ON LINEARIZING ALGEBRAIC TORUS ACTIONS

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Certain results of Bialynicki-Birula as well as other pieces of evidence suggest that an algebraic torus action on an affine space is always linear with respect to a suitably chosen coordinate system (cf. [1, 5]). With eventual proof of this conjecture in mind, we examine in this note torus actions on a smooth affine variety over an algebraically closed ground field. It turns out that, in what we have termed *unmixed* cases (see 2.3), the variety in question is a vector bundle over the fixed point variety and the torus action is linear along the fibers (see Theorem 2.5 below). We have noticed that this fact is already known in essence to Bialynicki-Birula [2; Th. 2.5]. But our proof is elementary and seems simpler than his, resting upon the smoothness of fixed point schemes (Fogarty [3]) and a version of Nakayama's Lemma for semigroup-graded rings and modules (see 1.2 and 1.4). It is clear that *definiteness* of a torus action as defined by Bialynicki-Birula (see [2; p. 482]) implies *unmixedness*, but the converse is true as well (see 1.6). Since *unmixedness* is a quite intrinsic condition, it seems worthwhile to point this out.

As an immediate consequence of the above result and the Quillen–Suslin Theorem [9, 12], a smooth affine variety with an unmixed torus action is actually an affine space with a *linear* torus action, provided the fixed point variety is isomorphic to some affine space (possibly a single point). By making use of recent results of Fujita, Miyanishi and Sugie [4, 8], we show this last to be the case indeed for any unmixed torus action on an affine space with a fixed point variety of dimension ≤2 (see Theorem 3.4).

Hyman Bass told one of the authors how a graded ring \( R = \bigoplus_{n \geq 0} R_n \) becomes isomorphic to a symmetric algebra over \( R_0 \) under a certain homological hypothesis (see 2.6). His remark was largely responsible for getting us started on the present investigation. His suggestions were also responsible for an improvement in our treat-
ment of semi-group gradings. We wish to express our thanks to him for generously sharing his ideas with us.

1. Lemmas on semigroup gradings

1.1. Let \((S, +, 0)\) be a commutative monoid. \(S\) has a canonical pre-order defined for \(\alpha, \beta \in S\) by
\[
\alpha \preceq \beta \iff \beta = \alpha + \gamma \text{ for some } \gamma \in S.
\]
(1)
We are interested in the following condition on \(S\): For any \(\alpha, \beta, \gamma \in S\)
\[
\alpha + \beta + \gamma = \alpha \Rightarrow \beta = \gamma = 0.
\]
(2)
Special cases of this are
\[
\beta + \gamma = 0 \Rightarrow \beta = \gamma = 0 \quad ("S\) has no units")
\]
(3)
and
\[
\alpha + \beta = \alpha \Rightarrow \beta = 0 \quad ("cancellation holds in \(S\))."
\]
(4)
Conversely, (3) and (4) imply (2). We note that (3) is equivalent to
\[
S^* := S - \{0\} \text{ is a subsemigroup of } S.
\]
(5)
It is easy to see that if (2) holds, then (1) defines an order on \(S\). The converse need not be true. (Take, for instance, \(S = \{0, 1, 2, \ldots, \infty\}\) with the natural order and the usual rules about computing with \(\infty\). \(S\) does not satisfy (4).)

1.2. Lemma (Graded Nakayama Lemma). Let \(R = \bigoplus_{a \in S} R_a\) be a graded commutative ring and \(M = \bigoplus_{a \in S} M_a\) a graded \(R\)-module, where both gradings are taken over a commutative monoid \(S\). Assume that the canonical pre-order of \(S\) satisfies (2) and that the resulting order satisfies the descending chain condition (dcc). Let \(I = \bigoplus_{a \in S} R_a\) and assume \(IM = M\). Then \(M = 0\).

Proof. Note that \(I\) is an ideal by (5). Assume there exists an \(m \in M_a, m \neq 0\). Then \(m \in IM\) is a sum of elements of the form \(am'\) with \(a \in R_\beta, m' \in M_{a'}, \alpha = \beta + \alpha', \beta \neq 0\). By (4), \(\alpha > \alpha'\). Repeat the process using \(m'\) instead of \(m\), and get \(\alpha' > \alpha''\). This can go on ad infinitum, violating the dcc.

