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Convergence of the semi-implicit Euler method for stochastic age-dependent population equations with Poisson jumps

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ABSTRACT

We consider semi-implicit methods for stochastic age-dependent population equations with Poisson jumps. The main purpose of this paper is to show the convergence of the numerical approximation solution to the true solution with strong order $p = \frac{1}{2}$.

1. Introduction

Stochastic partial functional differential equations are very important in stochastic models of biological, chemical, physical and economical systems, and the study of stochastic age-dependent populations has received a great deal of attention [1,2]. Zhang et al. [3] show the existence, uniqueness and exponential stability for stochastic age-dependent population equations. Pang et al. [4] give the convergence of the semi-implicit Euler method for a stochastic age-dependent population equations of the form

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial a} = -\mu(t,a)P + f(t,P) + g(t,P)\frac{dW_t}{dt}, \tag{1.1}$$

where $P(t,a)$ denotes the population density of age a at time t, $\beta(t,a)$ denotes the fertility rate of females of age a at time t, $\mu(t,a)$ denotes the mortality rate of age a at time t, $f(t,P)$ denotes effects of external environment for population system, $g(t,P)$ is a diffusion coefficient, W_t is a Brownian motion.

In the stochastic age-dependent population system, due to brusque variations from some rare events (for example, tsunami, earthquakes, impacts of extra terrestrial objects and so on), the size of the population systems increases or decreases drastically, so the jump-diffusion processes better describe the dynamics of population density. Such a generalization seems to be more appropriate for population system. Recently, Li et al. [5] studied the convergence of numerical solutions to stochastic delay differential equations with jumps. In the present paper we will further research this topic.

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The stochastic age-dependent population equation with Poisson jumps of the form

$$d_tP = \left[-\frac{\partial P}{\partial a} - \mu(t,a)P + f(t,P)\right]dt + g(t,P)dW_t + h(t,P)dN_t, \tag{1.2}$$

where $h(t,P)$ is a jump coefficient, N_t is a scalar Possion process with intensity λ_1.

Obviously, Eq. (1.2) can be regarded as a generalization of the stochastic age-dependent population equation (1.1). In general, stochastic age-dependent population equation with jumps rarely has an explicit solution. Thus, numerical approximation schemes are invaluable tools for exploring its properties. However, there is little work on the convergence of the semi-implicit Euler method for a stochastic age-dependent population equation with jumps. In this paper, we will develop the semi-implicit Euler method for stochastic age-dependent population equation of the type described by Eq. (1.2).

In [5], Li introduced explicit schemes that generate approximate solutions Q_t^n of (2.1) on the grid points t_n, they did not imposed the jump time τ_i in (2.1) which are violated by the discontinuous nature of the thinning construction. In this article the main distinction between our work and [5] is that we construct a discrete-time approximation to P by consider the jump time.

This paper can be organized as follows: in Section 2, we begin with some preliminary results which are essential for our analysis and define a semi-implicit Euler approximate solution to stochastic age-dependent population equation with jumps. In Section 3, we shall prove that the numerical solutions converge to the exact solutions and provide the order of convergence.

2. Preliminaries and semi-implicit approximation

At the beginning we introduce the following notation. Let

$$V = H^1([0,A]) \equiv \left\{\varphi|\varphi \in L^2([0,A]), \frac{\partial \varphi}{\partial x_i} \in L^2([0,A]) \quad \text{where} \quad \frac{\partial \varphi}{\partial x_i} \text{ is generalized partial derivatives}\right\},$$

V is a Sobolev space. $H = L^2([0,A])$ such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

V' is the dual space of V. We denote by $\|\cdot\|$, $\|\cdot\|$ and $\|\cdot\|_*$ the norms in V, H, and V', respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V and V', let $(\cdot,\cdot)$ indicate the scalar product in H.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i. e., it is increasing and right continuous with a left-hand side limit, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let W_t be a Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking its values in the separable Hilbert space M, with increment covariance operator W. For an operator $B \in L(M,H)$ be the space of all bounded linear operators from M into H, we denote by $\|B\|_2$ the Hilbert–Schmidt norm, i. e., $\|B\|_2 = \text{tr}(BWB)^T$. Let $C = C([0,T];H)$ be the space of all continuous function from $[0,T]$ into H with sup-norm $\|\psi\|_c = \sup_{0\leqslant s\leqslant T}|\psi(s)|$, $L_V^P = L^P([0,T];V)$ and $L_H^P = L^P([0,T];H)$.

We define $P_{t^-} = \lim_{s\to t^-} P(s,a)$, and $\frac{\partial P_{t^-}}{\partial a} = \lim_{s\to t^-} \frac{\partial P_s}{\partial a}$, then consider stochastic age-dependent population equation with jump of the form

$$\begin{cases} d_tP = \left[-\frac{\partial P_{t^-}}{\partial a} - \mu(t,a)P_{t^-} + f(t,P_{t^-})\right]dt \\ \qquad +g(t,P_{t^-})dW_t + h(t,P_{t^-})dN_t, & \text{in} \quad G; \\ P(0,a) = P_0(a), & \text{in} \quad [0,A]; \\ P(t,0) = \int_0^A \beta(t,a)P(t,a)da, & \text{in} \quad [0,T], \end{cases} \tag{2.1}$$

where $T > 0$, $A > 0$, and $G = (0,T) \times (0,A)$, let $f(t,\cdot), h(t,\cdot) : L_H^2 \to H$ be a family of nonlinear operators, $\mathscr{F}_t$-measurable almost surely in t. $g(t,\cdot) : L_H^2 \to L(M,H)$ is the family of nonlinear operator, $\mathscr{F}_t$-measurable almost surely in t.

