ and $\delta(X^-) = \sigma(h) - h$. Furthermore, it can be checked z is in the center of $R[z](\sigma, h+z)$, z is regular, and $R[z](\sigma, h+z)/(z) \cong R(\sigma, h)$.

By [Ek, Theorem 4.2] $R[z](\sigma, h+z)$ satisfies the Auslander–Gorenstein conditions. But z is a central regular element, so we can conclude from [ASZ, Proposition 2.1] that the Auslander–Gorenstein conditions carry over to the factor ring $R[z](\sigma, h+z)/(z) \cong R(\sigma, x) \cong A$. If A is a connected graded ring it satisfies the Artin–Schelter Gorenstein conditions by [L, Theorem 6.3]. $\square$

As noted in [Ba2] if R has finite global dimension it is possible that $A = R(\sigma, h)$ has infinite global dimension. But in cases such as [Ba2, Theorem 5.1], where R is an Auslander-regular ring but A has infinite global dimension, it follows from the preceding result that the injective dimension of A is finite (and the Auslander condition is satisfied), so these rings have some nice homological properties, even though their global dimensions are infinite. In [Ba2, Theorem 3.5] necessary and sufficient conditions are given for $R(\sigma, h)$ to have finite global dimension. Sufficient conditions for the global dimension of $R(\sigma, h)$ to be infinite are given in [Ba2, Lemma 3.6].

Next we consider automorphisms of generalized Weyl algebras that preserve the generalized Weyl structure. Let g be an automorphism of $A = R(\sigma, h)$ such that $g|_R$ is an automorphism of a K-algebra R. Let $V = \langle X^-, X^+\rangle$ be the K-vector space spanned by X^- and X^+. Suppose $g|_V$ is represented by the matrix $\left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$ for elements $w, x, y, z \in K$; that is, $g(X^-) = wX^- + yX^+$ and $g(X^+) = xX^- + zX^+$.

Since g must preserve the relation $X^-X^+ = h$, we must have

$$\begin{aligned} g(h) = g(X^-X^+) &= (wX^- + yX^+)(xX^- + zX^+) \\ &= wx(X^-)^2 + wzX^-X^+ + xyX^+X^- + yz(X^+)^2 \\ &= wx(X^-)^2 + wzh + xy\sigma(h) + yz(X^+)^2. \end{aligned} \tag{4}$$

Because A is $\mathbb{Z}$-graded in powers of X^- and X^+ and $g(h) \in R$, it follows that $wx = 0$ and $yz = 0$. Consequently,

$$g|_V = \begin{bmatrix} w & 0 \\ 0 & z \end{bmatrix} \quad \text{or} \quad g|_V = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}.$$

Case 1. Assume $g|_V = \left[\begin{smallmatrix} w & 0 \\ 0 & z \end{smallmatrix}\right]$.

From equation (4) we see that $g(h) = wzh$. Applying g to the relation $X^+r = \sigma(r)X^+$ yields $g(X^+r) = g(X^+)g(r) = zX^+g(r) = z\sigma(g(r))X^+$ and $g(\sigma(r)X^+) = g(\sigma(r)X^+) =$

$zg(\sigma(r))X^+$, so that $g(\sigma(r)) = \sigma(g(r))$ for all $r \in R$. The other relations yield no additional requirements. Conversely, if these two conditions hold, g will preserve the relations of the generalized Weyl algebra and hence define an automorphism.

Case 2. Assume $g|_V = \left[\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right]$.

We proceed in a manner analogous to that in Case 1, and obtain $g(h) = g(X^-X^+) = xyX^+X^- = xy\sigma(h)$. If $r \in R$, then $g(X^+r) = g(X^+)g(r) = xX^-g(r) = x\sigma^{-1}(g(r))X^-$ and $g(\sigma(r)X^+) = g(\sigma(r))g(X^+) = xg(\sigma(r))X^-$. Consequently, we must have $g(h) = xy\sigma(h)$ and $g(\sigma(r)) = \sigma^{-1}(g(r))$ for all $r \in R$. As in Case 1, these two conditions are necessary and sufficient.

Summarizing we have the following proposition.

