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Numerical range of linear pencils

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Abstract

Consider a linear pencil $A\lambda + B$, where A and B are $n \times n$ complex matrices. The numerical range of $A\lambda + B$ is defined as

$$W(A\lambda + B) = \left\{\lambda \in \mathbb{C}\colon\ x^*(A\lambda + B)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\right\}.$$

In this paper, we study the geometrical properties of $W(A\lambda + B)$, with emphasis to its boundary. An answer to the problem of the numerical approximation of $W(A\lambda + B)$, when one of the coefficients A and B is Hermitian, is presented. The numerical range of a matrix on an indefinite inner product space is also considered. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $A\lambda + B$ be a *linear pencil*, where A and B are $n \times n$ complex matrices and λ is a complex variable. If $A = I$, then the pencil $A\lambda + B$ is called *monic* and if the matrices A and B are Hermitian, then it is called *selfadjoint*. The study of linear pencils has a long history [1–3], usually in the context of their spectral analysis.

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PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 1 4 5 - 2

A complex number λ_0 is said to be an *eigenvalue* of $A\lambda + B$ if the equation

$$(A\lambda_0 + B)x = 0$$

has a nonzero solution $x_0 \in \mathbb{C}^n$. The vector x_0 is known as an *eigenvector* of $A\lambda + B$ corresponding to the eigenvalue λ_0. The set of all eigenvalues of the linear pencil $A\lambda + B$ is known as the *spectrum* of $A\lambda + B$, namely,

$$\sigma(A\lambda + B) = \{\lambda \in \mathbb{C}\colon \det(A\lambda + B) = 0\}.$$

The spectrum $\sigma(A\lambda + B)$ coincides with the complex plane $\mathbb{C}$ or it contains no more than n points. The multiplicity of λ_0, as a root of the equation $\det(A\lambda + B) = 0$, is called *algebraic multiplicity* of λ_0. The vectors $x_1, x_2, \ldots, x_m$ are said to be *associated* to the eigenvector x_0 if

$$Ax_{j-1} + (A\lambda_0 + B)x_j = 0, \quad j = 1, 2, \ldots, m.$$

The system of vectors $x_0, x_1, x_2, \ldots, x_m$ is known as a *Jordan chain* (*of length* $m + 1$) of $A\lambda + B$ corresponding to the eigenvalue λ_0 and it leads to a solution of the differential equation

$$Au'(t) + Bu(t) = 0.$$

The dimension of the kernel $\mathrm{Ker}(A\lambda_0 + B)$ is called *geometric multiplicity* of λ_0 and it is no greater than the algebraic one. If the geometric multiplicity of λ_0 is equal to the algebraic multiplicity, then the eigenvalue λ_0 is called *semisimple*. In this case, all the corresponding *elementary divisors* (see [3,4] for definitions) are linear and all the corresponding Jordan chains have length 1.

The *numerical range* of the pencil $A\lambda + B$ is defined by

$$W(A\lambda + B) = \left\{\lambda \in \mathbb{C}\colon x^*(A\lambda + B)x = 0 \text{ for some nonzero } x \in \mathbb{C}^n\right\} \tag{1}$$

and it always contains the spectrum $\sigma(A\lambda + B)$. In this paper, we assume that $\sigma(A\lambda + B) \neq \mathbb{C}$, i.e., the linear pencil $A\lambda + B$ is *regular*. The numerical range $W(A\lambda + B)$ in (1) is a generalization of the *classical numerical range* (*field of values*) of an $n \times n$ complex matrix A,

$$F(A) = \left\{x^*Ax \in \mathbb{C}\colon x \in \mathbb{C}^n \text{ with } x^*x = 1\right\}.$$

Indeed, it is obvious that $W(I\lambda - A) = F(A)$. One can find a complete survey of the properties of $F(A)$ in [5].

In Section 2, we study the boundary of the numerical range $W(A\lambda + B)$ in (1), and we investigate the interplay between the geometrical properties of $W(A\lambda + B)$ and the algebraic and analytic properties of the pencil $A\lambda + B$. Moreover, it is obtained that the eigenvalues of $A\lambda + B$ on the boundary of $W(A\lambda + B)$ are semisimple. In Section 3, we consider selfadjoint linear pencils and the real endpoints of their numerical range. In Section 4, we generate the boundary of $W(A\lambda + H + \mathrm{i}S)$, where the matrices A, H and S are Hermitian. Finally, in Section 5, connections are made with the notion of the *Krein space numerical range*.

2. Geometry and boundary

Consider an $n \times n$ linear pencil $A\lambda + B$ ($A \neq 0$) and its numerical range $W(A\lambda + B)$ in (1). Then it is easy to verify the following properties of $W(A\lambda + B)$ [6].

Proposition 1. *Let $A\lambda + B$ be an $n \times n$ linear pencil, where $A \neq 0$.*

(i) *$W(A\lambda + B)$ is a closed subset of $\mathbb{C}$.*

(ii) *For every $\mu \in \mathbb{C}$, $W(A(\lambda + \mu) + B) = W(A\lambda + B) - \mu$.*

(iii) *$W(A\lambda + B)\backslash\{0\} = \{\lambda^{-1} \in \mathbb{C}\colon \lambda \in W(B\lambda + A)\backslash\{0\}\}$.*

(iv) *For every $n \times r$ matrix Q of rank r, with $r \leqslant n$, we have that $W(Q^*(A\lambda + B)Q) \subseteq W(A\lambda + B)$. Equality holds if $r = n$.*

(v) *If the matrices A and B have a nonzero common isotropic vector $x_0 \in \mathbb{C}^n$, i.e., $x_0^* A x_0 = x_0^* B x_0 = 0$, then $W(A\lambda + B) \equiv \mathbb{C}$.*

(vi) *$W(A\lambda + B)$ is bounded if and only if $0 \notin F(A)$.*

(vii) *In general, $W(A\lambda + B)$ is connected. Only if A is an indefinite Hermitian matrix, then $W(A\lambda + B)$ may have two unbounded connected components.*

Note that $W(A\lambda + B)$ is not always bounded or connected and even if it is bounded and connected it is not always convex.

