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Twin solutions to singular semipositone problems ☆

Yansheng Liu

Department of Mathematics, Shandong Normal University, Jinan, 250014, PR China

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Abstract

In this paper, by a specially constructed cone and the fixed point index theory, we investigate the existence of multiple positive solutions for the following singular semipositone problem:

$$\begin{cases} y'' + \lambda f(t, y) = 0, & t \in (0, 1), \\ y(0) = y(1) = 0. \end{cases}$$

The nonlinear term $f(t, y)$ may be singular at $t = 0$, $t = 1$, and $y = 0$, also may be negative for some values of t and y; and λ is a positive parameter.

Keywords: Multiple positive solutions; Singular semipositone problem; Fixed point index; Cone

1. Introduction

This paper discusses existence of two positive solutions to the singular semipositone problem

$$\begin{cases} y'' + \lambda f(t, y) = 0, & t \in (0, 1), \\ y(0) = y(1) = 0. \end{cases} \tag{1.1$_\lambda$}$$

The nonlinear term $f(t, y)$ may be singular at $t = 0$, $t = 1$, and $y = 0$, also may be negative for some values of t and y; and λ is a positive parameter.

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E-mail address: ysliu6668@sohu.com.

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The boundary value problem (1.1_λ) arises in a variety of differential applied mathematics and physics. As to semipositone problem, we note that the well-known Proudman equation

$$\begin{cases} (p(t)y')' + \lambda u = 1, & t \in (0,1), \\ y'(0) = y(1) = 0 \end{cases}$$

(which was established by Proudman in 1926, see Proc. London Math. Soc. 24 (1926) 131–139) is of that kind, see [5, p. 2] for details. Also (1.1_λ) describes the movement of a particle which is acted by variable forces.

Recently, the case when $f(t,y) \geqslant 0$ (positone problem) for $(t,y) \in (0,1) \times (0,+\infty)$ in (1.1_λ) has received almost all the attention (for example, see [2,3,6,8,9] and references therein). And what is more, all the results for twin solutions of (1.1_λ) to our knowledge are concentrated on the positone problems (see, for instance, [3] etc.). Only few results exist for semipositone problem (see [1,4] and some references therein). In [1], (1.1_λ) has been considered when $y(0) = a > 0$, f is semipositone and singular at $t = 0$, $t = 1$, and $y = 0$. In [4] the problem similar to (1.1_λ) has been also studied when f is semipositone and singular only at $y = 0$. Unfortunately, what obtained in [1,4] are only the existence of one positive solution to (1.1_λ) when λ is sufficiently small.

Motivated by the works of [1,3,4], the present paper investigates the existence of multiple positive solutions to (1.1_λ) when f is negative for some values of t and y, and λ is small enough. At the same time, we improve and generalize the results obtained in [1,4], since the degree of singularity in [1,4] are lower than that of the present paper (for details, please see our examples and remarks). Our approaches are the approximation method, the fixed point index theory, and a new constructed cone. Also we would like to remark that the theory presented here for Dirichlet problem could be extended (in an obvious way) to general boundary value problems. The organization of this paper is as follows. We shall introduce some definitions and lemmas in the rest of this section. The main result will be stated and proved in Section 2. Finally, two examples are worked out to demonstrate our main result.

A map $y \in C[0,1]$ with $y(t) > 0$ for $t \in (0,1)$ is said to be a positive solution to BVP (1.1_λ) if it satisfies Eq. (1.1_λ).

For the remainder of this section, we present some results which will be used in Section 2. First from [7, Lemma 2.3.1, p. 88, and Lemma 2.3.3, p. 91] we can get the following lemma.

Lemma 1.1. *Let P be a cone of real Banach space E, Ω be a bounded open set of E, $\theta \in \Omega$, $A : P \cap \bar{\Omega} \to P$ be completely continuous.*

(i) *If $x \neq \mu Ax$ for $x \in P \cap \partial\Omega$ and $\mu \in [0,1]$, then*

$$i(A, P \cap \Omega, P) = 1.$$

(ii) *If $\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0$ and $Ax \neq \mu x$ for $x \in P \cap \partial\Omega$ and $\mu \in (0,1]$, then*

$$i(A, P \cap \Omega, P) = 0.$$

Lemma 1.2. *If $g \in C[(0,+\infty), R^+]$, then there exists a nondecreasing function $h \in C[R^+, R^+]$ such that $h(x) > 0$ as $x > 0$ and $g(x)h(x) \in C[R^+, R^+]$ (that is, $\lim_{x\to 0+} g(x)h(x)$ exists), where $R^+ = [0,+\infty)$.*

Proof. Without loss of generality, we assume $g(x) \not\equiv 0$ on $(0,+\infty)$. Then there exists $x_0 \in (0,+\infty)$ with $g(x_0) > 0$. Let

$$h_1(x) = \begin{cases} \max_{t\in[x,x_0]} g(t), & x \in (0, x_0), \\ g(x_0), & x \geqslant x_0. \end{cases}$$

Then $h_1(x) > 0$ and nonincreasing on $(0,+\infty)$.