1.3. Remark. If \(M\) is finitely generated, the conclusion of Lemma 1.2 holds valid for any monoid \(S\) as long as \(S^* := S - \{0\}\) is a subsemigroup. The standard proof works.

1.4. Corollary. Let \(R, I = \bigoplus_{a \in S} R_a\) and \(S\) be the same as in Lemma 1.2, and suppose that a set of homogeneous elements \(E \subseteq I\) generates \(I\) modulo \(I^2\) over \(R_0 = R/I\). Then \(E\) generates \(I\) as an \(R\)-module, and \(R\) as an \(R_0\)-algebra.
Proof. The first assertion follows from Lemma 1.2 in the usual manner. To prove \( R = R_0[E] \), let \( f \in R_\alpha \) with \( \alpha \neq 0 \), and write \( f = r_1e_1 + \cdots + r_ne_n \) with \( r_i \in R, 0 \neq e_i \in E \) for all \( 1 \leq i \leq n \). We may discard superfluous terms and assume that each \( r_i \) is homogeneous of grade \( \beta_i \), say. Then \( \alpha = \beta_i + e_i \), with \( e_i \) := the grade of \( e_i \), and \( \alpha > \beta_i \) by virtue of (4). If \( \beta_i = 0 \), stop there; if otherwise, express \( r_i \) as an \( R \)-linear combination of elements of \( E \). The coefficients again will be homogeneous of grade <\( \beta_i \). The dcc assures us that this process will end with coefficients in \( R_0 \). Thus, \( R \subseteq R_0[E] \) and we are done.

1.5. Unmixed semigroups. We now confine ourselves to finitely generated monoids \( S \) contained in \( \mathbb{Z}' \), where \( \mathbb{Z} \) denotes the additive group of integers. For such an \( S \) put \( S^* := S - \{0\} \) as before. We shall say \( S \) is unmixed if and only if \( S^* \) is a semigroup. This is equivalent to saying that \( \sum a_i \alpha_i = 0 \) with \( \alpha_i \in S \) and \( a_i \in \mathbb{Z}, a_i \geq 0 \), implies all \( a_i = 0 \).

1.6. Let \( V := \mathbb{Q} \otimes \mathbb{Z}' \), a vector space over the rational number field \( \mathbb{Q} \), and put

\[
\mathcal{C}(S) := \{ \sum q_i \alpha_i | q_i \in \mathbb{Q}, q_i \geq 0, \alpha_i \in S \} \subseteq V
\]

and

\[
\hat{\mathcal{C}}(S) := \{ l \in \hat{V} | l(x) \geq 0 \text{ for all } x \in \mathcal{C}(S) \}
\]

where \( \hat{V} \) denotes the vector space dual to \( V \).

Lemma. \( S \) is unmixed if and only if there exists a basis \( \{ u_1, \ldots, u_r \} \) for \( \mathbb{Z}' \) such that

\[
S \subseteq \left( \sum_{1 \leq i \leq r} a_i \mathcal{C}(S) \cap \mathbb{Z} \right).
\]

Proof. It is easy to see that \( S \) is unmixed if and only if \( \mathcal{C}(S) \) does not contain a \( \mathbb{Q} \)-linear subspace \( \neq 0 \) of \( V \), and it is well known (see [7; p. 61) that this is so if and only if \( \hat{\mathcal{C}}(S) \) is not contained in a hyperplane of \( \hat{V} \). Hence, if \( S \) is unmixed, there exist linearly independent elements \( l_1, \ldots, l_r \in \hat{\mathcal{C}}(S) \). Let \( e_1, \ldots, e_r \) be the standard basis of \( V = \mathbb{Q}' \). Our aim is to show that \( l_1, \ldots, l_r \) can be chosen so that

\[
a_{ij} := l_i(e_j)
\]