Proposition 2. *Let $A\lambda + B$ be an $n \times n$ linear pencil, where $A \neq 0$.*

(i) *$W(A\lambda + B) = \{\lambda_0\}$ if and only if $0 \notin F(A)$ and $B = -\lambda_0 A$.*

(ii) *If the matrices A and B are real, then the numerical range $W(A\lambda + B)$ is symmetric with respect to the $\mathbb{R}$-axis.*

Proof. (i) For the complex number λ_0, we have that $x^*(A\lambda_0 + B)x = 0$ for every $x \in \mathbb{C}^n$ if and only if $A\lambda_0 + B = 0$.

(ii) Consider a point $\lambda_0 \in W(A\lambda + B)$ and a nonzero vector $x_0 \in \mathbb{C}^n$ such that $x_0^*(A\lambda_0 + B)x_0 = 0$. By the conjucate of this equation, it follows that $x_0^{\mathrm{T}}(A\overline{\lambda}_0 + B)\overline{x}_0 = 0$ and consequently, $\overline{\lambda}_0 \in W(A\lambda + B)$. □

In [7], Maroulas and Psarrakos investigate the boundary and the sharp points of the numerical range of matrix polynomials of arbitary degree. A point $\lambda_0 \in \partial W(A\lambda + B)$ is called *sharp point* of $W(A\lambda + B)$ if there exist a disk $S(\lambda_0, r)$ ($r > 0$) and three angles $\varphi_1, \varphi_2, \psi_0 \in [0, 2\pi]$, with $0 \leqslant \varphi_2 - \varphi_1 \leqslant \psi_0 < \pi$ such that

$$\varphi_1 \leqslant \operatorname{Arg}(z - \lambda_0) \leqslant \varphi_2,$$

for every $z \in W(A\lambda + B) \cap S(\lambda_0, r)$.

Theorem 3 (see Theorem 1.1 in [7]). *If λ_0 is a boundary point of $W(A\lambda + B)$, then the origin is a boundary point of $F(A\lambda_0 + B)$.*

Theorem 4 (see Theorem 1.3 in [7]). *If λ_0 is a sharp point of $W(A\lambda + B)$, then the origin is a sharp point of $F(A\lambda_0 + B)$. In particular, λ_0 is an eigenvalue of the pencil $A\lambda + B$.*

If $W(A\lambda + B)$ is bounded, then we can estimate the length of the Jordan chains which correspond to eigenvalues of $A\lambda + B$ on the boundary of $W(A\lambda + B)$ (see also Theorem 1.6.6 in [5]).

Theorem 5. *Let $A\lambda + B$ be an $n \times n$ linear pencil and assume that its numerical range $W(A\lambda + B)$ is bounded. If $\lambda_0 \in \sigma(A\lambda + B) \cap \partial W(A\lambda + B)$, then every Jordan chain corresponding to λ_0 has length equal to 1, i.e., λ_0 is a semisimple eigenvalue of $A\lambda + B$.*

Proof. Since λ_0 is a boundary point of $W(A\lambda + B)$, there exist an angle φ_0 and a real $r > 0$ such that

$$\left\{\lambda_0 + \rho\, \mathrm{e}^{\mathrm{i}\varphi_0}\colon \rho \in [0, r]\right\} \cap W(A\lambda + B) = \{\lambda_0\}.$$

Suppose that for every angle $\varphi \in [0, 2\pi]\backslash\varphi_0$ there exists a real $\rho_\varphi > 0$ such that $\lambda_0 + \rho_\varphi\, \mathrm{e}^{\mathrm{i}\varphi} \in W(A\lambda + B)$. Hence,

$$0 \in F(A\lambda_0 + B) + \rho_\varphi\, \mathrm{e}^{\mathrm{i}\varphi} F(A) \quad \text{for all } \varphi \in [0, 2\pi]\backslash\varphi_0. \tag{2}$$

The numerical range $F(A\lambda_0 + B)$ is convex [5] and the origin belongs to its boundary (see Theorem 3). Moreover, $F(A)$ is convex and it does not contain the origin. Consequently, there exist infinitely many angles $\varphi \in [0, 2\pi]$ such that the range $F(A\lambda_0 + B) + \rho_\varphi\, \mathrm{e}^{\mathrm{i}\varphi} F(A)$ does not contain the origin. Thus, (2) is not true and there exists a cone

$$\mathscr{L} = \{z \in \mathbb{C}\colon \varphi_1 \leqslant \mathrm{Arg}(z - \lambda_0) \leqslant \varphi_2,\ 0 < \varphi_2 - \varphi_1 < \pi\}$$

such that

$$\mathscr{L} \cap W(A\lambda + B) = \{\lambda_0\}.$$

By Theorem 2 in [8], every Jordan chain corresponding to λ_0 has length equal to 1. □

Moreover, a sufficient condition for the pencil $A\lambda + B$ to be diagonalizable, can be formulated in terms of $W(A\lambda + B)$.

Proposition 6. *Let $A\lambda + B$ be an $n \times n$ linear pencil and assume that the numerical range $W(A\lambda + B)$ is bounded. If $\sigma(A\lambda + B) \subset \partial W(A\lambda + B)$, then there exist two $n \times n$ invertible matrices P and Q such that the pencil $P(A\lambda + B)Q$ is diagonal.*

Proof. Since the conditions of Theorem 5 hold, all the elementary divisors of $A\lambda + B$ are linear. Hence, by Theorem 1, Section 7.7, in [4], the proof is complete. □

Next we prove the converse statement of Theorem 3.