If $\lim_{x\to 0+}(g(x)/h_1(x))$ exists, then we can define $h(x) = 1/h_1(x)$. Evidently, $\lim_{x\to 0+} h(x)$ exists, so $h \in C[R^+, R^+]$ and nondecreasing on $(0,+\infty)$.

If $\lim_{x\to 0+}(g(x)/h_1(x))$ does not exist, since $0 \leqslant g(x)/h_1(x) \leqslant 1$ for $x \in (0, x_0)$, we may choose $h(x) = \ln(1 + x)/h_1(x)$ or $h(x) = x^\alpha/h_1(x)$ $(\alpha > 0)$, which satisfies our requirement. □

2. Main results

For convenience, we list the following assumptions:

(H_1) $f \in C[(0,1) \times (0,+\infty), R]$ and there exist a constant $M > 0$ and maps $q \in C[(0,1), R^+]$, $g \in C[(0,+\infty), R^+]$, which satisfy

$$0 \leqslant f(t, y) + M \leqslant q(t)g(y), \quad \forall t \in (0,1),\ y \in (0,+\infty),$$

and

$$0 < \int_0^1 s(1-s)q(s)\,ds < +\infty.$$

(H_2) There exists an interval $[\alpha, \beta] \subset (0,1)$ such that

$$\lim_{x\to 0+} f(t,x) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} \frac{f(t,x)}{x} = +\infty$$

both uniformly with respect to $t \in [\alpha, \beta]$.

The following theorem is our main result.

Theorem 2.1. *Suppose that conditions (H_1) and (H_2) hold. Then for each $r > 0$, there exists $\bar\lambda = \bar\lambda(r) > 0$ such that BVP (1.1_λ) has at least two positive solutions $x(t)$ and $y(t)$ satisfying $0 < \|x\| < r < \|y\|$ provided $\lambda \in (0, \bar\lambda)$.*

Before giving the proof of Theorem 2.1, we first list some preliminaries and prove some lemmas.

Let $E = C[0,1]$, $P = \{x \in E\colon\ x(t) \geqslant 0,\ t \in J\}$, $J = [0,1]$, and $Q = \{x \in P\colon\ x(t) \geqslant t(1-t)x(s),\ \forall t, s \in J\}$. Then E is a Banach space with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$; and obviously, P and Q are cones in E.

Set $\phi^*(t) = \int_0^1 G(t,s)\,ds$, where

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

For $\lambda \in (0,+\infty)$, $j \in N$, consider the following approximation problem of (1.1_λ):

$$\begin{cases} y''(t) + \lambda f_j^*\big(t, y(t) - \phi_\lambda(t) + \frac{1}{j}\big) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.1_j}$$

where

$$\phi_\lambda = \lambda M \phi^*, \qquad f_j^*\left(t, u + \frac{1}{j}\right) = \begin{cases} f\big(t, u + \frac{1}{j}\big) + M, & u > 0, \\ f\big(t, \frac{1}{j}\big) + M, & u \leqslant 0. \end{cases}$$

Since $\phi^*(t) \leqslant t(1-t)$ for $t \in J$, we have $\phi_\lambda(t) \leqslant \lambda M t(1-t)$ for $t \in J$. It is easy to see that if $y_j \in C^1[(0,1) \times (0,+\infty), R] \cap C[0,1]$ is a solution of (2.1_j) and $y_j(t) > \phi_\lambda(t)$ for $t \in (0,1)$, then $u_j(t) = y_j(t) - \phi_\lambda(t)$ is a positive solution of the following (BVP):

$$\begin{cases} u''(t) + \lambda f\big(t, u(t) + \frac{1}{j}\big) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.2_j}$$

For the sake of solving (2.1_j), we first consider the following operator:

$$\big(A_\lambda^j y\big)(t) =: \lambda \int_0^1 G(t,s) f_j^*\left(s, y(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds, \quad t \in J.$$

Lemma 2.1. *For each $\lambda \in (0,+\infty)$, $j \in N$, $A_\lambda^j : Q \to Q$ is a continuous and compact map.*

Proof. Since $G(t,\tau) \geqslant t(1-t)G(s,\tau)$ for $t, s, \tau \in J$, it is easy to see that $A_\lambda^j : Q \to Q$. Meanwhile, by $f_j^* \in C[(0,1) \times R^+, R^+]$ and (H_1), one can conclude A_λ^j is continuous and compact from Q to Q. □

To sum up, we can see that $y \in Q$ is a solution of (2.1_j) if y is a fixed point of A_λ^j on Q. Therefore, we next consider the existence of fixed point of A_λ^j on Q.

