



Classification of extensions of classifiable C^* -algebras

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Abstract

For a certain class of extensions $\epsilon : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ of C^* -algebras in which B and A belong to classifiable classes of C^* -algebras, we show that the functor which sends ϵ to its associated six term exact sequence in K -theory and the positive cones of $K_0(B)$ and $K_0(A)$ is a classification functor. We give two independent applications addressing the classification of a class of C^* -algebras arising from substitutional shift spaces on one hand and of graph algebras on the other. The former application leads to the answer of a question of Carlsen and the first named author concerning the completeness of stabilized Matsumoto algebras as an invariant of flow equivalence. The latter leads to the first classification result for nonsimple graph C^* -algebras.

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0. Introduction

The magnificent recent progress of the classification theory for simple C^* -algebras has few direct consequences for general C^* -algebras, even for those with finite ideal lattices. Furthermore, it is not even clear what kind of K -theoretical invariant to use in such a context.

When there is just one nontrivial ideal, however, there is a canonical choice of invariant. Associated to every extension $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ of nonzero C^* -algebras is the standard six term exact sequence of K -groups

$$\begin{array}{ccccc}
 K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\
 \uparrow & & & & \downarrow \\
 K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B)
 \end{array}$$

providing a necessary condition for two extensions to be isomorphic. For examples of classification results involving the six term exact sequence of K -groups see [31,25,24,43]. In each case, the extensions considered were extensions that can be expressed as inductive limits of simpler extensions. The classification results were achieved by using the standard intertwining argument.

In [40], Rørðam used a completely different technique to classify a certain class of extensions. He considered essential extensions of separable nuclear purely infinite simple C^* -algebras in \mathcal{N} , where \mathcal{N} is the bootstrap category of Rosenberg and Schochet [42]. Employing the fact that every invertible element of $KK(A, B)$ (where A and B are separable nuclear stable purely infinite simple C^* -algebras) lifts to a $*$ -isomorphism from A to B and that every essential extension of A by B is absorbing, Rørðam showed that the six term sequence is, indeed, a complete invariant in this case.

The purpose of this paper is to extend the above result to other classes of C^* -algebras that are classified via K -theoretical invariants. As we shall see, both the celebrated classification results of Kirchberg and Phillips [21] and of Lin [27] can be transferred to this setting under an assumption of *fullness* of the extension which is automatic in the case solved by Rørðam.

The motivation of our work was an application to a class of C^* -algebras introduced in the work of Matsumoto. In a case studied in [4,7] one gets that the Matsumoto algebra $\mathcal{O}_{\underline{X}}$ fits in a short exact sequence of the form

$$0 \longrightarrow \mathcal{K}^n \longrightarrow \mathcal{O}_{\underline{X}} \longrightarrow C(\underline{X}) \rtimes_{\sigma} \mathbb{Z} \longrightarrow 0.$$

Since $C(\underline{X}) \rtimes_{\sigma} \mathbb{Z}$ is a unital simple AT -algebra with real rank zero our results apply to classify $\mathcal{O}_{\underline{X}}$ by its K_0 -group with a scale consisting of n preferred elements.

The paper is organized as follows. In Section 1, we give basic properties and develop some notation concerning extensions of C^* -algebras. Section 2 gives notation (mainly from [40]) concerning the six term exact sequence of K -groups and extends work of Rørðam. Section 3 contains our main results (Theorem 3.9 and Theorem 3.12). In the last section we use these results to classify the C^* -algebras described in the previous paragraph. We also present an alternative application to *graph C^* -algebras* which fully employs the capacity of our classification result to handle C^* -algebras which have some subquotients which are stably finite, and some which are purely infinite.

An earlier version of this paper was included in the second named author’s PhD thesis.

1. Extensions

1.1. Notation

For a stable C^* -algebra B and a C^* -algebra A , we will denote the class of essential extensions

$$0 \rightarrow B \xrightarrow{\varphi} E \xrightarrow{\psi} A \rightarrow 0$$

by $\mathcal{E}xt(A, B)$.

Since the goal of this paper is to classify extensions of separable nuclear C^* -algebras, throughout the rest of the paper we will only consider C^* -algebras that are separable and nuclear.

Assumption 1.1. In the rest of the paper all C^* -algebras considered are assumed to be separable and nuclear unless stated otherwise. Note in particular that multiplier and corona algebras will be nonseparable.

Under the above assumption, if B is a stable C^* -algebra, then we may identify $\text{Ext}(A, B)$ with $KK^1(A, B)$ (for the definition of $\text{Ext}(A, B)$ and $KK^i(A, B)$ see Chapters 7 and 8 in [1]). So for x in $\text{Ext}(A, B)$ and y in $KK^i(B, C)$, the Kasparov product $x \times y$ is an element of $KK^{i+1}(A, C)$. For every element ϵ of $\mathcal{E}xt(A, B)$, we use $x_{A,B}(\epsilon)$ to denote the element of $\text{Ext}(A, B)$ that is represented by ϵ .

Definition 1.2. A homomorphism from an extension $0 \rightarrow B_1 \rightarrow E_1 \rightarrow A_1 \rightarrow 0$ to an extension $0 \rightarrow B_2 \rightarrow E_2 \rightarrow A_2 \rightarrow 0$ is a triple (β, η, α) such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1 & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \beta \downarrow & & \eta \downarrow & & \alpha \downarrow & & \\ 0 & \longrightarrow & B_2 & \longrightarrow & E_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

commutes.

This turns the class of extensions of C^* -algebras into a category in the canonical way.

We say that an extension $\epsilon_1: 0 \rightarrow B \rightarrow E_1 \rightarrow A \rightarrow 0$ is *congruent* to an extension $\epsilon_2: 0 \rightarrow B \rightarrow E_2 \rightarrow A \rightarrow 0$, if there exists an isomorphism of the form $(\text{id}_B, \eta, \text{id}_A)$ from ϵ_1 to ϵ_2 .

We will use the following notation from [40]. For each injective $*$ -homomorphism α from A_1 to A_2 and for each ϵ in $\mathcal{E}xt(A_2, B)$, there exists a unique extension $\alpha \cdot \epsilon$ in $\mathcal{E}xt(A_1, B)$ such that the diagram

$$\begin{array}{ccccccccc} \alpha \cdot \epsilon: & 0 & \longrightarrow & B & \longrightarrow & \tilde{E} & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \alpha \downarrow & & \\ \epsilon: & 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

is commutative. For each $*$ -isomorphism β from B_1 to B_2 and for each ϵ in $\mathcal{E}xt(A, B_1)$, there exists a unique extension $\epsilon \cdot \beta$ in $\mathcal{E}xt(A, B_2)$ such that the diagram

$$\begin{array}{ccccccc}
 \epsilon: & 0 & \longrightarrow & B_1 & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \downarrow \beta & & \parallel & & \parallel & & \\
 \epsilon \cdot \beta: & 0 & \longrightarrow & B_2 & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0
 \end{array}$$

is commutative.

Proposition 1.1 and Proposition 1.2 in [40] explain the interrelations between the concepts introduced above and will be crucial in our approach.

1.2. Full extensions

Let a be an element of a C^* -algebra A . We say that a is norm-full in A if a is not contained in any norm-closed proper ideal of A . The word “full” is also widely used, but since we will often work in multiplier algebras, we emphasize that it is the norm topology we are using, rather than the strict topology. The next lemma is a consequence of a result of L.G. Brown (see Corollary 2.6 in [2]); we leave the proof to the reader.

Lemma 1.3. *Let A be a separable C^* -algebra. If p is a norm-full projection in $A \otimes \mathbf{M}_n \subset A \otimes \mathcal{K}$, then there exists a $*$ -isomorphism φ from $A \otimes \mathcal{K}$ onto $p(A \otimes \mathcal{K})p \otimes \mathcal{K}$ such that $[\varphi(p)] = [p \otimes e_{11}]$.*

Definition 1.4. An extension ϵ is said to be full if the associated Busby invariant τ_ϵ has the property that $\tau_\epsilon(a)$ is a norm-full element of $\mathcal{Q}(B)$ for any $a \in A \setminus \{0\}$.