Proposition 3.3. *Let g be an automorphism of a K-algebra R. The nonsingular matrix $\left[\begin{smallmatrix} w & 0 \\ 0 & z \end{smallmatrix}\right]$ with $w, z \in K$ can be used to extend g to an automorphism (still called g) of $A = R(\sigma, h)$ that preserves the generalized Weyl algebra structure of A if and only if $g(h) = wzh$ and $g(\sigma(r)) = \sigma(g(r))$ for all $r \in R$. The nonsingular matrix $\left[\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right]$ with $x, y \in K$ can be used to extend g to an automorphism (still called g) of A that preserves the generalized Weyl algebra structure of A if and only if $g(h) = xy\sigma(h)$ and $g(\sigma(r)) = \sigma^{-1}(g(r))$ for all $r \in R$.*

Example 3.4. First note that a graded generalized Weyl algebra can have graded automorphisms that do not preserve the generalized Weyl structure, and hence are not as described in Proposition 3.3. For example, the graded automorphisms of Proposition 1.1, parts 1 and 2, satisfy the appropriate conditions of Proposition 3.3 but the automorphisms of Proposition 3.3, part 3, do not preserve the generalized Weyl structure of $A(0, 1, 0)$ or $A(-2, 1, 0)$.

Example 3.5. Some cases of the diagonal automorphisms described in Proposition 3.3 have been considered before. When A is an arbitrary generalized Weyl algebra $R(\sigma, h)$ and $g|_R$ is the identity on R then the commuting condition $g(\sigma(r)) = \sigma(g(r))$ of Proposition 3.3 is satisfied. If the condition $g(h) = h = wzh$ is also satisfied then the diagonal automorphisms of Proposition 3.3 are the class of automorphisms of generalized Weyl algebras studied by Jordan and Wells. More generally if $\lambda \in R$ is a central unit of R, it is shown in [BJ,JW] that the map $\Theta_\lambda : A \to A$ defined by $\Theta_\lambda(X^+) = X^+\lambda$, $\Theta_\lambda(X^-) = \lambda^{-1}X^-$, and $\Theta_\lambda(r) = r$ is an automorphism of $A = R(\sigma, h)$. When R is a commutative algebra over an algebraically closed field, $\lambda \in K$ is a primitive nth root of unity, and $G = \langle\Theta_\lambda\rangle$, then the ring of invariants A^G is calculated in [JW, Section 2.7] (and some specific examples are given in Section 2.8); in this case the fixed subring A^G is shown to be isomorphic to the generalized Weyl algebra $R(\sigma^n, h_n)$, where

$$h_n = \prod_{j=0}^{n-1} \sigma^{-j}(h).$$

It follows from Proposition 3.2 that when R satisfies the Auslander–Gorenstein conditions and $G = \langle \Theta_\lambda \rangle$ then A^G satisfies the Auslander–Gorenstein conditions. This extends results of Hodges [H, Theorem 2.1], who showed that a class of fixed rings of generalized Weyl algebras (including the ring of invariants of the usual Weyl algebra under the group generated by this diagonal automorphism Θ_λ) satisfy the Auslander–Gorenstein conditions. Furthermore, Hodges [H, Theorem 4.4] calculates the global dimensions of the class of generalized Weyl algebras with $R = K[x]$ and $\sigma(x) = x - 1$. It follows from this calculation (as well as from the fact that $A_1(K)$ is a simple ring) that the ring of invariants of the Weyl algebra under this cyclic group of diagonal automorphisms has global dimension 1. However, in general, the global dimension of the invariant subring under $G = \langle \Theta_\lambda \rangle$ will not be finite. In fact when R is a graded ring, σ is a graded automorphism, and h has positive degree, it follows from [Ba2, Lemma 3.6(ii)] that for $G = \langle \Theta_\lambda \rangle$ the global dimension of the fixed ring A^G is infinite. When A is a down–up algebra with $\gamma \neq 0$, since $\sigma(h) = \sigma(s) = t$ and $\sigma^{-1}(s) = \beta^{-1}(t - \alpha s - \gamma)$, it follows that h and $\sigma^{-1}(h)$ are factors of h_n with $Rh + R\sigma^{-1}(h) \neq R$, so that for $G = \langle \Theta_\lambda \rangle$ the fixed ring A^G has infinite global dimension by [Ba2, Lemma 3.6(ii)]. However, the results of [H] can be used to produce examples of rings of invariants with finite global dimension when A is not simple by taking R to be $K[x_1, \ldots, x_n]$, $\sigma(x_1) = x_1 - 1$, $\sigma(x_j) = x_j$ and choosing h as in [Ba2, Theorem 5.1]. Then for $G = \langle \Theta_\lambda \rangle$ both A and A^G will have global dimension n.