Lemma 2.2. *For each $r > 0$, there exists $\lambda(r) > 0$ such that*

$$i\big(A_\lambda^j, Q_r, Q\big) = 1, \quad \forall \lambda \in \big(0, \lambda(r)\big),\ j \in N,$$

where $Q_r = \{y \in Q\colon\ \|y\| < r\}$.

Proof. By Lemma 1.2 and (H_1), there exists a nondecreasing function $h \in C[R^+, R^+]$ such that $g(x)h(x) \in C[R^+, R^+]$ and $h(x) > 0$ as $x > 0$. For each $r > 0$, let

$$\lambda(r) =: \min\left\{\frac{r}{2M}, \frac{2}{ac(r)} \int_0^{r/2} h(s)\, ds\right\},$$

where

$$a =: 2\int_0^1 s(1-s)q(s)\, ds, \qquad c(r) =: \max_{y\in[0,r+1]} g(y)h(y).$$

We now claim that

$$y \neq \mu\big(A_\lambda^j y\big) \quad \text{for } \mu \in J = [0,1],\ y \in \partial Q_r,\ \text{and } \lambda \in \big(0, \lambda(r)\big). \tag{2.3}$$

Suppose this is false. Then there exist $y_0 \in \partial Q_r$ and $\mu_0 \in [0,1]$ with $y_0(t) = \mu_0(A_\lambda^j y_0)(t)$ for all $t \in [0,1]$. Since $y_0 \in Q$, we have

$$y_0(t) \geqslant t(1-t)\|y_0\| = rt(1-t) \quad \text{for } t \in J.$$

On the other hand,

$$\phi_\lambda(t) = \lambda M \int_0^1 G(t,s)\, ds \leqslant \lambda M t(1-t) \leqslant \frac{\lambda M}{r} y_0(t) \quad \text{for } t \in J.$$

Thus,

$$y_0(t) - \phi_\lambda(t) \geqslant \left(1 - \frac{\lambda M}{r}\right) y_0(t) \geqslant \frac{1}{2} y_0(t) \geqslant \frac{r}{2} t(1-t) \quad \text{for } t \in J. \tag{2.4}$$

This implies that

$$f_j^*\left(t, y_0(t) - \phi_\lambda(t) + \frac{1}{j}\right) = f\left(t, y_0(t) - \phi_\lambda(t) + \frac{1}{j}\right) + M. \tag{2.5}$$

Therefore, by (2.4), (2.5), and (H_1) we have

$$\begin{aligned} -y_0''(s) &= \lambda\mu f\left(t, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) + \lambda\mu M \leqslant \lambda q(s) g\left(y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) \\ &\leqslant \frac{\lambda c(r) q(s)}{h(y_0(s) - \phi_\lambda(s))} \leqslant \frac{\lambda c(r) q(s)}{h\big(\frac{1}{2} y_0(s)\big)} \quad \text{for } s \in (0,1). \end{aligned} \tag{2.6}$$

Since $y_0'' \leqslant 0$ on $(0,1)$ and $y_0(t) \geqslant rt(1-t)$ on J, there exists $t_0 \in (0,1)$ with $y_0' \geqslant 0$ on $(0, t_0)$ and $y_0' \leqslant 0$ on $(t_0, 1)$.