is an integer for \( i, j = 1, \ldots, r \), and that

\[
M := (a_{ij})
\]

is a unimodular matrix. We may assume, after multiplying each \( l_i \) by a suitable positive integer, that all entries of \( M \) are integers and that the first row \( M_1 \) is unimodular. After a unimodular change of basis in \( V \) we may assume \( M_1 = (1, 0, \ldots, 0) \). After now adding a suitable nonnegative integral multipole of \( l_1 \) to \( l_2 \) we may assume that \( d | a_{21} \), where \( d = \text{GCD}(a_{22}, \ldots, a_{2r}) \). Replacing \( l_2 \) by \( l_2/d \) we turn \( (a_{22}, \ldots, a_{2r}) \) into
a unimodular row and hence may assume \((a_{22}, ..., a_{2r}) = (1, 0, ..., 0)\) after a unimodular change of basis leaving \(M_1\) unaltered. It is clear how to continue this process until \(M\) is changed into a lower triangular matrix of determinant 1. This proves the 'only if' part of the lemma. The converse is obvious.

1.7. Clearly condition (2) holds for any unmixed monoid \(S\) contained in \(\mathbb{Z}^r\), and it follows readily from 1.6 that \(S\) has dcc if \(S\) is unmixed. Hence the assumptions of Lemma 1.2 and Corollary 1.4 hold for unmixed submonoids \(S\) of \(\mathbb{Z}^r\).

2. Torus action on an affine scheme

We work over a ground field \(k\) of arbitrary characteristic.

2.1. Torus action and grading. Let \(T\) be an \(r\)-dimensional algebraic torus split over \(k\): \(T = (G_m)^r\) with \(G_m = \text{GL}_1\) in the usual notations. The coordinate ring \(k[T]\) of \(T\) will be written as \(k[t_1, t_1^{-1}, ..., t_r, t_r^{-1}]\). A character \(\chi : T \to G_m\) is then given by a map \(t = (t_1, ..., t_r) \to \chi(t) = t_1^{\alpha(1)} \cdots t_r^{\alpha(r)}\) with \(\alpha(1), ..., \alpha(r) \in \mathbb{Z} = \text{the rational integers, and the group} \ \hat{T} \ \text{of all characters of} \ T \ \text{is isomorphic to} \ \mathbb{Z}^r \ \text{through the assignment} \ \chi \to (\alpha(1), ..., \alpha(r))\). We shall identify \(\hat{T}\) with \(\mathbb{Z}^r\) and regard \(\hat{T}\) as an additive group. Now let \(T\) act on an affine \(k\)-scheme \(X = \text{Spec} \ A\) as a group of automorphisms. To such an action corresponds a \(k\)-algebra homomorphism

\[
\phi_t : A \to A \otimes_k k[T] = A[t_1, t_1^{-1}, ..., t_r, t_r^{-1}],
\]

and we define \(a_x \in A\) for given \(a \in A\), \(x \in \hat{T}\) via the equality \(\phi_t(a) = \sum a_x \cdot \chi(t)\), the sum being taken over all \(\chi \in \hat{T}\). It is well known and easy to ascertain that \(a \to (..., a_x, ...)\) gives a direct sum decomposition indexed by \(x \in \hat{T} = \mathbb{Z}^r:\n
A = \bigoplus A_x \quad \text{where} \ A_x = \{a \in A : \phi_t(a) = \chi(t)a\}. \tag{7}

By definition, the nonzero members of the \(k\)-submodule \(A_x\) are isobaric of weight \(x\). The submodule \(A_0\) associated with the unit character 0 is actually a \(k\)-subalgebra of \(A\) and the decomposition (7) turns \(A\) into a graded \(A_0\)-algebra. Conversely, every grading of a \(k\)-algebra \(A\) indexed by \(\mathbb{Z}^r\) gives rise to an action of \(T\) on \(\text{Spec} \ A\) in the obvious manner, and the correspondence between \{\(T\)-actions on \(\text{Spec} \ A\)\} and \{\(\mathbb{Z}^r\)-gradings on \(A\)\} is thus one-to-one.

