ELSEVIER
Discrete Mathematics 260 (2003) 163–176
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

New types of integral sum graphs ☆

Jianqiang Wu, Jingzhong Mao*, Deying Li
Department of Mathematics, Central China Normal University, Wuhan, Hubei 430079, China

Received 7 April 1998; received in revised form 12 June 2001; accepted 10 January 2002

Abstract

Harary (Discrete Math. 124(1–3) (1994) 99) introduced the intergral sum graphs, and proved that the path P_n and matching mK_2 are integer sum graphs, and offered some conjectures. In this paper we enlarge the types of integral sum graphs. We prove that some caterpillars and mP_n are integral sum graphs and some conjectures of Harary are wrong. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Integral sum graph; Caterpillar; Comb-graph

1. Introduction

In general, we follow the graph-theoretic notation and terminology of [1].

The sum graph $G^+(S)$ of a finite subset $S \subset N = \{1, 2, 3, \dots,\}$ is the graph (V, E), where $V = S$ and $uv \in E$ if and only if $u + v \in S$. Then a *sum graph* G is isomorphic to the sum graph of some $S \subset N$. For each G, there is a minimum number $\sigma = \sigma(G)$ such that $G \cup \sigma K_1$ is a sum graph. This number $\sigma(G)$ is the sum number of G [2].

Following customary notation, write the set of all integers as

$$Z = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}.$$

The integral sum graph $G^+(S)$ is defined just as the sum graph, the difference being that $S \subset Z$ instead of $S \subset N$. The integral sum number $\zeta(G)$ is the smallest nonnegative integer s such that $G \cup sK_1$ is isomorphic to $G^+(S)$ for some $S \subset Z$, i.e., is an integral sum graph, written $\int \sum -$graph.

Obviously $\zeta(G) \leqslant \sigma(G)$ for all graphs G.

☆ Project 19871034 supported by NSFC.
* Corresponding author.
E-mail address: jzmao@mail.ccnu.edu.cn (J. Mao).

PII: S0012-365X(02)00456-9

Harary has pointed out that, for each positive integers n, the path P_n and matching nK_2 are integral sum graphs [2] and offered some unsolved problems and conjectures. In this paper we will enlarge the types of integral sum graphs by proving some caterpillars and $mP_n(m,n\in N)$ to be integral sum graphs, and show that some conjectures of Harary are wrong.

First, it is easy to show that, suppose $G=(V,E)$ is a sum graph over integer set $S=\{b_i \mid 1\leqslant i\leqslant n\}$, then:

(1) G is also a sum graph over integers set $mS=\{mb_i \mid 1\leqslant i\leqslant n\}$, for every integer $m\neq 0$.

(2) For an integer set $S_1=\{a_i \mid 1\leqslant i\leqslant n\}$, if we can induce $a_i+a_j=a_k$ from $b_i+b_j=b_k$, then there is an integer m such that, G is also a sum graph over integer set $T=\{a_i+mb_i \mid 1\leqslant i\leqslant n\}$. Actually, we can take

$$|m| = \max\{|t_{ij}| \, |b_i\pm b_j\neq b_k, 1\leqslant i,j,k\leqslant n\},$$

where

$$|t_{ij}| > \frac{\max\{|a_i+a_j-a_k| \, |1\leqslant k\leqslant n\}}{\min\{|b_i+b_j-b_k| \, |1\leqslant k\leqslant n\}}, \quad 1\leqslant i,\ j\leqslant n.$$

For $G_1=(V_1,E_1), G_2=(V_2,E_2)$, by the union of G_1 and G_2, we mean a graph $G=(V,E)$, where $V=V_1\cup V_2$, $E=E_1\cup E_2$ and $V_1\cap V_2=\emptyset$, we denote it by $G=G_1\cup G_2$. Similarly define $mG=\underbrace{G\cup G\cup\cdots\cup G}_{m\text{ times}}$.

If $G=(V,E)$, $|V|=n\geqslant 2$ and G is a sum graph over an integer set S, then it is easy to see that, $0\in S$ if and only if there is a vertex $x\in V$ with $d_G(x)=n-1$. This vertex is called 0-vertex of G.

Lemma 1. *Suppose G_1 and G_2 are $\int\sum$-graphs without 0-vertex, then $G_1\cup G_2$ is also an $\int\sum$-graph.*

Proof. Suppose G_1,G_2 are $\int\sum$-graphs over integer sets S_1,S_2 respectively. Then, we may relabel the vertices of $G=G_1\cup G_2$ by the integer set $S_1\cup mS_2$, where m is a sufficiently large number, say,

$$m = 10\max\{|a| \, |a\in S_1\}$$

in the following way:

If $v_i\in V_1$, v_i is labeled in the same way as in G_1; if $v_j\in V_2$, v_j is labeled by ma_j, where v_j is labeled by a_j in G_2. We can verify that $G_1\cup G_2$ is also an $\int\sum$-graph directly. □

Example 1. Fig. 1(a) shows that paths P_4 and P_8 are integral sum graphs, and Fig. 1(b) shows $P_4\cup P_8$ is an integral sum graph.

Corollary 2. *Suppose G_i $(i=1,2,\ldots,m)$ are $\int\sum$-graphs without 0-vertex, then $\bigcup_{i=1}^m G_i$ is also an $\int\sum$-graph.*

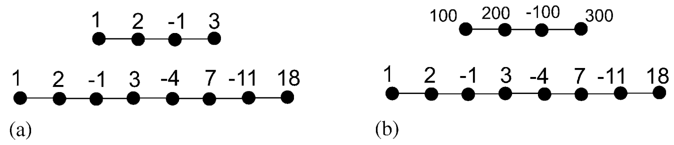


Fig. 1.

