Stochastic optimal control of ultradiffusion processes with application to dynamic portfolio management

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Abstract

We consider theoretical and approximation aspects of the stochastic optimal control of ultradiffusion processes in the context of a prototype model for the selling price of a European call option. Within a continuous-time framework, the dynamic management of a portfolio of assets is effected through continuous or point control, activation costs, and phase delay. The performance index is derived from the unique weak variational solution to the ultraparabolic Hamilton–Jacobi equation; the value function is the optimal realization of the performance index relative to all feasible portfolios. An approximation procedure based upon a temporal box scheme/finite element method is analyzed; numerical examples are presented in order to demonstrate the viability of the approach. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

The control of systems subject to exogenous sources of uncertainty is fundamental to many diverse fields. Although the scope of possibilities is endless, we mention by way of illustration: maintenance and quality control (e.g. modeling of breakdown phenomenon), management of information technologies (e.g. complexity and networks, tracking problems), production management (e.g. balancing global costs by shifting production from one source to another), portfolio management (e.g. option pricing), robotics and artificial intelligence, and genetic networks, to name only a few (cf. \cite{7,10,4,9}).

In this paper, we consider the approximation solvability of ultradiffusion stochastic optimal control in the context of an example from mathematical finance. To this end, an ultradiffusion process is isometric to a parameterized diffusion along a characteristic temporal trajectory (cf. \cite{23,39,28}). The conditional value function of the controlled ultradiffusion process measures the expected optimal performance relative to an indexed family of stochastic control problems and is characterized as the unique weak variational solution to the ultraparabolic Hamilton–Jacobi equation. The value function is the optimal realization of the performance index (derived from the conditional value function) relative to all feasible trajectories. Ultraparabolic equations have historically been of interest in connection with Brownian motion in phase space and have been studied in \cite{29,30,46}, among others; neither the strong maximum
principle nor interior \textit{a priori} estimates, for example, hold for ultraparabolic equations as they do for parabolic equations (cf. \cite{18,25,44,47}).

A review of stochastic optimal control theory of diffusion processes may be found in the seminal monographs (cf. \cite{7,8,15,31}). At the discrete level, one can approximate the original diffusion through an appropriate controlled Markov chain, which may be effected utilizing either deterministic (cf. \cite{32,33}) or simulated transitional states (cf. \cite{19}). Alternatively, when one is able to obtain a (deterministic) formulation and consequently apply energy methods (cf. \cite{21,20,31}), a full space/time discretization of the Hamilton–Jacobi equation is obtained from which one solves for the optimal performance and feedback control law at each time step through a marching scheme.

In terms of mathematical finance, we determine the selling price of a so-called European call option based on the value-maximizing exercise strategy in \cite{22,13}. Relative to a continuous-time model of a dynamically evolving portfolio of assets, we allow for both continuous and point control, activation costs, and phase delay. As such, our model is a generalization of the more familiar Black–Scholes framework. We follow Bensoussan and Julien \cite{6} and Aubin et al. \cite{4} in allowing friction of the more familiar Black–Scholes framework. We follow Bensoussan and Julien \cite{6} and Aubin et al. \cite{4} in allowing friction within the model to be implemented through the velocity of the wealth process.

In terms of control, Bensoussan and Julien \cite{6} formulate a model of control affected through the portfolio which accommodates certain activation costs (proportional transaction costs are not admissible, for example). This approach results in a noise controlled diffusion such that the value function satisfies a parabolic Hamilton–Jacobi equation. General transaction costs are considered in \cite{4} through the strategy control of a tychastic system and implemented in the context of a viability/capturability approach. Viability techniques formulate the tychastic optimal control problem as a deterministic dynamical game (cf. \cite{3}). A third more common, although less tractable, model offers the most direct conceptual link to Merton’s seminal work on portfolio theory (cf. \cite{40,41}). In particular, this approach maintains that dynamic management evolves through a reflecting diffusion process in which portfolio control derives from the reflection at an implicitly defined free-boundary (cf. \cite{35,13,17}).

The framework considered here admits general frictions within the context of ultradiffusion stochastic optimal control. The outline of this paper is as follows. In Section 2, the valuation problem is defined and characterized relative to continuous control involving activation costs. A temporal box scheme/finite element method forms the basis for the approximation solvability argument of Section 3. In Section 4, we extend the ultradiffusion to include point control and phase delay. Numerical results are presented in order to demonstrate the viability of the approach in Section 5. We present concluding remarks in Section 6. Generalization of the approach to include jump diffusions as well as early exercise features (cf. \cite{6}) proceeds without appreciable difficulty (cf. \cite{48,43}).

2. Valuation problem

We consider in this section the definition of the stochastic optimal control problem under continuous control subject to activation costs. In Section 2.1, we develop the model of the economy. In Section 2.2, we define the ultradiffusion and present a formal characterization of the conditional value function in terms of the Hamilton–Jacobi equation. We discuss issues related to the regularity of the stochastic optimal control problem in Section 2.3. The performance index and value function are defined in Section 2.4.

2.1. Model of the economy

We consider two financial assets in which the price per share of the bank account is denoted by $B(t)$ and that of a stock by $S(t)$; the financial assets represent a pair $(B(t), S(t)) \in (0, \infty)^2$. We suppose that the evolution of asset prices is governed by

$$dB(t) = rB(t)dt$$

(2.1)

and

$$dS(t) = \alpha S(t)dt + \sigma S(t)dw(t),$$

(2.2)

where $r > 0$ is the risk-free rate of return, $\alpha > r$ is the return and $\sigma > 0$ the volatility associated with the stock, and $dw(t)$ represents an exogenous source of uncertainty. A portfolio is a pair $(b(t), s(t))$ representing the number of shares of $B(t)$ and $S(t)$ held at time $t$, respectively. The value of the portfolio or wealth $W(t)$, such that

$$W(t) = b(t)B(t) + s(t)S(t),$$

(2.3)
is specified in terms of the assets and portfolio. We effect the wealth process through the control strategy \( \tilde{\beta}(t), \tilde{\xi}(t) \), which dictates the rate at which shares in the portfolio are bought and sold. To this end, let

\[
b(t) = b_+(t) - b_-(t), \quad s(t) = s_+(t) - s_-(t),
\]

and

\[
\tilde{\beta}(t) = (\beta_-(t), \beta_+(t)), \quad \tilde{\xi}(t) = (\xi_-(t), \xi_+(t)),
\]

such that

\[
db_+(t) = \beta_+(t)dt, \quad db_-(t) = \beta_-(t)dt,
\]

and

\[
ds_+(t) = \xi_+(t)dt, \quad ds_-(t) = \xi_-(t)dt,
\]

where \( 0 \leq \beta_\pm(t) \leq \gamma \) and \( 0 \leq \xi_\pm(t) \leq \gamma \), for some \( \gamma < \infty \). Given that \( \gamma \) is bounded, only finite transaction speeds are permissible within the model at this point (infinite transaction speeds are obtained by allowing \( \gamma \to \infty \) or as introduced in Section 4.1). For simplicity, we have considered the so-called “bang-bang” type control action in (2.4), although other models are admissible. (“Bang-bang” refers to the fact that the optimal action is either 0 or \( \gamma \); the control is either “on” or “off”.)