Lemma 1.5. *If B is stable and purely infinite, then any extension in $\text{Ext}(A, B)$ is full. If B is stable and A is unital and simple, then any unital extension in $\text{Ext}(A, B)$ is full.*

Proof. In the first case, the corona algebra is simple [28]. In the second, the image of the Busby map is a simple unital subalgebra of the corona algebra and hence cannot intersect an ideal nontrivially. \square

It seems reasonable to expect that the stabilized extension of a full extension is again full. We prove this under the added assumption that B is already stable:

Proposition 1.6. *Let $\epsilon : 0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$ be an essential extension where B is a stable C^* -algebra. If ϵ is full, then so is*

$$\epsilon^s: 0 \longrightarrow B \otimes \mathcal{K} \xrightarrow{\iota \otimes \text{id}_{\mathcal{K}}} E \otimes \mathcal{K} \xrightarrow{\pi \otimes \text{id}_{\mathcal{K}}} A \otimes \mathcal{K} \longrightarrow 0.$$

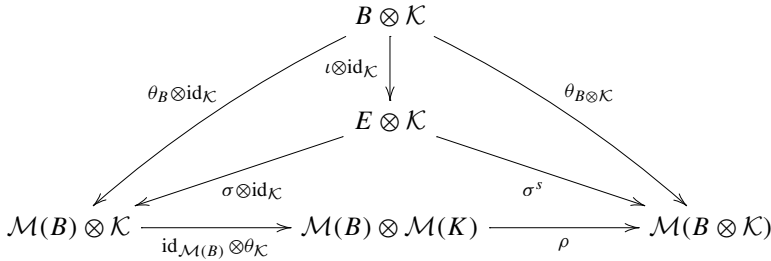
Proof. For any C^* -algebra C , denote the embedding of C into $C \otimes \mathcal{K}$ which sends c into $c \otimes e_{11}$ by ι_C and denote the canonical embedding of C as an essential ideal of the multiplier algebra $\mathcal{M}(C)$ of C by θ_C . We will first show that ι_B satisfies the following properties:

- (1) ι_B has an extension $\tilde{\iota}_B$ from $\mathcal{M}(B)$ to $\mathcal{M}(B \otimes \mathcal{K})$ (i.e. $\theta_{B \otimes \mathcal{K}} \circ \iota_B = \tilde{\iota}_B \circ \theta_B$), which maps $1_{\mathcal{M}(B)}$ to a norm-full projection in $\mathcal{M}(B \otimes \mathcal{K})$, and

(2) the map $\tilde{\iota}_B$ from $\mathcal{Q}(B)$ to $\mathcal{Q}(B \otimes \mathcal{K})$ induced by $\tilde{\iota}_B$ intertwines the Busby invariants of ϵ and ϵ^s , and the $*$ -homomorphism ι_A (i.e. $\tau_{\epsilon^s} \circ \iota_A = \tilde{\iota}_B \circ \tau_{\epsilon}$).

First note that there exist unique injective $*$ -homomorphisms σ from E to $\mathcal{M}(B)$ and σ^s from $E \otimes \mathcal{K}$ to $\mathcal{M}(B \otimes \mathcal{K})$ such that $\theta_B = \sigma \circ \iota$ and $\theta_{B \otimes \mathcal{K}} = \sigma^s \circ (\iota \otimes \text{id}_{\mathcal{K}})$. It is well known that we have a unique $*$ -homomorphism ρ from $\mathcal{M}(B) \otimes \mathcal{M}(\mathcal{K})$ to $\mathcal{M}(B \otimes \mathcal{K})$ such that $\theta_{B \otimes \mathcal{K}} = \rho \circ (\theta_B \otimes \theta_{\mathcal{K}})$ and that this map is injective and unital (see Lemma 11.12 in [34]).

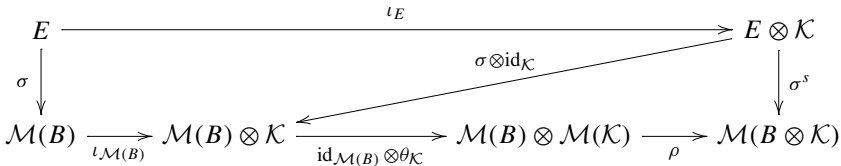
In the following diagram, all the maps are injective $*$ -homomorphisms



The bottom triangle commutes by the uniqueness of σ^s , so this is a commutative diagram.

Now let $\tilde{\iota}_B = \rho \circ (\text{id}_{\mathcal{M}(B)} \otimes \theta_{\mathcal{K}}) \circ \iota_{\mathcal{M}(B)}$. Clearly, $\theta_{B \otimes \mathcal{K}} \circ \iota_B = \tilde{\iota}_B \circ \theta_B$ and $p = \tilde{\iota}_B(1_{\mathcal{M}(B)})$ is a projection in $\mathcal{M}(B \otimes \mathcal{K})$. Note that $\iota_B(B) = B \otimes e_{11} \cong p \theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K}) p$. Therefore, $p \theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K}) p$ is a stable, hereditary, sub- C^* -algebra of $\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$ which is not contained in any proper ideal of $\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$. By Theorem 4.23 in [3], p is Murray–von Neumann equivalent to $1_{\mathcal{M}(B \otimes \mathcal{K})}$. Hence, $p = \tilde{\iota}_B(1_{\mathcal{M}(B)})$ is norm-full in $\mathcal{M}(B \otimes \mathcal{K})$.

Now we see that $\tilde{\iota}_B \circ \sigma = \sigma^s \circ \iota_E$ since the following diagram is commutative:



Let $\tilde{\iota}_B$ denote the $*$ -homomorphism from $\mathcal{Q}(B)$ to $\mathcal{Q}(B \otimes \mathcal{K})$ which is induced by $\tilde{\iota}_B$. Arguing as in the proof of Theorem 2.2 in [15], we have that the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\tau_{\epsilon}} & \mathcal{Q}(B) \\
 \iota_A \downarrow & & \downarrow \tilde{\iota}_B \\
 A \otimes \mathcal{K} & \xrightarrow{\tau_{\epsilon^s}} & \mathcal{Q}(B \otimes \mathcal{K})
 \end{array} \tag{1.1}$$

is commutative since $(\iota_B, \iota_E, \iota_A)$ is a morphism from ϵ to ϵ^s . This finishes the proof of the two claims (1) and (2) above.

We are now ready to prove the proposition. Let x be a nonzero positive element of $A \otimes \mathcal{K}$. Then there exist t and s in $A \otimes \mathcal{K}$ such that $s x^{\frac{1}{2}} t = \iota_A(y)$ for some nonzero positive element y

of A . Let ϵ be a strictly positive number. From (1) of our claim, there exist $x_1, \dots, x_n, y_1, \dots, y_n$ in $\mathcal{Q}(B \otimes \mathcal{K})$ such that

$$\left\| 1_{\mathcal{Q}(B \otimes \mathcal{K})} - \sum_{i=1}^n x_i \bar{t}_B(1_{\mathcal{Q}(B)}) y_i \right\| < \frac{\epsilon}{2}.$$

From our assumption on τ_ϵ , there exist $t_1, \dots, t_m, s_1, \dots, s_m$ in $\mathcal{Q}(B)$ such that

$$\left\| 1_{\mathcal{Q}(B)} - \sum_{j=1}^m s_j \tau_\epsilon(y) t_j \right\| < \frac{\epsilon}{2(\sum_{i=1}^n \|x_i\| \|y_i\| + 1)}.$$

An easy computation shows that

$$\left\| 1_{\mathcal{Q}(B \otimes \mathcal{K})} - \sum_{i=1}^n x_i \left(\sum_{j=1}^m \bar{t}_B(s_j \tau_\epsilon(y) t_j) \right) y_i \right\| < \epsilon.$$

By the commutativity of diagram (1.1), we have that

$$\left\| 1_{\mathcal{Q}(B \otimes \mathcal{K})} - \sum_{i=1}^n x_i \left(\sum_{j=1}^m \bar{t}_B(s_j) \tau_\epsilon^s(s x^{\frac{1}{2}} t) \bar{t}_B(t_j) \right) y_i \right\| < \epsilon.$$

Therefore, the ideal of $\mathcal{Q}(B \otimes \mathcal{K})$ generated by $\tau_{\epsilon^s}(x^{\frac{1}{2}})$ is equal to $\mathcal{Q}(B \otimes \mathcal{K})$. Since $x^{\frac{1}{2}}$ is contained in the ideal of $A \otimes \mathcal{K}$ generated by x , we have that x is norm-full in $\mathcal{Q}(B \otimes \mathcal{K})$.