Corollary 3. *If G is an $\int\sum$-graph without* 0*-vertex, then mG is also an $\int\sum$-graph for each $m \in N$.*

2. Comb graphs

Recall that given tree T, its pruned tree T' is obtained by removing all the end vertices from T. A tree T is a caterpillar if T' is a path which is called the spine of T. If the spine of caterpillar T is a path P_n with $n \geqslant 1$, and for each vertex $v_i (1 \leqslant i \leqslant n)$ of P_n, there are m_i end-vertices adjacent to v_i, then we denote the caterpillar T by $C_n(m_1, m_2, \ldots, m_n)$. Furthermore, if $m_1 = 1, m_2 = \cdots = m_k = 0$, $m_{k+1} = m_{k+2} = \cdots = m_n = 1$ for some k: $0 \leqslant k < n$, then we call this caterpillar a comb with $n - k$ teeth. Especially, for $n = 1$, caterpillar $C_1(m_1)$ is a star, for $n = 2$, caterpillar $C_2(m_1, m_2)$ is a double star.

Theorem 4. *Every comb is an $\int\sum$-graph.*

Proof. Suppose G is a comb. We denote the longest path of G by $a_1a_2 \cdots a_kb_1b_2 \cdots b_nc_n$, $0 < k < n$, where $d(a_1) = d(c_n) = 1$, $d(b_n) = 2$, $d(b_1) = \cdots = d(b_{n-1}) = 3$ if $n > 1$, and if $k \geqslant 2$, then we have $2 \leqslant d(a_2) \leqslant d(a_3) \leqslant \cdots \leqslant d(a_k) \leqslant 3$. For $1 \leqslant j \leqslant n$ we let b_j adjacent to c_j and we may think c_j as the teeth of the comb. We may take $L = k + n - 1$ as the length of the comb.

For the case $L \leqslant 5$, all combs are shown in Fig. 2. Their labels illustrate that all combs with length $L \leqslant 5$ are integral sum graphs.

For the case $L \geqslant 6$ we divide the discussions into four cases as follows: (In the following we use the notation $l(x)$ to denote the label of vertex x.)

Case 1: $k \geqslant 4$, and $d(a_2) = d(a_3) = \cdots = d(a_k) = 2$.

Label the vertices of the comb as follows:

$$l(a_1) = 4, \quad l(a_2) = 1,$$

$$l(a_i) = l(a_{i-2}) - l(a_{i-1}), \quad 3 \leqslant i \leqslant k,$$

$$l(b_1) = l(a_{k-1}) - l(a_k),$$

$$l(c_1) = l(a_k) - l(b_1),$$

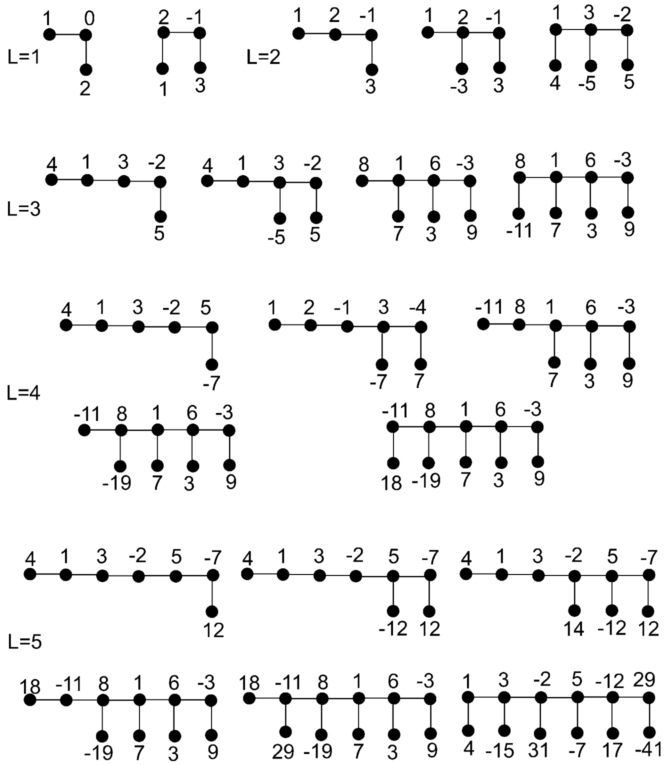


Fig. 2.

$$l(b_{j+1}) = l(c_j) - l(b_j), \quad 1 \leqslant j \leqslant n-1,$$

$$l(c_{j+1}) = l(b_j) - l(b_{j+1}), \quad 1 \leqslant j \leqslant n-1. \tag{1}$$

Thus we get a sequence

$$S = (l(a_1), l(a_2), \ldots, l(a_k), l(b_1), l(c_1), l(b_2), l(c_2), \ldots, l(b_{n-1}), l(c_{n-1}), l(b_n), l(c_n))$$

with the properties:

(1) For $1 \leqslant i \leqslant n$, $l(b_i) \cdot l(c_i) < 0$,
(2) For $1 \leqslant i \leqslant n-1$, $l(b_i) \cdot l(b_{i+1}) < 0$, $l(c_i) \cdot l(c_{i+1}) < 0$.
(3) The sequence

$$(|l(a_1)|, |l(a_2)|, \ldots, |l(a_k)|, |l(b_1)|, |l(c_1)|, \ldots, |l(b_{n-1})|, |l(c_{n-1})|, |l(b_n)|, |l(c_n)|)$$

is strictly increasing from the fourth term.

These properties can be proved by mathematical induction easily.

Now we claim that the comb is an $\int\sum$-graph over S by mathematical induction.