Integrate (2.6) from t to t_0 to obtain

$$y_0'(t) \leqslant \lambda c(r) \frac{\int_t^{t_0} q(s)\, ds}{h\big(\frac{1}{2} y_0(t)\big)} \quad \text{for } t \in (0, t_0).$$

Consequently,

$$h\left(\frac{1}{2}y_0(t)\right)y_0'(t) \leqslant \lambda c(r)\int_t^{t_0} q(s)\,ds \quad \text{for } t \in (0, t_0). \tag{2.7}$$

Integrate (2.7) from 0 to t_0 again to obtain

$$\int_0^r h\left(\frac{1}{2}s\right)ds = 2\int_0^{r/2} h(s)\,ds \leqslant \lambda c(r)\int_0^{t_0}\int_t^{t_0} q(s)\,ds\,dt = \lambda c(r)\int_0^{t_0} sq(s)\,ds.$$

Then

$$\frac{2}{\lambda c(r)}\int_0^{r/2} h(s)\,ds \leqslant \frac{1}{1-t_0}\int_0^{t_0} s(1-s)q(s)\,ds. \tag{2.8}$$

Similarly, integrate (2.6) from t_0 to t $(t \in (t_0, 1))$ and integrate it again from t_0 to 1 to get

$$\frac{2}{\lambda c(r)}\int_0^{r/2} h(s)\,ds \leqslant \frac{1}{t_0}\int_{t_0}^{1} s(1-s)q(s)\,ds.$$

This together with (2.8) guarantees that

$$\frac{2}{\lambda c(r)}\int_0^{r/2} h(s)\,ds \leqslant 2\int_0^{1} s(1-s)q(s)\,ds = a,$$

in contradiction with

$$0 < \lambda < \frac{2}{ac(r)}\int_0^{r/2} h(s)\,ds,$$

and consequently, the result of Lemma 2.2 follows. □

Lemma 2.3. *For each $\lambda \in (0, \lambda(r))$, there exists $R > r > 0$ with*

$$i\left(A_\lambda^j, Q_R, Q\right) = 0 \quad \textit{for } j \in N.$$

Proof. By (H_2) we know that, for each $\lambda \in (0, \lambda(r))$, there exists $L' > r > 0$ such that

$$\frac{f(t,x)}{x} \geqslant 2\left(\lambda\alpha(1-\beta)\min_{t\in[\alpha,\beta]}\int_\alpha^\beta G(t,s)\,ds\right)^{-1} \quad \text{for } x > L'. \tag{2.9}$$

Let

$$R = R(\lambda) > \max\left\{\frac{2L'}{\alpha(1-\beta)}, 2\lambda M\right\}.$$

We claim that

$$A_\lambda^j y \neq \mu y \quad \text{for } y \in \partial Q_R \text{ and } \mu \in (0,1].$$

In fact, if it is not true, then there exist $y_0 \in \partial Q_R$ and $\mu_0 \in (0,1]$ such that $A_\lambda^j y_0 = \mu_0 y_0$, that is,

$$y_0(t) \geqslant \left(A_\lambda^j y_0\right)(t) = \lambda \int_0^1 G(t,s) f_j^*\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds. \tag{2.10}$$

Since

$$y_0(t) - \phi_\lambda(t) \geqslant (R - \lambda M)t(1-t) > 0 \quad \text{for } t \in (0,1),$$

we have

$$y_0(t) - \phi_\lambda(t) \geqslant (R - \lambda M)\alpha(1-\beta) > \frac{R}{2}\alpha(1-\beta) > L' \quad \text{for } t \in [\alpha,\beta].$$

By (2.9) and (2.10), we get

$$\begin{aligned}
y_0(t) &\geqslant \lambda \int_0^1 G(t,s) f_j^*\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds \\
&= \lambda \int_0^1 G(t,s)\left[f\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) + M\right] ds \\
&> \lambda \int_\alpha^\beta G(t,s) f\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds \\
&\geqslant 2\lambda\left(\lambda\alpha(1-\beta) \min_{t\in[\alpha,\beta]} \int_\alpha^\beta G(t,s)\,ds\right)^{-1} (R - \lambda M)\alpha(1-\beta) \int_\alpha^\beta G(t,s)\,ds \\
&\geqslant 2(R - \lambda M) > R \quad \text{for } t \in [\alpha,\beta].
\end{aligned}$$

This is in contradiction with $y_0 \in \partial Q_R$ and our conclusion follows. □

Lemma 2.4. *For the above-mentioned $r > 0$ and $\lambda(r) > 0$, there exists $\bar{\lambda} = \bar{\lambda}(r) \in (0, \lambda(r)]$ satisfying that for each $\lambda \in (0, \bar{\lambda})$, there exists $r' = r'(\lambda) \in (0, r)$ such that*

$$i\left(A_\lambda^j, Q_{r'}, Q\right) = 0 \quad \textit{if } j \textit{ is sufficiently large}.$$

