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Defending the Roman Empire from multiple attacks

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Abstract

Motivated by articles by Stewart (Defend the Roman Empire!, Sci. Amer. (December 1999) 136–138) and ReVelle and Rosing (Defendens imperium Romanum:[2] A classical problem in military strategy, Amer. Math. Monthly 107 (7) (2000) 585–594), we explore a strategy of defending the Roman Empire from multiple attacks by stationing as few legions as possible.

Keywords: Domination number; k-Roman dominating function; Forest

1. Introduction

Motivated by articles in *Scientific American* by Stewart entitled “Defend the Roman Empire!” [13] and in *American Mathematical Monthly* by ReVelle and Rosing entitled “Defendens imperium Romanum: A classical problem in military strategy” [12], we present a graph theoretic approach to defending the Roman Empire from multiple attacks by stationing as few legions as possible. Graph theoretic models to defend the Roman Empire from single attacks have been studied, for example, in [3,4,6,10,11] and elsewhere.

Faced with reductions in the size of the Roman armies due to economic constraints, Emperor Constantine the Great, in the fourth century A.D., switched from a “forward

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defense" strategy to a "defense-in-depth" strategy. This strategy divided the Roman army into the *comitatensis*, the mobile fighting forces who could arrive at any location within 2–3 weeks, and the *limitanei*, or frontier troops stationed permanently on the borders of the empire. The limitanei were considered second class soldiers (usually recruited from the landless peasants and barbarians who had entered Rome to settle on land in exchange for imperial service) and were looked down on by the elite troops of the comitatensis who were paid regularly and were much better equipped.

Our mathematical model is a graph $G=(V,E)$ with vertex set V and edge set E. Each vertex in our graph represents a location in the Roman Empire, and two vertices are joined by an edge if the corresponding locations are adjacent. Let k be a positive integer and let f be a function $f:V \to \{0,1,\dots,k+1\}$. A location (vertex v) is considered *unsecured* if no legions are stationed there (i.e., $f(v)=0$) and *secured* otherwise (i.e., if $f(v)\in\{1,2,\dots,k+1\}$). For $i=0,1,\dots,k+1$, let V_i be the set of vertices assigned the values i under f. Note that there is a 1–1 correspondence between the functions $f:V \to \{0,1,\dots,k+1\}$ and the ordered partitions $(V_0,V_1,\dots,V_{k+1})$ of V. Thus we will write $f=(V_0,V_1,\dots,V_{k+1})$.

We say that a vertex in V_0 is *undefended with respect to* f, or simply *undefended* if the function f is clear from the context, if it is not adjacent to a vertex in V_i for any $i \geqslant 1$. We say that f *has no undefended vertex* if no vertex in V_0 is undefended with respect to f.

If a vertex $u\in V_0$ is adjacent to a vertex $v\in V_i$ for some i, $1\leqslant i\leqslant k+1$, then we say that the function $g:V \to \{0,1,\dots,k+1\}$, defined by $g(u)=1$, $g(v)=f(v)-1$ and $g(w)=f(w)$ if $w\in V-\{u,v\}$, *is obtained from* f *by one movement from* v *to* u, or simply that g *is obtained from* f *by one movement* if the vertices v and u is clear from context.

We call the function f a *k-Roman dominating function* (kRDF) if f has no undefended vertex and for any sequence $v_1,\dots,v_k$ of (not necessarily distinct) vertices, there exists a sequence of functions $f=f_0,f_1,\dots,f_k$ such that for $i=1,\dots,k$, (i) either $f_{i-1}(v_i)>0$, in which case $f_i=f_{i-1}$, or $f_{i-1}(v_i)=0$, in which case f_i is obtained from f_{i-1} by one movement to v_i, and (ii) f_i has no undefended vertex. If f_i is obtained from f_{i-1} by one movement for each $i=1,\dots,k$, then we say that f_k is *obtained from* f *by* k *movements*. We may assume in what follows that f_k is obtained from f by k movements. When $k=1$, a kRDF is a Roman dominating function which has been studied, for example, in [3,4,6,10,11].

We define the weight of f to be $w(f)=\sum_{v\in V} f(v)$, and for $S\subseteq V$ we define $f(S)=\sum_{v\in S} f(v)$, so $w(f)=f(V)=\sum_{i=1}^{k} i|V_i|$. The *k-Roman domination number*, denoted $\gamma_R^k(G)$, is the minimum weight of a kRDF in G; that is, $\gamma_R^k(G)=\min\{w(f)\mid f$ is a kRDF in $G\}$. A kRDF of weight $\gamma_R^k(G)$ we call a $\gamma_R^k(G)$-function.