One possible model of transaction costs holds that they are paid for through holdings of stock (this choice has been made without any sacrifice to generality in order to reduce the complexity of the model), in which case

\[
\beta(t)B(t) + \xi(t)S(t) = - (\tilde{\xi}(t) \cdot \tilde{f}^- (t)) S(t),
\]

where

\[
\beta(t) = \beta_+(t) - \beta_-(t), \quad \xi(t) = \xi_+(t) - \xi_-(t),
\]

\[
\tilde{f}^- (t) = (f^-_+(t), f^-_-(t)),
\]

\[
f^-_\pm(t) = f^-_\pm(S, s_-, s_+, t; \xi_-, \xi_+) \text{ such that } f^-_\pm(t) \geq 0 \geq f^-_\mp(t), \text{ and } \bullet \text{ refers to the } \mathbb{R}^2 \text{ inner product.}\]

The portfolio velocity \( dW \) is then

\[
dW(t) = \left[ rW(t) + \left[ (\alpha - r)s(t) - \tilde{\xi}(t) \cdot \tilde{f}^- (t) \right] S(t) \right] dt + \sigma s(t)S(t)dW(t),
\]

which follows from (2.1)–(2.5). When transaction costs are proportional to the rate at which shares are traded, it follows that

\[
f^-_\pm(t) = \pm \kappa,
\]

where \( \kappa \) is the transaction cost per unit share per unit time (cf. [4]). As a special case, we note that a portfolio incurring zero transaction costs (i.e. \( \kappa = 0 \)) is known as self-financing.

2.2. Hamilton–Jacobi equation

To be more precise, let \((\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space with filtration \( \{\mathcal{F}(t)\}_{t \geq 0} \) with right-continuous filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) such that \( \mathcal{F}_0 \) contains all sets of \( \mathbb{P} \)-measure zero (cf. [27]). Let \[\{w(t)\}_{t \geq 0}\] be an \( \mathcal{F}(t) \)-adapted standard Wiener process on \( \mathbb{R} \) and \( s_\pm(t), \xi_\pm(t), \) and \( f^-_\pm(t) \) measurable functions adapted to \( \mathcal{F}(t) \). We assume continuous trading over a finite time interval \([0, T]\) such that \( T > \tilde{T} \), where \( \tilde{T} \) denotes the contract life of the option. Summarizing Section 2.1, the dynamics of the economy are specified by the ultradiffusion process

\[
ds_+(t) = \xi_+(t; \tilde{\xi})dt,
\]

\[
ds_-(t) = -\xi_-(t; \tilde{\xi})dt,
\]

\[
ds(t) = \alpha S(t)dt + \sigma S(t)dW(t),
\]

\[
dW(t) = \left[ rW(t) + \left[ (\alpha - r)s(t) - \tilde{\xi}(t; \tilde{\xi}) \cdot \tilde{f}^- (t) \right] S(t) \right] dt + \sigma s(t)S(t)dW(t),
\]

\[
dK(t) = \gamma (K(t)) dt + \Delta K(t) d\tilde{W}(t),
\]

where \( \Delta K(t) \) is a non-negative payoff function adapted to \( \mathcal{F}(t) \). (The \( \tilde{W}(t) \) is a standard Wiener process adapted to \( \mathcal{F}(t) \).)
for $t \in (0, \tilde{T})$, such that $s_+(0) = s_+, s_-(0) = s_-, S(0) = S$, $W(0) = W$, and $\tilde{r} = (s_+, s_-)$. Here, the initial conditions $s_\pm$ are temporal, $S$ is natural, and $W$ is essential (neither temporal nor natural).

Let $\bar{x} = (S, W)$ and $\bar{t} = (s_-, s_+, t)$, then along with the process (2.7), we consider the expected value of the optimization criteria

$$J_{\bar{x}}(T; \bar{s}) = E_{\bar{x}; \bar{s}} \left[ W(T) - \psi(S(T)) \right]$$

(2.8a)

and the conditional value function

$$V(\bar{x}, \bar{t}) = \max_{\bar{x}(t) \in [0, T]^2} J_{\bar{x}}(T; \bar{s}),$$

(2.8b)

such that $\psi(S)$ is the pay-off of the option. For example, a so-called European call option with exercise price $E$ is defined with $\psi(S) = \max(S - E, 0)$, a passport option would require $\psi(S) = 0$ (cf. [34]), and so forth. The conditional value function attempts to maximize the expected excess wealth, predicated (indexed) upon the choice of the initial portfolio $\bar{s}$.

In order to characterize the conditional value function (2.8), we introduce $\Omega = \mathbb{R} \times (0, \infty)$ and the weighted Sobolev spaces $W^{d, p, \mu}(\Omega)$ equipped with the norm

$$||u||_{d, p, \mu} = \left\{ \sum_{k \leq d} \int_{\Omega} |D^k u(\bar{x})|^p \cdot e^{-\mu |\bar{x}|} d\bar{x} \right\}^{1/p},$$

where $D^k u \in L^p(\Omega, e^{-\mu |\bar{x}|} d\bar{x})$, for all multi-index $|\alpha| \leq d$. Let $Q = (0, \infty)^2 \times (0, \tilde{T})$, then the space $L^p(\Omega; W^{d, p, \mu}(\Omega))$ consists of the set of measurable functions $h : \Omega \rightarrow W^{d, p, \mu}(\Omega)$ such that $\int_{\Omega} ||h(\bar{t})||_{d, p, \mu}^p d\bar{t} < \infty$.

Finally, we set

$$\mathcal{W}^{2, 1, p, \mu}(\Omega \times Q) = \left\{ u | u \in L^2(\Omega; W^{2, p, \mu}(\Omega)) \cap L^2(\Omega; H^1_{lo}(\Omega)), \right.$$

$$\left. \nabla u \in \left[ L^2(\Omega; W^{0, p, \mu}(\Omega)) \cap L^2(\Omega; L^2_{lo}(\Omega)) \right]^3 \right\},$$

where $\nabla u = (\partial/\partial s_-, \partial/\partial s_+, \partial/\partial \bar{t})$.