The integral version of Eq. (2.1) is given by the equation

$$P_t = P_0 - \int_0^t \frac{\partial P_{s^-}}{\partial a}ds - \int_0^t \mu(s,a)P_{s^-}ds + \int_0^t f(s,P_{s^-})ds + \int_0^t g(s,P_{s^-})dW_s + \int_0^t h(s,P_{s^-})dN_s. \tag{2.2}$$

Let τ_j denotes the jth jump of N_s occurrence time. Suppose for example, that jumps arrive at distinct, ordered times $\tau_1 < \tau_2 < \cdots$, let $t_1, \ldots, t_m$ be a deterministic grid points of $[0,T]$. We construct approximate solutions to models of the form (2.1) at a discrete set of times $\{\tau_n\}$. This set is the superposition of the random jump times of a Poisson process on $[0,T]$ and a deterministic grid $t_1, \ldots, t_m$, and satisfy $\max\{|\tau_{i+1} - \tau_i| < \triangle$. It is clearer that the random Poisson jump times can be computed without any knowledge of the realized path of (2.1).

Let $\triangle_n = \tau_{n+1} - \tau_n$, $\triangle W_n = W(\tau_{n+1}) - W(\tau_n)$, and $\triangle N_n = N(\tau_{n+1}) - N(\tau_n)$ denote the increments of the time, Brownian motion and the Poisson processes, respectively. For system (2.1) the semi-implicit approximate solution on $\{\tau_n\}$ is defined by the iterative scheme

$$Q_{n+1} = Q_n + (1-\theta)\left[-\frac{\partial Q_{n+1}}{\partial a} - \mu(t,a)Q_n + f(t,Q_n)\right]\triangle_n + \theta\left[-\frac{\partial Q_{n+1}}{\partial a} - \mu(t,a)Q_{n+1} + f(t,Q_{n+1})\right]\triangle_n \\ + g(t,Q_n)\triangle W_n + h(t,Q_n)\triangle N_n, \tag{2.3}$$

with initial value $Q_0 = P(0,a)$, $Q_n(t,0) = \int_0^A \beta(t,a)Q_n da$, $n > 0$. Here, θ is a parameter with $0 \leqslant \theta \leqslant 1$. We first define step functions

$$Z_1(t) = Z_1(t,a) = \sum_{k=0}^{N-1} Q_k \mathrm{I}_{[\tau_k,\tau_{k+1})}(t), \tag{2.4}$$

$$Z_2(t) = Z_2(t,a) = \sum_{k=0}^{N-1} Q_{k+1} \mathrm{I}_{[\tau_k,\tau_{k+1})}(t), \tag{2.5}$$

where I_G is the indicator function for the set G, and $Q_k = P(t_k,a)$, $\overline{Q}_k = P(t_k^-,a) = \lim_{s\to t_k} P(s,a)$. Note that $Q_{k+1} = \overline{Q}_{k+1} + h(t,\overline{Q}_{k+1})$ at each jump time τ_k, and when $P(t,a)$ is continuous at t_k, then $Q_{k+1} = \overline{Q}_{k+1}$, so that $Z_1(t_k) = Q_k = Q(t_k,a)$, $Z_2(t_k) = Q_{k+1}$. We define

$$Q_t = P_0 + \int_0^t (1-\theta)\left[-\frac{\partial Q_s}{\partial a} - \mu(s,a)Z_1(s) + f(s,Z_1(s))\right]ds + \int_0^t \theta\left[-\frac{\partial Q_s}{\partial a} - \mu(s,a)Z_2(s) + f(s,Z_2(s))\right]ds \\ + \int_0^t g(s,Z_1(s))dW_s + \int_0^t h(s,Z_1(s))dN_s, \tag{2.6}$$

with $Q_0 = P(0,a)$, $Q(t,0) = \int_0^A \beta(t,a)Q_t da$, $Q_t = Q(t,a)$. Note the difference between Q_t and Q_k.

We note that the numerical solution in Eq. (2.3) is defined by a sime-implicit equation containing partial derivative and Poisson jumps. The Eq. (2.3) is reduced to the Euler approximation to Eq. (2.1) witch discussed in [4], if $\theta = 0$, so this paper is regarded as a generalization of the paper [4].

As the standing hypotheses we always assume that the following conditions are satisfied:

(i) $f(t,0) = 0, \quad g(t,0) = 0, \quad h(t,0) = 0$;

(ii) (Lipschitz condition) There exists a positive constant K such that

$$|f(t,y) - f(t,x)| \vee \|g(t,y) - g(t,x)\|_2 \vee |h(t,y) - h(t,x)| \leqslant K\|y-x\|_c, \tag{2.7}$$

(iii) $\mu(t,a), \beta(t,a)$ are continuous in $\overline{G}$ (the closure of G) such that

$$0 \leqslant \mu_0 \leqslant \mu(t,a) \leqslant \overline{\alpha} < \infty, \quad 0 \leqslant \beta(t,a) \leqslant \overline{\beta} < \infty, \tag{2.8}$$

(iv) (Coercivity condition) There exist constants $\alpha > 0$, $\xi > 0$, $\lambda \in \mathbb{R}$, and a nonnegative continuous function $\gamma(t)$, $t \in \mathbb{R}^+$, such that

$$2\langle f(t,v) + \lambda_1 h(t,v), v\rangle + \|g(t,v)\|_2^2 \leqslant -\alpha\|v\|^2 + \lambda|v|^2 + \gamma(t)e^{-\xi t},$$

where, λ_1 is intensity of scalar Poisson process N_t, for arbitrary $\delta > 0$, $\lim_{t\to\infty}\gamma(t)e^{-\delta t} = 0$.