Indeed, letting $S_1=\{l(a_1),l(a_2),\ldots,l(a_k),l(b_1),l(c_1)\}$, we can prove that $G[\{a_1,a_2,\ldots,a_k,b_1,c_1\}]$ is an $\int\sum$-graph over S_1 by arguments similar to that in [2] (cf. [2, Theorem 3.1]). By mathematical induction, suppose $G[\{a_1,a_2,\ldots,a_k,b_1,c_1,\ldots,b_m,c_m\}]$ is an $\int\sum$-graph over integer set $S_2=\{l(a_1),l(a_2),\ldots,l(a_k),l(b_1),l(c_1),\ldots,l(b_m),l(c_m)\}$. From property (3) of S we know $l(b_{m+1})$ is a number with the largest absolute value in the set $S_2\cup\{l(b_{m+1})\}$. Therefore, for any $x\in S_2$, if $x\cdot l(b_{m+1})>0$, then $|x+l(b_{m+1})|=|x|+|l(b_{m+1})|>|l(b_{m+1})|$; if $x\cdot l(b_{m+1})<0$ and $x\neq l(b_m)$, then $|x|<|l(b_m)|$, thus we obtain:

$|l(b_{m+1})|>|x+l(b_{m+1})|=|l(b_{m+1})|-|x|>|l(b_{m+1})|-|l(b_m)|=|l(b_{m+1})+l(b_m)|=|l(c_m)|$; If $x=l(b_m)$, then $x+l(b_{m+1})=l(b_{m+1})+l(b_m)=l(c_m)$. This means $x+l(b_{m+1})\in S_2\cup\{l(b_{m+1})\}$ and $x\in S_2$ if and only if $x=l(b_m)$.

Take any two numbers x and y from S_2, if at least one of x and y is not $l(b_m)$ or $l(c_m)$, then $|x+y|\leqslant|x|+|y|<|l(c_m)|+|l(b_m)|=|l(b_{m+1})|$. By the induction assumption, if $x+y\notin S_2$, then $x+y\notin S_2\cup\{l(b_{m+1})\}$. Thus, we have that $G[\{b_{m+1}\}\cup\{a_1,a_2,\ldots,a_k,b_1,c_1,\ldots,b_m,c_m\}]$ is an $\int\sum$-graph over the set $S_2\cup\{l(b_{m+1})\}$.

Now we prove $G[\{a_1,a_2,\ldots,a_k,b_1,c_1,\ldots,b_m,c_m\}\cup\{b_{m+1},c_{m+1}\}]$ is an $\int\sum$-graph over the set $S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$. From property (3) of S we know that $l(c_{m+1})$ is a number with the largest absolute value in the set $S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$. Therefore, for any $x\in S_2\cup\{l(b_{m+1})\}$, if $x\cdot l(c_{m+1})>0$, then: $|x+l(c_{m+1})|=|x|+|l(c_{m+1})|>|l(c_{m+1})|$; if $x\cdot l(c_{m+1})<0$ and $x\neq l(b_{m+1})$, then $|x|<|l(b_{m+1})|$. By properties (1)–(3) of S we can induce that $x+l(c_{m+1})$, $l(c_{m+1})$ and $l(b_m)$ have the same sign, and the numbers in $S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$ with the absolute value between $|l(b_m)|$ and $|l(c_{m+1})|$ are $l(c_m)$ and $l(b_{m+1})$, but, the signs of $l(c_m)$ and $l(b_{m+1})$ are different from the signs of $x+l(c_{m+1})$, $l(c_{m+1})$ and $l(b_m)$. Therefore $x+l(c_{m+1})\notin S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$. Thus, $x\in S_2\cup\{l(b_{m+1})\}$ and $x+l(c_{m+1})\in S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$ if and only if $x=l(b_{m+1})$.

Suppose $x,y\in S_2\cup\{l(b_{m+1})\}$, because $|x|<|l(c_{m+1})|$, $|y|<|l(c_{m+1})|$, it follows that if $x+y=l(c_{m+1})$, then, the signs of x, y must be the same with the sign of $l(c_{m+1})$, but in this case we have

$$|x+y|\leqslant|x|+|y|<|l(b_m)|+|l(b_{m+1})|=|-l(b_{m+1})+l(b_m)|=|l(c_{m+1})|$$

Therefore $x+y\neq l(c_{m+1})$.

It follows that, $G[\{b_{m+1},c_{m+1}\}\cup\{a_1,a_2,\ldots,a_k,b_1,c_1,\ldots,b_m,c_m\}]$ is an $\int\sum$-graph over the integer set $S_2\cup\{l(b_{m+1}),l(c_{m+1})\}$. By mathematical induction we know G is an $\int\sum$-graph over integer set S.

Case 2: $k=4$ and $d(a_2)=d(a_3)=2$, $d(a_4)=3$. Let a_4 be adjacent to a tooth denoted by c_{n+1}.

Let $S=\{l(a_1),l(a_2),l(a_3),l(a_4),l(b_1),l(c_1),l(b_2),l(c_2),\ldots,l(b_n),l(c_n)\}$, the values of the terms in S are given by formula (1). We take the value of $l(c_{n+1})$ as follows:

$$l(c_{n+1})=\begin{cases} l(c_n)+2 & \text{if } l(c_n)>0,\\ l(b_n)+2 & \text{if } l(c_n)<0. \end{cases} \tag{2}$$

It is straightforward that c_{n+1} is only adjacent to a_4 and $x+y\neq l(c_{n+1})$ for all x, $y\in S$. Therefore, in this case we get that G is an $\int\sum$-graph over the set $S\cup\{l(c_{n+1})\}$.

Case 3: $k=4$ and $d(a_2)=2$, $d(a_3)=d(a_4)=3$. We denote the teeth adjacent to a_3 and a_4 by c_{n+2} and c_{n+1}, respectively.