For $p \geq 2, \mu \geq 0$, we seek to determine the conditional value function $V \in \mathcal{W}^{2, 1, p, \mu}(\Omega \times Q)$, which is the unique solution of the ultraparabolic Hamilton–Jacobi equation

$$\frac{\partial V}{\partial \bar{t}} + A(\bar{t}) V + \max_{\xi(\bar{t}) \in [0, \bar{T}]} \left\{ L^\xi(\bar{t}) V + \xi_+(\bar{t}; \bar{s}) \frac{\partial V}{\partial s_+} + \xi_-(\bar{t}; \bar{s}) \frac{\partial V}{\partial s_-} \right\} = 0 \quad \text{a.e. in } \Omega \times Q,$$

(2.9a)

such that

$$V|_{\bar{t}=T} = W - \psi(S) \quad \text{in } \Omega \times \partial Q$$

(2.9b)

and

$$V|_{\bar{t}=0} = W \quad \text{in } \partial \Omega \times Q,$$

(2.9c)

where

$$A(\bar{t}) V = \frac{1}{2} \sigma^2 \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} s(t) \sigma^2 S^2 \frac{\partial^2 V}{\partial W \partial S} + \frac{1}{2} s(t) \sigma^2 S^2 \frac{\partial^2 V}{\partial S \partial W} + \frac{1}{2} s^2(t) \sigma^2 S^2 \frac{\partial^2 V}{\partial W^2},$$

$$L^\xi(\bar{t}) V = \alpha s \frac{\partial V}{\partial S} + \left[ r W + [(\alpha - r) s(t) - \xi(\bar{t}; \bar{s})] \right] \frac{\partial V}{\partial W}. $$

The optimal feedback control law $\xi^* = (\xi^+_*(t; \bar{s}), \xi^+_*(t; \bar{s}))$ satisfies

$$-\xi^*(t; \bar{s}) \cdot \bar{J}^\xi^*(t) S \frac{\partial V}{\partial W} + \xi^+_*(t; \bar{s}) \frac{\partial V}{\partial s_+} + \xi^+_*(t; \bar{s}) \frac{\partial V}{\partial s_-} + \max_{\xi(t) \in [0, \bar{T}]} \left\{ -\xi(t; \bar{s}) \cdot \bar{J}^\xi(t) S \frac{\partial V}{\partial W} + \xi_+(t; \bar{s}) \frac{\partial V}{\partial s_+} + \xi_-(t; \bar{s}) \frac{\partial V}{\partial s_-} \right\}.$$

(2.10)
For example, in the case of a self-financing portfolio, we may write (2.9a) explicitly as the nonlinear partial differential equation

\[ \{ \mathcal{H}(\bar{t}) + A(\bar{t}) + L(\bar{t}) \} V(\bar{x}, \bar{t}) = 0 \quad \text{a.e. in } \Omega \times Q, \]

where

\[ \mathcal{H}(\bar{t})V = a(V_{s+}) \frac{\partial V}{\partial s+} + a(V_{s-}) \frac{\partial V}{\partial s-} + \frac{\partial V}{\partial t}, \]

\[ L(\bar{t})V = \alpha S \frac{\partial V}{\partial S} + \sigma W + (\alpha - r)s(t)S \frac{\partial V}{\partial W}, \]

such that

\[ a(z) = \begin{cases} \gamma, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0. \end{cases} \]

We note in particular that the operator \( A(\bar{t}) + L(\bar{t}) \) is elliptic and \( \mathcal{H}(\bar{t}) \) is hyperbolic. In particular, the ultraparabolic problem (2.9) and (2.10) (resp., ultradiffusion (2.7)) reduces to a 2-parameter family of parabolic equations (resp., diffusions) along the characteristic temporal directions of \( \mathcal{H}(\bar{t}) \), which when represented in nonparametric form furnishes a solution to (2.9) (cf. [36,42,47]).

2.3. Elliptic regularization

Potential degeneracy of the ultradiffusion (2.7) may be addressed by perturbing the state equations and the control set. To this end, let \( d\tilde{w}(t) \) denote a second independent source of exogenous uncertainty, then in lieu of (2.7c) and (2.7d) we consider

\[ dS(t) = \alpha S(t)dt + \sigma S(t)dw(t) + \epsilon d\tilde{w}(t), \]

\[ dW(t) = \left\{ rW(t) + [(\alpha - r)s(t) - \xi(t; \tilde{s}) \cdot \tilde{f}(\tilde{s}(t))]S(t) \right\} dt + \sigma s(t)S(t)dw(t) + \epsilon d\tilde{w}(t), \]

for any \( \epsilon > 0 \) arbitrarily small. As such, the perturbed diffusion generator is

\[ A_\epsilon(\bar{t}) = A(\bar{t}) + \frac{1}{2} \epsilon^2 \left\{ \frac{\partial^2}{\partial \tilde{S}^2} + \frac{\partial^2}{\partial \tilde{S} \partial \tilde{W}} + \frac{\partial^2}{\partial \tilde{W}^2} \right\} \]

for which the coercivity condition holds, namely,

\[ \frac{1}{2} \left[ \sigma^2 S^2 \xi_1^2 + \epsilon^2 \right] + \left[ s(t)\sigma^2 S^2 + \epsilon^2 \right] \xi_1 \xi_2 + \frac{1}{2} \left[ s^2(t)\sigma^2 S^2 + \epsilon^2 \right] \xi_2^2 \geq \frac{1}{2} \epsilon^2 |\tilde{\xi}|^2, \]

for all \( \tilde{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2 \). We then seek the perturbed conditional value function \( V^\epsilon \) satisfying

\[ \frac{\partial V^\epsilon}{\partial t} + A_\epsilon(\bar{t})V^\epsilon + \max_{\xi(t; \tilde{s}) \in [\xi, \gamma]^2} \left\{ L(\bar{t}) \tilde{V}^\epsilon + \xi_{\tilde{s}}(t; \tilde{s}) \frac{\partial V^\epsilon}{\partial s_+} + \xi_{-\tilde{s}}(t; \tilde{s}) \frac{\partial V^\epsilon}{\partial s_-} \right\} = 0 \quad \text{a.e. in } \Omega \times Q. \]  

(2.11)

Utilizing stochastic optimal control arguments for diffusion processes applied along the temporal characteristics, it follows that there exists a unique solution \( V^\epsilon \) of (2.11). Singular perturbation arguments validate the convergence of \( V^\epsilon \) to the solution \( V \) of (2.9) as \( \epsilon \to 0^+ \) (cf. [5,14]).