With the similar proof of Theorem 3.1 in [3], we can get

Theorem 2.1. *Under the assumptions (i)–(iv), then Eq. (2.1) has a unique strong solution on* $t \in [0,T]$.

3. Stochastic population system under Possion jumps

Throughout this work, we use $C_1, C_2, \ldots$, to denote a generic constant (independent of $\triangle$) that may change from line to line.

Lemma 3.1 [6]. *Assuming* $f(t,x) \in C^{1,2}([0,+\infty) \times R)$, *and*

$$dX_s = a(s,X_{s^-})dt + b(s,X_{s^-})dW_s + c(s,X_{s^-})dN_s,$$

or

$$X_t = X_0 + \int_0^t a(s,X_{s^-})ds + \int_0^t b(s,X_{s^-})dW_s + \int_0^t c(s,X_{s^-})dN_s.$$

Then we have

$$f(t,X_t) = f(0,X_0) + \int_0^t L_0 f(s,X_s)ds + \int_0^t L_1 f(s,X_s)dW_s + \int_0^t L_{-1} f(s,X_s)dN_s, \tag{3.1}$$

where

$$L_0 f(t,x) = f'_t(t,x) + a(t,x)f'_x(t,x) + \frac{1}{2}b^2(t,x)f''_{xx}(t,x),$$
$$L_1 f(t,x) = b(t,x)\frac{\partial f(t,x)}{\partial x} = b(t,x)f'_x(t,x),$$
$$L_1 f(t,x) = f(t,x+c(t,x)) - f(t,x).$$

The next lemma shows that the continuous approximation has bounded second moments in a strong sense.

Lemma 3.2. *Under the assumptions (i)–(iv) above, then*

$$\mathbb{E}\sup_{t\in[0,T]}|Q_t|^2 \leqslant C_1. \tag{3.2}$$

Proof. From Eq. (2.6), one can obtain

$$dQ_t = -\frac{\partial Q_t}{\partial a}dt + (1-\theta)[f(t,Z_1(t)) - \mu(t,a)Z_1(t)]dt + g(t,Z_1(t))dW_t + \theta[f(t,Z_2(t)) - \mu(t,a)Z_2(t)]dt + h(t,Z_1(t))dN_t. \tag{3.3}$$

Applying Lemma 3.1 to $|Q_t|^2$ yields

$$\begin{aligned}|Q_t|^2 &= |Q_0|^2 + 2\int_0^t\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds + 2\int_0^t((1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s)), Q_s)ds \\ &\quad - 2\int_0^t(\mu(s,a)[(1-\theta)Z_1(s) + \theta Z_2(s)], Q_s)ds \\ &\quad + 2\int_0^t(Q_s, g(s,Z_1(s))dW_s) + \int_0^t\|g(s,Z_1(s))\|_2^2 ds + \int_0^t\left[|Q_s + h(s,Z_1(s))|^2 - |Q_s|^2\right]dN_s \\ &\leqslant |Q_0|^2 + 2\int_0^t\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds + 2\int_0^t|Q_s||(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))|ds \\ &\quad + 2\mu_0\int_0^t|Q_s||(1-\theta)Z_1(s) + \theta Z_2(s)|ds + \int_0^t\|g(s,Z_1(s))\|_2^2 ds + 2\int_0^t(Q_s, g(s,Z_1(s))dW_s) \\ &\quad + 2\int_0^t(Q_s, h(s,Z_1(s)))d\widetilde{N}_s + \int_0^t|h(s,Z_1(s))|^2 d\widetilde{N}_s + \lambda_1\int_0^t|h(s,Z_1(s))|^2 ds + 2\lambda_1\int_0^t|Q_s||h(s,Z_1(s))|ds,\end{aligned} \tag{3.4}$$

where $\widetilde{N}_t = N_t - \lambda_1 t$ is a compensated Poisson process. Since

$$\begin{aligned}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle &= -\int_0^A Q_s d_a(Q_s) = \frac{1}{2}\left(\int_0^A \beta(s,a)Q_s da\right)^2 \\ &\leqslant \frac{1}{2}\int_0^A\beta^2(s,a)da\int_0^A Q_s^2 da \leqslant \frac{1}{2}A^2\overline{\beta}^2|Q_s|^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&2\int_0^t|Q_s||(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))|ds \\ &\quad\leqslant \int_0^t|Q_s|^2 ds + 2\int_0^t[|f(s,Z_1(s))|^2 + |f(s,Z_2(s))|^2]ds \\ &\quad\leqslant \int_0^t|Q_s|^2 ds + 2K^2\int_0^t[|Z_1(s)|^2 + |Z_2(s)|^2]ds,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&2\mu_0\int_0^t|Q_s||(1-\theta)Z_1(s) + \theta Z_2(s)|ds \\ &\quad\leqslant \mu_0\int_0^t|Q_s|^2 ds + 2\mu_0\int_0^t[|Z_1(s)|^2 + |Z_2(s)|^2]ds\end{aligned} \tag{3.7}$$

$$\begin{aligned}2\lambda_1\int_0^t|Q_s||h(s,Z_1(s))|ds &\leqslant \lambda_1\int_0^t|Q_s|^2 ds + \lambda_1\int_0^t|h(s,Z_1(s))|^2 ds \\ &\leqslant \lambda_1\int_0^t|Q_s|^2 ds + \lambda_1 K^2\int_0^t|Z_1(s)|^2 ds.\end{aligned} \tag{3.8}$$