Theorem 7. *Let $A\lambda + B$ be an $n \times n$ linear pencil and let the numerical range $W(A\lambda + B)$ in* (1) *be bounded. If λ_0 is a point of $W(A\lambda + B)$ such that the origin is a boundary point of $F(A\lambda_0 + B)$, then $\lambda_0 \in \partial W(A\lambda + B)$.*

Proof. Suppose that λ_0 is an interior point of $W(A\lambda + B)$. Then there exists a real number $\varepsilon > 0$ such that $S(\lambda_0, \varepsilon) \subset W(A\lambda + B)$. Consequently, for every complex number z_0 with $|z_0| < \varepsilon$, there exists a nonzero vector x_0 such that

$$x_0^*(A\lambda_0 + B)x_0 = -z_0\left(x_0^* A x_0\right). \tag{3}$$

Moreover, $0 \in \partial F(A\lambda_0 + B),\ 0 \notin F(A)$ and the numerical ranges $F(A)$ and $F(A\lambda_0 + B)$ are convex. So, there exist five angles $\varphi_1, \varphi_2, \psi_0, \vartheta_1, \vartheta_2 \in [0, 2\pi)$, with $0 \leqslant \varphi_2 - \varphi_1 \leqslant \psi_0 < \pi$ and $0 \leqslant \vartheta_2 - \vartheta_1 \leqslant \pi$ such that

$$F(A) \subset \{z \in \mathbb{C}\colon\ \varphi_1 \leqslant \operatorname{Arg} z \leqslant \varphi_2\}$$

and

$$F(A\lambda_0 + B) \subset \{z \in \mathbb{C}\colon\ \vartheta_1 \leqslant \operatorname{Arg} z \leqslant \vartheta_2\}.$$

By Eq. (3),

$$\operatorname{Arg}\left[x_0^*(A\lambda_0 + B)x_0\right] - \operatorname{Arg}\left(x_0^* A x_0\right) = \operatorname{Arg}(-z_0)$$

cannot be true for every $z_0 \in \mathbb{C}$ with $|z_0| \leqslant \varepsilon$. Thus, λ_0 is a boundary point of $W(A\lambda + B)$. □

3. Selfadjoint pencils

In this section, we consider selfadjoint pencils $A\lambda + B$, i.e., the matrices A and B are Hermitian. In this case, the numerical range $W(A\lambda + B)$ in (1) is a subset of $\mathbb{R}$-axis or it coincides with the complex plane $\mathbb{C}$. If $W(A\lambda + B) \neq \mathbb{C}$, then an interesting extension of Proposition 6 follows from Theorem 1.7.17 in [5].

Proposition 8. *Let $A\lambda + B$ be an $n \times n$ linear selfadjoint pencil with numerical range $W(A\lambda + B) \neq \mathbb{C}$. Then there exists an invertible matrix Q such that the pencil $Q^*(A\lambda + B)Q$ is diagonal.*

Note that in the previous proposition, all the elementary divisors of $A\lambda + B$ are linear and $W(Q^*(A\lambda + B)Q) = W(A\lambda + B)$.

The shape of $W(A\lambda + B)$ is described in Theorem 4.1 in [6].

Theorem 9. *Let $A\lambda + B$ be an $n \times n$ selfadjoint pencil with $W(A\lambda + B) \neq \mathbb{C}$. Then we have exactly one of the following cases*:

(i) *If the matrix A is (positive or negative) definite, then* $W(A\lambda + B)$ *is a bounded closed interval in* $\mathbb{R}$.

(ii) *If A is semidefinite, then* $W(A\lambda + B)$ *is an unbounded interval of the form* $[a, +\infty)$ *or* $(-\infty, a]$.

(iii) *If A is indefinite and B is definite, then* $W(A\lambda + B)$ *is the union of two distinct unbounded intervals in* $\mathbb{R}$ *such that* $0 \notin W(A\lambda + B)$.

(iv) *If A is indefinite and B is semidefinite, then* $W(A\lambda + B)$ *is the union of two distinct unbounded intervals in* $\mathbb{R}$ *such that* $0 \in W(A\lambda + B)$.

(v) *If A and B are both indefinite, then* $W(A\lambda + B) \equiv \mathbb{R}$.

In all cases, the finite endpoints of the intervals are eigenvalues of the pencil $A\lambda + B$.

A question, which arises in a natural way, is what one can say about the *real boundary* of $W(A\lambda + B)$, i.e.,

$$\partial_{\mathbb{R}} W(A\lambda + B) = W(A\lambda + B) \cap [\overline{\mathbb{R}\backslash W(A\lambda + B)}].$$

In fact, if we consider the real boundary of $F(A)$, namely,

$$\partial_{\mathbb{R}} F(A) = F(A) \cap [\overline{\mathbb{R}\backslash F(A)}],$$

then a statement similar to Theorems 3 and 7 can be obtained.

Theorem 10. *Let* $A\lambda + B$ *be an* $n \times n$ *selfadjoint pencil with* $W(A\lambda + B) \neq \mathbb{C}$. *If* λ_0 *is a nonzero point of* $W(A\lambda + B)$, *then* $\lambda_0 \in \partial_{\mathbb{R}} W(A\lambda + B)$ *if and only if* $0 \in \partial_{\mathbb{R}} F(A\lambda_0 + B)$.

Proof. Since $\lambda_0 \in \partial_{\mathbb{R}} W(A\lambda + B)$, there exists a sequence of points $\{\lambda_k\}_{k\in\mathbb{N}} \in \mathbb{R}\backslash W(A\lambda + B)$ converging to λ_0. Hence, for every $k \in \mathbb{N}$, the origin does not belong to $F(A\lambda_k + B)$ and without lost of generality, we can assume that all the matrices

$$A\lambda_k + B, \quad k \in \mathbb{N},$$

are positive definite. The sequence of the numerical ranges $F(A\lambda_k + B) = [a_k, b_k]$ with $0 < a_k < b_k$, $k \in \mathbb{N}$, converges to $F(A\lambda_0 + B)$ and $0 \in F(A\lambda_0 + B)$. So, the matrix $A\lambda_0 + B$ is positive semidefinite, i.e., $0 \in \partial_{\mathbb{R}} F(A\lambda_0 + B)$.