Let

$$S=\{l(a_1),l(a_2),l(a_3),l(a_4),l(b_1),l(c_1),l(b_2),l(c_2),\dots,l(b_n),l(c_n)\},$$

their values are given by formula (1). Then, we take the value of $l(c_{n+1})$ according to formula (2), and the value of $l(c_{n+2})$ as follows:

$$l(c_{n+2})=\begin{cases} l(b_n)-3 & \text{if } l(c_n)>0,\\ l(c_n)-3 & \text{if } l(c_n)<0. \end{cases} \tag{3}$$

From property (3) of the sequence

$$S=(l(a_1),l(a_2),\dots,l(a_k),l(b_1),l(c_1),\dots,l(b_{n-1}),l(c_{n-1}),l(b_n),l(c_n))$$

we know that we cannot find out two elements x and y of S satisfying the condition that, $x+y=l(c_{n+1})$ or $l(c_{n+2})$. From formulas (2) and (3) we know, if $l(c_n)>0$, then we have $l(c_{n+1})+l(a_4)=l(c_{n+1})-2=l(c_n)$, $l(c_{n+2})+l(a_3)=l(c_{n+2})+3=l(b_n)$. These tell us that c_{n+1} and c_{n+2} are only adjacent to a_4 and a_3, respectively. If $l(c_n)<0$, then we have the similar results. Therefore, in this case we still get that G is an $\int\sum$-graph over the set $S\cup\{l(c_{n+1}),l(c_{n+2})\}$.

Case 4: $k=4$ and $d(a_2)=d(a_3)=d(a_4)=3$. We denote the teeth adjacent to a_2, a_3 and a_4 by c_{n+3}, c_{n+2}, c_{n+1}, respectively. Let

$$S=\{l(a_1),l(a_2),l(a_3),l(a_4),l(b_1),l(c_1),l(b_2),l(c_2),\dots,l(b_n),l(c_n)\}.$$

Their values are given by formula (1). Then, we take the value of $l(c_{n+1})$ according to formula (2), the value of $l(c_{n+2})$ according to formula (3), and the value of $l(c_{n+3})$ as follows:

$$l(c_{n+3})=\begin{cases} l(b_n)-1 & \text{if } l(c_n)>0;\\ l(c_n)-1 & \text{if } l(c_n)<0. \end{cases} \tag{4}$$

First, we notice that, from $L\geqslant 6$ and $k=4$ we know $n=L+1-k\geqslant 3$. Then it is completely similar to case 3. We can prove that c_{n+1},c_{n+2}, and c_{n+3} are only adjacent to a_4,a_3 and a_2, respectively. Therefore, in this case we still have that G is a $\int\sum$-graph over the set $S\cup\{l(c_{n+1}),l(c_{n+2}),l(c_{n+3})\}$.

Since in case 3 the vertex a_1 can be considered to be a tooth, we have proved that a comb is an $\int\sum$-graph in any case. □

Example 2. In Fig. 3 we give the comb with (a) $k=5$, $n=5$; (b) $k=4$, $n=5$ and there are 6 teeth; (c) $k=4$, $n=6$ and there are 8 teeth.

Suppose we have a caterpillar T, its spine T' is a path denoted by

$$P=a_{1,1}a_{1,2}\cdots a_{1,k_1}b_{1,1}b_{1,2}\cdots b_{1,n_1}a_{2,1}a_{2,2}\cdots a_{2,k_2}b_{2,1}b_{2,2}\cdots b_{2,n_2}\cdots$$
$$a_{m,1}a_{m,2}\cdots a_{m,k_m}b_{m,1}b_{m,2}\cdots b_{m,n_m}, \tag{5}$$

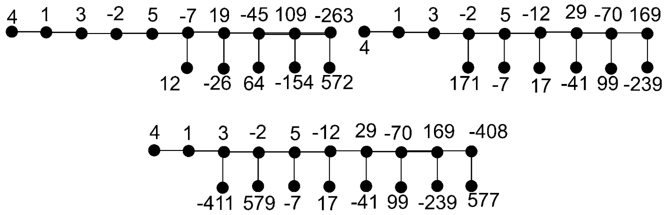


Fig. 3.

where each $b_{i,j}$ $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n_i)$ is only adjacent to one end-vertex $c_{i,j}$, and each $a_{i,j}$ $(1 \leqslant j \leqslant k_i,\ 1 \leqslant i \leqslant m)$ is of degree two except vertex $a_{1,1}$ $(d(a_{1,1}) = 1)$. Then we call this tree T as a quasi-comb. By cutting it at vertices $b_{1,n_1}, b_{2,n_2}, \ldots, b_{m-1,n_{m-1}}$, we get m combs, each of them has vertex set

$$\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}, b_{i,1}, b_{i,2}, \ldots, b_{i,n_i}, c_{i,1}, c_{i,2}, \ldots, c_{i,n_i}\} = S_i, \quad 1 \leqslant i \leqslant m.$$

Then we call the subgraph generated by the set S_i the ith comb of original quasi-comb. We have

Theorem 5. *If the spine (path) P of quasi-comb T can be written as in (5) with $k_1 \geqslant 4$, then T is an $\int \sum$-graph.*

Proof. We only give a method to label its vertices.

For the first comb with vertex set

$$S_1 = \{a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, b_{1,1}, c_{1,1}, b_{1,2}, c_{1,2}, \ldots, b_{1,n_1}, c_{1,n_1}\}$$

the labels are given by formula (1) in Theorem 4.

Generally, suppose we have labeled the jth comb with vertex set $S_j = \{a_{j,1}, a_{j,2}, \ldots, a_{j,k_j}, b_{j,1}, c_{j,1}, b_{j,2}, c_{j,2}, \ldots, b_{j,n_j}, c_{j,n_j}\}$, then the $(j+1)$th comb with vertex set

$$S_{j+1} = \{a_{j+1,1}, a_{j+1,2}, \ldots, a_{j+1,k_{j+1}}, b_{j+1,1}, c_{j+1,1}, b_{j+1,2}, c_{j+1,2}, \ldots, b_{j+1,n_{j+1}}, c_{j+1,n_{j+1}}\}$$

is labeled as follows:

$$l(a_{j+1,1}) = l(c_{j,n_j}) - l(b_{j,n_j}),$$

$$l(a_{j+1,2}) = l(b_{j,n_j}) - l(a_{j+1,1}),$$

$$l(a_{j+1,t}) = l(a_{j+1,t-2}) - l(a_{j+1,t-1}), \quad 3 \leqslant t \leqslant k_{j+1},$$

$$l(b_{j+1,1}) = l(a_{j+1,k_{j+1}-1}) - l(a_{j+1,k_{j+1}}), \quad \text{if } k_{j+1} \geqslant 2,$$

$$l(b_{j+1,1}) = l(b_{j,n_j}) - l(a_{j+1,1}), \quad \text{if } k_{j+1} = 1,$$

$$l(c_{j+1,1}) = l(a_{j+1,k_{j+1}}) - l(b_{j+1,1}),$$

$$l(b_{j+1,\lambda}) = l(c_{j+1,\lambda-1}) - l(b_{j+1,\lambda-1}), \quad 2 \leqslant \lambda \leqslant n_{j+1},$$

$$l(c_{j+1,\lambda}) = l(b_{j+1,\lambda-1}) - l(b_{j+1,\lambda}), \quad 2 \leqslant \lambda \leqslant n_{j+1}. \tag{6}$$