For an arbitrary nonzero element x of $A \otimes \mathcal{K}$, consider the positive nonzero element x^*x of $A \otimes \mathcal{K}$ and apply the result on positive elements to conclude that $\tau_{\epsilon^s}(x^*x)$ is norm-full in $\mathcal{Q}(B \otimes \mathcal{K})$. Therefore, $\tau_{\epsilon^s}(x)$ is norm-full in $\mathcal{Q}(B \otimes \mathcal{K})$ since x^*x is contained in the ideal of $A \otimes \mathcal{K}$ generated by x . \square

2. Six term exact sequence in K -theory

We need to extend some results by Rørdam [40] to a more general setting. First we recall some of the definitions there.

2.1. For every ϵ in $\mathcal{E}xt(A, B)$ we denote the cyclic six term exact sequence associated to ϵ by $K_{\text{six}}(\epsilon)$. Let $\mathcal{H}ext(A, B)$ denote the class of all cyclic six term exact sequences arising from elements of $\mathcal{E}xt(A, B)$. A homomorphism between such cyclic six term exact sequences is a 6-periodic chain homomorphism. We will frequently denote a homomorphism from $K_{\text{six}}(\epsilon_1)$, with $\epsilon_1 : 0 \rightarrow B_1 \rightarrow E_1 \rightarrow A_1 \rightarrow 0$, to $K_{\text{six}}(\epsilon_2)$, with $\epsilon_2 : 0 \rightarrow B_2 \rightarrow E_2 \rightarrow A_2 \rightarrow 0$, by a triple $(\beta_*, \eta_*, \alpha_*)$, where β_* from $K_*(B_1)$ to $K_*(B_2)$, η_* from $K_*(E_1)$ to $K_*(E_2)$, and α_* from $K_*(A_1)$ to $K_*(A_2)$ are homomorphisms (making the obvious diagrams commutative).

We say that two elements h_1 and h_2 of $\mathcal{H}ext(A, B)$ are *congruent* if there is an isomorphism from h_1 to h_2 of the form $(\text{id}_{K_*(B)}, \eta_*, \text{id}_{K_*(A)})$. Let $\text{Hex}t(A, B)$ be the set of all congruence classes of $\mathcal{H}ext(A, B)$. For every element h of $\mathcal{H}ext(A, B)$ we let $\mathbf{x}_{A, B}(h)$ denote the congruence class in $\text{Hex}t(A, B)$ containing h . According to [40, Proposition 2.1], there is a unique map \mathbf{K}_{six}

from $\text{Ext}(A, B)$ to $\text{Hext}(A, B)$ such that $\mathbf{x}_{A,B}(K_{\text{six}}(\epsilon)) = \mathbf{K}_{\text{six}}(x_{A,B}(\epsilon))$ for every element ϵ of $\mathcal{E}\text{xt}(A, B)$.

We let K_* denote the map in the Universal Coefficient Theorem from $KK^i(A, B)$ to $\text{Hom}(K_*(A), K_{*+i}(B))$, for $i = 0, 1$. Moreover, we set

$$\begin{aligned} \text{Hext}(A, B, \delta_*) &= \{h \in \text{Hext}(A, B) \mid K_*(h) = \delta_*\}, \\ \text{Ext}_{\delta_*}^1(A, B) &= \left\{x \in \text{Ext}(A, B) \mid \begin{array}{l} \ker \delta_j \subseteq \ker(K_j(x)) \\ \text{image}(K_j(x)) \subseteq \text{image}(\delta_j) \end{array}, j = 0, 1\right\}. \end{aligned}$$

In the following, we will need the maps

$$\begin{aligned} (\sigma_{\delta_*} =) \sigma_{A,B,\delta_*} : \text{Hext}(A, B; \delta_*) &\rightarrow \text{Ext}_{\mathbb{Z}}^1(\ker(\delta_*), \text{coker}(\delta_{*+1})), \\ (s_{\delta_*} =) s_{A,B,\delta_*} : \text{Ext}_{\delta_*}^1(A, B) &\rightarrow \text{Ext}_{\mathbb{Z}}^1(\ker(\delta_*), \text{coker}(\delta_{*+1})) \end{aligned}$$

introduced on pages 101–103 in [40].

Lemma 2.1. *Let A, B , and C be separable nuclear C^* -algebras with B stable and let δ_* be an element of $\text{Hom}(K_*(C), K_{*+1}(B))$. Suppose C is in \mathcal{N} and suppose x in $KK(A, C)$ is a KK -equivalence.*

Set $\lambda_ = \delta_* \circ K_*(x)$ in $\text{Hom}(K_*(A), K_{*+1}(B))$. Then*

- (1) $x \times (\cdot)$ is an isomorphism from $\text{Ext}_{\delta_*}^1(C, B)$ onto $\text{Ext}_{\lambda_*}^1(A, B)$.
- (2) x induces an isomorphism $[K_*(x)]$ from $\text{Ext}_{\mathbb{Z}}^1(\ker(\delta_*), \text{coker}(\delta_{*+1}))$ onto $\text{Ext}_{\mathbb{Z}}^1(\ker(\lambda_*), \text{coker}(\lambda_{*+1}))$.
- (3) Moreover, if A and B are in \mathcal{N} and if $x = KK(\alpha)$ for some injective $*$ -homomorphism α from A to C , then the diagram

$$\begin{array}{ccc} \text{Ext}_{\delta_*}^1(C, B) & \xrightarrow{x \times (\cdot)} & \text{Ext}_{\lambda_*}^1(A, B) \\ \downarrow s_{\delta_*} & \cong & \downarrow s_{\lambda_*} \\ \text{Ext}_{\mathbb{Z}}^1(\ker(\delta_*), \text{coker}(\delta_{*+1})) & \xrightarrow{[K_*(x)]} & \text{Ext}_{\mathbb{Z}}^1(\ker(\lambda_*), \text{coker}(\lambda_{*+1})) \end{array}$$

is commutative.

Proof. Since x is a KK -equivalence, $x \times (\cdot)$ is an isomorphism from $\text{Ext}(C, B)$ onto $\text{Ext}(A, B)$. Therefore, to prove (1) it is enough to show that $x \times (\cdot)$ maps $\text{Ext}_{\delta_*}^1(C, B)$ to $\text{Ext}_{\lambda_*}^1(A, B)$ and $x^{-1} \times (\cdot)$ maps $\text{Ext}_{\lambda_*}^1(A, B)$ to $\text{Ext}_{\delta_*}^1(C, B)$.

Note that $K_*(x)$ is an isomorphism and $K_j(x \times z) = K_j(z) \circ K_j(x)$ for $j = 0, 1$ and z in $\text{Ext}(C, B)$. Hence, $\text{image}(K_j(z) \circ K_j(x)) = \text{image}(K_j(z))$ and $\text{image}(\delta_j) = \text{image}(\delta_j \circ K_j(x))$. By definition, if z is in $\text{Ext}_{\delta_*}^1(C, B)$, then $\text{image}(K_j(z)) \subset \text{image}(\delta_j)$ for $j = 0, 1$. Therefore, for $j = 0, 1$,

$$\text{image}(K_j(z) \circ K_j(x)) \subset \text{image}(\delta_j \circ K_j(x)) = \text{image}(\lambda_j).$$

A straightforward computation shows that $\ker(\lambda_j) \subset \ker(K_j(z) \circ K_j(x))$. Hence, $x \times z$ is an element of $\text{Ext}_{\lambda_*}(A, B)$ for all z in $\text{Ext}_{\delta_*}(C, B)$. A similar computation shows that $x^{-1} \times (\cdot)$ maps $\text{Ext}_{\lambda_*}(A, B)$ to $\text{Ext}_{\delta_*}(C, B)$. We have just proved (1).

Since $K_*(x)$ is an isomorphism, $\text{image}(\lambda_*) = \text{image}(\delta_*)$. It is straightforward to show that $K_j(x)$ is an isomorphism from $\ker(\lambda_j)$ onto $\ker(\delta_j)$ for $j = 0, 1$. Therefore, $[K_*(x)]$ is the isomorphism induced by $K_*(x)$. This proves (2).

We now prove (3). Let z be in $\text{Ext}_{\lambda_*}(C, B)$. Let ϵ be an element of $\mathcal{E}\text{xt}(C, B)$ such that $x_{C,B}(\epsilon) = z$. Since $x = KK(\alpha)$ for some injective $*$ -homomorphism α from A to C , there exist $\alpha \cdot \epsilon$ in $\mathcal{E}\text{xt}(A, B)$ and a homomorphism $(\text{id}_B, \eta, \alpha)$ from $\alpha \cdot \epsilon$ to ϵ . By [40, Proposition 1.1]

$$x_{A,B}(\alpha \cdot \epsilon) = KK(\alpha) \times x_{C,B}(\epsilon) = x \times z.$$

By the Five Lemma, $(K_*(\text{id}_B), K_*(\eta), K_*(\alpha))$ is an isomorphism from $\mathbf{K}_{\text{six}}(z)$ onto $\mathbf{K}_{\text{six}}(x \times z)$ since $K_*(x)$ and $K_*(\text{id}_B)$ are isomorphisms.