An interesting feature of Example 3.5 is that the fixed ring of $R(\sigma, h)$ under the group of automorphisms is again a generalized Weyl algebra. One can view this fact as a noncommutative analogue of the Shephard–Todd–Chevalley theorem (see e.g. [ST,C], [S, p. 44–49], [AP]), which states that the algebra of invariants of a finite group G acting on the symmetric algebra of a finite-dimensional vector space V over a field K ($= \mathbb{R}$ or $\mathbb{C}$) is a polynomial algebra over K if (and only if) G is generated by pseudo-reflections (linear homomorphisms that can be diagonalized with exactly one eigenvalue (of multiplicity one) that is not equal to 1). Next we create other examples of groups acting on generalized Weyl algebras with fixed rings that are also generalized Weyl algebras.

Example 3.6. Let $R = \mathbb{C}[s, t]$ be a commutative polynomial ring, let $h = s$ and let σ be a diagonal map $\sigma(s) = \sigma_1 s$ and $\sigma(t) = \sigma_2 t$ for scalars σ_i in $\mathbb{C}$. Let $G = \mathbb{Z}_n \times \mathbb{Z}_n$ be the group generated by the pseudo-reflections

$$g_1 = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix},$$

where ω is a primitive nth root of 1. Let $g_i(s) = \omega s$ and $g_i(t) = t$. Then g_i satisfy the conditions of Propositions 3.3 and so define automorphisms of $A = R(\sigma, h)$. The fixed ring A^G is isomorphic to $R^G(\sigma^n, h_n)$, where

$$h_n = \prod_{j=0}^{n-1} \sigma^{-j}(h).$$

The ring A is isomorphic to the iterated Ore extension $K[t][X^+,\sigma][X^-,\sigma^{-1}]$; when $\sigma_i^{n-1}=1$ the rings A and A^G are isomorphic.

Example 3.7. The usual Weyl algebra $A_1(\mathbb{C})$ also has automorphisms of the skew-diagonal type that preserve the generalized Weyl structure. If the Weyl algebra is given by $X^-X^+ - X^+X^- = 1$ it is a generalized Weyl algebra with $R=\mathbb{C}[t], h=t$, and $\sigma(t)=t-1$. Define g as the linear automorphism of R with $g(t)=1-t$ and $g(X^-)=aX^+, g(X^+)=-a^{-1}X^-$ for any $a\neq 0\in\mathbb{C}$. Then $g(t)=a(-a^{-1})\sigma(t)$, and $\sigma(g(\sigma(r)))=g(r)$ for all $r\in R$, $g^4=1$, and hence these automorphisms satisfy the conditions of Proposition 3.3. As in Example 1.4 the elements $T_1=X^- - iaX^+$ and $T_2=X^- + iaX^+$ generate $A_1(\mathbb{C})$ and have $g(T_1)=iT_1, g(T_2)=-iT_2$. Furthermore, replacing T_1 by $T_1/(\sqrt{-2ia})$ and T_2 by $T_2/(\sqrt{-2ia})$ these generators satisfy both the relation $T_2T_1-T_1T_2=1$ and $g(T_1)=iT_1$, $g(T_2)=-iT_2$. Hence by [JW] the fixed ring A^G under the cyclic group G generated by g is the ring generated by T_2T_1, T_1^4 and T_2^4, and it is the generalized Weyl algebra $\mathbb{C}[s](\sigma^4,h_4)$, where $s=T_2T_1$, $\sigma(s)=s-1$ and $h_4=(s+3)(s+2)(s+1)s$.