For the converse, suppose that $\lambda_0 \in W(A\lambda + B)$ and $0 \in \partial_{\mathbb{R}} F(A\lambda_0 + B)$. Then without lost of generality, we can assume that the matrix $A\lambda_0 + B$ is positive semidefinite with $F(A\lambda_0 + B) = [0, b]$ and investigate the following cases.

(i) If the matrix A is positive definite or positive semidefinite and $x_0 \in \mathbb{C}^n$ is a vector such that $x_0^* A x_0 \neq 0$ and $\lambda_0 = -(x_0^* B x_0)/(x_0^* A x_0)$, then there exists a real number $r_0 > 0$ such that $x^* A x > 0$ for every $x \in S(x_0, r_0)$. Moreover,

$$\lambda_0 - \left(-\frac{x^* B x}{x^* A x}\right) = \frac{x^*(A\lambda_0 + B)x}{x^* A x} \geqslant 0,$$

i.e., for every $x \in S(x_0, r_0)$, the root of equation $x^*(A\lambda + B)x = 0$ is not greater than λ_0. By the continuity of the root $\lambda_0 = -(x^* B x)/(x^* A x)$ $(x^* A x \neq 0)$ with respect to x, it follows that $\lambda_0 \in \partial_{\mathbb{R}} W(A\lambda + B)$.

(ii) Suppose that A is indefinite and B is positive definite or positive semidefinite. Since $\lambda_0 \neq 0$, for every nonzero vector $x_0 \in \mathbb{C}^n$ such that $x_0^*(A\lambda_0 + B)x_0 = 0$, there exists a real number $r_0 > 0$ such that for every $x \in S(x_0, r_0)$ the ratio $-(x^*Bx)/\lambda_0$ has constant sign. Working exactly as in (i), we obtain that $\lambda_0 \in \partial_{\mathbb{R}} W(A\lambda + B)$.

(iii) If the matrices A and B are both indefinite, then $W(A\lambda + B) \equiv \mathbb{R}$ and the matrix $A\mu + B$ is indefinite for every $\mu \in \mathbb{R}$.

The rest of the cases are similar to (i) or (ii). □

4. Pencils with one Hermitian coefficient

Let $A\lambda + B$ be an $n \times n$ linear pencil and assume that A is a Hermitian matrix. The matrix B is written as

$$B = H + \mathrm{i}S,$$

where the matrices $H = (B + B^*)/2$ and $S = (B - B^*)/(2\mathrm{i})$ are Hermitian. In this section, the numerical range $W(A\lambda + B) \equiv W(A\lambda + H + \mathrm{i}S)$ and its boundary are investigated.

It is worth noting that if A is positive definite, then there exists an invertible matrix M such that $A = M^*M$ and

$$\begin{aligned} W(A\lambda + B) &= \left\{ -\frac{x^*Bx}{x^*Ax} \in \mathbb{C}\colon\ x \in \mathbb{C}^n,\ x \neq 0 \right\} \\ &= \left\{ -\frac{(x^*M^*)\left[\left(M^{-1}\right)^* BM^{-1}\right](Mx)}{(x^*M^*)(Mx)} \in \mathbb{C}\colon\ x \in \mathbb{C}^n,\ x \neq 0 \right\} \\ &= F\left(-\left(M^{-1}\right)^* BM^{-1} \right). \end{aligned}$$

Thus, in this case, $W(A\lambda + B)$ coincides with the (compact and convex) numerical range of the matrix $-(M^{-1})^*BM^{-1}$.

In our discussion, we need the *joint numerical range*

$$\mathrm{JNR}(A, H, S) = \left\{ \left(x^*Ax, x^*Hx, x^*Sx\right) \in \mathbb{R}^3 \colon x \in \mathbb{C}^n \text{ with } x^*x = 1 \right\} \tag{4}$$

of the triple (A, H, S). It is well known that $\mathrm{JNR}(A, H, S)$ is a compact subset of $\mathbb{R}^3$. Moreover, for $n \geqslant 3$ it is convex and for $n = 2$ it is either convex or the surface of an ellipsoid [9]. Using this characteristic property of $\mathrm{JNR}(A, H, S)$, Li and Rodman describe in [10] an algorithm which illustrates the boundary $\partial\mathrm{JNR}(A, H, S)$.

Consider a point $(u_0, v_0, w_0) \in \mathbb{R}^3$ and the corresponding equation

$$u_0\lambda + v_0 + \mathrm{i}w_0 = 0.$$

Obviously, every point of the open halfline

$$\epsilon^+ = \left\{ t(u_0, v_0, w_0) \in \mathbb{R}^3\colon\ t \in (0, +\infty) \right\}$$

corresponds to a linear equation with the same root $\lambda_0 = -(v_0 + \mathrm{i}w_0)/u_0$ ($u_0 \neq 0$). So, if we define the *supporting cone* of $\mathrm{JNR}(A, H, S)$,

$$\mathscr{K} = \bigcup_{t>0} t\mathrm{JNR}(A, H, S), \tag{5}$$

then $\mathscr{K}$ is always convex and

$$\begin{aligned} W(A\lambda + B) &= \left\{\lambda \in \mathbb{C}\colon \left(x^*Ax\right)\lambda + x^*(H + \mathrm{i}S)x = 0,\ x \in \mathbb{C}^n,\ x \neq 0\right\} \\ &= \{\lambda \in \mathbb{C}\colon u\lambda + (v + \mathrm{i}w) = 0,\ (u, v, w) \in \mathrm{JNR}(A, H, S)\} \\ &= \{\lambda \in \mathbb{C}\colon u\lambda + (v + \mathrm{i}w) = 0,\ (u, v, w) \in \mathscr{K}\}. \end{aligned}$$

Consequently, a complex number λ_0 belongs to $W(A\lambda + B)$ if and only if the line $\epsilon = \{t(-1,\ \mathrm{Re}\,\lambda_0,\ \mathrm{Im}\,\lambda_0) \in \mathbb{R}^3\colon t \in \mathbb{R}\}$ intersects $\mathrm{JNR}(A, H, S)$ in (4). It is also obvious that $(0, 0, 0) \in \mathrm{JNR}(A, H, S)$ if and only if $(0, 0, 0) \in \mathscr{K}$, and then $W(A\lambda + B) \equiv \mathbb{C}$.