Proof. By condition (H_2), we know $\lim_{x\to 0+} f(t,x) = +\infty$ uniformly with respect to $t \in [\alpha,\beta]$. Therefore, for each $L > M(\min_{t\in[\alpha,\beta]} \int_\alpha^\beta G(t,s)\,ds)^{-1}$, there exists $\delta > 0$ such that

$$f(t,x) > L \quad \text{for } x \in (0,\delta) \text{ and } t \in [\alpha,\beta]. \tag{2.11}$$

Let

$$\bar{\lambda} = \min\left\{\frac{\delta}{M}, \lambda(r)\right\} \quad \text{and} \quad l = L \min_{t\in[\alpha,\beta]} \int_{\alpha}^{\beta} G(t,s)\,ds.$$

Then $M\lambda < \delta$ for $\lambda \in (0, \bar{\lambda})$. Fix $\lambda \in (0, \bar{\lambda})$, choose $r' = r'(\lambda) \in (0, \delta)$ and j sufficiently large such that

$$\frac{r'}{l} < \lambda < \frac{r'}{M} \quad \text{and} \quad r' + \frac{1}{j} < \delta.$$

We claim that

$$A_\lambda^j y \neq \mu y \quad \text{for } y \in \partial Q_{r'} \text{ and } \mu \in (0,1].$$

Suppose this is false. Then there exist $y_0 \in \partial Q_{r'}$ and $\mu_0 \in (0,1]$ with $A_\lambda^j y_0 = \mu_0 y_0$, that is,

$$y_0(t) \geqslant \big(A_\lambda^j y_0\big)(t) = \lambda \int_0^1 G(t,s) f_j^*\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds \quad \text{for } t \in J. \tag{2.12}$$

From

$$y_0(t) \geqslant r' t(1-t) \quad \text{and} \quad \phi_\lambda(t) \leqslant M\lambda t(1-t) \quad \text{for } t \in J,$$

it follows that

$$y_0(t) - \phi_\lambda(t) \geqslant (r' - M\lambda)t(1-t) > 0 \quad \text{for } t \in (0,1). \tag{2.13}$$

Consequently,

$$f_j^*\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) = f\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) + M. \tag{2.14}$$

On the other hand, by (2.13) we know

$$0 < y_0(s) - \phi_\lambda(s) + \frac{1}{j} \leqslant y_0(s) + \frac{1}{j} \leqslant r' + \frac{1}{j} < \delta \quad \text{for } s \in J.$$

Therefore, by (2.11) and (2.14) we obtain

$$f_j^*\left(t, y_0(t) - \phi_\lambda(t) + \frac{1}{j}\right) > L + M > L \quad \text{for } t \in [\alpha, \beta]. \tag{2.15}$$

At last, combining (2.12) with (2.14) and (2.15), it follows that

$$y_0(t) \geqslant \lambda \int_0^1 G(t,s) f\left(s, y_0(s) - \phi_\lambda(s) + \frac{1}{j}\right) ds > \lambda L \int_{\alpha}^{\beta} G(t,s)\,ds \geqslant \lambda l > r' \quad \text{for } t \in [\alpha, \beta].$$

This is in contradiction with $y_0 \in \partial Q_{r'}$ and immediately our result follows. □

Lemma 2.5. *For each $\lambda \in (0, \bar{\lambda})$ and sufficiently large j, BVP (2.1_j) has at least two positive solutions x_j and y_j satisfying*

$$r' < \|x_j\| < r < \|y_j\| < R,$$

where r, r', and $\bar{\lambda}$ are the same as in Lemma 2.4.

Proof. By Lemmas 2.2–2.4 and additivity of the fixed point index, we know

$$i\big(A_\lambda^j, Q_R \setminus \bar{Q}_r, Q\big) = i\big(A_\lambda^j, Q_R, Q\big) - i\big(A_\lambda^j, Q_r, Q\big) = 0 - 1 = -1,$$
$$i\big(A_\lambda^j, Q_r \setminus \bar{Q}_{r'}, Q\big) = i\big(A_\lambda^j, Q_r, Q\big) - i\big(A_\lambda^j, Q_{r'}, Q\big) = 1 - 0 = 1.$$

It follows from solution property of the fixed point index that there exist $x_j \in Q_r \setminus \bar{Q}_{r'}$ and $y_j \in Q_R \setminus \bar{Q}_r$ such that

$$A_\lambda^j x_j = x_j \quad \text{and} \quad A_\lambda^j y_j = y_j.$$

Consequently, BVP (2.1_j) has at least two positive solutions. □

Proof of Theorem 2.1. Let $\{x_j\}$, $\{y_j\}$ $(j \geqslant j_0)$ be the positive solutions of (2.1_j) obtained in Lemma 2.5.