Taking Eqs. (3.5)–(3.8) into Eq. (3.4), we compute that for some positive $K_1 = A^2\overline{\beta}^2 + 1 + \mu_0 + \lambda_1$, $K_2 = 3K^2 + 2\lambda_1 K^2 + 2\mu_0$, $K_3 = 2(K^2 + \mu_0)$,

$$|Q_t|^2 \leqslant |Q_0|^2 + K_1 \int_0^t |Q_s|^2 ds + K_2 \int_0^t |Z_1(s)|^2 ds + K_3 \int_0^t |Z_2(s)|^2 ds + 2\int_0^t (Q_s, g(s, Z_1(s)) dW_s) + 2\int_0^t (Q_s, h(s, Z_1(s))) d\widetilde{N}_s + \int_0^t |h(s, Z_1(s))|^2 d\widetilde{N}_s,$$

Now, it follows that for any $t_1 \in [0, T]$

$$\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] \leqslant \mathbb{E}|Q_0|^2 + (K_1 + K_2 + K_3) \int_0^{t_1} \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |Q_s|^2\right] dt + 2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t (Q_s, g(s, Z_1(s)) dW_s)\right] + 2\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t (Q_s, h(r, Z_1(s)) d\widetilde{N}_s\right] + \mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t |h(s, Z_1(s))|^2 d\widetilde{N}_s\right], \tag{3.9}$$

By Burkholder–Davis–Gundy's inequality (see for example [7]), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t (Q_s, g(s, Z_1(s)) dW_s)\right] &\leqslant \frac{1}{6}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_4 \int_0^{t_1} \|g(t, Z_1(t))\|_2^2 dt \\ &\leqslant \frac{1}{6}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_4 \cdot K^2 \int_0^{t_1} \mathbb{E}\|Z_1(t\|_c^2 dt,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t (Q_r, h(r, Z_1(r))) d\widetilde{N}_r\right] &\leqslant \frac{1}{6}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_5 \int_0^{t_1} |h(t, Z_1(t))|^2 dt \\ &\leqslant \frac{1}{6}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_5 \cdot K^2 \int_0^{t_1} \mathbb{E}\|Q_t\|_c^2 dt,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,t_1]} \int_0^t |h(s, Z_1(s))|^2 d\widetilde{N}_s\right] &\leqslant C\mathbb{E}\left[\int_0^{t_1} |h(t, Z_1(t))|^4 dt\right]^{\frac{1}{2}} \\ &\leqslant CK^2\mathbb{E}\left[\|Z_1\|^2 \int_0^{t_1} \|Q_t\|^2 dt\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{6}\mathbb{E}\left[\sup_{t\in[0,t_1]} |Q_t|^2\right] + K_6 \int_0^{t_1} \mathbb{E}\|Q_t\|_c^2 dt,\end{aligned} \tag{3.12}$$

for some positive constant K_4, K_5, $K_6 > 0$. Substituting (3.10)–(3.12) into (3.9) yields, again for a possibly different K,

$$\mathbb{E}\sup_{t\in[0,t_1]} |Q_t|^2 \leqslant 6\mathbb{E}|Q_0|^2 + K\int_0^{t_1} \mathbb{E}\sup_{t\in[0,t_1]} |Q_t|^2 dt \quad \forall t_1 \in [0, T].$$

Now, Gronwall's lemma implies the required result. □

Lemma 3.3. *Under the assumptions (i)-(iv) and* $\int_0^T \mathbb{E}|\frac{\partial Q_s}{\partial a}|^2 ds < \infty$, *then*

$$\mathbb{E}\sup_{t\in[0,T]} |Q_t - Z_1(t)|^2 \leqslant C_3\triangle, \tag{3.13}$$

and

$$\mathbb{E}\sup_{t\in[0,T]} |Q_t - Z_2(t)|^2 \leqslant C_4\triangle. \tag{3.14}$$

Proof. For given $t \in [0, T]$, there exists an integer k such that $t \in [\tau_k, \tau_{k+1})$, we have

$$
\begin{aligned}
Q_t - Z_1(t) &= Q_t - Q_k \\
&= -\int_{\tau_k}^t \frac{\partial Q_s}{\partial a} ds - \int_{\tau_k}^t [(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))]ds + \int_{\tau_k}^t \mu(s,a)[(1-\theta)Z_1(s)) + \theta Z_2(s)]ds \\
&\quad + \int_{\tau_k}^t g(s,Z_1(s))dW_s + \int_{\tau_k}^t h(s,Z_1(s))dN_s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|Q_t - Z_1(t)|^2 &\leqslant 5\left|\int_{\tau_k}^t \frac{\partial Q_s}{\partial a} ds\right|^2 + 5\left|\int_{\tau_k}^t [(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))]ds\right|^2 + 5\left|\int_{\tau_k}^t \mu(s,a)[(1-\theta)Z_1(s) + \theta Z_2(s)]ds\right|^2 \\
&\quad + 5\left|\int_{\tau_k}^t g(s,Z_1(s))dW_s\right|^2 + 5\left|\int_{\tau_k}^t h(s,Z_1(s))dN_s\right|^2.
\end{aligned}
$$