Theorem 11. *Let $A\lambda + H + \mathrm{i}S$ be an $n \times n$ linear pencil* (*where A, H, S are Hermitian matrices*) *with $W(A\lambda + H + \mathrm{i}S) \neq \mathbb{C}$ and let $\mathscr{K}$ be the cone in* (5). *Suppose that $\lambda_0 \in W(A\lambda + H + \mathrm{i}S)$ and $(u_0, v_0, w_0) \in \mathscr{K}$ such that $\lambda_0 = -(v_0 + \mathrm{i}w_0)/u_0$ ($u_0 \neq 0$). Then $\lambda_0 \in \partial W(A\lambda + H + \mathrm{i}S)$ if and only if $(u_0, v_0, w_0) \in \partial\mathscr{K}$.*

Proof. Since $W(A\lambda + H + \mathrm{i}S) \neq \mathbb{C}$, $(0, 0, 0) \notin \mathrm{JNR}(A, H, S)$ and if $\lambda_0 \in \partial W(A\lambda + H + \mathrm{i}S)$, then there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \in \mathbb{C}\backslash W(A\lambda + H + \mathrm{i}S)$ converging to the point λ_0. Moreover, the sequence of lines

$$\epsilon_k = \left\{t(-1,\ \mathrm{Re}\,\lambda_k,\ \mathrm{Im}\,\lambda_k) \in \mathbb{R}^3\colon\ t \in \mathbb{R}\right\}, \quad k \in \mathbb{N},$$

converges to the line

$$\epsilon_0 = \left\{t(u_0, v_0, w_0) \in \mathbb{R}^3\colon\ t \in \mathbb{R}\right\} = \left\{t(-1,\ \mathrm{Re}\,\lambda_0,\ \mathrm{Im}\,\lambda_0) \in \mathbb{R}^3\colon t \in \mathbb{R}\right\}.$$

Since $\epsilon_k \cap \mathrm{JNR}(A, H, S) = \emptyset$, for every $k \in \mathbb{N}$, the line ϵ_0 is a supporting line of $\mathrm{JNR}(A, H, S)$ and consequently, $(u_0, v_0, w_0) \in \partial\mathscr{K}$.

Conversely, assume that $(u_0, v_0, w_0) \in \partial\mathscr{K}$ and consider the line

$$\epsilon_0 = \{t(u_0, v_0, w_0) \in \mathbb{R}\colon\ t \in \mathbb{R}\}.$$

Then there exists a sequence of lines

$$\epsilon_k = \left\{t(-1, v_k, w_k) \in \mathbb{R}^3\colon\ t \in \mathbb{R}\right\}, \quad k \in \mathbb{N},$$

converging to ϵ_0 such that $\epsilon_k \cap \mathrm{JNR}(A, H, S) = \emptyset$ for every $k \in \mathbb{N}$. Thus, the sequence

$$\{\lambda_k = v_k + \mathrm{i}w_k\}_{k\in\mathbb{N}} \in \mathbb{C}\backslash W(A\lambda + H + \mathrm{i}S)$$

converges to λ_0, and $\lambda_0 \in \partial W(A\lambda + H + \mathrm{i}S)$. □

Corollary 12. *Let $A\lambda + H + \mathrm{i}S$ be an $n \times n$ linear pencil, $\lambda_0 \in W(A\lambda + H + \mathrm{i}S)$ and $(u_0, v_0, w_0) \in \mathrm{JNR}(A, H, S)$ as in Theorem* 11. *Then $\lambda_0 \in \partial W(A\lambda + H + \mathrm{i}S)$ if and only if $(u_0, v_0, w_0) \in \partial\mathrm{JNR}(A, H, S) \cap \partial\mathscr{K}$.*

If A is a scalar matrix of the form $A = \mu_0 I$ ($\mu_0 \in \mathbb{R},\ \mu_0 \neq 0$), then

$$\mathrm{JNR}(A, H, S) = \left\{(\mu_0, h, s) \in \mathbb{R}^3 \colon h + \mathrm{i}s \in F(H + \mathrm{i}S)\right\}.$$

Hence, the curve $\partial\mathrm{JNR}(A, H, S) \cap \partial\mathscr{K}$ is just the boundary of $\mu_0^{-1} F(H + \mathrm{i}S)$, and Corollary 12 is verified.

Assume that $W(A\lambda + H + \mathrm{i}S) \neq \mathbb{C}$. Using the algorithm of Li and Rodman [10], we can compute boundary points (u, v, w) of $\mathrm{JNR}(A, H, S)$. The solutions of the equations $u\lambda + v + \mathrm{i}w = 0$ ($u \neq 0$) are points of the numerical range $W(A\lambda + H + \mathrm{i}S)$ (see Fig. 1). Furthermore, if we choose points (u, v, w) on the boundary of the supporting cone $\mathscr{K}$, then we approximate $\partial W(A\lambda + H + \mathrm{i}S)$.

The algorithm has three steps:

Step 1: Construct a grid on the unit sphere in $\mathbb{R}^3$ using the spherical coordinates

$$(\sin r\ \cos t,\ \sin r\ \sin t,\ \cos r),$$

with

$$r = \pi/m, 2\pi/m, \ldots, \pi \quad \text{and} \quad t = \pi/m, 2\pi/m, \ldots, 2\pi,$$

for some positive integer m.