First we show $\{x_j\}_{j \geqslant j_0}$ is a bounded, equicontinuous family on $[0, 1]$.

Since $\|x_j\| > r'$, $\phi_\lambda(t) \leqslant \lambda M t(1 - t) \leqslant (M\lambda/r')x_j(t)$ for $j \geqslant j_0$, and $M\lambda < r'$, we have

$$x_j(t) - \phi_\lambda(t) \geqslant \left(1 - \frac{M\lambda}{r'}\right)x_j(t) \quad \text{for } t \in (0, 1). \tag{2.16}$$

Return to (2.6) (with y_0 replaced by x_j) to obtain

$$-x_j''(s) \leqslant \lambda q(s) g\left(x_j(s) - \phi_\lambda(s) + \frac{1}{j}\right) \leqslant \frac{\lambda c(r) q(s)}{h(x_j(s) - \phi_\lambda(s))} \leqslant \frac{\lambda c(r) q(s)}{h\big(\big(1 - \frac{M\lambda}{r'}\big)x_j(s)\big)} \quad \text{for } s \in (0, 1), \tag{2.17}$$

where h is the same as in the proof of Lemma 2.2. Since $x_j''(t) \leqslant 0$ on $(0, 1)$, there exists $t_j \in (0, 1)$ with $x_j' \geqslant 0$ on $(0, t_j)$ and $x_j' \leqslant 0$ on $(t_j, 1)$. Integrate (2.17) from t $(t < t_j)$ to t_j to obtain

$$h\left(\left(1 - \frac{M\lambda}{r'}\right)x_j(t)\right)x_j'(t) \leqslant \lambda c(r) \int_t^{t_j} q(s)\, ds. \tag{2.18}$$

On the other hand, integrate (2.17) from t_j to t $(t > t_j)$ to obtain

$$-h\left(\left(1 - \frac{M\lambda}{r'}\right)x_j(t)\right)x_j'(t) \leqslant \lambda c(r) \int_{t_j}^{t} q(s)\, ds. \tag{2.19}$$

We now claim that there exist a_0 and a_1 with $0 < a_0 < a_1 < 1$ such that

$$a_0 < \inf\{t_j\colon\ j \geqslant j_0\} \leqslant \sup\{t_j\colon\ j \geqslant j_0\} < a_1. \tag{2.20}$$

First we show $\inf\{t_j\colon\ j \geqslant j_0\} > 0$. If this is not true, then there is a subsequence S of N with $t_j \to 0+$ as $j \to +\infty$ in S. Now integrate (2.18) from 0 to t_j to obtain

$$\int_0^{x_j(t_j)} h\left(\left(1 - \frac{M\lambda}{r'}\right)s\right) ds \leqslant \lambda c(r) \int_0^{t_j} s q(s)\, ds \quad \text{for } j \in S. \tag{2.21}$$

Since $t_j \to 0+$ as $j \to +\infty$ in S, we have from (2.21) that $x_j(t_j) \to 0$ as $j \to +\infty$ in S. However, since the maximum of x_j on $[0,1]$ occurs at t_j we have $x_j \to 0$ in $C[0,1]$ as $j \to +\infty$ in S. This is in contradiction with $\|x_j\| > r'$. Consequently, $\inf\{t_j\colon\ j \geqslant j_0\} > 0$.

A similar argument shows $\sup\{t_j\colon\ j \geqslant j_0\} < 1$.

Moreover, by condition (H_1), we know that

$$\lim_{t \to 0+} t \int_t^1 (1-s) q(s)\, ds = 0 \tag{2.22}$$

and

$$\lim_{t \to 1-0} (1-t) \int_0^t s q(s)\, ds = 0. \tag{2.23}$$

Let $I : R^+ \to R^+$ be defined by

$$I(u) = \int_0^u h\left(\left(1 - \frac{\lambda M}{r'}\right)s\right) ds.$$

Note that I is an increasing and continuous map from R^+ onto R^+ (notice $I(\infty) = \infty$ since $h > 0$ is nondecreasing on $(0, +\infty)$). We claim that

$$\{x_j\}_{j \geqslant j_0} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.24}$$