A second distinct paradigm involves approximating (2.7) through a fully stochastic system, namely

\[ ds_+(t) = \xi_+(t; \tilde{s}) \cdot \tilde{f}(\tilde{s}(t)) dt + \epsilon d\tilde{w}(t), \]

\[ ds_-(t) = \xi_-(t; \tilde{s}) \cdot \tilde{f}(\tilde{s}(t)) dt + \epsilon d\tilde{w}(t), \]

\[ dS(t) = \alpha S(t)dt + \sigma S(t)dw(t) + \epsilon d\tilde{w}(t), \]

\[ dW(t) = \left\{ rW(t) + [(\alpha - r)s(t) - \xi(t; \tilde{s}) \cdot \tilde{f}(\tilde{s}(t))]S(t) \right\} dt + \sigma s(t)S(t)dw(t) + \epsilon d\tilde{w}(t). \]
for $t \in (0, \bar{T})$ and $\varepsilon > 0$ arbitrarily small, where again $d\tilde{w}(t)$ denotes a second independent exogenous source of uncertainty. In this case, the perturbed Hamilton–Jacobi equation becomes

$$
\frac{\partial V^\varepsilon}{\partial t} + A_\varepsilon(\tilde{\tau}) V^\varepsilon + \max_{\xi(t;\tilde{s}) \in [0,\gamma]} \left\{ L^2(\tilde{\tau}) V^\varepsilon + \zeta_+(t; \tilde{s}) \frac{\partial V^\varepsilon}{\partial s_+} + \zeta_-(t; \tilde{s}) \frac{\partial V^\varepsilon}{\partial s_-} \right\} = 0 \text{ a.e. in } \Omega \times Q.
$$

(2.12)

such that

$$
A_\varepsilon(\tilde{\tau}) = A(\tilde{\tau}) + \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial^2}{\partial s_-^2} + \frac{\partial^2}{\partial s_- \partial s_+} + \frac{\partial^2}{\partial s_- \partial \tilde{s}} + \frac{\partial^2}{\partial s_- \partial W} + \frac{\partial^2}{\partial s_+ \partial s_-} + \frac{\partial^2}{\partial s_+ \partial \tilde{s}} + \frac{\partial^2}{\partial s_+ \partial W} + \cdots \right\}
$$

is uniformly parabolic (cf. [15]). The functions $V^\varepsilon$ are known as a vanishing viscosity solutions in deference to their fluid mechanics origins. Singular perturbation arguments confirm the existence of a viscosity solution $V$ satisfying (2.9) such that $V^\varepsilon \to V$ as $\varepsilon \to 0^+$ (cf. [12,26,16]).

### 2.4. System optimization

We consider now the dependence of the stochastic optimal control problem (2.7) and (2.8) on the initial portfolio distribution $\tilde{s}$. To this end, let $\mathcal{M}(\tilde{x}, T) \subseteq [0, \infty)^2$ be a sufficiently regular manifold. For given $\tilde{x} \in \Omega$, a feasible trajectory (e.g. portfolio) $(s_-(t), s_+(t))$ satisfies: $s_-(0) = s_-, s_+(0) = s_+$, the deterministic equations (2.7a) and (2.7b) such that $\tilde{x}^\varepsilon(t; \tilde{s})$ is a solution of (2.10), and the terminal constraint $(s_-(T), s_+(T)) \in \mathcal{M}(\tilde{x}, T)$. Relative to a given terminal set $\mathcal{M}(\tilde{x}, T)$, we denote the set of all feasible initial portfolio distributions $\tilde{s}$ by $\mathcal{F}(\tilde{x}, T)$. We are now in a position to define the value function as an extended real-valued function. To this end, let the performance index be given by

$$
\tilde{U}(S, t; \tilde{s}) = \min\{W \mid V(S, W, s_-, s_+, t) \geq 0\}
$$

(2.13a)

and the value function be such that

$$
U(S, t) = \min\{\tilde{U}(S, t; \tilde{s}) \mid \tilde{s} \in \mathcal{F}(\tilde{x}, T)\},
$$

(2.13b)

or, by convention, $U(S, t) = \infty$ if $\mathcal{F}(\tilde{x}, T)$ is empty.

Relative to the option valuation problem, we stipulate that the option writer be fully invested in the market at expiry. To this end, let

$$
g_-(S, W) = 0
$$

and

$$
g_+(S, W) = \begin{cases} 
1, & \text{if } S > E \\
0, & \text{if } S \leq E,
\end{cases}
$$

where again $E$ is the exercise price of the option. We then suppose the terminal set $\mathcal{M}(\tilde{x}, T)$ is specified by

$$
\mathcal{M}(\tilde{x}, T) = \{(g_-(S, W), g_+(S, W)) \mid (S, W) \in \Omega\}.
$$

The value function $U(S, t) \in L^2(0, T; H^2(\Omega))$ represents the expected minimum wealth necessary at time $t$ required to meet the pay-off $\psi(S)$ at time $T$ and as such is the present value or price of the European option.

### 3. Constructive approximation

Our objective is to obtain a constructive approximation of the conditional value function. Without loss of generality, we shall do so only for self-financing European call options. We begin in Section 3.1 by examining the effect of exhausting the state and temporal spaces through a series of bounded domains. In Section 3.2, we develop the variational form of the Hamilton–Jacobi equation and consider a semi-discretization of the approximate state space through the Galerkin finite element method. We obtain a full-discretization in Section 3.3 through the application of a box type finite difference scheme to the hyperbolic operator. Estimates are provided in Section 3.4 to quantify the approximation.
3.1. Approximation on exhausting domains

To obtain a constructive approximation of $V$, we consider an exhausting sequence of bounded open domains $\{\Omega_k\}$ such that $\Omega_k \subset \Omega_{k+1}$ and $\bigcup \Omega_k = \Omega$. Moreover, let $T_k < \infty$ such that $T_k \to \infty$ monotonically as $k \to \infty$ and $Q_k := (0, T_k)^2 \times (0, T)$, then $Q_k \subset Q_{k+1}$ and $\bigcup Q_k = Q$. For $p \geq 2$, and $k \in \mathbb{N}$ sufficiently large, we seek $V^k \in \mathcal{W}^{2,1,p,0}(\Omega_k \times Q_k)$ satisfying

$$\{ \mathcal{H}(\tilde{t}) + A(\tilde{t}) + L(\tilde{t}) \} V^k = 0 \quad \text{a.e. in } \Omega_k \times Q_k, \quad (3.1a)$$

subject to the terminal conditions

$$V^k|_{\tau = \tau_k} = V^k|_{\tau = \tau_k} = V^k|_{\tau = \tau} = W - \psi(S) \quad \text{in } \overline{Q}_k \times \partial Q_k, \quad (3.1b)$$

and boundary data

$$V^k = W - \psi(S) \quad \text{in } \partial \Omega_k \times Q_k. \quad (3.1c)$$

We note that as a consequence of the approximation to a domain of finite extent, it has become necessary to impose additional “terminal” conditions relative to $s_{\pm} = \tau_k$ in (3.1b) as well as artificial boundary condition in (3.1c). The function $V^k$ approximates $V$ in the sense that

$$\int_{\Gamma} \|V - V^k\|_{L^\infty(\mathcal{G})} \to 0 \quad \text{as } k \to \infty, \quad (3.2)$$

where $\Gamma$ is any sufficiently regular curve contained in $Q$ and $\overline{\mathcal{G}}$ is any fixed compact set in $\Omega$.

Remark. If required (e.g. jump processes), one can extend $V^k$ to $\Omega \times Q$ trivially; that is, let $V_k$ be defined in $\Omega \times Q$ such that $V_k = V^k$ in $\Omega_k \times Q_k$ and $V_k = W - \psi(S)$ otherwise.