Now, the Cauchy–Schwarz inequality and the assumptions give

$$
\begin{aligned}
|Q_t - Z_1(t)|^2 &\leqslant 5\triangle \int_{\tau_k}^t \left|\frac{\partial Q_s}{\partial a}\right|^2 ds + 5\triangle \int_{\tau_k}^t |(1-\theta)f(s,Z_1(s)) + \theta f(s,Z_2(s))|^2 ds + 5\bar{\alpha}^2 \triangle \int_{\tau_k}^t |(1-\theta)Z_1(s) + \theta Z_2(s)|^2 ds \\
&\quad + 5\left|\int_{\tau_k}^t g(s,Z_1(s))dW_s\right|^2 + 10\left|\int_{\tau_k}^t h(s,Z_1(s))d\widetilde{N}_s\right|^2 + 10\lambda_1^2\left|\int_{\tau_k}^t h(s,Z_1(s))ds\right|^2 \\
&\leqslant 5\triangle \int_{\tau_k}^t \left|\frac{\partial Q_s}{\partial a}\right|^2 ds + 10\bar{\alpha}^2\left[\int_{\tau_k}^t |Z_1(s)|^2 ds + \int_{\tau_k}^t |Z_2(s)|^2 ds\right] + 10(1+\lambda_1^2)K^2\triangle \int_{\tau_k}^t |Z_1(s)|^2 ds \\
&\quad + 10K^2\triangle \int_{\tau_k}^t |Z_2(s)|^2 ds + 5\left|\int_{\tau_k}^t g(s,Z_1(s))dW_s\right|^2 + 10\left|\int_{\tau_k}^t h(s,Z_1(s))d\widetilde{N}_s\right|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} |Q_t - Z_1(t)|^2 &\leqslant 5\triangle \int_0^T \left|\frac{\partial Q_s}{\partial a}\right|^2 ds + 20\bar{\alpha}^2 C_1\triangle + (20+10\lambda_1^2)K^2\triangle TC_1 + 5\mathbb{E}\sup_{t\in[0,T]} \max_{k=0,1,\dots,N-1}\left|\int_{\tau_k}^t g(s,Z_1(s))dW_s\right|^2 \\
&\quad + 10\mathbb{E}\sup_{t\in[0,T]} \max_{k=0,1,\dots,N-1}\left|\int_{\tau_k}^t h(s,Z_1(s))d\widetilde{N}_s\right|^2.
\end{aligned}
$$

Using the (2.7), (3.2) and Doob inequality yield

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} |Q_t - Z_1(t)|^2 &\leqslant 5\triangle \int_0^T \left|\frac{\partial Q_s}{\partial a}\right|^2 ds + 20\bar{\alpha}^2 C_1\triangle + (20+10\lambda_1^2)K^2\triangle TC_1 + 5\max_{k=0,1,\dots,N-1}\int_{\tau_k}^{(k+1)\triangle} \mathbb{E}|g(s,Z_1(s))|^2 ds \\
&\quad + 10\lambda_1 \max_{k=0,1,\dots,N-1}\int_{\tau_k}^{(k+1)\triangle} \mathbb{E}|h(s,Z_1(s))|^2 ds \\
&\leqslant 5\triangle \int_0^T \left|\frac{\partial Q_s}{\partial a}\right|^2 ds + 20\bar{\alpha}^2 C_1\triangle + (20+10\lambda_1^2)K^2\triangle TC_1 + 5K^2C_1\triangle + 10K^2\lambda_1 C_1\triangle.
\end{aligned}
$$

So the inequality (3.13) holds, a similar analysis gives (3.14). □

Theorem 3.1. *Under the assumptions in Lemma 3.3, then*

$$
\mathbb{E}\sup_{t\in[0,T]} |P_t - Q_t|^2 \leqslant C_5\triangle. \tag{3.15}
$$

Proof. Combining (2.2) with (2.6) has

$$
\begin{aligned}
Q_t - P_t &= -\int_0^t \frac{\partial(Q_s - P_{s^-})}{\partial a} ds - \int_0^t \mu(s,a)[(1-\theta)(Z_1(s) - P_{s^-}) + \theta(Z_2(s) - P_{s^-})]ds + \int_0^t [(1-\theta)(f(s,Z_1(s)) - f(P_{s^-}))) \\
&\quad + \theta(f(s,Z_2(s)) - f(P_{s^-})))]ds + \int_0^t (g(s,Z_1(s)) - g(s,P_{s^-}))dW_s + \int_0^t (h(s,Z_1(s)) - h(s,P_{s^-})))dN_s.
\end{aligned}
$$

Therefore using the generalized Itô formula, along with the Cauchy–Schwarz inequality and (2.7) yields