Step 2: For each choice of $(\sin r\ \cos t,\ \sin r\ \sin t,\ \cos r)$, compute the largest eigenvalue d of the matrix

$$H(r, t) = (\sin r\ \cos t)A + (\sin r\ \sin t)H + (\cos r)S.$$

Step 3: Compute a unitary eigenvector $y \in \mathbb{C}^n$ of $H(r, t)$, corresponding to the eigenvalue d. The plane

$$P = \left\{(u, v, w) \in \mathbb{R}^3 \colon (\sin r\ \cos t)u + (\sin r\ \sin t)v + (\cos r)w = d\right\}$$

is a supporting plane of $\mathrm{JNR}(A, H, S)$ on the point $(y^*Ay, y^*Hy, y^*Sy) \in \partial\mathrm{JNR}(A, H, S)$. If $y^*Ay \neq 0$, then plot the point

$$\lambda = -\frac{y^*Hy + \mathrm{i}y^*Sy}{y^*Ay} \in W(A\lambda + H + \mathrm{i}S).$$

The boundary of the cone $\mathscr{K}$ is constructed by all the supporting planes of $\mathrm{JNR}(A, H, S)$, which contain the origin. So, to approximate the boundary of the numerical range $W(A\lambda + H + \mathrm{i}S)$, modify Step 3 to the following.

Step 3′: If $d = 0$ (or $d \cong 0$), then compute a unitary eigenvector $y \in \mathbb{C}^n$ of $H(r, t)$ corresponding to d. The plane

$$P = \left\{(u, v, w) \in \mathbb{R}^3 \colon (\sin r\ \cos t)u + (\sin r\ \sin t)v + (\cos r)w = d\right\}$$

is a supporting plane of $\mathrm{JNR}(A, H, S)$ on the point $(y^*Ay, y^*Hy, y^*Sy) \in \partial\mathrm{JNR}(A, H, S) \cap \partial\mathscr{K}$. If $y^*Ay \neq 0$, then plot the point

$$\lambda = -\frac{y^*Hy + \mathrm{i}y^*Sy}{y^*Ay} \in \partial W(A\lambda + H + \mathrm{i}S).$$

It is worth noting that if A is invertible, then each connected component of $W(A\lambda + H + \mathrm{i}S)$ is convex (see Theorem 2.4 in [11]).

Remark 1. If the origin $(0, 0, 0)$ lies in the convex hull of the interior of JNR (A, H, S), then $\mathscr{K} \equiv \mathbb{R}^3$ and $W(A\lambda + H + \mathrm{i}S) \equiv \mathbb{C}$. In this case, the picture $W(A\lambda + H + \mathrm{i}S)$ generated by the algorithm can be quite chaotic.

Remark 2. Assume that $A\lambda + B$ is an $n \times n$ linear pencil such that A is a non-Hermitian matrix and B is a Hermitian matrix. In this case, the numerical range $W(B\lambda + A)$ is approximated by the previous algorithm and the equation

$$W(A\lambda + B)\backslash\{0\} = \left\{\lambda^{-1} \in \mathbb{C}: \lambda \in W(B\lambda + A)\backslash\{0\}\right\}$$

from Proposition 1(iii). Moreover, Theorem 11 and Corollary 12 are also true for linear pencils of the form $(H + \mathrm{i}S)\lambda + A$, where the matrices A, H and S are Hermitian.

Remark 3. Suppose that $\alpha + \beta v + \gamma w = 0$ is the equation of a supporting line of the numerical range $F(H + \mathrm{i}S)$ (where v, w are orthogonal coordinates in the (v, w)-plane). Following a method in [5, Section 1.5], it is easy to see that the Hermitian matrix $\alpha I + \beta H + \gamma S$ is semidefinite and

$$\det(\alpha I + \beta H + \gamma S) = 0.$$

It follows that the boundary of $F(H + \mathrm{i}S)$ can be considered as the set of real points of the algebraic curve whose equation in line coordinates is

$$\det(uI + vH + wS) = 0$$

(see [12–14]). Furthermore, consider the linear pencil $A\lambda + H + \mathrm{i}S$ with numerical range $W(A\lambda + H + \mathrm{i}S) \neq \mathbb{C}$. Let $\lambda_0 \in \partial W(A\lambda + H + \mathrm{i}S)$ and assume that $\alpha u + \beta v + \gamma w = 0$ is the equation of a supporting plane of JNR(A, H, S) (where u, v, w are orthogonal coordinates in $\mathbb{R}^3$), which contains the line $\epsilon = \{t(-1, \ \mathrm{Re}\ \lambda_0, \ \mathrm{Im}\ \lambda_0) \in \mathbb{R}^3: t \in \mathbb{R}\}$. Then the Hermitian matrix $\alpha A + \beta H + \gamma S$ is also semidefinite and

$$\det(\alpha A + \beta H + \gamma S) = 0.$$

Consider the equation $\alpha u + \beta v + \gamma w = 0$ as a homogeneous equation of a line (see [14] for definitions and background). Then the boundary of $W(A\lambda + H + \mathrm{i}S)$ may be viewed as the set of real points of the algebraic curve whose equation in line coordinates is

$$\det(uA + vH + wS) = 0.$$

Next we generate $W(A\lambda + H + \mathrm{i}S)$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -2\mathrm{i} \\ 2\mathrm{i} & 0 \end{bmatrix}.$$

(see Fig. 1.) It is unbounded and the boundary $\partial W(A\lambda + H + \mathrm{i}S)$, in Fig. 2, is a branch of hyperbola (see Section 3 in [10]).

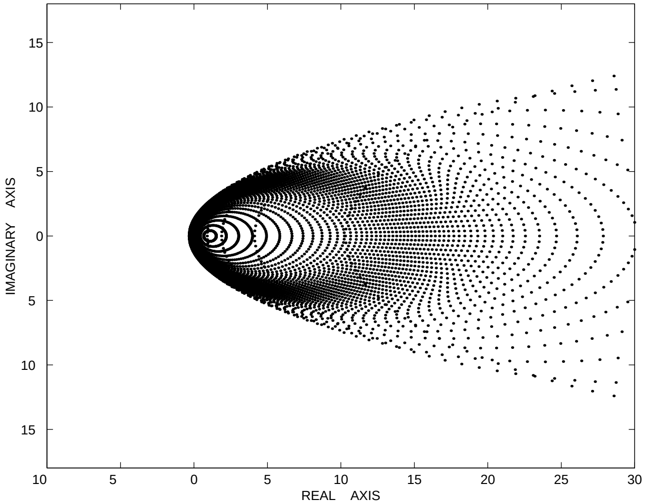

Fig. 1. The numerical range $W(A\lambda + H + \mathrm{i}S)$.