Since the boundedness is obvious, we need to prove only the equicontinuity. In fact, for $0 < t_1 < t_2 < a_0$, by (2.18) we know that

$$\begin{aligned}
\bigl|I\bigl(x_j(t_2)\bigr) - I\bigl(x_j(t_1)\bigr)\bigr| &= \left| \int_{t_1}^{t_2} h\left(\left(1 - \frac{\lambda M}{r'}\right)x_j(\tau)\right) x_j'(\tau)\, d\tau \right| \\
&\leqslant \lambda c(r) \int_{t_1}^{t_2} ds \int_s^{a_1} q(\tau)\, d\tau = \lambda c(r) \left(\int_{t_1}^{t_2} (s - t_1) q(s)\, ds + (t_2 - t_1) \int_{t_2}^{a_1} q(s)\, ds \right) \\
&\leqslant \lambda c(r) \left(\int_{t_1}^{t_2} s q(s)\, ds + (1 - a_1)^{-1} (t_2 - t_1) \int_{t_2}^{a_1} (1-s) q(s)\, ds \right) \\
&\leqslant \frac{\lambda c(r)}{1 - a_1} \left(\int_{t_1}^{t_2} s(1-s) q(s)\, ds + (t_2 - t_1) \int_{t_2 - t_1}^{1} s(1-s) q(s)\, ds \right).
\end{aligned} \tag{2.25}$$

Thus, by (2.22) and (2.25) we obtain $\{I(x_j(t))\}_{j\geqslant j_0}$ is equicontinuous on $[0,a_0]$.

In addition, by the uniform continuity of I^{-1} on $[0, I(r)]$ and

$$\left|x_j(t_2)-x_j(t_1)\right| = \left|I^{-1}\big(I\big(x_j(t_2)\big)\big) - I^{-1}\big(I\big(x_j(t_1)\big)\big)\right|,$$

now the equicontinuity of $\{x_j(t)\}_{j\geqslant j_0}$ on $[0,a_0]$ is established.

Similarly, by (2.19) and (2.23) we can show the equicontinuity of $\{x_j(t)\}_{j\geqslant j_0}$ on $[a_1,1]$.

On the other hand, for $t\in[a_0,a_1]$, it is easy to see

$$\left|x_j'(t)\right| \leqslant \frac{\lambda c(r)\int_{a_0}^{a_1} q(s)\,ds}{h\big(\big(1-\frac{M\lambda}{r'}\big)x_j(t)\big)}. \tag{2.26}$$

Note $x_j(t)\in Q$ and

$$x_j(t) \geqslant \|x_j\| t(1-t) \geqslant r' a_0(1-a_1) \quad \text{for } t\in[a_0,a_1]. \tag{2.27}$$

Therefore, by (2.26) and (2.27), we know that

$$\left|x_j'(t)\right| \leqslant \frac{\lambda c(r)\int_{a_0}^{a_1} q(s)\,ds}{h\big(\big(1-\frac{M\lambda}{r'}\big)r' a_0(1-a_1)\big)} \quad \text{for } t\in[a_0,a_1].$$

Consequently, the equicontinuity of $\{x_j(t)\}_{j\geqslant j_0}$ on $[a_0,a_1]$ follows, and now establish (2.24).

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_0 of N and a function $x\in C[0,1]$ with x_j converging uniformly on J to x as $j\to+\infty$ through N_0. Also $x(0)=x(1)=0$, $r'\leqslant\|x\|\leqslant r$, and $x(t)\geqslant t(1-t)r'$ for $t\in J$. In particular, $x(t)>0$ on $(0,1)$. Fix $t\in(0,1)$ (without loss of generality assume $t\neq 1/2$). Now x_j, $j\in N_0$, satisfies the integral equation

$$\begin{aligned} x_j(t) &= x_j\left(\frac{1}{2}\right) + x_j'\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right) \\ &\quad + \lambda\int_{1/2}^{t}(s-t)\left[f\left(s, x_j(s)-\phi_\lambda(s)+\frac{1}{j}\right)+M\right]ds \end{aligned} \tag{2.28}$$

for $t\in(0,1)$. Notice (take $t=2/3$) that $\{x_j'(1/2)\}_{j\in N_0}$ is a bounded sequence since $r's(1-s)\leqslant x_j(s)\leqslant r$ for $s\in J$. Thus $\{x_j'(1/2)\}_{j\in N_0}$ has a convergent subsequence; for convenience, let $\{x_j'(1/2)\}_{j\in N_0}$ denote this subsequence also, and let $r_0\in R$ be its limit. Now for the above fixed t, let $j\to+\infty$ through N_0 in (2.28) to obtain

$$x_0(t) = x_0\left(\frac{1}{2}\right) + r_0\left(t-\frac{1}{2}\right) + \lambda\int_{1/2}^{t}(s-t)\big[f\big(s,x_0(s)-\phi_\lambda(s)\big)+M\big]\,ds.$$