$$
\begin{aligned}
& d|Q_t - P_{t^-}|^2 \\
& \quad = -2\left\langle Q_t - P_t, \frac{\partial (Q_t - P_{t^-})}{\partial a}\right\rangle dt + \|g(t, Z_1(t)) - g(t, P_{t^-})\|_2^2 dt \\
& \qquad - 2(Q_t - P_{t^-}, \mu(t,a)[(1-\theta)(Z_1(t) - P_{t^-}) + \theta(Z_2(t) - P_{t^-})])dt \\
& \qquad + 2(Q_t - P_{t^-}, (1-\theta)(f(t, Z_1(t)) - f(t, P_{t^-})) + \theta(f(t, Z_2(t)) - f(t, P_{t^-})))dt \\
& \qquad + \lambda_1 |h(t, Z_1(t)) - h(t, P_{t^-}))|^2 dt + 2\lambda_1 (Q_t - P_{t^-}, (h(t, Z_1(t)) - h(t, P_{t^-}))))dt \\
& \qquad + 2(Q_t - P_{t^-}, (g(t, Z_1(t)) - g(t, P_{t^-}))dW_t) \\
& \qquad + 2(Q_t - P_{t^-}, (h(t, Z_1(t)) - h(t, P_{t^-})))d\widetilde{N}_t + |h(t, Z_1(t)) - h(t, P_{t^-}))|^2 d\widetilde{N}_t \\
& \quad \leqslant (1 + A^2\overline{\beta}^2 + \mu_0 + \lambda_1)|Q_t - P_{t^-}|^2 dt \\
& \qquad + [(3 + 2\lambda_1)K^2 + 2\mu_0]\|Z_1(t) - P_{t^-}\|_c^2 dt + 2(K^2 + \mu_0)\|Z_2(t) - P_{t^-})\|_c^2 dt \\
& \qquad + 2(Q_t - P_{t^-}, (g(t, Z_1(t)) - g(t, P_{t^-}))dW_t) \\
& \qquad + 2(Q_t - P_{t^-}, (h(t, Z_1(t)) - h(t, P_{t^-}))))d\widetilde{N}_t + |h(t, Z_1(t)) - h(t, P_{t^-}))|^2 d\widetilde{N}_t.
\end{aligned}
$$

Hence, for any $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_s|^2\right] \leqslant {} & (1 + A^2\overline{\beta}^2 + \mu_0 + \lambda_1)\int_0^t \mathbb{E}\sup_{r\in[0,s]} |Q_r - P_{r^-}|^2 ds + [(3 + 2\lambda_1)K^2 + 2\mu_0]\int_0^t \mathbb{E}\|Z_1(s) - P_{s^-}\|_c^2 ds \\
& + 2(K^2 + \mu_0)\int_0^t \mathbb{E}\|Z_2(s) - P_{s^-}\|_c^2 ds + 2\mathbb{E}\sup_{s\in[0,t]}\int_0^s (Q_r - P_{r^-}, (g(r, Z_1(r)) - g(r, P_{r^-})dW_r) \\
& + 2\mathbb{E}\sup_{s\in[0,t]}\int_0^s (Q_r - P_{r^-}, (h(r, Z_1(r)) - h(r, P_{r^-})))d\widetilde{N}_r \\
& + 2\mathbb{E}\sup_{s\in[0,t]}\int_0^s |h(r, Z_1(r)) - h(r, P_{r^-})|^2 d\widetilde{N}_r. \qquad (3.16)
\end{aligned}
$$

By Burkholder–Davis–Gundy's inequality, we have

$$
\begin{aligned}
& 2\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^s (Q_r - P_{r^-}, g(r, Z_1(r)) - g(r, P_{r^-}))dW_r)\right] \\
& \quad \leqslant C\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|\left(\int_0^t \|g(s, Z_1(s)) - g(s, P_{s^-})\|_2^2\right)^{1/2}\right] \\
& \quad \leqslant \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|^2\right] + k_1\int_0^t \mathbb{E}\|Z_1(s) - P_{s^-}\|_c^2 ds \\
& \quad \leqslant \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|^2\right] + 2k_1\int_0^t \left[\mathbb{E}\|Q_s - P_{s^-}\|_c^2 + \mathbb{E}\|Q_s - Z_1(s)\|_c^2\right] ds, \qquad (3.17) \\
& 2\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^s (Q_r - P_{r^-}, (h(r, Z_1(r)) - h(r, P_r)))d\widetilde{N}_r\right] \\
& \quad \leqslant C\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|\left(\int_0^t |h(r, P_{r^-}) - h(r, Z_1(r))|^2\right)^{1/2}\right] \\
& \quad \leqslant \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|^2\right] + k_2\int_0^t \mathbb{E}\|Z_1(s) - P_{s^-}\|_c^2 ds \\
& \quad \leqslant \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|^2\right] + 2k_2\int_0^t \left[\mathbb{E}\|Q_s - P_{s^-}\|_c^2 + \mathbb{E}\|Q_s - Z_1(s)\|_c^2\right] ds, \qquad (3.18) \\
& 2\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^s |h(t, Z_1(t)) - h(t, P_{t^-})|^2 d\widetilde{N}_t\right] \\
& \quad \leqslant C\mathbb{E}\left[\int_0^{t_1} |h(t, Z_1(t)) - h(t, P_{t^-}))|^4 dt\right]^{\frac{1}{2}} \\
& \quad \leqslant CK^2\mathbb{E}\left[\|Z_1(t) - P_{t^-}\|^2\int_0^{t_1} \|Z_1(t) - P_{t^-}\|^2 dt\right]^{\frac{1}{2}} \\
& \quad \leqslant \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - P_{s^-}|^2\right] + \frac{1}{6}\mathbb{E}\left[\sup_{s\in[0,t]} |Q_s - Z_1(s)|^2\right] \\
& \quad + 2k_3\int_0^t \left[\mathbb{E}\|Q_s - P_{s^-}\|_c^2 + \mathbb{E}\|Q_s - Z_1(s)\|_c^2\right] ds, \qquad (3.19)
\end{aligned}
$$

where k_1, k_2, k_3 are three positive constants. Therefore, inserting (3.17)–(3.19) into (3.16) has

$$\mathbb{E}\left[\sup_{s\in[0,t]}|Q_s-P_{s^-}|^2\right]\leqslant 2d_2\int_0^t\mathbb{E}\sup_{r\in[0,s]}|P_r-Q_r|^2ds+\frac{1}{3}\mathbb{E}\left[\sup_{s\in[0,t]}|Q_s-Z_1(s)|^2\right]$$
$$+2d_1\int_0^t\mathbb{E}[\|Q_s-Z_1(s)\|_c^2ds+4(K^2+\mu_0)\int_0^t\mathbb{E}\|Q_s-Z_2(s)\|_c^2ds,$$

where $d_1=2[3K^2+2\lambda_1K^2+2\mu_0+k_1+k_2+k_3]$, and $d_2=1+A^2\overline{\beta}^2+\mu_0+\lambda_1+2(d_1+2K^2+2\mu_0)$.