It is clear from the observations made in the previous paragraph and from the definition of s_{δ_*} , s_{λ_*} , $x \times (\cdot)$, and $[K_*(x)]$ that the above diagram is commutative. \square

Rørddam conjectured that Proposition 3.1 and Theorem 3.2 in [40] are true for all separable nuclear C^* -algebras in \mathcal{N} . We establish this under the added assumption of stability:

Theorem 2.2. *Let A and B be separable nuclear C^* -algebras in \mathcal{N} with B stable. Let $\delta_* = (\delta_0, \delta_1)$ be an element of $\text{Hom}(K_*(A), K_{*+1}(B))$.*

(1) *The map*

$$s_{\delta_*} = s_{A,B,\delta_*} : \text{Ext}_{\delta_*}(A, B) \rightarrow \text{Ext}_{\mathbb{Z}}^1(\ker(\delta_*), \text{coker}(\delta_{*+1}))$$

is a group homomorphism.

- (2) *If x is in $\text{Ext}(A, B)$ and if $K_*(x) = \delta_*$, then $s_{\delta_*}(x) = \sigma_{\delta_*}(\mathbf{K}_{\text{six}}(x))$.*
- (3) *If z is in $\text{Ext}_{\mathbb{Z}}^1(K_*(A), K_*(B))$, then $s_{\delta_*}(\epsilon(z)) = \zeta_{\delta_*}(z)$, where ϵ is the canonical embedding of $\text{Ext}_{\mathbb{Z}}^1(K_*(A), K_*(B))$ into $\text{Ext}(A, B)$.*

Proof. (2) and (3) are clear from the definition of s_{δ_*} and ζ_{δ_*} .

We now prove (1). We claim that it is enough to prove (1) for the case that A is a unital separable nuclear purely infinite simple C^* -algebra in \mathcal{N} . Indeed, by the range results in [39] and [19], there exists a unital separable nuclear purely infinite simple C^* -algebra A_0 in \mathcal{N} such that $K_i(A)$ is isomorphic to $K_i(A_0)$. Denote this isomorphism by λ_i . Suppose A is unital. Then, by Theorem 6.7 in [26], there exists an injective $*$ -homomorphism ψ from A to A_0 which induces λ_* . Suppose A is not unital. Let ϵ be the embedding of A into the unitization of A , which we denote by \tilde{A} . It is easy to find a homomorphism $\tilde{\lambda}_i$ from $K_i(\tilde{A})$ to $K_i(A_0)$ such that $\tilde{\lambda}_i \circ K_i(\epsilon) = \lambda_i$. Note that \tilde{A} is a separable unital C^* -algebra in \mathcal{N} . By Theorem 6.7 in [26], there exists an injective $*$ -homomorphism $\tilde{\psi}$ from \tilde{A} to A_0 which induces $\tilde{\lambda}_*$. Hence, $\psi = \tilde{\psi} \circ \epsilon$ is an injective $*$ -homomorphism from A to A_0 which induces λ_* . Therefore, in both the unital or the nonunital case, we have an injective $*$ -homomorphism ψ which induces an isomorphism from $K_i(A)$ to $K_i(A_0)$. An easy consequence of the Universal Coefficient Theorem [42] and the Five Lemma shows that $KK(\psi)$ is a KK -equivalence. Therefore by Lemma 2.1 our claim is true.

Let A be a unital separable nuclear purely infinite simple C^* -algebra in \mathcal{N} . By the range results of [39] and [19], there exist separable nuclear purely infinite simple C^* -algebras A_0 and B_0 in \mathcal{N} such that A_0 is unital, B_0 is stable, and

$$\begin{aligned} \alpha_j: K_j(A_0) &\cong \ker(\delta_j : K_j(A) \rightarrow K_{j+1}(B)), \\ \beta_j: K_j(B_0) &\cong \operatorname{coker}(\delta_{j+1} : K_{j+1}(A) \rightarrow K_j(B)) \end{aligned}$$

for $j = 0, 1$. Since A and A_0 are unital separable nuclear purely infinite simple C^* -algebras satisfying the Universal Coefficient Theorem, by Theorem 6.7 in [26] there exists an injective $*$ -homomorphism φ from A_0 to A such that for $j = 0, 1$ the map $K_j(A_0) \xrightarrow{\alpha_j} \ker(\delta_j) \hookrightarrow K_j(A)$ is equal to $K_j(\varphi)$. Choose b in $KK(B, B_0)$ such that for $j = 0, 1$ the map from $K_j(B)$ to $\operatorname{coker}(\delta_{j+1})$ is equal to $\beta_j \circ K_j(b)$. Now, using the same argument as Proposition 3.1 in [40], we have that the map s_{δ_*} is a group homomorphism. \square

Replacing Proposition 3.1 in [40] by the above theorem and arguing as in Theorem 3.2 in [40], we get the following result.

Theorem 2.3. *Let A and B be separable nuclear C^* -algebras in \mathcal{N} with B stable. Suppose x_1 and x_2 are elements of $\operatorname{Ext}(A, B)$. Then $\mathbf{K}_{\text{six}}(x_1) = \mathbf{K}_{\text{six}}(x_2)$ in $\operatorname{Hext}(A, B)$ if and only if there exist elements a of $KK(A, A)$ and b of $KK(B, B)$ with $K_*(a) = K_*(\operatorname{id}_A)$ and $K_*(b) = K_*(\operatorname{id}_B)$ such that $x_1 \times b = a \times x_2$.*

3. Classification results

We will now use the results of the previous sections to generalize Rørdam’s results in [40].

Since in the sequel we will be mostly interested in C^* -algebras that are classified by $(K_0(A), K_0(A)_+, K_1(A))$, we will not state the Elliott invariant in its full generality.

Definition 3.1. For a C^* -algebra A of real rank zero, the *Elliott invariant* (which we denote by $K_*^+(A)$) consists of the triple

$$K_*^+(A) = (K_0(A), K_0(A)_+, K_1(A)).$$

It is well known that the canonical embedding of a C^* -algebra A into its stabilization $A \otimes \mathcal{K}$ induces an isomorphism from $K_*^+(A)$ to $K_*^+(A \otimes \mathcal{K})$ (this follows easily from Theorem 6.3.2 and the proof of Proposition 4.3.8 in [41]).

Suppose A and B are separable nuclear C^* -algebras in \mathcal{N} . Let x be an element of $KK(A, B)$. We say that x induces a homomorphism from $K_*^+(A)$ to $K_*^+(B)$ if $K_*(x)$ is a homomorphism from $K_*^+(A)$ to $K_*^+(B)$. If, moreover $K_0(x)([1_A]) = [1_B]$, then we say x induces a homomorphism from $(K_*^+(A), [1_A])$ to $(K_*^+(B), [1_B])$.

Definition 3.2. We will be interested in classes \mathcal{C} of separable nuclear unital simple C^* -algebras in \mathcal{N} satisfying the following properties:

- (I) Any element of \mathcal{C} is either purely infinite or stably finite.
- (II) \mathcal{C} is closed under tensoring with \mathbf{M}_n , where \mathbf{M}_n is the C^* -algebra of n by n matrices over \mathbb{C} .

- (III) If A is in \mathcal{C} , then any unital hereditary sub- C^* -algebra of A is in \mathcal{C} .
- (IV) For all A and B in \mathcal{C} and for all x in $KK(A, B)$ which induce an isomorphism from $(K_*^+(A), [1_A])$ to $(K_*^+(B), [1_B])$, there exists a $*$ -isomorphism α from A to B such that $KK(\alpha) = x$.

Remark 3.3. (1) The class of all unital separable nuclear purely infinite simple C^* -algebras satisfying the Universal Coefficient Theorem satisfies the properties in Definition 3.2 (see [20] and [35]).

(2) The class of all unital separable nuclear simple C^* -algebras satisfying the Universal Coefficient Theorem and with tracial topological rank zero satisfies the properties in Definition 3.2 (see Corollary 3.26 in [30]). This supersedes earlier work by Kishimoto and Kumjian (see Corollary 3.13 in [22]) and by Lin (see Theorem 1.1 in [11]).

The proof of the following lemma is left to the reader.