This definition of a kRDF is motivated as follows. Using notation introduced earlier, we define a location to be *undefended* if the location and every location adjacent to it is unsecured (i.e., have no legion stationed there). Since an undefended location is vulnerable to an attack, we require that every unsecure location be adjacent to a secure location in such a way that the movement of a legion from the secure location to the unsecure location does not create an undefended location. Hence every unsecure location can be defended without creating an undefended location. We make the assumption that multiple attacks, if any, are consecutive (and do not occur simultaneously). Further, we make the assumption that a secure location can defend against an attack. Hence in our model we may assume that any attacked location is an unsecure location. Thus in the event of k consecutive attacks, we require that there exist k (consecutive) movements of a legion(s) from secure locations to unsecure locations so that each movement does not create an undefended location. In this way Emperor Constantine the Great can save substantial costs of maintaining legions while still defending the Roman Empire from k consecutive attacks. Such a placement of legions corresponds to a kRDF and a minimum such placement of legions corresponds to a minimum kRDF.

This graph–theoretic model of defending the Roman Empire from multiple attacks has applications in, among others, facility location problems. For example, this model can be used to deploy fire-engines in a cost effective manner while guaranteeing a rapid response should there be an outbreak of any k consecutive fires in the region.

2. Notation

For notation and graph theory terminology we in general follow [8]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v)=\{u\in V \mid uv\in E\}$ and the closed neighborhood of v is $N[v]=\{v\}\cup N(v)$. For a set $S\subseteq V$, its *open neighborhood* $N(S)=\bigcup_{v\in S} N(v)$ and its *closed neighborhood* $N[S]=N(S)\cup S$. A vertex u is called a *private neighbor of* v *with respect to* S, or simply an S*-pn of* v, if $N[u]\cap S=\{v\}$. The set $\mathrm{pn}(v,S)=N[v]-N[S-\{v\}]$ of all S-pns of v is called the *private neighbor set of* v *with respect to* S. We define the *external private neighbor set of* v *with respect to* S by $\mathrm{epn}(v,S)=\mathrm{pn}(v,S)-\{v\}$. Hence the set $\mathrm{epn}(v,S)$ consists of all S-pns of v that belong to $V-S$.

For ease of presentation, we mostly consider *rooted trees*. If a vertex v in a rooted tree T is adjacent to u and u lies in the level below v, then u is called a *child* of v, and v is the *parent* of u. We let $C(v)$ denote the set of children of v in T. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. We call a support vertex adjacent to at least r leaves a r*-support vertex*. In this paper, we denote the set of all r-support vertices of T by $S_r(T)$.

The *corona* $\mathrm{coro}(G)$ of a graph G is that graph obtained from G by adding a pendant edge to each vertex of G.

Let $G=(V,E)$ be a graph and let $S\subseteq V$. A set S dominates a set U, denoted $S\succ U$, if every vertex in U is adjacent to a vertex of S. Furthermore, if $S=\{v\}$ for some $v\in V$, then we simply write $v\succ U$. If $S\succ V-S$, then S is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ we call a $\gamma(G)$-set. The *independence number* $\beta(G)$ is the maximum cardinality of an independent set of vertices in G. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [8,9].

In this paper we show that for any graph G and any integer $k \geqslant 1$, $\gamma(G) \leqslant \gamma_R^k(G) \leqslant (k+1)\gamma(G)$. In Section 3, we investigate graphs G with $\gamma(G) = \gamma_R^k(G)$ and we characterize trees T for which $\gamma(T) = \gamma_R^k(T)$. In Section 4, we investigate graphs G with $\gamma(G)=(k+1)\gamma_R^k(G)$ and we characterize trees T for which $\gamma(T)=(k+1)\gamma_R^k(T)$. We show in Section 5 that the decision problem to determine whether a graph G has a kRDF of weight at most j for any given integer j where $j \leqslant (k+1)|V(G)|$ is NP-complete even for bipartite or chordal graphs.

3. Graphs with small *k*-Roman domination number

Our aim in this section is threefold: First we show that for any graph G, $\gamma(G) \leqslant \gamma_R^k(G)$. Second, we give a necessary and sufficient condition for a graph G to satisfy $\gamma(G) = \gamma_R^k(G)$. Third, we give a characterization of trees T for which $\gamma(T) = \gamma_R^k(T)$.