3.2. Semi-discretization in space

The semi-discrete analogue of (3.1) will be based on the variational formulation for the initial value problem. To this end, let $\rho = \bar{T}_k - s_-, \xi = \bar{T}_k - s_+, \tau = T_k - t$, and $\bar{\tau} = (\rho, \xi, \tau)$, and consider a perturbation from the optimization criteria

$$\tilde{V}^k(S, W, \rho, \xi, \tau) = V^k(S, W, \bar{T}_k - \rho, \bar{T}_k - \xi, T_k - \tau) - (W - \psi(S)).$$

Formally, the variational form is obtained by multiplying (3.1a) through by a test function $v \in H^1_0(\Omega_k) \cap H^2(\Omega_k)$, integrating over the domain $\Omega_k$, and applying the Green’s formula (cf. [7]). In particular, for $p \geq 2$, we seek a strong variational solution $\tilde{V}^k \in \mathcal{W}^{2,1,p,0}(\Omega_k \times Q_k)$ satisfying

$$- \left( \tilde{H}(\bar{\tau}) \tilde{V}^k, v \right)_k + a_k(\bar{\tau}; \tilde{V}^k, v) = (f, v)_k \quad \text{a.e. in } Q_k, \quad (3.3a)$$

for all $v \in H^1_0(\Omega_k) \cap H^2(\Omega_k)$, subject to the initial conditions

$$\tilde{V}^k|_{\rho = 0} = \tilde{V}^k|_{\xi = 0} = \tilde{V}^k|_{\tau = 0} = 0 \quad \text{in } \overline{Q}_k \times \partial Q_k, \quad (3.3b)$$

where

$$\tilde{H}(\bar{\tau}) \tilde{V}^k = a(\tilde{V}_\rho^k) \frac{\partial \tilde{V}^k}{\partial \rho} + a(\tilde{V}_\xi^k) \frac{\partial \tilde{V}^k}{\partial \xi} + \frac{\partial \tilde{V}^k}{\partial \tau},$$

$$a_k(\bar{\tau}; \tilde{V}^k, v) = \int_{\Omega_k} \left\{ \frac{1}{2} \sigma^2 S \frac{\partial \tilde{V}^k}{\partial S} \frac{\partial v}{\partial S} + \frac{1}{2} \xi^2 \frac{\partial \tilde{V}^k}{\partial \xi} \frac{\partial v}{\partial W} + \frac{1}{2} \tau^2 \frac{\partial \tilde{V}^k}{\partial \tau} \frac{\partial v}{\partial W} + L(\tilde{t}) \tilde{V}^k \cdot v \right\} d\bar{\tau},$$

and
\[ f = L(\tilde{r})(W - \psi(S)), \]

such that \((\cdot, \cdot)_k\) denotes the \(L^2\) inner product over \(\Omega_k\). The variational formulation (3.1) and (3.3) are equivalent in the sense of distributions.

Let \(\{S_h\}_k\) denote a family of finite-dimensional subspaces for which \(\cup_h S_h = H^1_0(\Omega_k)\). In particular, we suppose that \(\Omega_k\) is rectangular and that \(S_h\) consists of continuous, piecewise linear (componentwise) functions on a uniform triangulation of \(\Omega_k\) with mesh size \(h\), which vanish on \(\partial\Omega_k\) (that is, linear finite elements \([11,45]\)). Replacing \(H^1_0(\Omega_k) \cap H^2(\Omega_k)\) with \(S_h\), we obtain the following semi-discrete analogue of (3.3): determine \(\tilde{V}_h^k \in L^2(\tilde{Q}_k; S_h)\), such that

\[
- \left( \tilde{H}(\tilde{r}) \tilde{V}_h^k, v_h \right)_k + a_k(\tilde{r}, \tilde{V}_h^k, v_h) = (f, v_h)_k \quad \text{a.e. in } Q_k, \tag{3.4a}
\]

for all \(v_h \in S_h\), subject to the initial conditions

\[
\tilde{V}_h^k|_{\rho=0} = \tilde{V}_h^k|_{\zeta=0} = \tilde{V}_h^k|_{\tau=0} = 0 \quad \text{in } T_k \times \partial Q_k, \tag{3.4b}
\]

where \(\tilde{V}_h^k\) is the semi-discrete finite element approximation of \(\tilde{V}^k\). The quality of the approximation (3.4) to (3.3), for fixed \(k\), satisfies the estimate

\[
\int_{Q_k} \| \tilde{V}_h^k - \tilde{V}_h^k \|_{L^2(\Omega_k)} \leq C_k h^2,
\]

where \(C_k = C_k(\tilde{V}^k) > 0\), as \(h \to 0\) (cf. [2]).

### 3.3. Fully discrete scheme

We consider now the approximation of \(\tilde{V}_h^k\) by (implicit) semi-discretization of the hyperbolic part of (3.4a). In particular, we introduce \(\delta := T/N > 0\) such that \(T_k = \delta \cdot M_k\), for some \(M_k, N \in \mathbb{N}\), and define \(Q_k = \{1, \ldots, M_k\} \times \{1, \ldots, M_k\} \times \{1, \ldots, N\}\), \(\bar{Q}_k = \{0, \ldots, M_k\} \times \{0, \ldots, M_k\} \times \{0, \ldots, N\}\), \(\rho_l = l\delta, \zeta_m = m\delta\), and \(\tau_n = n\delta\), for \((l, m, n) \in \bar{Q}_k\). We shall denote the approximation on the grid by \(\tilde{V}_{k,h} = \tilde{V}_{k,h}(\bar{x}) = \tilde{V}_{k,h}(\rho_l, \zeta_m, \tau_n)\) and the corresponding averages such that

\[
\tilde{V}_{k,h}^1 = \frac{1}{2} \left( \tilde{V}_{k,h}(l, m, n) + \tilde{V}_{k,h}(l-1, m, n) \right),
\]

\[
\tilde{V}_{k,h}^2 = \frac{1}{2} \left( \tilde{V}_{k,h}(l, m-1, n) + \tilde{V}_{k,h}(l, m-1, n-1) \right),
\]

\[
\tilde{V}_{k,h}^3 = \frac{1}{2} \left( \tilde{V}_{k,h}(l, m, n) + \tilde{V}_{k,h}(l, m, n-1) \right),
\]

and

\[
\tilde{V}_{k,h}^4 = \frac{1}{2} \left( \tilde{V}_{k,h}(l-1, m-1, n-1) + \tilde{V}_{k,h}(l-1, m-1, n) \right).
\]

We shall also use the backward difference quotients

\[
\tilde{a}_1 \tilde{V}_{k,h} = \frac{1}{\delta} \left( \tilde{V}_{k,h}(l, m, n) - \tilde{V}_{k,h}(l-1, m, n) \right),
\]

\[
\tilde{a}_2 \tilde{V}_{k,h} = \frac{1}{\delta} \left( \tilde{V}_{k,h}(l, m-1, n) - \tilde{V}_{k,h}(l, m-1, n-1) \right),
\]

and

\[
\tilde{a}_3 \tilde{V}_{k,h} = \frac{1}{\delta} \left( \tilde{V}_{k,h}(l, m, n-1) - \tilde{V}_{k,h}(l, m, n) \right).
\]