Since the vertex set S_1 is labeled by formula (1), we have the sequence

$$LS_1 = (l(a_{1,1}), l(a_{1,2}), \ldots, l(a_{1,k_1}), l(b_{1,1}), l(c_{1,1}), l(b_{1,2}),$$
$$l(c_{1,2}), \ldots, l(b_{1,n_1}), l(c_{1,n_1}))$$

satisfies:

(1) for $1 \leqslant i \leqslant n_1 - 1$ we have $l(b_{1,i}) \cdot l(b_{1,i+1}) < 0$, $l(c_{1,i}) \cdot l(c_{1,i+1}) < 0$;
(2) for $1 \leqslant i \leqslant n_1$ we have $l(b_{1,i}) \cdot l(c_{1,i}) < 0$;
(3) the sequence

$$(|l(a_{1,1})|, |l(a_{1,2})|, \ldots, |l(a_{1,k_1})|, |l(b_{1,1})|, |l(c_{1,1})|, |l(b_{1,2})|,$$
$$|l(c_{1,2})|, \ldots, |l(b_{1,n_1})|, |l(c_{1,n_1})|)$$

is strictly increasing from the fourth term.

Now we consider the vertex set S_2 which is labeled by formula (6). By using mathematical induction we may obtain that:

(1) for $1 \leqslant j \leqslant k_2 - 1$ we have

$$l(a_{2,j}) \cdot l(a_{2,j+1}) < 0, \quad l(a_{2,1}) \cdot l(b_{1,n_1}) < 0;$$

(2) for $1 \leqslant j \leqslant n_2 - 1$ we have

$$l(b_{2,j}) \cdot l(b_{2,j+1}) < 0, \quad l(c_{2,j}) \cdot l(c_{2,j+1}) < 0;$$

(3) for $1 \leqslant j \leqslant n_2$ we have

$$l(b_{2,j}) \cdot l(c_{2,j}) < 0, \quad l(a_{2,k_2}) \cdot l(b_{2,1}) < 0;$$

(4) $|l(c_{1,n_1})| < |l(a_{2,1})| < |l(a_{2,2})| < \cdots < |l(a_{2,k_2})| < |l(b_{2,1})| < |l(c_{2,1})| < |l(b_{2,2})| < |l(c_{2,2})| < \cdots < |l(b_{2,n_2})| < |l(c_{2,n_2})|$.

From these facts, by almost the same process, we can prove that, the quasi-comb T is an $\int\sum$-graph. □

Example 3. Fig. 4 shows that a quasi-comb is an $\int\sum$-graph.

3. Caterpillar

Now we will prove that some of the caterpillars are integral sum graphs.
First, we give the following result.

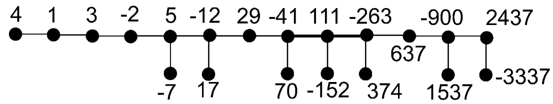


Fig. 4.

Lemma 6. *For all positive integers n, the star $S_n = K_{1,n}$ is an $\int\sum$-graph.*

Proof. Denote the center-vertex of star S_n by v_0, the end-vertices by $v_1, v_2, \ldots, v_n$. Then let $l(v_0)=0$, $l(v_i)=2i-1$, $1 \leqslant i \leqslant n$. It is straightforward to verify that it is an $\int\sum$-graph (see Fig. 5). □

Lemma 7. *For all $m, n \in N$, the double-star $S(m,n)$ is an $\int\sum$-graph.*

Proof. Suppose two centers of a double star $S(m,n)$ are vertices v_1, v_2, and there are m end-vertices $a_1, a_2, \ldots, a_m$ adjacent to v_1, and n end-vertices $b_1, b_2, \ldots, b_n$ adjacent to v_2. Now label these vertices as follows:

Let

$$l(v_1)=1, \quad l(a_i)=-(m+i+1), \ 1 \leqslant i \leqslant m,$$

$$l(v_2)=-(m+1), \quad l(b_1)=-m, \ l(b_j)=(j-1)m+j, \ 2 \leqslant j \leqslant n. \tag{7}$$

It is straightforward to verify that $S(m,n)$ is an $\int\sum$-graph by mathematical induction. □

Example 3. Fig. 6 shows that double stars $S(4,5)$ and $S(3,8)$ are $\int\sum$-graphs.

Remark (Harary [2]). Harary mentioned that neither $S(1,3)$ nor $S(2,2)$ is an $\int\sum$-graph. But, this conclusion is not correct. Indeed, we have proved that every $S(m,n)$ is an $\int\sum$-graph. Fig. 7 shows that $S(1,3)$ and $S(2,2)$ are $\int\sum$-graphs.