Theorem 1. *For any graph $G = (V,E)$ and for $k \geqslant 1$,*

$$\gamma(G) \leqslant \gamma_R^k(G)$$

with equality if and only if there exists a $\gamma(G)$-set S such that for any sequence $v_1,\dots,v_k$ of vertices of V, there exists a sequence $S_0,S_1,\dots,S_k$ of $\gamma(G)$-sets such that $S_0 = S$, and for $i = 1,\dots,k$, either $v_i \in S_{i-1}$, in which case $S_i = S_{i-1}$, or $v_i \notin S_{i-1}$, in which case there exists a vertex $u_i \in S_{i-1}$ adjacent to v_i and $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$.

Proof. Let $G = (V,E)$ and let $f = (V_0,V_1,\dots,V_{k+1})$ be a $\gamma_R^k(G)$-function. Let $D = \bigcup_{i=1}^{k+1} V_i$. Then, since $D \succ V_0$, D is a dominating set of G, and so

$$\gamma(G) \leqslant |D| = \sum_{i=1}^{k+1} |V_i| \leqslant \sum_{i=1}^{k+1} i \cdot |V_i| = w(f) = \gamma_R^k(G).$$

Hence $\gamma(G) \leqslant \gamma_R^k(G)$. Suppose that $\gamma(G)=\gamma_R^k(G)$. Then we must have equality throughout the above inequality chain. In particular, it follows that $V_i = \emptyset$ for $2 \leqslant i \leqslant k+1$. Thus, $S = V_1$ is a $\gamma(G)$-set.

Suppose that $v_1,\dots,v_k$ is any sequence of vertices of G. Since f is a $\gamma_R^k(G)$-function and since $S = V_1$, there exists a sequence of functions $f = f_0, f_1,\dots,f_k$ such that for $i = 1,\dots,k$, (i) either $f_{i-1}(v_i) = 1$, in which case $f_i = f_{i-1}$, or $f_{i-1}(v_i) = 0$, in which case f_i is obtained from f_{i-1} by one movement to v_i, and (ii) f_i has no undefended vertex. Each of the functions $f_0, f_1,\dots,f_k$ have equal weight, namely $|V_1|=|S|=\gamma(G)$, and the weight of every vertex under f_i is either 0 or 1. For $i=0,1,\dots,k$, let S_i denote the set of vertices that have weight 1 under f_i. In particular, $S_0 = S$. By the way in which f_i is constructed, either $v_i \in S_{i-1}$, in which case $S_i=S_{i-1}$, or $v_i \notin S_{i-1}$ and there exists a vertex $u_i \in S_{i-1}$ adjacent to v_i such that $S_i=(S_{i-1}-\{u_i\})\cup\{v_i\}$. Further, since f_i has no undefended vertex, each set S_i is a dominating set of G and is therefore a $\gamma(G)$-set. This establishes the necessity.

To prove the sufficiency, suppose there exists a $\gamma(G)$-set S satisfying the hypothesis of the theorem. Let $f = (V_0, V_1,\dots,V_{k+1})$ be the function defined by $V_1 = S$ and $V_i = \emptyset$

for $2 \leqslant i \leqslant k+1$. Since S is a dominating set of G, f has no undefended vertex. Let $v_1, \ldots, v_k$ be any sequence of vertices in G. By hypothesis, there exists a sequence $S_0, S_1, \ldots, S_k$ of $\gamma(G)$-sets such that $S_0 = S$, and for $i = 1, \ldots, k$, either $v_i \in S_{i-1}$, in which case $S_i = S_{i-1}$, or $v_i \notin S_{i-1}$, in which case there exists a vertex $u_i \in S_{i-1}$ adjacent to v_i and $S_i = (S_{i-1} - \{u_i\}) \cup \{v_i\}$. For $i = 1, \ldots, k$, let f_i be the characteristic function of S_i, i.e., $f(v) = 1$ if $v \in S_i$, otherwise $f(v) = 0$. Then $f = f_0, f_1, \ldots, f_k$ is a sequence of functions such that for $i = 1, \ldots, k$, (i) either $f_{i-1}(v_i) = 1$, in which case $f_i = f_{i-1}$, or $f_{i-1}(v_i) = 0$, in which case f_i is obtained from f_{i-1} by one movement to v_i, and (ii) f_i has no undefended vertex. Hence f is a kRDF of T. Thus, $\gamma_R^k(G) \leqslant w(f) = |S| = \gamma(G)$. Since $\gamma(G) \leqslant \gamma_R^k(G)$ for all graphs G, $\gamma(G) = \gamma_R^k(G)$ as desired. □

As an immediate consequence of Theorem 1, we have the following result.

