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On Kirkman triple systems of order 33

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Abstract

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Twenty-eight non-isomorphic KTS(33) with an automorphism of order 11 are constructed from the 84 cyclic STS(33).

1. Introduction

We assume familiarity with the basic facts and notions from design theory (cf., e.g. Cameron and van Lint [2], or [7]). Recall that two Kirkman systems are isomorphic if there exists an isomorphism between the underlying Steiner triple systems that maps the parallel classes of the resolution of the first system into parallel classes of the resolution of the second one. Consequently, two different resolutions of a given Steiner triple system $S = \mathrm{STS}(v)$ define isomorphic Kirkman systems if and only if there is an automorphism of S mapping one of these resolutions into the other.

The bound for the number of known non-isomorphic Kirkman triple system of order 33 according to Mathon and Rosa's table of designs [6] is $\geqslant 1$. What these authors apparently meant is that it has not been checked how many of the three KTS(33) listed in [1, 4, 5] are pairwise non-isomorphic. In fact, the system from [4] coincides with that from Ball paper [1], and the one listed by Kageyama [5] is non-isomorphic to the Ball system. For, Kageyama's design is invariant under a cyclic group of order 32 fixing one point and Ball design has a fixed point free automorphism of order 11, and any permutation group of degree 33 containing

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both such permutations of order 32 and 11 must be doubly-transitive, hance primitive and consequently too big to act on an STS(33).

Although the number of non-isomorphic STS(33) is known to be at least 2,000,000 [6], finding out whether a particular STS(33) is resolvable or not is computationally very difficult. The problem is that the number of parallel classes through a given triple is usually within the thousands which makes finding a resolution computationally hopeless unless one assumes some extra symmetry.

The Kirkman system constructed by Ball admits a resolution invariant under an automorphism of order 11 acting transitively on 11 parallel classes and fixing every of the remaining 5 parallel classes. It is the aim of this note to show that the same type of automorphism can be used to produce more Kirkman systems from the cyclic triple systems of order 33. As generating automorphism of order 11 we use the third power of the permutation of order 33 acting cyclically on the points.

Table 1
Kirkman triple systems KTS(33)

1: (13)	2: (15)	3: (16)	4: (18)	5: (20)	6: (26)
1 2 4	1 2 4	1 2 4	1 2 4	1 2 4	1 2 4
5 6 8	8 9 11	14 15 17	8 9 11	20 21 23	17 18 20
18 19 21	18 19 21	30 31 33	24 25 27	12 13 15	24 25 27
10 16 28	13 25 31	10 22 28	10 16 28	10 22 28	10 16 28
11 26 32	14 26 32	5 11 26	5 17 32	5 11 26	5 11 23
9 24 30	12 24 30	6 12 27	12 18 30	3 18 30	3 9 21
7 14 23	5 22 29	7 16 23	7 14 31	7 14 31	7 14 31
3 12 29	3 20 27	3 20 29	3 20 29	8 17 24	8 15 32
15 22 31	7 16 33	9 18 25	6 15 22	9 16 33	6 13 30
13 17 27	10 15 23	19 24 32	19 23 33	6 25 29	22 26 12
20 25 33	6 17 28	8 13 21	13 21 26	19 27 32	29 33 19
2 6 16	1 5 15	1 5 15	2 6 16	2 6 16	3 7 26
3 7 17	2 6 16	2 6 16	3 7 17	3 7 17	1 6 14
1 6 14	3 7 17	3 7 17	1 6 26	1 6 26	2 7 15
3 8 16	2 7 15	3 8 16	2 7 27	2 7 27	3 8 16
1 12 23	3 8 16	1 12 23	1 12 23	1 12 23	1 12 23
7: (27)	8: (28)	9: (30)	10: (46)	11: (47a)	12: (47b)
1 2 4	1 2 4	1 2 4	1 2 5	1 2 5	1 2 5
8 9 11	14 15 17	17 18 20	8 9 12	14 15 18	20 21 24
18 19 21	30 31 33	24 25 27	24 25 28	9 10 13	9 10 13
13 25 31	22 28 10	10 16 28	4 6 18	12 31 33	6 25 27
14 26 32	5 11 26	5 11 23	17 19 31	4 23 25	17 19 31
12 24 30	6 12 27	3 9 21	11 30 32	8 27 29	11 30 32
5 22 29	16 23 7	7 14 31	7 13 22	7 16 22	7 16 22
3 20 27	29 3 20	8 15 32	14 20 29	11 20 26	14 23 29
7 16 33	18 25 9	6 13 30	3 21 27	3 21 30	3 12 18
10 15 23	19 24 32	22 26 12	10 15 23	19 24 32	8 28 33
28 6 17	8 13 21	29 33 19	16 26 33	6 17 28	4 15 26
1 5 24	1 5 24	3 7 26	2 7 15	3 8 16	3 8 16
2 6 25	2 6 25	1 6 26	3 8 16	2 7 15	2 7 15
3 7 26	3 7 26	2 7 27	1 8 24	1 8 18	1 8 18
2 7 15	3 8 16	3 8 28	3 10 26	2 9 19	2 9 19
3 8 16	1 12 23	1 12 23	1 12 23	3 10 20	3 10 20

Table 1 (continued)