Lemma 3.4. *Let \mathcal{C} be a class of C^* -algebras satisfying the properties in Definition 3.2. Let A and B be in \mathcal{C} . Suppose there exists x in $KK(A \otimes \mathcal{K}, B \otimes \mathcal{K})$ such that x induces an isomorphism from $K_*^+(A \otimes \mathcal{K})$ onto $K_*^+(B \otimes \mathcal{K})$ and $K_0(x)([1_A \otimes e_{11}]) = [1_B \otimes e_{11}]$. Then there exists a $*$ -isomorphism α from $A \otimes \mathcal{K}$ onto $B \otimes \mathcal{K}$ such that $KK(\alpha) = x$.*

Lemma 3.5. *Let $A_1, A_2, B_1,$ and B_2 be unital separable nuclear C^* -algebras and let*

$$\epsilon: 0 \rightarrow B_1 \otimes \mathcal{K} \rightarrow E_1 \rightarrow A_1 \otimes \mathcal{K} \rightarrow 0$$

be an essential extension. Let α_ from $K_*^+(A_1 \otimes \mathcal{K})$ to $K_*^+(A_2 \otimes \mathcal{K})$ and β_* from $K_*^+(B_1 \otimes \mathcal{K})$ to $K_*^+(B_2 \otimes \mathcal{K})$ be isomorphisms. Suppose there exist a norm-full projection p in $\mathbf{M}_n(A_1)$ and a norm-full projection q in $\mathbf{M}_r(B_1)$ such that $\alpha_0([p]) = [1_{A_2} \otimes e_{11}]$, and $\beta_0([q]) = [1_{B_2} \otimes e_{11}]$.*

Then there exist $$ -isomorphisms φ from $p\mathbf{M}_n(A_1)p \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ and ψ from $q\mathbf{M}_r(B_1)q \otimes \mathcal{K}$ to $B_1 \otimes \mathcal{K}$ such that $\varphi \cdot \epsilon$ is isomorphic to ϵ via the isomorphism $(\text{id}_{B_1 \otimes \mathcal{K}}, \text{id}_{E_1}, \varphi)$ with $(\alpha_0 \circ K_0(\varphi))([p \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$ and ϵ is isomorphic to $\epsilon \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \text{id}_{E_1}, \text{id}_{A_1 \otimes \mathcal{K}})$ with $(\beta_0 \circ K_0(\psi))([q \otimes e_{11}]) = [1_{B_2} \otimes e_{11}]$.*

Moreover, ϵ is isomorphic to $\varphi \cdot \epsilon \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \text{id}_{E_1}, \varphi)$.

Proof. By Lemma 1.3, there exists a $*$ -isomorphism φ from $p(A_1 \otimes \mathcal{K})p \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ such that $[\varphi(p \otimes e_{11})] = [p]$. By the definition of $\varphi \cdot \epsilon$, we have that $\varphi \cdot \epsilon$ is isomorphic to ϵ via the isomorphism $(\text{id}_{B_1 \otimes \mathcal{K}}, \text{id}_{E_1}, \varphi)$. Also note that $(\alpha_0 \circ K_0(\varphi))([p \otimes e_{11}]) = \alpha_0([p]) = [1_{A_2} \otimes e_{11}]$.

Using Lemma 1.3 again, there exists a $*$ -isomorphism ψ from $q(B_1 \otimes \mathcal{K})q \otimes \mathcal{K}$ to $B_1 \otimes \mathcal{K}$ such that $[\psi(q \otimes e_{11})] = [q]$. By the definition of $\epsilon \cdot \psi^{-1}$, we have that ϵ is isomorphic to $\epsilon \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \text{id}_{E_1}, \text{id}_{A_1 \otimes \mathcal{K}})$. Note that $(\beta_0 \circ K_0(\psi))([q \otimes e_{11}]) = \beta_0([q]) = [1_{B_2} \otimes e_{11}]$.

Note that the composition of $(\text{id}_{B_1 \otimes \mathcal{K}}, \text{id}_{E_1}, \varphi)$ with $(\psi^{-1}, \text{id}_{E_1}, \text{id}_{A_1 \otimes \mathcal{K}})$ gives an isomorphism $(\psi^{-1}, \text{id}_{E_1}, \varphi)$ from ϵ onto $\varphi \cdot \epsilon \cdot \psi^{-1}$. \square

The next lemma is well known and we omit the proof.

Lemma 3.6. *Let ϵ_1 and ϵ_2 be in $\text{Ext}(A, B)$ and let τ_1 and τ_2 be the Busby invariant of ϵ_1 and ϵ_2 respectively. If τ_1 is unitarily equivalent to τ_2 with implementing unitary coming from the multiplier algebra of B , then ϵ_1 is isomorphic to ϵ_2 .*

A key component used by Rørdam in [40] was Kirchberg’s absorption theorem. Elliott and Kucerovsky in [18] give a criterion for when extensions are absorbing and call such extensions *purely large*. By Kirchberg’s theorem, every essential extension of separable nuclear C^* -algebras by stable purely infinite simple C^* -algebras is purely large. Kucerovsky and Ng (see [32] and [23]) proved that for C^* -algebras satisfying the corona factorization property, any full and essential extension is purely large, and hence absorbing. Properties similar to the corona factorization property were also studied by Lin [29].

Definition 3.7. Let B be a separable stable C^* -algebra. Then B is said to have the *corona factorization property* if every norm-full projection in $\mathcal{M}(B)$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(B)}$.

The following key results are due to Kucerovsky and Ng (see [32] and [23]):

Theorem 3.8. Let A be a unital separable simple C^* -algebra.

- (1) If A is exact, A has real rank zero and stable rank one, and $K_0(A)$ is weakly unperforated, then $A \otimes \mathcal{K}$ has the corona factorization property.
- (2) If A is purely infinite, then $A \otimes \mathcal{K}$ has the corona factorization property.

The following theorem is one of two main results in this paper. Using terminology introduced by Elliott in [17], the next result shows that the six term exact sequence together with certain positive cones is a classification functor for certain essential extensions of simple strongly classifiable C^* -algebras.

Theorem 3.9. Let \mathcal{C}_1 and \mathcal{C}_Q be classes of C^* -algebras satisfying the properties of Definition 3.2. Let A_1 and A_2 be in \mathcal{C}_Q and let B_1 and B_2 be in \mathcal{C}_1 with $B_1 \otimes \mathcal{K}$ and $B_2 \otimes \mathcal{K}$ satisfying the corona factorization property. Let

$$\mathfrak{e}_1: 0 \rightarrow B_1 \otimes \mathcal{K} \rightarrow E_1 \rightarrow A_1 \otimes \mathcal{K} \rightarrow 0,$$

$$\mathfrak{e}_2: 0 \rightarrow B_2 \otimes \mathcal{K} \rightarrow E_2 \rightarrow A_2 \otimes \mathcal{K} \rightarrow 0$$

be *essential* and *full* extensions. Then the following are equivalent:

- (1) E_1 is isomorphic to E_2 .
- (2) \mathfrak{e}_1 is isomorphic to \mathfrak{e}_2 .
- (3) There exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\text{six}}(\mathfrak{e}_1)$ to $K_{\text{six}}(\mathfrak{e}_2)$ such that β_* is an isomorphism from $K_*^+(B_1 \otimes \mathcal{K})$ onto $K_*^+(B_2 \otimes \mathcal{K})$ and α_* is an isomorphism from $K_*^+(A_1 \otimes \mathcal{K})$ onto $K_*^+(A_2 \otimes \mathcal{K})$.

Proof. Since A_1 , A_2 , B_1 , and B_2 are simple C^* -algebras, by [40, Proposition 1.2] E_1 is isomorphic to E_2 if and only if \mathfrak{e}_1 is isomorphic to \mathfrak{e}_2 . It is clear that an isomorphism from \mathfrak{e}_1 onto \mathfrak{e}_2 induces an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\text{six}}(\mathfrak{e}_1)$ onto $K_{\text{six}}(\mathfrak{e}_2)$ such that β_* is an isomorphism from $K_*^+(B_1 \otimes \mathcal{K})$ onto $K_*^+(B_2 \otimes \mathcal{K})$ and α_* is an isomorphism from $K_*^+(A_1 \otimes \mathcal{K})$ onto $K_*^+(A_2 \otimes \mathcal{K})$.