Moreover, the fixed ring of $A_1(\mathbb{C})$ under the binary dihedral group G of order $4n$ generated by matrices

$$g_1=\begin{bmatrix}\omega & 0\\ 0 & \omega^{-1}\end{bmatrix}\quad\text{and}\quad g_2=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix},$$

where ω is a $2n$th root of unity (see [AHV, p. 84]) can be described as follows. First form the fixed ring B under the cyclic group G_1 generated by g_1; by [JW] B is the $\mathbb{C}$-algebra generated by t, $(X^+)^{2n}$, and $(X^-)^{2n}$, and B can be written as a generalized Weyl algebra $B=\mathbb{C}[t](\sigma^{2n},h_{2n})$, where $h_{2n}=\prod_{j=0}^{2n-1}(t+j)$. The automorphism g_2 restricts to an automorphism on $\mathbb{C}[t]$ where $g_2(t)=1-t$, and g_2 restricts to an automorphism of B, where $g_2((X^+)^{2n})=(-1)^n(X^-)^{2n}$ and $g_2((X^-)^{2n})=(-1)^n(X^+)^{2n}$. One can check that $\sigma^{2n}(g_2(\sigma^{2n}(r)))=g_2(r)$ for all $r\in\mathbb{C}[t]$ and $g_2(h_{2n})=(-1)^n(-1)^n\sigma^{2n}(h_{2n})$, so that g_2 restricts to an automorphism of B that preserves the generalized Weyl structure of B. Since $A_1(\mathbb{C})^G=B^{G_2}$ it suffices to describe B^{G_2}. First we claim that the fixed ring $\mathbb{C}[t]^{G_2}$ is $\mathbb{C}[t^2-t]$, the commutative polynomial ring in t^2-t. To see this note that $t^m=t^{m-2}(t^2-t)+t^{m-1}$ for $m\geqslant 2$, so by induction any polynomial in $\mathbb{C}[t]$ can be written in the form $a_1(t^2-t)+a_2(t^2-1)t$ where $a_i(t^2-t)$ are polynomials in $\mathbb{C}[t^2-t]$. If $a_1(t^2-t)+a_2(t^2-1)t$ is fixed under g_2 then $a_2(t^2-t)=0$, so $\mathbb{C}[t]^{G_2}$ is $\mathbb{C}[t^2-t]$. The generalized Weyl algebra $A_1(\mathbb{C})^{G_1}$ is graded in powers of $(X^+)^{2n}$ and $(X^-)^{2n}$, with zero degree component $\mathbb{C}[t]$, so it follows that its fixed ring under G_2 is generated over $\mathbb{C}[t^2-t]$ by $((X^+)^{2n})^m+((X^-)^{2n})^m$ and $t((X^+)^{2n})^m+(1-t)((X^-)^{2n})^m$ for $m\geqslant 1$. By induction one can show that B^{G_2} is generated over $\mathbb{C}[t^2-t]$ by $(X^+)^{2n}+(X^-)^{2n}$ and $t(X^+)^{2n}+(1-t)(X^-)^{2n}$.

Any element of $SL(\mathbb{C},2)$ acts on the Weyl algebra, so there are also finite order automorphisms of $A_1(\mathbb{C})$ that do not preserve its generalized Weyl structure.

Next we compute the homological determinant of the graded automorphisms described above. These results extend the results of Theorem 1.5 for graded down–up algebras when the automorphisms preserve the generalized Weyl structure ($R = K[du, ud]$, $h = ud$, $\sigma(ud) = du$, and $\sigma(du) = \alpha du + \beta(ud)$).

Theorem 3.8. *Let R be a commutative graded ring of finite global dimension d_1, σ be a graded automorphism of R, and h be a homogeneous element of R of degree 2. Then by defining degree X^+ and degree X^- to be 1, then $A = R(\sigma, h)$ is a graded ring. If A is a Artin–Schelter regular ring of dimension d_2, then $d_2 = d_1 + 1$. The graded automorphism $g(X^+) = wX^+$ and $g(X^-) = zX^-$ of case 1 of Proposition 3.3 has* $\operatorname{hdet} g = \operatorname{hdet} g|_R$. *The graded homomorphism $g(X^+) = yX^-$ and $g(X^-) = xX^+$ as in case 2 of Proposition 3.3 has* $\operatorname{hdet} g = -\operatorname{hdet} g|_R$.