The fully discrete approximation of (3.4) is defined as follows. We seek \(\tilde{V}_{k,h}^k \in S_h\) satisfying

\[
- \left( \tilde{H}(l, m, n) \tilde{V}_{k,h}^k(l, m, n), v_h \right)_k + a_k(\rho_l, \zeta_m, \tau_n; \tilde{V}_{k,h}^k(l, m, n), v_h) = (f, v_h)_k, \tag{3.5a}
\]
for all \( v_h \in \mathcal{S}_h \) and \((l, m, n) \in Q_k\), such that
\[
\widehat{V}^{k,h}_{(0,m,n)} = \widehat{V}^{k,h}_{(l,0,n)} = \widehat{V}^{k,h}_{(l,m,0)} = 0,
\]  
(3.5b)
for \((l, m, n) \in \overline{Q}_k\), where
\[
\overline{H}(l, m, n) \widehat{V}^{k,h}_{(l,m,n)} = a(-\partial_1 \widehat{V}^{k,h}_{(l,m,n)})\partial_{\Gamma} \widehat{V}^{k,h}_{(l,m,n)} + a(-\partial_2 \widehat{V}^{k,h}_{(l,m,n)})\partial_{\Gamma} \widehat{V}^{k,h}_{(l,m,n)} + \partial_3 \widehat{V}^{k,h}_{(l,m,n)}.
\]
The quality of the approximation is provided by the estimate
\[
\|\hat{V} - \widehat{V}^{k,h}_{(l,m,n)}\|_k \leq C_k(\delta^2 + h^2) \tag{3.6}
\]
as \( \delta, h \to 0 \), \( C_k = C_k(\hat{V}) > 0 \), where
\[
\|v\|_k = \left( \delta \sum_\{\rho_l, \zeta_m, \tau_n \} \|v(\cdot, \rho_l, \zeta_m, \tau_n)\|_{L^2(\Omega_h)}^2 \right)^{1/2},
\]
such that the summation is taken over all mesh points \( \{\rho_l, \zeta_m, \tau_n\} \in \Gamma \) (cf. \([2,38]\)).

3.4. Approximation solvability

The approximation to the solution \( V \) of (2.9) is then defined as follows. For \( k > 1 \), let \( \Omega_k \subset \Omega_{k+1}, \ Q_k \subset Q_{k+1}, \ h_k > h_{k+1}, \delta_k > \delta_{k+1}, \) and define
\[
V^{k,h}_{(l_k,m_k,n_k)}(S, W) = \widehat{V}^{k,h}_{(M_k-l_k,M_k-m_k,N_k-n_k)}(S, W) + (W - \psi(S));
\]
then follows from (3.2) and (3.6) that
\[
\|V - V^{k,h}_{(l_k,m_k,n_k)}\|_{(\overline{G}, \Gamma)} = O(\delta_k^2 + h_k^2) + o(\pi_k^{-1} + H_k^{-1}) \quad \text{as} \quad k \to \infty, \tag{3.7}
\]
where
\[
\|v\|_{(\overline{G}, \Gamma)} = \left( \delta_k \sum_\{\rho_{l_k}, \zeta_{m_k}, \tau_{n_k}\} \|v(\cdot, \rho_l, \zeta_m, \tau_n)\|_{L^2(\overline{G})}^2 \right)^{1/2},
\]
such that \( \overline{G} \subset \Omega \) is compact, \( \Gamma \) is any piecewise smooth curve contained in \( Q \), \( H_k = \text{diam} \{\Omega_k\} \), and \( \pi_k = \text{diam} \{Q_k\} \). We note that in (3.7) the first term in the estimate bounds the temporal discretization error, the second the spatial discretization error, the third the error due to the truncation of the temporal domain, and the fourth term the error due to the truncation of the spatial domain. Specifically, asymptotic performance may only be realized on the approximating region \((\overline{G}, \Gamma)\) and is suboptimal when a mesh is refined relative to a fixed computational domain \( \Omega_k \times Q_k \) (cf. \([37, 38]\)).

4. Extension

We briefly consider extensions of the theory which allow the control to be applied at discrete moments of time as well as to possess a finite reaction speed. Our intent is to highlight only those aspects of the theory that are substantively different from that presented in the preceding sections and so, without loss of generality, we consider only self-financing portfolios. In Section 4.1, we introduce point control, which leads to a characterization of the conditional value function that is temporally weak. In the context of our portfolio model, this is known as block trading. In Section 4.2, we consider the phase delay, which results in the control parameter being defined on the temporal space; the effect is that the control action is based upon “dated” information.

4.1. Point control

Let \( \{t_1, \ldots, t_N\}, 0 < t_i < T \) denote \( N \) distinct monitoring times. Given the asset class \((B(t), S(t))\) of Section 2.1, point (strategy) control manifests itself within the model through the portfolio dynamics.
\[ \dot{b}(t) = \beta(t) \sum_{i=1}^{N} \delta(t - t_i) dt, \quad \ddot{b}(t) = \beta(t) \sum_{i=1}^{N} \delta(t - t_i) dt, \] (4.1a)

and

\[ \dot{s}(t) = \zeta(t) \sum_{i=1}^{N} \delta(t - t_i) dt, \quad \ddot{s}(t) = \zeta(t) \sum_{i=1}^{N} \delta(t - t_i) dt, \] (4.1b)

where we suppose \( 0 \leq \beta(t) \leq \gamma \) and \( 0 \leq \zeta(t) \leq \gamma \), for some \( \gamma < \infty \); control velocities under point control are infinite. The conditional value function is then characterized as the solution of the Hamilton–Jacobi equation in weak variational form (cf. [7]). Formally, we seek \( V \) satisfying

\[ \int_{Q} \left\{ (H_{\delta}(\bar{\ell}), V) + a(\bar{\ell}; V, v) \right\} d\bar{\ell} = 0, \] (4.2a)

for all test functions \( v \), such that

\[ V|_{t=T} = W - \psi(S) \quad \text{a.e. in } \bar{\ell} \times \partial Q, \] (4.2b)

and

\[ V|_{S=0} = W \quad \text{a.e. in } \partial \Omega \times Q, \] (4.2c)

where

\[ H_{\delta}(\bar{\ell})V = \frac{\partial V}{\partial t} + \max_{\zeta(t) \in [0, \gamma]} \left[ \zeta(t) \frac{\partial V}{\partial s_{\pm}} + \zeta(t) \frac{\partial V}{\partial s_{-}} \right] \sum_{i=1}^{N} \delta(t - t_i) \]

and

\[ a(\bar{\ell}; V, v) = \int_{\Omega} \left\{ \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \frac{\partial v}{\partial S} + \frac{1}{2} \sigma(t) \frac{\partial^2 V}{\partial S^2} \frac{\partial v}{\partial S} + \frac{1}{2} \sigma(t) \frac{\partial^2 V}{\partial S^2} \frac{\partial v}{\partial S} + L(\bar{\ell}) V \right\} d\bar{\ell}. \]