Corollary 2. *Let $k \geqslant 1$ and let $G = (V, E)$ be a graph with $\gamma(G) = \gamma_R^k(G)$. If S is a $\gamma(G)$-set satisfying the hypothesis of Theorem* 1, *then*

(1) $\mathrm{epn}(v, S) \cup \{v\}$ *induces a clique for every* $v \in S$;
(2) *for every vertex $u \in V - S$ that is not a private neighbor of any vertex of S, there exists a vertex $v \in S$ such that* $u \succ \mathrm{epn}(v, S) \cup \{v\}$.

Proof. (1) Let $u \in \mathrm{epn}(v, S)$. Then, by Theorem 1, $(S - \{v\}) \cup \{u\}$ is a $\gamma(G)$-set. Since v is the only vertex of S adjacent to each vertex of $\mathrm{epn}(v, S)$, u must be adjacent to every other vertex of $\mathrm{epn}(v, S)$.

(2) Suppose $u \in V - S$ is not a private neighbor of any vertex of S. Then, by Theorem 1, there exists a vertex $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is a $\gamma(G)$-set. Hence, as in the proof of (1), $u \succ \mathrm{epn}(v, S) \cup \{v\}$. □

We now provide a characterization of trees T for which $\gamma(T) = \gamma_R^k(T)$. For this purpose, we introduce a family $\mathscr{T}$ of trees as follows: Let a be a positive integer. For $i = 1, \ldots, a$, let $T_i = K_{1,n_i}$ where $n_i \geqslant 2$, and let v_i be the center of T_i. Let $S_A = \{v_1, \ldots, v_a\}$ and let L_A denote the set of all leaves of these a stars. Let b be an integer satisfying $b \geqslant (\sum_{i=1}^{a} n_i) - a + 1$. Let $T_0 = bK_2$ and let S_B be an independent set of b vertices in T_0 (one from each copy of K_2 in T_0). Let T be a tree obtained from the disjoint union $\bigcup_{i=0}^{a} T_i$ of $T_0, T_1, \ldots, T_a$ by adding $a + b - 1$ edges such that (i) each added edge joins vertices of $L_A \cup S_B$, (ii) each vertex of L_A is adjacent to at least one vertex of S_B, and (iii) each vertex of S_B is incident with at least one added edge (and so has degree at least 2 in T). Let $\mathscr{T}$ be the family of all such trees T. An example of a tree in the family $\mathscr{T}$ is shown in Fig. 1.

Theorem 3. *For $k \geqslant 1$, a tree T satisfies $\gamma(T) = \gamma_R^k(T)$ if and only if $k = 1$ and $T \in \mathscr{T}$, or $k \geqslant 1$ and T is the corona of a tree.*

Proof. If T is the corona of a tree, then the function that assigns to each leaf of T the weight 1 and to every other vertex of T the weight 0 is a kRDF of weight

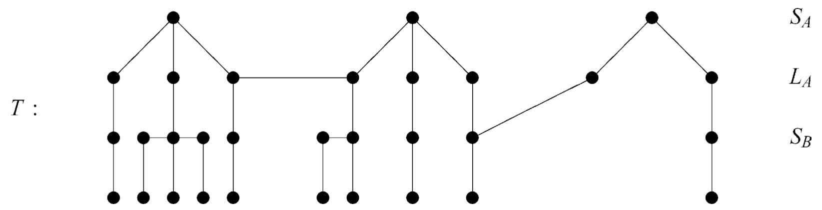

Fig. 1. A tree $T \in \mathscr{T}$.

$\gamma(T) = |V(T)|/2$, and so $\gamma_R^k(T) \leqslant \gamma(T)$. By Theorem 1, $\gamma(T) \leqslant \gamma_R^k(T)$. Consequently, $\gamma(T) = \gamma_R^k(T)$.

Suppose $k = 1$ and $T \in \mathscr{T}$. Using the notation introduced earlier when constructing the tree T, we note that $S = S_A \cup S_B$ is a $\gamma(T)$-set and S_A is a packing in T (i.e., a set of vertices pairwise at distance at least 3 apart in T). Furthermore, if $v \in S_B$, then $\text{epn}(v, S)$ consists only of the leaf adjacent to v in T, while if $v \in S_A$, then $\text{pn}(v, S) = \{v\}$. Let $u \in V(T) - S$. If u is a leaf adjacent to a vertex $v \in S_B$, then $(S - \{v\}) \cup \{u\}$ is a $\gamma(T)$-set. On the other hand, if $u \in L_A$ and v is the vertex of S_A adjacent to u, then $(S - \{v\}) \cup \{u\}$ is a $\gamma(T)$-set. Hence by Theorem 1, $\gamma(T) = \gamma_R^1(T)$. This establishes the sufficiency.