Proof. First consider the diagonal map $g(X^+) = wX^+$ and $g(X^-) = zX^-$. We will show that $\operatorname{hdet} g = (-1)^{d_2-d_1-1} \operatorname{hdet} g|_R$ for any graded automorphism g. It follows from [Ba2, Theorem 2.7] that $d_2 - d_1 = 0$ or 1. Then applying the result to the identity map, which has $\operatorname{hdet} g = \operatorname{hdet} g|_R = 1$, we conclude $d_2 - d_1 = 1$. Then we shall show that in the skew diagonal case $\operatorname{hdet} g = -\operatorname{hdet} g|R$. Since A is Artin–Schelter regular, the homological determinant of g can be computed using the trace of g.

Under the given grading we have $A_0 = R_0$, $A_1 = R_1 \oplus R_0X^+ \oplus R_0X^-$, $A_2 = R_2 \oplus R_0(X^+)^2 \oplus R_1X^+ \oplus R_1X^- \oplus R_0(X^-)^2$, and in general

$$A_n = R_n \oplus \left[\bigoplus_{i=0}^{n-1} R_i (X^+)^{n-i}\right] \oplus \left[\bigoplus_{j=0}^{n-1} R_j (X^-)^{n-j}\right].$$

Computing the trace of the linear map $g|_{A_n}$ we get

$$\operatorname{tr}(g|_{A_n}) = \operatorname{tr}(g|_{R_n}) + \left(\sum_{i=0}^{n-1} \operatorname{tr} g|_{R_i} w^{n-i}\right) + \left(\sum_{j=0}^{n-1} \operatorname{tr} g|_{R_j} z^{n-i}\right).$$

Hence the trace function of g as a map on A is

$$\begin{aligned}
\operatorname{Tr}(g,t) &= \operatorname{Tr}(g|_R, t) + \operatorname{tr}(g|_{R_0})wt + \big[\operatorname{tr}(g|_{R_0})w^2 + \operatorname{tr}(g|_{R_1})w\big]t^2 + \cdots \\
&\quad + \left[\sum_{i=0}^{n-1} \operatorname{tr}(g|_{R_i})w^{n-i}\right]t^n + \cdots + \operatorname{tr}(g|_{R_0})zt + \big[\operatorname{tr}(g|_{R_0})z^2 + \operatorname{tr}(g|_{R_1})z\big]t^2 + \cdots \\
&\quad + \left[\sum_{j=0}^{n-1} \operatorname{tr}(g|_{R_j})z^{n-j}\right]t^n + \cdots \\
&= \operatorname{Tr}(g|_R, t) + wt\big[\operatorname{tr}(g|_{R_0}) + \operatorname{tr}(g|_{R_1})t + \operatorname{tr}(g|_{R_2})t^2 + \cdots\big]\big[1 + wt + w^2t^2 + \cdots\big] \\
&\quad + zt\big[\operatorname{tr}(g|_{R_0}) + \operatorname{tr}(g|_{R_1})t + \operatorname{tr}(g|_{R_2})t^2 + \cdots\big]\big[1 + zt + z^2t^2 + \cdots\big] \\
&= \operatorname{Tr}(g|_R, t) + \operatorname{Tr}(g|_R, t)\frac{wt}{1 - wt} + \operatorname{Tr}(g|_R, t)\frac{zt}{1 - zt}
\end{aligned}$$

$$= \mathrm{Tr}(g|_R, t)\left[1 + \frac{-1}{1 - t^{-1}/w} + \frac{-1}{1 - t^{-1}/z}\right]$$
$$= \mathrm{Tr}(g|_R, t)\left[-1 + *t^{-1} + *t^{-2} + \cdots\right].$$