2.2. Fixed point scheme. Let \(X = \text{Spec} \ A\) be acted upon by \(T\) in a certain way. For any \(k\)-algebra \(R\), an \(R\)-valued point \(p \in \text{X}(R)\) of \(X\) is a \(k\)-homomorphism \(p : A \to R\), and likewise \(g : k[T] \to R\) for \(g \in T(R)\). Then the transform \(g(p) \in X(R)\) of \(p\) by \(g\) is \(\mu(g \otimes p)\phi_t\) where \(\mu : R \otimes_k R \to R\) is the multiplication of \(R\). From this follows, clearly, that \(g(p) = p\) for all \(g\) if and only if \(A_x \subseteq \text{Ker}(p)\) for all \(x \neq 0\). Consequently, the fixed point subscheme \(X^T\) of \(X\) under the given action of \(T\) is given by the ideal in \(A\) generated by all the \(A_x\), \(x \in \hat{T}\), \(x \neq 0\). One notes that \(X^T\) may not be reduced.
even when $A$ is a factorial domain finitely generated over $k$ — see Wagreich's example in Fogarty's paper [3; §6, p. 48]. On the other hand, if $X$ is algebraic and smooth over $k$, then $X^T$ is smooth over $k$ too by virtue of Fogarty's theorem [3; Th. 5.2, p. 45]. (If $K$ is a field containing $k$, then $A \otimes K = \bigoplus_{\chi \in S} A_{\chi} \otimes K$ and hence $(A \otimes K)_\chi = A_{\chi} \otimes K$. It follows in particular that $(X^T)_k = (X_k)^T$. So we may assume that $k$ is algebraically closed when checking the smoothness of $X^T$.)

2.3. Unmixed torus action. From now on we deal exclusively with the situation where $T = (\mathbb{G}_m)^r$ acts on an affine algebraic variety $X = \text{Spec} \ A$ (i.e. $A$ is a geometrically integral domain finitely generated over $k$). Given this, let $A = \bigoplus A_{\chi}$ be the corresponding decomposition as in (7). Define

$$S := \{ \chi \in \hat{T} = \mathbb{Z}^r : A_{\chi} \neq \{0\} \}$$

which is clearly a submonoid of $\mathbb{Z}^r$. Furthermore, $S$ is finitely generated as a semigroup because $A = k[z_1, \ldots, z_r]$ and the weights occurring in the decompositions of the $z_i$ generate $S$. We shall refer to $S$ as the monoid attached to the action of $T$ on $X$.

An action of $T$ on $X$ is said to be unmixed if and only if one of the following equivalent conditions is satisfied (see 1.6):

(a) The monoid $S$ attached to the action is unmixed in the sense of 1.5.
(b) $I := \bigoplus_{\chi \neq 0} A_{\chi}$ is an ideal of $A$.
(c) An isomorphism $T \cong \mathbb{G}_m^r$ can be chosen so that all components of all $\chi \in S$ are nonnegative. (This says that the action is definite in the sense of [2; p. 482].)

When that is so, the fixed point scheme $X^T$ is nonempty as it is defined by the ideal $I$ (see 2.2). Moreover, the composition of natural maps $A_0 \subset A \to A/I = A_0$ is the identity map on $A_0$. Consequently, the fixed point scheme $X^T = \text{Spec}(A/I)$ is a variety (irreducible and reduced) and the closed immersion $X^T \subset X$ has a retraction $X \to X^T = \text{Spec}(A_0)$ which makes $X^T$ a quotient $X/T$ in the category of affine $k$-schemes. (This last means merely that every $k$-morphism from $X$ to an affine $k$-scheme constant on the $T$-orbits must factor uniquely through $X \to X^T = X^T$.)

Note, however, that an unmixed $T$-action has no closed orbits of positive dimensions. This precludes any possibility of the existence of a good (geometric) quotient $X/T$.

2.4. Factoriality. It is an obvious consequence of the retractability (see 2.3) that the fixed point variety $X^T$ is factorial if the action is unmixed and $X$ is factorial (meaning $A$ is a UFD).