So we only need to prove (3) implies (2). Using the fact that the canonical embedding of A_i into $A_i \otimes \mathcal{K}$ induces an isomorphism between $K_j(A_i)$ and $K_j(A_i \otimes \mathcal{K})$ and since A_i is simple,

by Lemma 3.5 we may assume $\beta_0([1_{B_1} \otimes e_{11}]) = [1_{B_2} \otimes e_{11}]$ and $\alpha_0([1_{A_1} \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$. Hence, by Lemma 3.4 and the Universal Coefficient Theorem, there exist $*$ -isomorphisms β from $B_1 \otimes \mathcal{K}$ to $B_2 \otimes \mathcal{K}$ and α from $A_1 \otimes \mathcal{K}$ to $A_2 \otimes \mathcal{K}$ such that $K_*(\beta) = \beta_*$ and $K_*(\alpha) = \alpha_*$.

By [40, Proposition 1.2], ϵ_1 is isomorphic to $\epsilon_1 \cdot \beta$ and ϵ_2 is isomorphic to $\alpha \cdot \epsilon_2$. It is straightforward to check that $(K_*(\text{id}_{B_2 \otimes \mathcal{K}}), \eta_*, K_*(\text{id}_{A_1 \otimes \mathcal{K}}))$ gives a congruence between $K_{\text{six}}(\epsilon_1 \cdot \beta)$ and $K_{\text{six}}(\alpha \cdot \epsilon_2)$. Therefore, by Proposition 2.1 in [40]

$$\mathbf{K}_{\text{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\epsilon_1 \cdot \beta)) = \mathbf{K}_{\text{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\alpha \cdot \epsilon_2)).$$

Let $x_j = x_{A_j \otimes \mathcal{K}, B_j \otimes \mathcal{K}}(\epsilon_j)$ for $j = 1, 2$. By [40, Proposition 1.1],

$$\begin{aligned} \mathbf{K}_{\text{six}}(x_1 \times KK(\beta)) &= \mathbf{K}_{\text{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\epsilon_1 \cdot \beta)) \\ &= \mathbf{K}_{\text{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\alpha \cdot \epsilon_2)) \\ &= \mathbf{K}_{\text{six}}(KK(\alpha) \times x_2). \end{aligned}$$

By Theorem 2.3, there exist invertible elements a of $KK(A_1 \otimes \mathcal{K}, A_1 \otimes \mathcal{K})$ and b of $KK(B_2 \otimes \mathcal{K}, B_2 \otimes \mathcal{K})$ such that

- (1) $K_*(a) = K_*(\text{id}_{A_1 \otimes \mathcal{K}})$ and $K_*(b) = K_*(\text{id}_{B_2 \otimes \mathcal{K}})$ and
- (2) $x_1 \times KK(\beta) \times b = a \times KK(\alpha) \times x_2$.

Since A_1 is in \mathcal{C}_Q and B_2 is in \mathcal{C}_I , by Lemma 3.4 there exist $*$ -automorphisms ρ on $A_1 \otimes \mathcal{K}$ and γ on $B_2 \otimes \mathcal{K}$ such that $KK(\rho) = a$ and $KK(\gamma) = b$.

Using [40, Proposition 1.2] once again, $\epsilon_1 \cdot \beta$ is isomorphic to $\epsilon_1 \cdot \beta \cdot \gamma$ and $\alpha \cdot \epsilon_2$ is isomorphic to $\rho \cdot \alpha \cdot \epsilon_2$. By [40, Proposition 1.1],

$$\begin{aligned} x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\epsilon_1 \cdot \beta \cdot \gamma) &= x_1 \times KK(\beta) \times KK(\gamma) = x_1 \times KK(\beta) \times b \\ &= a \times KK(\alpha) \times x_2 = KK(\rho) \times KK(\alpha) \times x_2 \\ &= x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\rho \cdot \alpha \cdot \epsilon_2). \end{aligned}$$

By assumption, $B_2 \otimes \mathcal{K}$ satisfies the corona factorization property and $\epsilon_1 \cdot \beta \cdot \gamma$ and $\rho \cdot \alpha \cdot \epsilon_2$ are full extensions. The above equation shows that $\epsilon_1 \cdot \beta \cdot \gamma$ and $\rho \cdot \alpha \cdot \epsilon_2$ give the same element of $\text{Ext}(A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K})$, so by Theorem 3.2(2) in [32] (see also Corollary 1.9 in [23]) these extensions are unitarily equivalent with the implementing unitary coming from the multiplier algebra of $B_2 \otimes \mathcal{K}$. So by Lemma 3.6, $\epsilon_1 \cdot \beta \cdot \gamma$ is isomorphic to $\rho \cdot \alpha \cdot \epsilon_2$. Hence ϵ_1 is isomorphic to ϵ_2 . \square

We now extend our results to the case of an ideal which is nonsimple under the added assumption that the ideal is AF but the quotient is not. We first need:

Lemma 3.10. *Let A be a unital AF -algebra. Then $A \otimes \mathcal{K}$ has the corona factorization property.*

Proof. Suppose p is a norm-full projection in $\mathcal{M}(A \otimes \mathcal{K})$. Then, by Corollary 3.6 in [29], there exists z in $\mathcal{M}(A \otimes \mathcal{K})$ such that $zpz^* = 1_{\mathcal{M}(A \otimes \mathcal{K})}$. Therefore, $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is Murray–von Neumann equivalent to a sub-projection of p . Since $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is a properly infinite projection, p is a

properly infinite projection. By the results of Cuntz in [10] and the fact that $K_0(\mathcal{M}(A \otimes \mathcal{K})) = 0$, we have that $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is Murray–von Neumann equivalent to p . \square

Lemma 3.11. *Let A be a separable stable C^* -algebra satisfying the corona factorization property. Let q be a norm-full projection in $\mathcal{M}(A)$. Then qAq is isomorphic to A and hence qAq is stable.*

Proof. Since q is norm-full in $\mathcal{M}(A)$ and since A has the corona factorization property, there exists a partial isometry v in $\mathcal{M}(A)$ such that $v^*v = 1_{\mathcal{M}(A)}$ and $vv^* = q$. Therefore v induces a $*$ -isomorphism from A onto qAq . Since A is stable, qAq is stable. \square

Theorem 3.12. *Let \mathcal{C} be a class of C^* -algebras satisfying the properties of Definition 3.2 with the further property that it is disjoint from the class of AF -algebras. Let A_1 and A_2 be in \mathcal{C} and let B_1 and B_2 be unital AF -algebras. Suppose*

$$\begin{aligned} \epsilon_1: \quad 0 &\rightarrow B_1 \otimes \mathcal{K} \xrightarrow{\varphi_1} E_1 \xrightarrow{\psi_1} A_1 \rightarrow 0, \\ \epsilon_2: \quad 0 &\rightarrow B_2 \otimes \mathcal{K} \xrightarrow{\varphi_2} E_2 \xrightarrow{\psi_2} A_2 \rightarrow 0 \end{aligned}$$

are unital essential extensions. Let ϵ_1^s and ϵ_2^s be the extensions obtained by tensoring ϵ_1 and ϵ_2 with the compact operators. Then the following are equivalent:

- (1) $E_1 \otimes \mathcal{K}$ is isomorphic to $E_2 \otimes \mathcal{K}$.
- (2) ϵ_1^s is isomorphic to ϵ_2^s .
- (3) There exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\text{six}}(\epsilon_1)$ to $K_{\text{six}}(\epsilon_2)$ such that β_* is an isomorphism from $K_*^+(B_1)$ to $K_*^+(B_2)$ and α_* is an isomorphism from $K_*^+(A_1)$ to $K_*^+(A_2)$.

Proof. First we show that (1) implies (2). Suppose that there exists a $*$ -isomorphism η from $E_1 \otimes \mathcal{K}$ onto $E_2 \otimes \mathcal{K}$. Note that for $i = 1, 2$, $A_i \otimes \mathcal{K}$ is not an AF -algebra by assumption. Since $[(\psi_2 \otimes \text{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \text{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K})$ is an ideal of $A_1 \otimes \mathcal{K}$ and $A_1 \otimes \mathcal{K}$ is a simple C^* -algebra, $[(\psi_2 \otimes \text{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \text{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K})$ is either zero or $A_1 \otimes \mathcal{K}$. Since the image of an AF -algebra is again an AF -algebra, $[(\psi_2 \otimes \text{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \text{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K}) = 0$. Hence, η induces an isomorphism from ϵ_1^s onto ϵ_2^s .