The optimal feedback control pair \( \tilde{\xi}^*(t) = (\xi_{+}^*(t), \xi_{-}^*(t)) \) solves

\[ \int_{Q} \left( \frac{\partial V}{\partial s_{+}} + \frac{\partial V}{\partial s_{-}} \right) \sum_{i=1}^{N} \delta(t - t_i), v \right) d\bar{\ell} \]

\[ = \int_{Q} \left( \max_{\zeta(t) \in [0, \gamma]} \left[ \zeta(t) \frac{\partial V}{\partial s_{+}} + \zeta(t) \frac{\partial V}{\partial s_{-}} \right] \sum_{i=1}^{N} \delta(t - t_i), v \right) d\bar{\ell}. \]

It is possibly more informative to represent (4.2) in its strong form on the intervals \( (t_i, t_{i+1}) \), subject to jump conditions across the control instances \( t_i \). As such, we seek \( V \) satisfying the parabolic equation

\[ \frac{\partial V}{\partial t} + A(\bar{\ell}) V + L(\bar{\ell}) V = 0, \]

for \( t \in (t_i, t_{i+1}) \), such that

\[ V|_{t=T} = W - \psi(S) \]

and

\[ V|_{S=0} = W, \]

subject to the jump condition across the control instant \( t_{i+1} \),

\[ \lim_{t \to t_{i+1}^-} V(\bar{x}, \bar{\ell}) = \frac{1}{2} \lim_{t \to t_{i+1}^+} \left[ V(\bar{x}, \bar{\ell} + \delta_{s_{+}}) + V(\bar{x}, \bar{\ell} + \delta_{s_{-}}) \right], \]
where \( \delta_{s-} = (\gamma, 0, 0) \) if \( V_{s-} \geq 0 \) and \( \delta_{s-} = (0, 0, 0) \) otherwise, and analogously \( \delta_{s+} = (0, \gamma, 0) \) if \( V_{s+} \geq 0 \) and \( \delta_{s+} = (0, 0, 0) \) otherwise.

**Remark.** Within the portfolio control paradigm (cf. [6]), the stochastic system is noise controlled, a situation that is computationally challenging. An alternative is to recognize that in the limit as the number of discrete control instances \( N \to \infty \), we recover the portfolio control model through drift control of the stochastic system.

### 4.2. Phase delay

We suppose now that the control does not act instantaneously, but on information that is “dated”; that is, we suppose that our reaction speed is limited. To this end, let \( \Delta > 0 \) denote a deterministic lag time. To this end, we consider the ultradiffusion

\[
\begin{align*}
\text{dr}(t) &= dt, \\
\text{ds}_+(t) &= \zeta_+ \circ \tau(t) dt, \\
\text{ds}_-(t) &= \zeta_- \circ \tau(t) dt, \\
\text{dS}(t) &= \alpha S(t) dt + \sigma S(t) dw(t), \\
\text{dW}(t) &= \{ rW(t) + (\alpha - \gamma)s(t)S(t) \} dt + \sigma S(t) dw(t),
\end{align*}
\]

for \( t \in (0, \widetilde{T}) \), \( \tau(0) = \tau, s_+(0) = s_+, s_-(0) = s_-, S(0) = S, W(0) = W, \) and \( \zeta_\pm : [0, T] \to [0, \gamma] \).

Letting \( \bar{x} = (S, W), \bar{t} = (s_+, s_-, \tau, t) \), and \( \bar{T} = (s_-(T), s_+(T), \tau(T), T) \), we define the performance index as

\[
J_{\bar{x}}(\bar{T}) = \mathbb{E}_{\bar{x}}[W(T) - \psi(S(T))],
\]

and the conditional value function such that

\[
V(\bar{x}, \bar{t}) = \max_{\xi(\bar{t}) \in [0, \gamma]^2} J_{\bar{x}}(\bar{T}).
\]

Let \( Q = (0, \infty)^2 \times (0, T)^2 \), then \((4.4)\) is formally characterized as the solution to the Hamilton–Jacobi equation

\[
\frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial \tau} + A(\bar{t})V + L^5(\bar{t})V = 0 \quad \text{a.e. in } \Omega \times Q,
\]

such that

\[
V|_{\tau=T} = V|_{\tau=T} = W - \psi(S) \text{ in } \bar{\Omega} \times \partial Q
\]

and

\[
V|_{S=0} = W \text{ in } \partial \Omega \times Q,
\]

where

\[
A(\bar{t})V = 1 \frac{\partial^2 V}{\partial S^2} + 1 \frac{S(t)}{\sigma} \frac{\partial^2 V}{\partial S \partial W} + \frac{\sigma^2}{\partial S^2} \frac{\sigma^2}{\partial W^2} \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2}{\partial S^2} \frac{\sigma^2}{\partial W^2} \frac{\partial^2 V}{\partial S^2}
\]

and

\[
L^5(\bar{t})V = \max_{\xi(\bar{t}) \in [0, \gamma]^2} \left\{ \zeta_+(\bar{t}) \frac{\partial V}{\partial s_+} + \zeta_-(-\bar{t}) \frac{\partial V}{\partial s_-} \right\} \text{ for } t \in [\Delta, T],
\]

and \( L^5(\bar{t})V = 0 \) otherwise. The optimal control feedback law \( \xi^*(\bar{t}) = (\xi^*_+(\bar{t}), \xi^*_-(\bar{t})) \) satisfies

\[
\zeta_+(\bar{t}) \frac{\partial V}{\partial s_+} + \zeta_-(-\bar{t}) \frac{\partial V}{\partial s_-} = \max_{\xi(\bar{t}) \in [0, \gamma]^2} \left\{ \zeta_+(\bar{t}) \frac{\partial V}{\partial s_+} + \zeta_-(-\bar{t}) \frac{\partial V}{\partial s_-} \right\},
\]

for all \( t \in [\Delta, T], \) and \( \xi^*(\bar{t}) = (0, 0) \) otherwise.
We specify the phase delay through the definition of the terminal set. To this end, let \( \overline{M}(\bar{x}, T) \subseteq \overline{Q} \) be defined such that

\[
\overline{M}(\bar{x}, T) = \{(g^-(\bar{x}), g^+(\bar{x}), T - \Delta, T) \mid \bar{x} \in \Omega\},
\]

where

\[
g^-(S, W) = 0
\]

and

\[
g^+(S, W) = \begin{cases} 
1, & \text{if } S > E \\
0, & \text{if } S \leq E.
\end{cases}
\]

A feasible trajectory \((s_-(\bar{t}), s_+(\bar{t}), \tau(t), t)\) satisfies: \(s_-(0) = s_-, s_+(0) = s_+, \) and \(\tau(0) = \tau, \) the deterministic equations (4.3a)-(4.3c) such that \(\xi^\ast(\bar{t})\) is a solution of (4.6), and the terminal constraint \((s_-(T), s_+(T), \tau(T), T) \in \overline{M}(\bar{x}, T)\). We then define the performance index such that

\[
\tilde{U}(\bar{x}, \bar{t}) = \min\{W \mid V(\bar{x}, \bar{t}) \geq 0\}
\]

and the value function as

\[
(4.7a)
\]
(a) Option price $U(S, t)$.  
(b) Asset shares, $s(t)$.  
(c) Buying activity, $\zeta^+(t)$.  
(d) Selling activity, $\zeta^-(t)$.  