2.5. Theorem. Let an $r$-dimensional $k$-split torus $T = (\mathbb{G}_m)^r$ act on a smooth affine variety $X$ as a group of automorphisms. Assume that the action is unmixed. Then, through the natural retraction $\pi : X \to X^T$ (see 2.3), $X$ gets a structure of a vector bundle over the fixed point variety $X^T$. Furthermore, each fiber of $\pi$ is stable under the $T$-action and $T$ acts linearly on it.
Proof. Keeping the notations of 2.3, let us first observe that $X^T = \text{Spec}(A/I) = \text{Spec}(A_0)$ is $k$-smooth by virtue of Fogarty's theorem cited in 2.2. So its conormal sheaf $I/I^2$ is locally free, hence projective, and finite over $A_0 = A/I$. The rank of $I/I^2$ is constant and equals $\dim X - \dim X^T$. Now since both $I = \bigoplus_{\chi \neq 0} A_{\chi}$ and $I^2$ are homogeneous ideals (in the sense that if $a \in I$ or $I^2$ then $a_{\chi} \in I$ or $I^2$, respectively, for any $\chi$), $I/I^2$ is a graded $A_0$-module and the canonical $A_0$-linear map $I \to I/I^2$ preserves weights. Hence, for any $\chi \in \mathfrak{S}^*, A_{\chi}$ is mapped onto $(I/I^2)_{\chi}$, the component of $I/I^2$ of weight $\chi$. Being a direct summand of $I/I^2$, $(I/I^2)_{\chi}$ is $A_0$-projective. Therefore, $A_{\chi} \cong (I/I^2)_{\chi}$ has a section $s_{\chi} : (I/I^2)_{\chi} \to A_{\chi}$. Choose for each $\chi$ a set of generators for $(I/I^2)_{\chi}$ over $A_0$, and let $E_{\chi}$ be the image $GA_{\chi}$ of the chosen generating set under $s_{\chi}$. Then the union $E$ of all $E_{\chi}$ is a finite set of isobaric elements in $I$ and $I$ is generated by $E$ modulo $I^2$. It follows from Corollary 1.4 that $I$ is generated by $E$ as an ideal and $A = A_0[E]$. Therefore, the $A_0$-algebra homomorphism $\text{Sym}_{A_0}(I/I^2) \to A$ arising from the $A_0$-module map $s = \bigoplus s_{\chi} : I/I^2 \to I \subset A$ is surjective. (Here $\text{Sym}_R M$ denotes the symmetric $R$-algebra of the $R$-module $M$.) On the other hand, $\text{Sym}_{A_0}(I/I^2)$ is an integral domain over $k$ of transcendence degree equal to trans. deg. $(A_0) + \text{rank}(I/I^2) = \dim X^T + (\dim X - \dim X^T) = \dim X$. Therefore the map $\text{Sym}_{A_0}(I/I^2) \to A$ is injective as well. Thus an $A_0$-isomorphism

$$\text{Sym}_{A_0}(I/I^2) = A_0[E] = A$$

(9)

is obtained, establishing $X = \text{Spec} A$ as a vector bundle over $X^T = \text{Spec} A_0$. Since $T$ acts as a group of $A_0$-linear automorphisms on the projective module $I/I^2$ and since $s : I/I^2 \to I \subset A$ is $T$-equivariant (because $s$ preserves weights), the isomorphism (9) is $T$-equivariant, which proves the last assertion of the theorem.

2.6. Remarks. (1) One should compare this theorem with Bialynicki-Birula's theorem [2; Th. 2.5, p. 486].

(2) Hyman Bass has pointed out to us that if $A = \bigoplus_{\chi \in S} A_{\chi}$ with $S$ unmixed, and if furthermore $A$ is noetherian and $A_0$-projective, and if the $A$-projective dimension of $A_0 = A/I$ is finite, then $I/I^2$ is $A_0$-projective and again a graded $A_0$-linear section $I/I^2 \to I$ induces an $A_0$-algebra isomorphism $\text{Sym}_{A_0}(I/I^2) \to A$. This means that a kind of relative smoothness condition is enough to prove Theorem 2.5.