Clearly (2) implies both (1) and

- (3') there exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\text{six}}(\epsilon_1^s)$ onto $K_{\text{six}}(\epsilon_2^s)$ such that β_* is an isomorphism from $K_*^+(B_1 \otimes \mathcal{K} \otimes \mathcal{K})$ onto $K_*^+(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ and α_* is an isomorphism from $K_*^+(A_1 \otimes \mathcal{K})$ onto $K_*^+(A_2 \otimes \mathcal{K})$

and as noted in Definition 3.1, (3') is equivalent to (3). We now prove (3') implies (2). By Lemma 3.5, we may assume that $\alpha_0([1_{A_1} \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$. Using strong classification for AF -algebras and for the elements in \mathcal{C} , we get $*$ -isomorphisms α from $A_1 \otimes \mathcal{K}$ to $A_2 \otimes \mathcal{K}$ and β from $B_1 \otimes \mathcal{K} \otimes \mathcal{K}$ to $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ such that $K_*(\alpha) = \alpha_*$ and $K_*(\beta) = \beta_*$.

By [40, Proposition 1.2], ϵ_1^s is isomorphic to $\epsilon_1^s \cdot \beta$ and ϵ_2^s is isomorphic to $\alpha \cdot \epsilon_2^s$. It is straightforward to check that $K_{\text{six}}(\epsilon_1^s \cdot \beta)$ is congruent to $K_{\text{six}}(\alpha \cdot \epsilon_2^s)$. Hence, by Theorem 2.3 there exist invertible elements a of $KK(A_1 \otimes \mathcal{K}, A_1 \otimes \mathcal{K})$ and b of $KK(B_2 \otimes \mathcal{K} \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ such that

- (i) $K_*(a) = K_*(\text{id}_{A_1 \otimes \mathcal{K}})$ and $K_*(b) = K_*(\text{id}_{B_2 \otimes \mathcal{K} \otimes \mathcal{K}})$; and
- (ii) $x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\epsilon_1^s \cdot \beta) \times b = a \times x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\alpha \cdot \epsilon_2^s)$.

By the Universal Coefficient Theorem, $b = KK(\text{id}_{B_2 \otimes \mathcal{K} \otimes \mathcal{K}})$ since $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ is an AF -algebra. By property (3') again combined with Lemma 3.4 there exists a $*$ -isomorphism ρ from $A_1 \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ such that $KK(\rho) = a$.

By [40, Propositions 1.1 and 1.2], $\rho \cdot \alpha \cdot \epsilon_2^s$ is isomorphic to $\alpha \cdot \epsilon_2^s$ and

$$\begin{aligned} x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\epsilon_1^s \cdot \beta) &= x_1 \times KK(\beta) \\ &= KK(\rho) \times KK(\alpha) \times x_2 \\ &= x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\rho \cdot \alpha \cdot \epsilon_2^s), \end{aligned}$$

where $x_i = x_{A_i \otimes \mathcal{K}, B_i \otimes \mathcal{K} \otimes \mathcal{K}}(\epsilon_i^s)$.

Let τ_1 be the Busby invariant of $\epsilon_1^s \cdot \beta$ and let τ_2 be the Busby invariant of $\rho \cdot \alpha \cdot \epsilon_2^s$. Then, $[\tau_1] = [\tau_2]$ in $\text{Ext}(A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K})$. Note that ϵ_i is full by Lemma 1.5, so by Proposition 1.6, so is ϵ_i^s . Using this observation and the fact that β, α , and ρ are $*$ -isomorphisms, it is clear that $\epsilon_1^s \cdot \beta$ and $\rho \cdot \alpha \cdot \epsilon_2^s$ are full extensions.

Note that by Lemma 3.10, $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ has the corona factorization property. Therefore, by the observations made in the previous paragraph one can apply Theorem 3.2(2) in [32] to get a unitary u in $\mathcal{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ such that

$$\pi(u)\tau_1(x)\pi(u)^* = \tau_2(x)$$

for all a in $A_1 \otimes \mathcal{K}$. Hence, by Lemma 3.6, $\epsilon_1^s \cdot \beta$ and $\rho \circ \alpha \cdot \epsilon_2^s$ are isomorphic. Therefore, ϵ_1^s is isomorphic to ϵ_2^s . \square

Remark 3.13. Examples of extensions with AF ideals and quotients which are simple AD -algebras of real rank zero are given in [12] to demonstrate the need of K -theory with coefficients. These examples show that there is no generalization of the previous theorem to general extensions; one needs to arrange for fullness for the methods to work. It would be interesting to investigate if, as suggested by this example, having K -theory with coefficients as part of the invariant could reduce the requirements on the extension.

4. Applications

Clearly, Theorem 3.9 applies to essential extensions of separable nuclear purely infinite simple stable C^* -algebras in \mathcal{N} (and gives us the classification obtained by Rørdam in [40]). We present here two other examples of classes of special interest, to which our results apply.

4.1. Matsumoto algebras

The results of the previous section apply to a class of C^* -algebras introduced in the work by Matsumoto which was investigated in recent work by the first named author and Carlsen [4, 9,6–8]. Indeed, as seen in [4] we have for each minimal shift space \underline{X} with a certain technical property (***) introduced in Definition 3.2 in [7] that the Matsumoto algebra $\mathcal{O}_{\underline{X}}$ fits in a short exact sequence of the form

$$0 \longrightarrow \mathcal{K}^n \longrightarrow \mathcal{O}_{\underline{X}} \longrightarrow C(\underline{X}) \rtimes_{\sigma} \mathbb{Z} \longrightarrow 0 \tag{4.1}$$

where n is an integer determined by the structure of the so-called *special words* of \underline{X} . Clearly the ideal is an *AF*-algebra and by the work of Putnam [36] the quotient is a unital simple *AT*-algebra with real rank zero which falls in the class mentioned in Remark 3.3 (2). Let us record a couple of consequences of this:

Corollary 4.1. *Let \underline{X}_{α} denote the Sturmian shift space associated to the parameter α in $[0, 1] \setminus \mathbb{Q}$ and $\mathcal{O}_{\underline{X}_{\alpha}}$ the Matsumoto algebra associated to \underline{X}_{α} . If α and β are elements of $[0, 1] \setminus \mathbb{Q}$, then*

$$\mathcal{O}_{\underline{X}_{\alpha}} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{X}_{\beta}} \otimes \mathcal{K}$$

if and only if $\mathbb{Z} + \alpha\mathbb{Z} \cong \mathbb{Z} + \beta\mathbb{Z}$ as ordered groups.

Proof. The extension (4.1) has the six term exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} + \alpha\mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} + \alpha\mathbb{Z} \\ \parallel & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

by Example 5.3 in [8]. Now apply Theorem 3.12. \square

The bulk of the work in the papers [5–9] is devoted to the case of shift spaces associated to primitive, aperiodic substitutions. As a main result, an algorithm is devised to compute the ordered group $K_0(\mathcal{O}_{\underline{X}_{\tau}})$ for any such substitution τ , thus providing new invariants for such dynamical systems up to flow equivalence (see [33]). The structure result of [4] applies in this case as well, and in fact, as noted in Section 6.4 of [6], the algorithm provides all the data in the six term exact sequence associated to the extension (4.1). This is based on computable objects $n_{\tau}, p_{\tau}, \mathbf{A}_{\tau}, \tilde{\mathbf{A}}_{\tau}$ of which the latter two are square matrices with integer entries. For each such matrix, say A in $\mathbf{M}_n(\mathbb{Z})$, we define a group

$$DG(A) = \varinjlim (\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \dots)$$

which, when A has only nonnegative entries, may be considered as an ordered group which will be a dimension group. We get:

Theorem 4.2. *Let τ_1 and τ_2 be basic substitutions, see [7], over the alphabets \mathfrak{a}_1 and \mathfrak{a}_2 , respectively. Then*

$$\mathcal{O}_{\underline{X}_{\tau_1}} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{X}_{\tau_2}} \otimes \mathcal{K}$$

if and only if there exist group isomorphisms ϕ_1, ϕ_2, ϕ_3 with ϕ_1 and ϕ_3 order isomorphisms, making the diagram

$$\begin{CD}
 \mathbb{Z} @>{p_{\tau_1}}>> \mathbb{Z}^{n_{\tau_1}} @>{Q_1}>> DG(\tilde{\mathbf{A}}_{\tau_1}) @>{R_1}>> DG(\mathbf{A}_{\tau_1}) \\
 @| @V{\phi_1}VV @V{\phi_2}VV @V{\phi_3}VV \\
 \mathbb{Z} @>{p_{\tau_2}}>> \mathbb{Z}^{n_{\tau_2}} @>{Q_2}>> DG(\tilde{\mathbf{A}}_{\tau_2}) @>{R_2}>> DG(\mathbf{A}_{\tau_2})
 \end{CD} \tag{4.2}$$

commutative. Here the finite data n_{τ_i} in \mathbb{N} , p_{τ_i} in $\mathbb{Z}^{n_{\tau_i}}$, \mathbf{A}_{τ_i} in $\mathbf{M}_{|\alpha_i|}(\mathbb{N}_0)$, $\tilde{\mathbf{A}}_{\tau_i}$ in $\mathbf{M}_{|\alpha_i|+n_{\tau_i}}(\mathbb{Z})$ are as described in [6], the Q_i are defined by the canonical map to the first occurrence of $\mathbb{Z}^{n_{\tau_i}}$ in the inductive limit, and R_i are induced by the canonical map from $\mathbb{Z}^{|\alpha_i|+n_{\tau_i}}$ to $\mathbb{Z}^{|\alpha_i|}$.