Theorem 8. *Let $n_1, n_2, \ldots, n_m \in N$ with $m \geqslant 3$, $m \in N$. Suppose that, either $n_2 \geqslant 3$, or $n_2 = 2$ and $n_1 = 1$ or 2, then caterpillar $C_m(n_1, n_2, \ldots, n_m)$ is an $\int\sum$-graph.*

Proof. Denote the spine of caterpillar $C_m(n_1, n_2, \ldots, n_m)$ by $P_m = v_1 v_2 \cdots v_m$, the end-vertices adjacent to v_i are denoted by $a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}$, and we call the subgraph generated by the vertex subset

$$S_i = \{v_i, a_{i,1}, a_{i,2}, \ldots, a_{i,n_i}\}$$

as ith star for $1 \leqslant i \leqslant m$. Now we label the vertices of $\bigcup_{i=1}^{m} S_i$ as follows:

For $i = 1, 2$,

$$l(v_1)=1, \quad l(a_{1,j})=-(n_1+j+1), \ 1 \leqslant j \leqslant n_1,$$

$$l(v_2)=-(n_1+1), \quad l(a_{2,1})=-n_1,$$

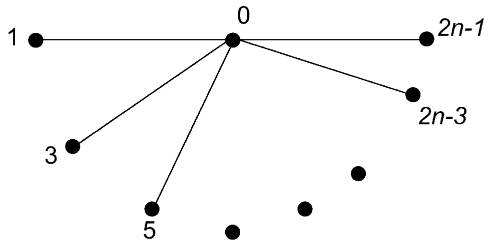


Fig. 5.

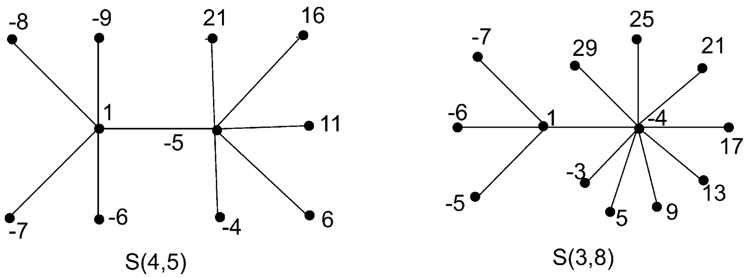


Fig. 6.

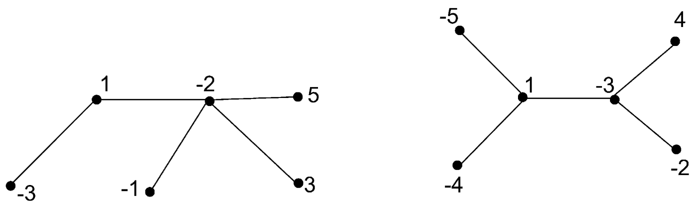


Fig. 7.

$$l(a_{2,j}) = (j-1)n_1 + j, \quad 2 \leqslant j \leqslant n_2. \tag{8}$$

Generally, suppose the sets S_k have been labeled for $k \geqslant 2$, and the number with largest absolute value of the labels of S_k is p, then

$$l(v_{k+1}) = p - l(v_k), \quad l(a_{k+1,j}) = -j(p - l(v_k)) + l(v_k), \quad 1 \leqslant j \leqslant n_{k+1}. \tag{9}$$

According to this formula we can successively label all vertices of the caterpillar $C_m(n_1, n_2, \ldots, n_m)$. Now we are going to prove that it is an $\int\sum$-graph. We suppose $m \geqslant 3$ (for $m = 1, 2$ see Lemmas 6 and 7).

In the following, we use the same symbol S_i to represent the labels set of the vertices of S_i.

First, it is easy to verify that, according to formula (7), each $a_{1,j}$ is only adjacent to v_1 for $1 \leqslant j \leqslant n_1$, each $a_{2,j}$ is only adjacent to v_2 for $1 \leqslant j \leqslant n_2$ and v_1 is adjacent to

v_2. For the subgraph generated by $S_1 \cup S_2$, there is no other pair of adjacent vertices.

Now, consider the subgraph generated by $S_1 \cup S_2 \cup S_3$. Since $l(v_3) + l(v_2) = p \in S_2$, v_2 is adjacent to v_3.

For $2 \leqslant j \leqslant n_3$, since $l(a_{3,j}) + l(v_3) = [-j(p - l(v_2)) + l(v_2)] + (p - l(v_2)) = -(j-1)(p - l(v_2)) + l(v_2) \in S_2 \cup S_3$ (formula (8)), it follows that $a_{3,j}$ is adjacent to v_3.

From $l(a_{3,1}) + l(v_3) = -(p - l(v_2)) + l(v_2) + p - l(v_2) = l(v_2)$, we obtain that $a_{3,1}$ is adjacent to v_2.

If $n_2 \geqslant 3$, then the largest absolute value of the labels of S_2 is $p = (n_2 - 1)n_1 + n_2$. Therefore, for $1 \leqslant j \leqslant n_2$, we have $l(a_{2,j}) + l(v_3) = (j-1)n_1 + j + p - l(v_2) = (j-1)n_1 + j + (n_2 - 1)n_1 + n_2 - (-(n_1+1)) = (n_2 + j - 1)(n_1 + 1) \notin S_1 \cup S_2 \cup S_3$. It follows that $a_{2,j}$ is not adjacent to v_3.

If $n_2 = 2$ and $n_1 = 1$ or 2, in a similar way, we can induce that $a_{2,1}$ is not adjacent to v_3. Similarly, we can show that, $a_{1,j}$ is not adjacent to v_3 for $1 \leqslant j \leqslant n_1$.

Next assume $1 \leqslant j_1 < j_2 \leqslant n_3$. Since $l(a_{3,j_1}) + l(a_{3,j_2}) = [-j_1(p - l(v_2)) + l(v_2)] + [-j_2(p - l(v_2)) + l(v_2)] = -(j_1 + j_2)(p - l(v_2)) + 2l(v_2) \notin S_1 \cup S_2 \cup S_3$, thus a_{3,j_1} is not adjacent to a_{3,j_2}.