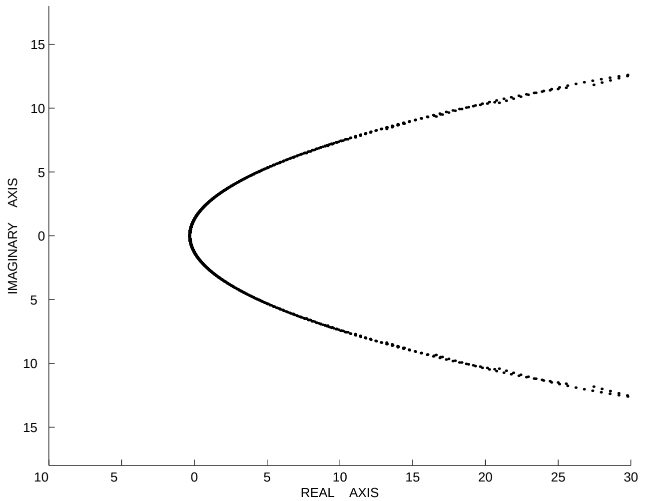

Fig. 2. The boundary $\partial W(A\lambda + H + \mathrm{i}S)$.

5. Krein space numerical range

Suppose that A is an $n \times n$ indefinite Hermitian invertible matrix. Then we can define the *indefinite inner product*

$$[x, y] = y^*(Ax), \quad x, y \in \mathbb{C}^n.$$

The indefinite inner product space $(\mathbb{C}^n, [\cdot,\cdot])$ is known as a *finite* (*complex*) *Krein space* and for any complex matrix *B*, the *Krein space numerical range* (with respect to $[\cdot,\cdot]$) is defined by

$$W_A^+(B) = \{[Bx, x]/[x, x] \in \mathbb{C}\colon\ x \in \mathbb{C}^n \text{ with } [x, x] > 0\}.$$

Moreover, the *A-numerical range* of *B* is defined by

$$W_A(B) = \{[Bx, x]/[x, x] \in \mathbb{C}\colon\ x \in \mathbb{C}^n \text{ with } [x, x] \neq 0\}$$

and it is easy to verify that

$$W_A(B) = W_A^+(B) \cup W_{(-A)}^+(B).$$

The sets $W_A^+(B)$ and $W_A(B)$ have been studied in [10,11,15], with emphasis on the convexity properties and the geometric shapes of $W_A^+(B)$ and $W_A(B)$. If *B* is not a scalar matrix, then $W_A^+(B)$ and $W_A(B)$ are unbounded (see Proposition 2.1 in [10]). The problem of the numerical approximation of $W_A(B)$ and $W_A^+(B)$ was stated in [11] and it was partially solved in [10]. Using the two versions of the algorithm in Section 4, one can approximate $W_A(B)$, particularly the boundary.

The study of the numerical range of linear pencils, as a special case of the numerical range of matrix polynomials, gives a new approach to numerical ranges in indefinite inner product spaces. Consider the Hermitian matrices

$$H = \frac{AB + (AB)^*}{2}, \qquad S = \frac{AB - (AB)^*}{2\mathrm{i}}$$

and the supporting cone $\mathscr{K}$, in (5), for the joint numerical range $\mathrm{JNR}(A, H, S)$. Since $\mathscr{K}$ is convex, it is clear that the sets

$$W_A^+(B) = \{\lambda \in \mathbb{C}\colon (1,\ \mathrm{Re}\,\lambda,\ \mathrm{Im}\,\lambda) \in \mathscr{K}\}$$

and

$$W_{(-A)}^+(B) = \{\lambda \in \mathbb{C}\colon (1,\ \mathrm{Re}\,\lambda,\ \mathrm{Im}\,\lambda) \in -\mathscr{K}\}$$

are also convex. If $(0, 0, 0) \notin \partial\mathrm{JNR}(A, H, S)$, then it follows immediately that

$$W_A(B) = W(A\lambda - AB).$$

Note also that the curves $\partial\mathscr{K} \cap \{(u, v, w) \in \mathbb{R}^3\colon u = 1\}$ and $\partial\mathscr{K} \cap \{(u, v, w) \in JNR(A, H, S)\colon u > 0\}$ are homotopic relative to the boundary of the cone $\mathscr{K}$. As a consequence, we can have a second proof of Theorem 11. If $(0, 0, 0) \in \partial\mathrm{JNR}(A, H, S)$, then $W(A\lambda - AB) = \mathbb{C}$, but $W_A(B)$ need not be the entire complex plane (see Section 2 in [10]).

An interesting question is when the spectral containment $\sigma(B) \subset W_A(B)$ holds (see Section 4 in [11]). Since *A* is invertible, it is clear that for any $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$,

$$\det(A\mu - AB) = \det A \,\det(I\mu - B)$$

and

$$(A\mu - AB)x = 0 \Leftrightarrow (I\mu - B)x = 0.$$

Hence, $\sigma(B) = \sigma(A\lambda - AB)$, and every eigenvalue has the same algebraic multiplicity and the same corresponding eigenspace, for the matrix B and the linear pencil $A\lambda - AB$. Moreover, $\sigma(A\lambda - AB) \subset W(A\lambda - AB)$ and thus, an eigenvalue $\mu \notin W_A(B)$ if and only if $\mu \in W(A\lambda - AB) \backslash W_A(B)$. So, if $(0, 0, 0) \notin \partial\mathrm{JNR}(A, H, S)$, then

$$\sigma(B) \subset W_A(B).$$

Suppose that $(0, 0, 0) \in \partial\mathrm{JNR}(A, H, S)$ (this is the case when $W(A\lambda - AB) = \mathbb{C}$ and $W_A(B) \neq \mathbb{C}$). Then it is clear that an eigenvalue $\mu \in \sigma(B)$ does not belong to $W_A(B)$ if and only if for every nonzero $y \in \mathbb{C}^n$ such that $y^*(A\mu - AB)y = 0$, we have that $y^*Ay = y^*Hy = y^*Sy = 0$.