Applying Lemma 3.3 we obtain a bound of the form

$$\mathbb{E}\sup_{s\in[0,t]}|Q_s-P_{s^-}|^2\leqslant D_1\triangle+2C_2\int_0^t\mathbb{E}\sup_{r\in[0,s]}|P_r-Q_r|^2ds, \tag{3.20}$$

where $D_1=2d_1C_3T+\frac{1}{3}C_3+4(K^2+\mu_0)C_4T$. The result (3.15) then follows from the continuous Gronwall inequality with $C_5=D_1\exp(2C_2T)$. □

It is easy to have the following theorem.

Theorem 3.2. *Assume the preceding hypotheses hold, the approximate solution (2.6) will converge to the true solution of Eq. (2.1) in the sense*

$$\lim_{\triangle\to 0}\mathbb{E}[\sup_{t\in[0,T]}|Q_t-P_{t^-}|^2]=0. \tag{3.21}$$

4. One example

Let us consider a stochastic age-dependent population equation with Possion jumps of the form

$$\begin{cases} d_tP=\left[-\frac{\partial P_{t^-}}{\partial a}-\frac{1}{1-a}P_{t^-}\right]dt+\varphi(P)dW_t-P_{t^-}dN_t, & \text{in} \quad Q,\\ P(0,a)=1-a, & \text{in} \quad [0,1],\\ P(t,0)=2\int_0^1P(t,a)da, & \text{in} \quad [0,T]. \end{cases} \tag{4.1}$$

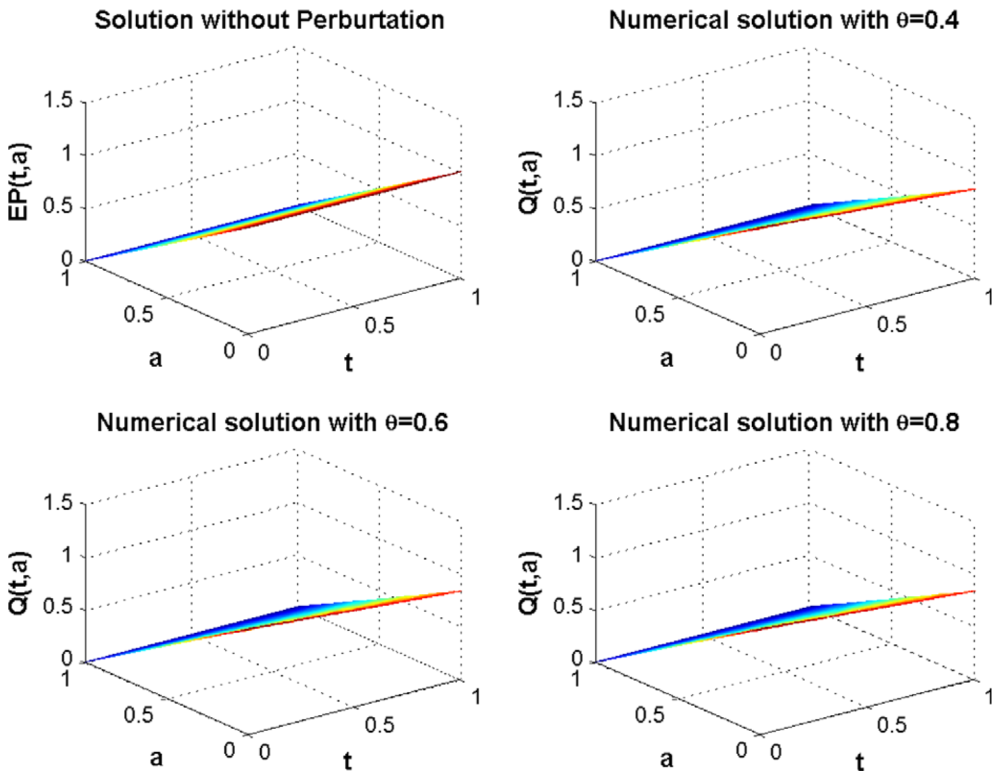

Fig. 1. Numerical simulations of stochastic population equation.