Suppose $n_3 \geqslant 3$, for $1 \leqslant j_1 \leqslant n_1$, $1 \leqslant j_2 \leqslant n_3$. Since $l(a_{1,j_1}) + l(a_{3,j_2}) = -(n_1 + j_1 + 1) - j_2(p - l(v_2)) + l(v_2) = -n_1 - j_1 - 1 - j_2[(n_2 - 1)n_1 + n_2] + j_2(-n_1 - 1) - n_1 - 1 = -(j_2 n_2 + 2)n_1 - (j_2 n_2 + 2) - (j_1 + j_2)$, it is easy to show that $l(a_{1,j_1}) + l(a_{3,j_2}) \notin S_1 \cup S_2 \cup S_3$. Similarly, we can prove that $l(a_{2,j_1}) + l(a_{3,j_2}) \notin S_1 \cup S_2 \cup S_3$ for $1 \leqslant j_1 \leqslant n_2$, $1 \leqslant j_2 \leqslant n_3$, and $l(a_{1,j_1}) + l(a_{2,j_2}) \notin S_1 \cup S_2 \cup S_3$ for $1 \leqslant j_1 \leqslant n_1$, $1 \leqslant j_2 \leqslant n_2$. It follows that, the subgraph generated by $S_1 \cup S_2 \cup S_3$ is surely a caterpillar.

Now we turn to the case $k \geqslant 3$. Suppose we have proven that the subgraph generated by $\bigcup_{i=1}^{k} S_i$ is a caterpillar $C_k(n_1, n_2, \ldots, n_k)$. We consider the subgraph generated by $\bigcup_{i=1}^{k} S_i \cup S_{k+1}$.

First we note that: (1) for each i: $1 \leqslant i \leqslant k+1$, $i \neq 2$, $1 \leqslant j \leqslant n_i$, we have $l(v_i) \cdot l(a_{i,j}) < 0$;

(2) Now consider the sequence

$$(l(v_1), l(a_{1,1}), \ldots, l(a_{1,n_1}), l(v_2), l(a_{2,1}), l(a_{2,2}), \ldots, l(a_{2,n_2}), l(v_3),$$
$$l(a_{3,1}), l(a_{3,2}), \ldots, l(v_{k+1}), l(a_{k+1,1}), \ldots, l(a_{k+1,n_{k+1}})).$$

We can find out the absolute value of terms which are strictly increasing from the $(n_1 + 4)$th term, i.e. $l(a_{2,2})$. Notice that, p is the number with the largest absolute value in S_k.

(i) For $1 \leqslant j \leqslant n_{k+1}$, $l(a_{k+1,j}) + l(v_{k+1}) = [-j(p - l(v_k)) + l(v_k)] + (p - l(v_k)) = -(j-1)(p - l(v_k)) + l(v_k) \in S_k \cup S_{k+1}$, so we have $a_{k+1,j}$ is only adjacent to v_{k+1}. Indeed, for any other $x \in \bigcup_{i=1}^{k+1} S_i$, since $|x| < |l(v_{k+1})|$, we have

$$\begin{aligned} |x + l(a_{k+1,j})| &\leqslant |x| + |l(a_{k+1,j})| = |x| + |-j(p - l(v_k)) + l(v_k)| \\ &< |l(v_{k+1})| + |-j(p - l(v_k)) + l(v_k)| \\ &= |-(j+1)(p - l(v_k)) + l(v_k)| \end{aligned}$$

and

$$|x + l(a_{k+1,j})| \geqslant |l(a_{k+1,j})| - |x| > |l(a_{k+1,j})| - |l(v_{k+1})|$$
$$= | - (j-1)(p - l(v_k)) + l(v_k)|$$

this means $x + l(a_{k+1,j}) \notin \bigcup_{i=1}^{k+1} S_i$.

(ii) Take any two elements x and y from $\bigcup_{i=1}^{k} S_i \cup \{l(v_{k+1})\}$. Suppose $x \cdot l(v_{k+1}) > 0$ and $y \cdot l(v_{k+1}) > 0$, then $(x+y) \cdot l(a_{k+1,j}) < 0$, for $1 \leqslant j \leqslant n_{k+1}$, it follows that $x + y \neq l(a_{k+1,j})$, $1 \leqslant j \leqslant n_{k+1}$. Suppose $x \cdot l(v_{k+1}) < 0$. If $x \cdot y > 0$, then $|x+y| = |x| + |y| < |l(v_k)| + |l(v_{k+1})| = |l(v_k) - l(v_{k+1})|$; If $x \cdot y < 0$, then $|x+y| < |x| + |y| \leqslant |l(v_k)| + |l(v_{k+1})| = |l(v_k) - l(v_{k+1})|$, it follows that $x + y \notin S_{k+1}$ for all $x, y \in \bigcup_{i=1}^{k} S_i$.

If $x \in \bigcup_{i=1}^{k} S_i$, $y = l(v_{k+1})$, then from the fact that $x \cdot l(v_{k+1}) < 0$ we know $|x + l(v_{k+1})| < |l(v_{k+1})|$, from the fact that $x \cdot l(v_{k+1}) > 0$ we know $|x + l(v_{k+1})| > |l(v_{k+1})|$, thus we have $x + l(v_{k+1}) \notin S_{k+1}$.

Therefore, if the subgraph generated by $\bigcup_{i=1}^{k} S_i$ is a caterpillar $C_k(n_1, n_2, \ldots, n_k)$, then the subgraph generated by $\bigcup_{i=1}^{k+1} S_i$ is also a caterpillar $C_{k+1}(n_1, n_2, \ldots, n_{k+1})$. By mathematical induction, we obtain that, for the caterpillar $C_m(n_1, n_2, \ldots, n_m)$ with $m \geqslant 3$ either $n_2 \geqslant 3$, or $n_2 = 2$ and $n_1 = 1$ or 2, then it is an $\int\sum$-graph. □

4. A generalization of matching

Harary pointed out [2] that the matching $mK_2 (m \in N)$ is an $\int\sum$-graph. Now we generalize this result.