13: (48a)	14: (48b)	15: (55a)	16: (55b)	17: (59a)	18: (59b)
1 2 5	1 2 5	1 2 5	1 2 5	1 2 5	1 2 5
4 15 18	20 21 24	17 18 21	8 9 12	20 21 24	8 9 12
9 10 13	9 10 13	9 10 13	24 25 28	15 16 19	24 25 28
31 33 12	25 27 6	28 30 16	31 33 19	7 9 28	31 33 19
23 25 4	17 19 31	23 25 11	11 13 32	11 13 32	11 13 32
27 29 8	30 32 11	27 29 15	18 20 6	6 8 27	18 20 6
16 22 7	16 22 7	31 4 22	16 22 7	31 4 22	16 22 7
20 26 11	23 29 14	8 14 32	23 29 14	23 29 14	23 29 14
30 3 21	12 18 3	33 6 24	30 3 21	12 18 3	30 3 21
19 24 32	28 33 8	7 12 20	10 17 27	25 30 17	10 17 27
28 6 17	4 15 26	19 26 3	4 15 26	26 33 10	4 15 26
3 8 16	3 8 16	3 8 16	3 8 16	3 8 28	3 8 28
2 7 15	2 7 15	2 7 15	2 7 15	2 7 27	2 7 27
1 8 24	1 8 24	2 9 19	1 6 14	1 8 18	1 6 26
2 9 25	2 9 25	3 10 20	3 10 20	3 10 20	3 10 20
3 10 26	3 10 26	1 12 23	2 9 19	1 12 23	2 9 19

19: (66)	20: (68a)	21: 68(b)	22: (68c)	23: (69)	24: (70a)
1 3 16	1 3 16	1 3 16	1 3 16	1 3 16	1 3 16
8 10 23	2 4 17	2 4 17	8 26 28	11 13 26	17 19 32
2 33 15	9 11 24	27 29 9	9 11 24	15 17 30	9 11 24
4 25 28	7 10 19	7 10 19	19 22 31	19 22 10	22 25 13
14 17 26	8 29 32	8 11 20	2 5 14	32 2 23	5 8 29
6 27 30	15 18 27	12 15 24	6 27 30	21 24 12	27 30 18
7 13 21	22 28 14	22 28 14	7 13 32	25 31 6	31 4 23
5 11 19	20 26 12	26 32 18	15 23 29	14 20 28	14 20 6
18 24 32	6 25 33	31 5 21	4 12 18	27 33 8	15 21 7
12 22 29	5 21 31	23 30 13	10 17 33	4 5 9	28 2 12
9 20 31	13 23 30	33 6 25	20 21 25	7 18 29	26 33 10
3 10 26	3 10 26	1 2 6	3 10 26	3 10 20	3 10 20
2 9 25	1 2 6	2 3 7	2 9 25	2 9 19	1 2 6
1 2 6	2 3 7	3 4 8	1 2 6	1 8 18	2 3 7
2 3 7	3 4 8	1 12 23	3 4 8	3 4 8	3 4 8
3 4 8	1 12 23	3 10 26	1 12 23	2 3 7	1 12 23

25: (70b)	26: (70c)	27: (70d)	28:(71)	Ball	Kageyama
1 3 16	1 3 16	1 3 16	1 3 21	1 13 19	$*$ 0 16
23 25 5	5 7 20	20 22 2	26 28 13	23 11 5	1 10 15
33 2 15	15 17 30	30 32 12	18 20 5	3 15 21	2 22 23
28 31 19	28 31 19	7 10 31	7 10 19	2 4 7	3 5 9
8 11 32	11 14 2	26 29 17	29 32 8	12 14 17	4 12 29
6 9 30	21 24 12	15 18 6	12 15 24	28 30 33	6 7 18
4 10 29	4 10 29	19 25 11	16 22 30	6 16 25	8 27 30
20 26 12	26 32 18	8 14 33	17 23 31	10 20 29	11 14 24
21 27 13	33 6 25	21 27 13	33 6 14	8 18 27	13 20 28
7 14 24	22 23 27	4 5 9	4 11 27	24 31 32	17 26 31
17 18 22	8 9 13	23 24 28	2 9 25	9 22 26	19 21 25
3 10 20	3 10 20	3 10 20	3 10 26	25 32 33	(mod 32)
2 9 19	2 9 19	2 9 19	1 2 6	26 33 1	Resolution:
1 2 6	1 8 18	1 8 18	2 3 7	10 23 27	$B1+i, \ldots,$
3 4 8	3 4 8	3 4 8	3 4 8	11 24 28	$B11+i,$
1 12 23	1 12 23	1 12 23	1 12 23	1 12 23	$i=0, \ldots, 15$

The 176 blocks are partitioned into 16 orbits under the automorphism of order 11. Clearly, in order this automorphism to preserve a resoltuion there have to be at least 5 orbits of triples being also parallel calsses.

There are precisely 84 non-isomorphic cyclic STS(33) [3], and only 40 of those turned out to possess the last property. Further analysis shows that only 19 of those 40 cyclic STS(33) admit resolutions invariant under an automorphism of order 11, producing 28 non-isomorphic Kirkman triple systems.

2. The Kirkman systems

The 19 cyclic STS(33) which possess resolutions invariant under an automorphism of order 11 all have full automorphism group of order 33. Therefore the equivalence classes of Kirkman triple systems derived from a given cyclic STS(33) are obtained by the action of the automorphism of order 3 being the 11th power of the cyclic permutation of order 33 on the set of resolutions.

The 28 Kirkman systems thus obtained are listed in Table 1 together with the Ball and Kageyama systems. After the number of an KTS in Table 1 we give in parentheses the number of the corresponding cyclic STS from [3].

Remark. The order 33 is the smallest but one for which the existence of a doubly-resolvable STS is unknown [8]. We have checked by computer that none of the KTS listed in Table 1 possesses a second resolution orthogonal to the given one.

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