Proof. We have already noted above that Theorem 3.12 applies, proving “if”. For “only if”, we use that any $*$ -isomorphism between $\mathcal{O}_{\underline{X}_{\tau_1}} \otimes \mathcal{K}$ and $\mathcal{O}_{\underline{X}_{\tau_2}} \otimes \mathcal{K}$ must preserve the ideal in (4.1) and hence induce isomorphisms on the corresponding six term exact sequence which are intertwined by the maps of this sequence as indicated. And since the vectors p_{τ_i} both have all entries positive, the isomorphism $x \mapsto -x$ between \mathbb{Z} and \mathbb{Z} can be ruled out by positivity of ϕ_1 . \square

The following reformulation, suggested to us by Takeshi Katsura, improves the usability of the result:

Corollary 4.3. *The triple $[K_0(\mathcal{O}_{\underline{X}_{\tau}}), K_0(\mathcal{O}_{\underline{X}_{\tau}})_+, \Sigma_{\tau}]$ with Σ_{τ} the scale consisting of a multiset in $K_0(\mathcal{O}_{\underline{X}})$ given by*

$$[Q(e_1), \dots, Q(e_{n_{\tau}})]$$

(e_i the canonical generators of $\mathbb{Z}^{n_{\tau}}$, and Q the map defined in Theorem 4.2), is a complete invariant up to stable isomorphism for the class of Matsumoto algebras $\mathcal{O}_{\underline{X}_{\tau}}$ associated to basic substitutions.

Proof. Assume first that a triple (ϕ_1, ϕ_2, ϕ_3) is given as in (4.2). By [7], we may conclude from the fact that ϕ_3 is an order isomorphism that the same is true for ϕ_2 . We also note that $n_{\tau_1} = n_{\tau_2}$, and that ϕ_1 must permute the e_i to be an order isomorphism. Thus, $\phi_2(\Sigma_{\tau_1}) = \Sigma_{\tau_2}$.

In the other direction, assume that $\phi_2 : K_0(\mathcal{O}_{\underline{X}_{\tau_1}}) \rightarrow K_0(\mathcal{O}_{\underline{X}_{\tau_2}})$ preserves both the positive cone and the scale. Again, $n_{\tau_1} = n_{\tau_2}$, and by permuting the generators according to the identification of Σ_{τ_1} and Σ_{τ_2} we get an order isomorphism ϕ_1 with $Q_2 \circ \phi_1 = \phi_2 \circ Q_1$. Consequently, an isomorphism ϕ_3 is induced, and it will be an order isomorphism by [7]. Finally we see that

$$\phi_1(\mathbb{Z}p_{\tau_1}) = \phi_1(\ker Q_1) = \ker Q_2 = \mathbb{Z}p_{\tau_2}$$

whence we must have $\phi_1(p_{\tau_1}) = \pm p_{\tau_2}$, and the negative sign is impossible by positivity. \square

Such a classification result puts further emphasis on the question raised in Section 6.4 in [6] of what relation stable isomorphism of the Matsumoto algebras induces on the shift spaces. We note here that this relation must be strictly coarser than flow equivalence:

Example 4.4. Consider the substitutions

$$\tau(0) = 10101000, \quad \tau(1) = 10100$$

and

$$\nu(0) = 10100100, \quad \nu(1) = 10100.$$

We have that $\mathcal{O}_{\underline{X}_\tau} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{X}_\nu} \otimes \mathcal{K}$ although \underline{X}_τ and \underline{X}_ν are not flow equivalent.

Proof. Since both substitutions are chosen to be basic, computations using the algorithm from [9] (for instance using the program [5]) show that the invariant reduces to $[DG(\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}), [0]]$ for both substitutions (see Corollary 5.20 in [7]). Hence by Theorem 4.2, the C^* -algebras $\mathcal{O}_{\underline{X}_\tau}$ and $\mathcal{O}_{\underline{X}_\nu}$ are stably isomorphic. However, the *configuration data* (see [9]) are different, namely



respectively, and since this is a flow invariant, the shift spaces \underline{X}_τ and \underline{X}_ν are not flow equivalent. \square

4.2. Graph algebras

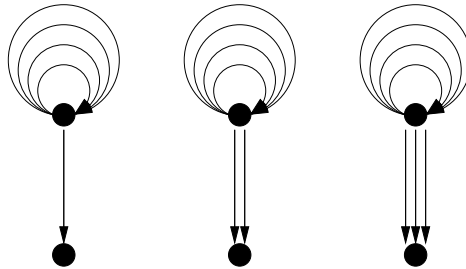
A completely independent application is presented by the first named author and Tomforde in a forthcoming paper [14] and we sketch a basic instance of it here. By the work of many hands (see [37] and the references therein) a *graph C^* -algebra* may be associated to any directed graph (countable, but possibly infinite). When such C^* -algebras are simple, they are always nuclear and in the bootstrap class \mathcal{N} , and either purely infinite or AF . They are hence, by appealing to either [20] or [16], classifiable by the Elliott invariant. Our first main result Theorem 3.9 applies to prove the following:

Theorem 4.5. *Let E and E' be unital graph algebras with exactly one nontrivial ideal B and B' , respectively. Then $E \otimes \mathcal{K} \cong E' \otimes \mathcal{K}$ if and only if there exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ between the six term exact sequences associated with E and E' such that α_0 from $K_0(E/B)$ to $K_0(E'/B')$ and β_0 from $K_0(B)$ to $K_0(B')$ are order isomorphisms.*

Sketch of proof. Known structure results for graph C^* -algebras establish that all of $B, B', E/B$ and E'/B' are themselves graph C^* -algebras, but to invoke Theorem 3.9 we furthermore need to know that B and B' are stable and of the form $J \otimes \mathcal{K}$ for J a unital graph algebra. This is a nontrivial result which is established in [14].

With this we can choose as \mathcal{C} in Theorem 3.9 the union of the set of unital, simple, separable, nuclear and purely infinite algebras with UCT and the unital simple AF -algebras. Then it is easy to check that properties (I)–(IV) are satisfied, as is the corona factorization property. \square

Example 4.6. Consider the three graphs



which all define graph algebras with precisely one ideal, and with vanishing K_1 everywhere. The remaining part of the six term exact sequence is, up to equivalence in $\text{Ext}(\mathbb{Z}/3, \mathbb{Z})$,

$$\mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/3, \quad \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{2} \mathbb{Z}/3, \quad \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/3 \longrightarrow \mathbb{Z}/3$$

respectively. Hence the graph algebras corresponding to the two first graphs are stably isomorphic to each other, but not to the one associated to the latter.

This example confirms a special case of a conjecture by Tomforde which we shall discuss in [14]. Note also that with substantially more work, Theorem 4.5 above is generalized to the nonunital case in [14]. An application of Theorem 3.12 is also presented there.

5. Concluding discussion

5.1. Extended invariants

As may be seen by invoking more sophisticated invariants, our assumption of fullness of the extensions considered in Theorem 3.9 is necessary. As is to be expected, one cannot ignore the order on K_0 of the middle algebra in general, but also invariants such as the tracial simplex or K -theory with coefficients are useful in a nonfull context.

It is an interesting and, at present, open question if it is possible to classify all real rank zero extensions of simple classifiable C^* -algebras by the six-term exact sequence of total K -theory.

5.2. Improved classification results

It is often of great importance in applications of classification results to know that the given isomorphism of K -groups lifts to a $*$ -isomorphism, or to have direct classification by involving a scale or a class of the unit in K -theory.

The authors in [13] and [38] have resolved this question in the case considered by Rørdam, but the methods used there do not readily extend to the generality of the present paper.

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