Theorem 9. *For all $m, n \in N$, mP_n is an $\int\sum$-graph.*

Proof. Consider the following cases:

Case 1: $n = 1$. mP_n is m isolated vertices which can be labeled by the numbers $1, 3, 5, \ldots, 2m-1$. It is easy to see that mP_1 is an $\int\sum$-graph over set $\{1, 3, 5, \ldots, 2m-1\}$.

Case 2: $n = 2$. $mP_n = mK_2$ see [2].

Case 3: $n = 3$.

Subcase 3.1: $m = 2k, k \in N$. It is easy to show that $2P_3$ is an $\int\sum$-graph without 0-vertex, its label can be as in Fig. 9. Then by Lemma 1 and Corollary 3, we may get the label of mP_3 by multiplying the labels of $2P_3$ by the numbers: $10, 10^2, \ldots, 10^{k-1}$. It is easy to show that $mP_3 = 2kP_3$ is an $\int\sum$-graph over this number set. For example, we can label $8P_3$ as in Fig. 8.

Subcase 3.2: $m = 2k+1$, $k \in N$. First we label the vertices of $2kP_3$ as described in the subcase 3.1, then, label the vertices of last P_3 as in Fig. 10, where t is a sufficient large natural number and $t \geqslant k$. It is straightforward to verify that $mP_3 = (2k+1)P_3$ is an $\int\sum$-graph.

Case 4: $n \geqslant 4$. First we verify that P_n is an $\int\sum$-graph [2] without 0-vertex, then mP_n is an $\int\sum$-graph by Lemma 1 and Corollary 3. □

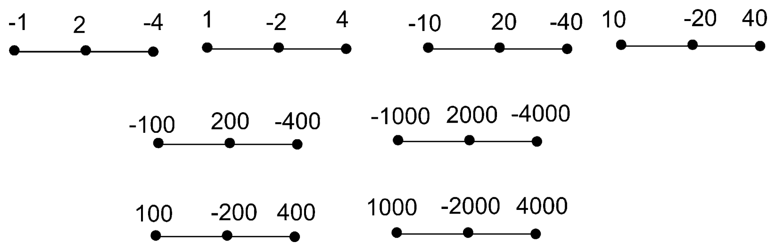


Fig. 8.

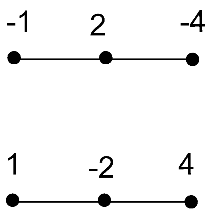


Fig. 9.

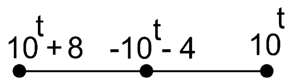
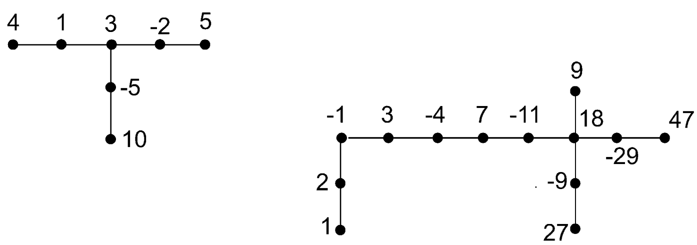


Fig. 10.

5. Remarks

1. In [2], Harary offered a conjecture that every tree T with $\zeta(T)=0$ is a caterpillar. But this is not true. The counter-examples are given in Fig. 11.
   All these graphs are not caterpillars, but $\zeta(T)=0$.
2. Harary also gave another conjecture that for n-cycle C_n, which satisfies the equality $\zeta(C_n)=\sigma(C_n)$ iff $n\neq 3,\ 5$ [2]. But this is also not true. Here are counter-examples in Fig. 12.
   From Fig. 12, we know $\zeta(C_7)\leqslant 1$, but $\sigma(C_7)=2$, therefore $\zeta(C_7)<\sigma(C_7)$. From Fig. 12, we know $\zeta(C_8)\leqslant 1$, but $\sigma(C_8)=2$, therefore $\zeta(C_8)<\sigma(C_8)$. In general we have

Theorem 10. *If* $n\in N$, $n\geqslant 3$ *and* $n\neq 4,6$, *then* $\zeta(C_n)\leqslant 1$.



(a) (b)

Fig. 11.

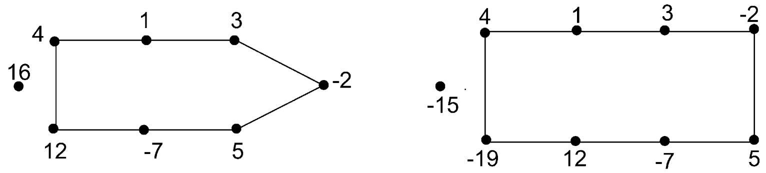


Fig. 12.

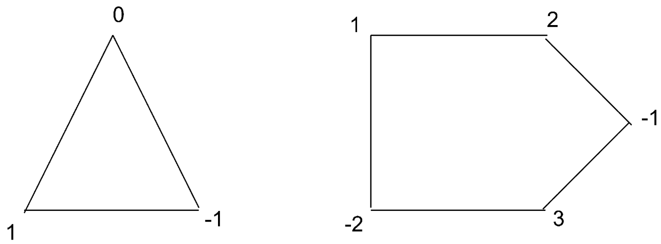


Fig. 13.

Actually, suppose the vertices of C_n successively are $a_1, a_2, \ldots, a_n$ if $n \geqslant 7$, then we let $l(a_1) = 4$, $l(a_2) = 1$, $l(a_k) = l(a_{k-2}) - l(a_{k-1})$, $3 \leqslant k \leqslant n$. Now it is easy to verify that, $C_n \cup K_1$ is an $\int \sum$-graph over the set $S = \{l(a_1), l(a_2), \ldots, l(a_n), l(a_n) + l(a_1)\}$. For $n = 3$, 5 we have $\zeta(C_3) = \zeta(C_5) = 0$, see Fig. 13.

Acknowledgements

The authors wish to thank the referee for useful comments and suggestions that have allowed them to improve the presentation of this paper.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan Press, New York, 1976.
[2] F. Harary, Sum graphs over all the integers, Discrete Math. 124(1–3) (1994) 99–105.