Fig. 2. Continuous control with activation costs.

$$U(S, t) = \min \{ \bar{U}(\bar{x}, \bar{t}) \mid (s_-(\bar{t}), s_+(\bar{t}), \tau(t), t) \in \mathcal{F}_{(\bar{x}, \bar{T})} \}$$

where $\mathcal{F}_{(\bar{x}, \bar{T})}$ is the set of all feasible trajectories.

5. Numerical experiments

Experiments were performed utilizing continuous control, with and without transaction costs, and with point control without transaction costs; the phase delay case was not attempted owing to the computational complexity of the problem relative to the computing platform. Computations were performed using an Intel 2.8 GHz P4 processor with 2 GB RAM. We suppose $\alpha = 0.10$ and $r = 0.05$ per year, $\sigma^2 = (0.3)^2$ per year, an option life of $T = 1 \times 10^{-4}$ years, and an exercise price for the European call option of $E = 1$. The mesh was $h = \Delta S = \Delta W = 5 \times 10^{-4}$, $\Delta \tau = 1 \times 10^{-6}$, and $\Delta \rho = \Delta \zeta = 1 \times 10^{-2}$. The strategy was bounded above by $\gamma = 5 \times 10^{-3}$ for continuous control and $\gamma = 1 \times 10^{-2}$ for point control. When assessed, transaction costs amounted to $\kappa = $0.001/year. In the point control scenario, monitoring times were every $10 \Delta \tau$. The computational domain was $[0.99, 1.01]^2 \times [0.0, 0.4]^2 \times [0, 10^{-4}]$.

For each $(l, m, n) \in Q_k$, the nonlinear system (3.5) was solved by successive-under-relaxation (parameter $\omega = 0.045$) coupled to a preconditioned biconjugate gradient algorithm. Relative to a self-financing portfolio subject to continuous control, let $\tilde{V}_{(l,m,n)}^{k,h}(0)$ denote the initial estimate of the solution. Given $\tilde{V}_{(l,m,n)}^{k,h}(t)$, we obtain $\tilde{V}_{(l,m,n)}^{k,h}(t+1)$ by solving the linear system.
(a) Option price \( U(S, t) \).
(b) Asset shares, \( s(t) \).
(c) Buying activity, \( \zeta_+(t) \).
(d) Selling activity, \( \zeta_-(t) \).

**Fig. 3.** Point control, self-financing portfolio.

\[
\begin{align*}
& a(-\partial_1 \tilde{V}^{k,h}_{(l,m,n)}(t)) \cdot \partial_1 \tilde{V}^{k,h}_{(l,m,n)}(t + 1) + a(-\partial_2 \tilde{V}^{k,h}_{(l,m,n)}(t)) \cdot \partial_2 \tilde{V}^{k,h}_{(l,m,n)}(t + 1) \\
& + \partial_3 \tilde{V}^{k,h}_{(l,m,n)}(t + 1) + a_k(\rho_l, \zeta_m, \tau_n; \tilde{V}^{k,h}_{(l,m,n)}(t + 1, v_h)) = (f, v_h)_k,
\end{align*}
\]

and setting

\[
\tilde{V}^{k,h}_{(l,m,n)}(t + 1) \leftarrow \tilde{V}^{k,h}_{(l,m,n)}(t) + \omega \left( \tilde{V}^{k,h}_{(l,m,n)}(t + 1) - \tilde{V}^{k,h}_{(l,m,n)}(t) \right),
\]

for \( 0 < t \in \mathbb{N} \).

Results are organized as follows. In **Fig. 1**, we consider the continuous control case of a self-financing portfolio, **Fig. 2** introduces transaction costs into the continuous control environment, and **Fig. 3** demonstrates the effect of point control on a self-financing portfolio. Figure subcase (a) presents the valuation as a function of time and stock price, (b) shows the number of shares held of the stock as a function of time and stock price, while (c) and (d) demonstrate the activation sequence of buying and selling orders for shares of stock, again as a function of time and stock price, respectively.

In the context of mathematical finance, friction would be expected to inflate the selling price of the European call option. This was observed in the point control case; the result in the case of transaction costs is less clear as the computation does not appear to have evolved sufficiently or to be sufficiently resolved. However, it is apparent that control activity is economized in the presence of friction. Moreover, given the appropriate choice of monitoring times and strategy bounds, point control appears to represent a viable approximation to continuous control.
In terms of regularization of the method (cf. Section 2.3), we did not encounter any numerical difficulties in setting $\epsilon = 0$ in (2.11); the magnitude of the coercivity constant $\epsilon^2/2$ potentially effects the stability and accuracy of the associative approximation method. Additionally, in our admittedly limited experience, Eq. (2.11) appears to be computationally more tractable than (2.12), which is an advection–diffusion equation. Relative to the ultraparabolic framework, the principal difficulty involves the lack of advanced temporal solvers. With viscosity techniques, we require $\epsilon \ll 1$ in order not to corrupt the approximation, a circumstance that can be somewhat mitigated through the use of semi-implicit integrators (cf. [24]) and specially weighted quadrature techniques (cf. [1]).

6. Conclusion

We have developed a theoretical and approximation framework for the stochastic optimal control of ultradiffusion processes relative to a given performance index and expectation. The conditional value function and indexed control feedback law satisfy an ultraparabolic Hamilton–Jacobi equation in weak variational form, which is isometric to a parameterized family of parabolic equations along temporal characteristics. We have considered a model for the selling price of a European call option based on a value-maximizing exercise strategy; we allowed for both continuous and point control, activation costs, and phase delays. The valuation procedure admits well-defined numerical procedures. In particular, we found that control activity is economized in the presence of friction.

In addition, we observe that point control appears to represent a viable approximation to continuous control, resulting in piecewise in time, parabolic equations subject to jump constraints across monitoring times. As such, existing software for parabolic partial differential equations may be incorporated into general point control packages. Another potential application of strategy point control appears to be as a viable alternative to computationally expensive portfolio control.

In terms of approximation theory, we note that asymptotic performance of the method is only realizable on the approximating region and not on the entire computational domain; corruption due to artificial boundary data is localized. Our admittedly limited computational experience indicates that an ultraparabolic representation for continuous control appears to be more tractable than elliptic regularization, which results in an advection–diffusion equation with vanishing coercivity constant.

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