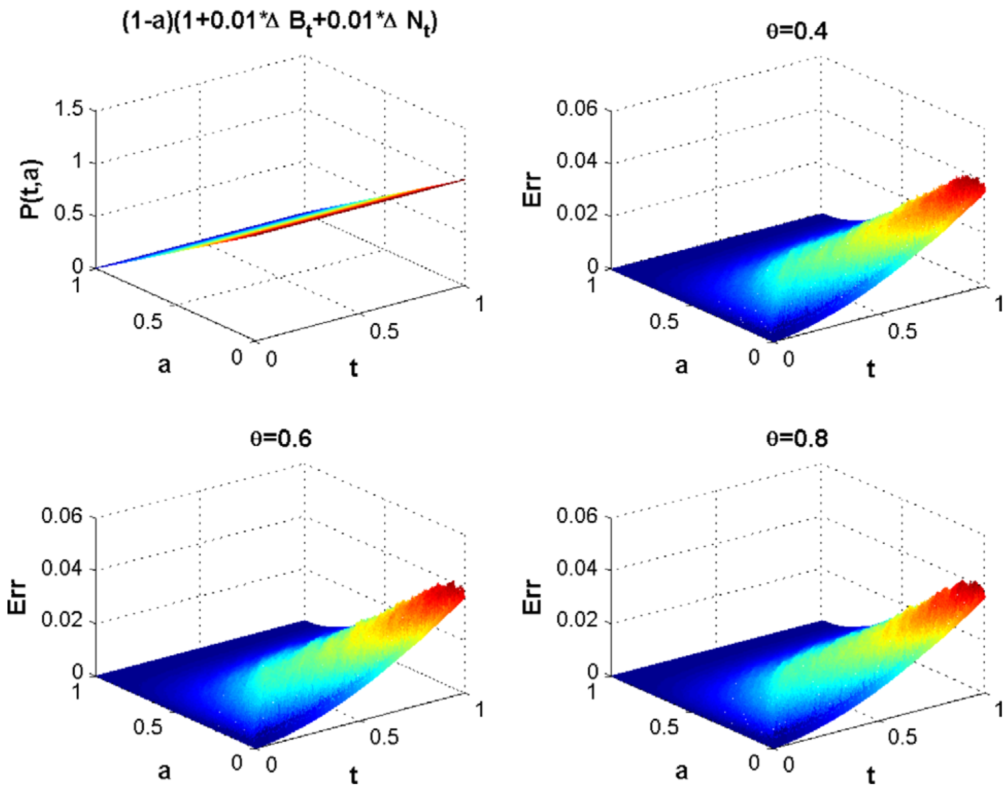

Fig. 2. Mean-square error of simulation.

This example is the modification of example in [4]. Where W_t is a real standard Brownian motion, N_t is a Possion jumps with intensity λ_1, and $\varphi(.) : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $\varphi(0) = 0$. Let $H = L^2([0,1])$, and $V = W_0^1([0,1])$ (a Sobolev space with elements satisfying the boundary conditions above), $M = \mathbb{R}$, $\mu(a) = \frac{1}{1-a}$, $\beta(t,a) = 2$, $f(t,p) = 0$, $h(t,p) = -p$, $g(t,a) = \varphi(a)$ and $P(0,a) = 1 - a$. Clearly, the operators f, g and h satisfy conditions (i) and (ii), $\mu(t,a)$ and $\beta(t,a)$ satisfy condition (iii). that is for arbitrary $v \in V$

$$2 < f(t,v) + \lambda_1 h(t,v), v > + \|g(t,v)\|_2^2 \leqslant -\alpha\|v\|^2 + k^2|v|^2 + \gamma(t)e^{-\xi t},$$

where $\alpha > 0$ is small enough, k is the Lipschitz constant for the function φ.

Therefore, it follows that condition (iv) is satisfied. Consequently, the semi-implicit approximate solution will converge to the true solution of (4.1) for any $(t,a) \in (0,T) \times (0,1)$ in the sense of Theorem 3.2, provided $\mathbb{E}(\frac{\partial P}{\partial a})^2$ is bounded in almost sure t.

Take $\varphi(p) = p$, $T = 1$ in Eq. (4.1). First, we fix the step sizes $\triangle_t = 0.002$, $\triangle_a = 0.005$ and let $\lambda_1 = 0.2$, and change the parameter θ in Fig. 1. The upper in the left is the explicit solution to Eq. (4.1) without perturbation, that is $\mathbb{E}P(t,a) = 1 - a$. The other three pictures are numerical simulations of the stochastic age-dependent population equations with $\theta = 0.4, \theta = 0.6$ and $\theta = 0.8$, respectively, where $Q(t,a) = \frac{1}{5000}\sum_{k=1}^{5000} Q_k(t,a,\omega)$, It clearly reveals the fact that the numerical approximation will tend to the true solution in the mean sense. Since the analytic explicit solution to Eq. (4.1) is not obtained, so the explicit solution $P(t,a)$ to Eq. (4.1) can be replaced by $(1-a) \times (1 + 0.001\triangle W_t + 0.001\triangle N_t)$. Fig. 2 shows the computer simulation for the differences between $(1-a) \times (1 + 0.001\triangle W_t + 0.001\triangle N_t)$ and the Euler approximation solution $Q(t,a)$ with $\theta = 0.4, \theta = 0.6$ and $\theta = 0.8$, respectively. The maximum value of the error square is not greater than 0.042. Clearly the numerical approximation will tend to the true solution in the mean square sense.

5. Conclusion

In this paper, we proposed new method for the numerical solution of stochastic age-dependent population equations with Poisson jumps. The approach is based on constructing a discrete-time approximation to exact solution by consider the jump time. The error analysis has been presented for approximate solutions and exact solutions, It was proved that the semi-implicit Euler methods were convergent with strong order $p = 1/2$. Finally, the efficiency of this method has illustrated by a simple example of stochastic age-dependent population equations with Poisson jumps. Our results have generalized and improved some known results, it can be probably extended to obtain approximate numerical solutions of stochastic population equations with Markovian switching arising in mathematical biological.

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