3. The case of the affine space

_In this section the ground field $k$ is assumed to be algebraically closed._

3.1. The situation. We now restrict ourselves to the situation in which an $r$-dimensional torus $T = \langle \mathbb{G}_m^r \rangle$ acts on an affine $n$-space $\mathbb{A}^n := \text{Spec} k[x_1, \ldots, x_n]$, where $k[x_1, \ldots, x_n] = k[t^n]$ is an $n$-variable polynomial ring over $k$. Since the quotient of a torus by any subgroup is again a torus we may assume that $T$ acts effectively, i.e. that no proper subgroup of $T$ acts neutrally on $\mathbb{A}^n$. 
The known facts in this situation are as follows: By an argument due to Shafarevich and Bialynicki-Birula [1; Th. 1, p. 177], any $T$-action on $\mathbb{A}^n$ has a fixed point. Moreover, thanks to Bialynicki-Birula [1; Th. 2, p. 177 and II; Th., p. 123], we know that the action is linearizable in the cases of $r = n$ or $r = n - 1$. This means that one can find isobaric elements $y_1, \ldots, y_n$ in $k[x_1, \ldots, x_n]$ such that $k[y_1, \ldots, y_n]$.

3.2. Standard generators. In the situation of 3.1, first pick a fixed (closed) point as the origin and then diagonalize the induced action of $T$ on the tangent space of $\mathbb{A}^n$ at the origin. That way the action of $T$ is given in the following form relative to the notations of 2.1:

$$\phi_i(x_i) = \chi_i(t)x_i + f_i(t, t^{-1}, x) \quad \text{for all } 1 \leq i \leq n$$

(10)

where $\chi_i \in \bar{T} = \mathbb{Z}$, $f_i \in (k[T])[x_1, \ldots, x_n]$ and the total degree of every term of $f_i$, with respect to the variables $x_j$, is greater than one for each $i$. We claim:

The monoid $S \subseteq \mathbb{Z}$ attached to the given action of $T$ (see 2.3) is generated by $x_1, \ldots, x_n$.

Proof. Let $S'$ be the monoid generated by the $\chi_i$ inside $\mathbb{Z}$. First we show $S' \subseteq S = \{ \chi: \text{there exists an isobaric polynomial } \neq 0 \text{ of weight } \chi \}$. For each $i$, let $x_i = x_{i,0} + \cdots + x_{i,p}$ be the isobaric decomposition of $x_i$. One of the polynomials in $x_{i,0}$, say $x_{i,0}$, must contain a term $x_i$. Thus, $x_{i,0} = c_i x_i + P_i(x)$, $P_i(x) \in k^{[n]}$, $c_i \in k^*$, and $x_{i,0} \in (k^{[n]})_{\lambda_i}$, with $\lambda_i \in \bar{T}$. Applying $\phi_i$ (see 2.1, (6)) to both sides of the equality, we obtain

$$\phi_i(x_{i,0}) = \lambda_i(t)x_{i,0} = c_i \lambda_i(t)x_i + \lambda_i(t)P_i(x) = \phi_i(c_i x_i + P_i(x)) = c_i \chi_i(t)x_i + c_i f_i(t, t^{-1}, x) + P_i(x + f(t, t^{-1}, x))$$

Since the linear terms of $P_i$ do not include $x_i$, comparison of the third and the last polynomials tells us that $\lambda_i = \chi_i$. Hence all $\chi_i \in S$ and $S' \subseteq S$. To see $S \subseteq S'$, let $\chi \in S$ and take $Q(x) \in k^{[n]}$ isobaric of weight $\chi$. Write $Q(x) = bx_1^{p_1} \cdots x_n^{p_n} + (\text{other terms of total degree } \geq p_1 + \cdots + p_n)$. Apply $\phi_i$ to either side to get

$$\phi_i(Q(x)) = \chi(t)Q(x) = b(\chi_1(t)x_1 + f_1)(\chi_2(t)x_2 + f_2) \cdots (\chi_n(t)x_n + f_n)^{p_n} + (\text{other terms}) = b(\chi_1(t))^{p_1} \cdots x_1^{p_1} \cdots x_n^{p_n} + (\text{other terms})$$

and clearly the term $b(p_1 \chi_1 + \cdots + p_n \chi_n)(t)x_1^{p_1} \cdots x_n^{p_n}$ in the last expression is unique and uncancellable there. So this term must match $\chi(t)bx_1^{p_1} \cdots x_n^{p_n}$ from $\chi(t)Q(t)$. This proves $\chi = p_1 \chi_1 + \cdots + p_n \chi_n \in S'$.