Therefore, $x_0(t)$ is a solution of

$$\begin{cases} x''(t)+\lambda[f(t,x(t)-\phi_\lambda(t))+M]=0, & t\in(0,1),\\ x(0)=x(1)=0, \end{cases} \tag{2.29}$$

since

$$x_0(t) \geqslant \|x_0\| t(1-t) \geqslant r' t(1-t) > M\lambda t(1-t) \geqslant \phi_\lambda(t) \quad \text{for } t \in (0,1).$$

Let $x(t) = x_0(t) - \phi_\lambda(t)$ for $t \in [0,1]$, then $x(t) \in Q$. It is easy to see by (2.29) that $x(t)$ is a positive solution of (1.1_λ) which satisfies $\|x\| \geqslant r' - \lambda M > 0$.

Completely similar to the above discussion, from $\{y_j(t)\}_{j \geqslant j_0}$ we can obtain the existence of positive solution $y_0(t)$ to (2.29) and $y_0(t) \geqslant \|y_0\| t(1-t) \geqslant rt(1-t) > \phi_\lambda(t)$ for $t \in (0,1)$. Consequently we get that $y(t) = y_0(t) - \phi_\lambda(t)$ is also a solution of (1.1_λ).

Now we prove $x \neq y$. We need to prove only $x_0 \neq y_0$. Since $r' \leqslant \|x_0\| \leqslant r$, $r \leqslant \|y_0\| \leqslant R$, we shall show (2.29) has no solution on ∂Q_r. Suppose this is not true. Then there exists $x \in \partial Q_r$ satisfying (2.29). Then

$$\left|x''(t)\right| \leqslant \lambda q(t) g\big(x(t) - \phi_\lambda(t)\big) \quad \text{for } t \in (0,1).$$

Similarly to (2.7) and (2.8), we can get

$$\frac{2}{\lambda c(r)} \int_0^{r/2} h(s)\,ds \leqslant 2 \int_0^1 s(1-s) q(s)\,ds.$$

This is in contradiction with

$$0 < \lambda < \frac{2}{ac(r)} \int_0^{r/2} h(s)\,ds.$$

To sum up, the Theorem 2.1 is proved. □

Example. Consider the following BVP:

$$\begin{cases} y'' + \lambda f(t,y) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.30}$$

where

$$f(t,y) = \frac{1}{\sqrt{t(1-t)}} \left(\frac{a}{y^\alpha} + be^y \sin t \right) - M\cos t, \quad a > 0,\ \alpha > 0,\ b > 0,\ M > 0.$$

It is easy to see that (H_1) and (H_2) are satisfied for (2.30). Then, by Theorem 2.1, we know (2.30) has at least two positive solutions when λ is sufficiently small.

Remark 2.1. Theorem 2.1 in [1] cannot be used to study (2.30), since $f(t,y)$ in (2.30) cannot satisfy conditions (2.4) and (2.6) of [1] when $\alpha > 1$.

On the other hand, it is easy to see that for each $\theta > 0$, there does not exist $p_\theta \in L^1[0,1]$ such that $f(t,y) \leqslant p_\theta(t)$ for every $y \in C[0,1]$ with $y(t) \geqslant \theta l(t)$ when $\alpha > 1$, where $l(t) = \min\{t, 1-t\}$. This implies that condition (A_7) in [4] is not satisfied if [4, Theorem 3.1] is used to consider (2.30).

Therefore, from the results in [1,4], we cannot derive the existence of solutions for (2.30) when $\alpha > 1$. However, by using Theorem 2.1 of the present paper, not only the existence of solution but also twin solutions are obtained for (2.30).

Remark 2.2. When (1.1_λ) reduces to positone problem, Theorem 2.1 can also be used. Comparing with [3, Theorem 2.3] (where $f(t,y)=\phi(t)[g(y)+h(y)]\geqslant 0$ and requiring that $g>0$ is nonincreasing on $(0,+\infty)$ and h/g is nondecreasing on $(0,+\infty)$), our result is more general. Please see the following example:

$$\begin{cases} y''+\frac{\lambda}{t(1-t)}(\sin ty+y^{-\alpha}e^{y})=0, & t\in(0,1),\\ y(0)=y(1)=0, & \alpha>0. \end{cases} \tag{2.31}$$

It is easy to see by Theorem 2.1 that (2.31) has at least two positive solutions when λ is sufficiently small. But [3, Theorem 2.3] is not applicable to study (2.31), since the nonlinear term does not possess any monotonicity in y.

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