Assume that $W(A\lambda - AB) \neq \mathbb{C}$. Then $(0, 0, 0) \notin \mathrm{JNR}(A, H, S)$ and

$$W_A^+(B) \cap W_{(-A)}^+(B) = \emptyset.$$

Consequently, λ_0 is a boundary point of $W_A^+(B)$ if and only if it is a boundary point of $W_A(B)$, and λ_0 is a sharp point of $W_A^+(B)$ if and only if it is a sharp point of $W_A(B)$. In connection with the results in Section 2, we obtain the following two theorems.

Theorem 13. *Suppose that $W_A(B) \neq \mathbb{C}$ and $\lambda_0 \in W_A^+(B)$. Then λ_0 is a boundary point of $W_A^+(B)$ if and only if the origin is a boundary point of the numerical range $F(A\lambda_0 - AB)$.*

Proof. Let λ_0 be a boundary point of $W_A^+(B)$. If $W(A\lambda - AB) \neq \mathbb{C}$, then λ_0 is also a boundary point of $W_A(B) = W(A\lambda - AB)$. Thus, by Theorem 3, the origin is a boundary point of $F(A\lambda_0 - AB)$. If $W(A\lambda - AB) = \mathbb{C}$, then the arguments in the proof of Theorem 1.1 in [7] apply to obtain that $0 \in \partial F(A\lambda_0 - AB)$.

For the converse, assume that $0 \in \partial F(A\lambda_0 - AB)$ and $\lambda_0 \in \mathrm{Int}\, W_A^+(B)$. Then there is a real $\varepsilon > 0$ such that $S(\lambda_0, \varepsilon) \subset W_A^+(B)$. Hence, for every $\mu \in S(\lambda_0, \varepsilon)$, there exists a nonzero vector $x_\mu \in \mathbb{C}^n$ such that

$$x_\mu^* A x_\mu > 0 \quad \text{and} \quad \mu = \frac{x_\mu^* A B x_\mu}{x_\mu^* A x_\mu}.$$

Thus,

$$\mu - \lambda_0 = -\frac{x_\mu^*(A\lambda_0 - AB)x_\mu}{x_\mu^* A x_\mu}$$

and consequently, for every $\mu \in S(\lambda_0, \varepsilon)$,

$$\mathrm{Arg}(\mu - \lambda_0) = \mathrm{Arg}\big[-x_\mu^*(A\lambda_0 - AB)x_\mu\big].$$

Since the origin is a boundary point of the convex range $F(A\lambda_0 - AB)$, this is a contradiction and the proof is complete. $\square$

Theorem 14. *Suppose that* $W_A(B) \neq \mathbb{C}$ *and* $\lambda_0 \in W_A^+(B)$*. Then* λ_0 *is a sharp point of* $W_A^+(B)$ *if and only if the origin is a sharp point of the numerical range* $F(A\lambda_0 - AB)$*.*

Proof. Let λ_0 be a sharp point of $W_A^+(B)$. If $W(A\lambda - AB) \neq \mathbb{C}$, then λ_0 is also a sharp point of $W_A(B) = W(A\lambda - AB)$. Thus, by Theorem 4, the origin is a sharp point of $F(A\lambda_0 - AB)$. If $W(A\lambda - AB) = \mathbb{C}$, then the proof of Theorem 1.3 in [7] yields that 0 is a sharp point of $F(A\lambda_0 - AB)$.

For the converse, assume that the origin is a sharp point of the convex range $F(A\lambda_0 - AB)$. Then there exist three angles $\varphi_1, \varphi_2, \psi_0 \in [0, 2\pi]$, with $0 \leqslant \varphi_2 - \varphi_1 \leqslant \psi_0 < \pi$ and

$$\varphi_1 \leqslant \operatorname{Arg}\left[x^*(A\lambda_0 - AB)x\right] \leqslant \varphi_2$$

for every nonzero vector $x \in \mathbb{C}^n$. For every $\mu \in W_A^+(B)$, there is a nonzero vector x_μ such that $x_\mu^* A x_\mu > 0$ and $x_\mu^*(A\mu - AB)x_\mu = 0$. Since

$$\begin{aligned}\operatorname{Arg}\left[x_\mu^*(A\lambda_0 - AB)x_\mu\right] &= \operatorname{Arg}\left\{x_\mu^*[A(\lambda_0 - \mu + \mu) - AB]x_\mu\right\} \\ &= \operatorname{Arg}(\lambda_0 - \mu),\end{aligned}$$

it is clear that

$$\varphi_1 \leqslant \operatorname{Arg}(\lambda_0 - \mu) \leqslant \varphi_2.$$

Thus, λ_0 is a sharp point of $W_A^+(B)$. □

By the previous theorem, we can verify easily a known result (see Theorem 3.1 in [15]).

Corollary 15. *Suppose that* $\lambda_0 \in W_A^+(B)$ *is a sharp point of* $W_A^+(B)$*. Then* λ_0 *is an eigenvalue of B and every* $x_0 \in \mathbb{C}^n$ *such that* $\lambda_0 = (x_0^* ABx_0)/(x_0^* Ax_0)$ (*where* $x_0^* Ax_0 \neq 0$) *is a corresponding eigenvector of B.*

Proof. By Theorem 14, the origin is a sharp point of $F(A\lambda_0 - AB)$. Thus, $0 \in \sigma(A\lambda_0 - AB)$ and $(A\lambda_0 - AB)x_0 = 0$ (see Theorem 1.6.3 in [5] and its proof). □

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