We shall call $\{\chi_1, \ldots, \chi_n\}$ appearing in (10) a set of standard generators for $S$.

3.3. Remark. The diagonal matrix with entries $\chi_1(t), \ldots, \chi_n(t)$ represents the induced action of $T$ on the tangent space to $\mathbb{A}^n$ at the origin. Therefore, if $x_1, \ldots, x_{n-m}$ are nonzero and the rest are zero, then the dimension of the fixed point variety is precisely $m$. This follows at once from Fogarty's theorem [3; Th. 5.2, p. 45].

3.4. Theorem. Let $T = (\mathbb{G}_m)^r$ act on $\mathbb{A}^n$. Assume that the action is unmixed and the
dimension of the fixed point variety \((\mathbb{A}^n)^T\) is \(\leq 2\). Then the action is linearizable (see 3.1 for the terminology here).

Proof. Let us put \(F := (\mathbb{A}^n)^T = \text{Spec} \ A_0\) with \(A := k[\lambda]\). We already know from 2.2, 2.4 and 2.5 that \(F\) is smooth, that \(A_0\) is factorial and that \(\mathbb{A}^n\) is generically separable over \(F\). Moreover, \(m := \dim F \leq 2\) by assumption. By the affine Castelnuovo theorem of Fujita, Miyanishi and Sugie, \(A_0 = k[\lambda]\) (see [4] and [8]; also compare Kambayashi [6] and Russell [10]). By Seshadri's theorem (see [11] or [9], [12]), the \(A_0\)-projective module \(I/I^2\) is free of rank \(n - m\). Choose a base \(\{u_1, \ldots, u_{n-m}\}\) of \(I/I^2\) over \(A_0\) consisting of homogeneous elements. Now as shown in the proof of Theorem 2.5, the image set \(E = \{s(u_1), \ldots, s(u_{n-m})\}\) under the lifting \(s: I/I^2 \to I\) generates \(A = k[\lambda]\) over \(A_0\). Writing \(A_0 = k[y_1, \ldots, y_m]\) and \(z_i = s(u_i)\) for \(1 \leq i \leq n - m\), we find \(A = k[y_1, \ldots, y_m, z_1, \ldots, z_{n-m}]\) with all \(y_i\) and \(z_j\) isobaric.

3.5. Remark. Consider an effective torus action on \(\mathbb{A}^3\). If \(T = (\mathbb{G}_m)^r\), we must have \(r \leq 3\) (see Kambayashi [5; §3, (6). p. 447]). If \(r = 3\) or \(2\), the action is linearizable by Bialynicki-Birula's results quoted in 3.1, and all unmixed cases were settled just now in Theorem 3.4. Thus the only remaining case to be investigated is that of a \(\mathbb{G}_m\)-action on \(\mathbb{A}^3\) of mixed type. Write \(A^3 = \text{Spec} k[x, y, z]\) and \(\phi_i(x) = t^i x + f_i, \phi_i(y) = t^i y + g_i, \phi_i(z) = t^i z + h_i\) as done in (10). Then, in view of what was shown in 3.2, the monoid \(S\) attached here is generated by \(a, b\) and \(c\) inside \(\mathbb{Z}\). The open problem boils down to deciding whether or not the \(\mathbb{G}_m\)-action is linearizable in two cases, namely when the signature of \(\{a, b, c\}\) is either \(+, +, -\) or \(+, -, 0\). One should consult Bialynicki-Birula [2; §4] for interesting results in this connection.

References