Hence

$$\mathrm{Tr}(g, t) = \mathrm{Tr}(g|_R, t)[-1 + \cdots],$$

and

$$(-1)^{d_2}(\operatorname{hdet} g)^{-1} t^{-\ell_2} + \cdots = \left[(-1)^{d_1}(\operatorname{hdet} g|_R)^{-1} t^{-\ell_1} + \cdots\right][-1 + \cdots].$$

Hence

$$(-1)^{d_2}(\operatorname{hdet} g)^{-1} = (-1)^{d_1+1}(\operatorname{hdet} g|_R)^{-1},$$

so that $\operatorname{hdet} g = (-1)^{d_2-d_1-1}(\operatorname{hdet} g|_R)$. When g is the identity $\operatorname{hdet} g = 1$ so that $d_2 = d_1 + 1$, and hence $\operatorname{hdet} g = \operatorname{hdet} g|_R$ as claimed.

In the second case

$$\mathrm{tr}(g|_{A_n}) = \mathrm{tr}(g|_{R_n}),$$

so $\mathrm{Tr}(g, t) = \mathrm{Tr}(g|_R, t)$ and hence

$$(-1)^{d_1+1}(\operatorname{hdet} g)^{-1} t^{-\ell_2} + \cdots = (-1)^{d_1}(\operatorname{hdet} g|_R)^{-1} t^{-\ell_1} + \cdots,$$

so

$$(-1)^{d_1+1}(\operatorname{hdet} g)^{-1} = (-1)^{d_1}(\operatorname{hdet} g|_R)^{-1}$$

and hence $\operatorname{hdet} g = -\operatorname{hdet} g|R$. □

Example 3.9. Let A be the coordinate ring of quantum 2×2 matrices, i.e. the K-algebra generated by a, b, c, d satisfying the following relations:

$$ab = qba, \qquad bd = qdb, \qquad ac = qca, \qquad cd = qdc,$$
$$bc = cb, \qquad ad - da = (q - q^{-1})bc$$

for $q \in K^*$. The ring A is an AS-regular and Auslander-regular ring of global dimension 4. It is an iterated Ore extension of the form $K[b, c][a : \tau][d; \tau^{-1}, \delta]$ and a generalized Weyl algebra $R(\sigma, h)$ with R the commutative polynomial ring $R = K[b, c, h]$, $X^- = a$, $X^+ = d$, $h = ad$, $\sigma(b) = q^{-1}b$, $\sigma(c) = q^{-1}c$, and $\sigma(h) = h + (q^{-1} - q)bc$ (σ is a graded homomorphism of R where we give h degree 2). The map g with $g(b) = -c$, $g(c) = -b$, $g(a) = a$, and $g(d) = d$ is a graded automorphism of A of order 2. The map $g|_{K[b,c,h]}$ is a pseudo-reflection (there is only one eigenvalue not equal to 1), so the fixed subring of $K[b,c,h]$ under the group generated by $g|_{K[b,c,h]}$ is the commutative polynomial ring $K[b^2+c^2,bc,h]$. The fixed ring of A under the group G generated by g is the ring generated by a, b^2+c^2, bc, d; it can be described as the ring generated by a, b', c', d satisfying the relations:

$$ab' = q^2 b'a, \qquad b'd = q^2 db', \qquad ac' = q^2 c'a, \qquad c'd = q^2 dc',$$
$$b'c' = c'b', \qquad ad - da = \left(q - q^{-1}\right)c'$$

for $q \in K^*$. The ring A^G can be graded by taking generators a, b', d in degree 1. This ring A^G can also be described as the generalized Weyl algebra $K[b', c', h](\sigma', h)$ where σ' is the graded automorphism with $\sigma'(b) = q^{-2}b$, $\sigma'(c) = q^{-2}c$, and $\sigma'(h) = h + (q^{-1} - q)c'$. The ring A^G is also an iterated Ore extension of a commutative polynomial ring so it has finite global dimension and satisfies the AS–Gorenstein condition (though by Theorem 3.8 hdet $g = -1$). Hence the fixed ring is an AS-regular ring of dimension 4. The ring A^G is not isomorphic to A when $q = -1$ since A^G is commutative but A is not.

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