To prove the necessity, suppose that $\gamma(T) = \gamma_R^k(T)$. Then there exists a $\gamma(T)$-set S satisfying the hypothesis of Theorem 1. We may assume that S contains no leaf of T, for otherwise we can simply replace a leaf in S with its neighbor. Thus we may assume each vertex of S has degree at least 2 in T.

Let $S_B = \{v \in S \mid \text{epn}(v, S) \neq \emptyset\}$ and let $S_A = S - S_B$. By Corollary 2, $\text{epn}(v, S)$ induces a clique for every $v \in S$, and so since T is a tree, $|\text{epn}(v, S)| = 1$ for every $v \in S_B$. Let $S'_B = \{\text{epn}(v, S) \mid v \in S_B\}$. Then, $|S'_B| = |S_B|$.

We proceed further with six claims.

Claim 1. *S_A is a packing in T.*

Proof. For every $v \in S_A$, $\text{epn}(v, S) = \emptyset$ and so $\text{pn}(v, S) = \{v\}$. Hence each vertex of S_A is isolated in $\langle S \rangle$ (otherwise, $S - \{v\}$ should be a dominating set of T). Thus, $N(S_A) \subseteq V - S$. If $d(v_1, v_2) = 2$ where $v_1, v_2 \in S_A$, then there exists a unique vertex $u \in N(S_A)$ adjacent to both v_1 and v_2. But then $(S - \{v_1, v_2\}) \cup \{u\}$ is a dominating set of T of cardinality less than $\gamma(T)$, which is impossible. Hence S_A is a packing in T. $\square$

Claim 2. $V - S - S'_B = N(S_A)$.

Proof. Let $u \in V - S - S'_B$. Then by Theorem 1, there exists a vertex $v \in S$ adjacent to u such that $(S - \{v\}) \cup \{u\}$ is a $\gamma(T)$-set. Since $u \notin S'_B$ and there exists at least one vertex u' in S'_B adjacent to v, v cannot be in S_B (otherwise, u' should not be dominated by $(S - \{v\}) \cup \{u\}$); therefore, $v \in S_A$ and $u \in N(S_A)$. Thus, $V - S - S'_B \subseteq N(S_A)$. Since S_A is a packing in T, $N(S_A) \subseteq V - S - S'_B$. The desired result follows. $\square$

Claim 3. *Each vertex of $N(S_A)$ is adjacent to a unique vertex of S_A and to at least one vertex of S_B.*

Proof. By Claim 1, S_A is a packing in T, and so each vertex of $N(S_A)$ is adjacent to a unique vertex of S_A. Since $\text{epn}(v,S)=\emptyset$ for each $v\in S_A$, each vertex of $N(S_A)$ is adjacent to at least one vertex of S_B. □

Claim 4. *S'_B is an independent set.*

Proof. Let $v_1,v_2\in S_B$. For $i=1,2$, let $\text{epn}(v_i,S)=\{u_i\}$. Suppose $u_1u_2\in E(T)$. Since $\deg_T v_1\geqslant 2$, there exists a vertex $x\in N(v_1)-\{u_1\}$. If $x\in S$, then $(S-\{v_1,v_2\})\cup\{u_2\}$ is a dominating set of T of cardinality less than $\gamma(T)$, which is impossible. Hence, $x\in V-S-S'_B=N(S_A)$. Let v be the vertex of S_A adjacent to x. Then, $(S-\{v,v_1,v_2\})\cup\{u_2,x\}$ is a dominating set of T of cardinality less than $\gamma(T)$, which is impossible. Hence, $u_1u_2\notin E(T)$. The desired result follows. □

Claim 5. *Each vertex of S'_B is a leaf.*

Proof. Let $v_1\in S_B$ and let $\text{epn}(v_1,S)=\{u_1\}$. Suppose u_1 is not a leaf. Let $u_2\in N(u_1)-\{v_1\}$. Since S'_B is an independent set, it follows that $u_2\in N(S_A)$. Let v_2 be the vertex of S_A adjacent to u_2. Since $\deg_T v_1\geqslant 2$, there exists a vertex $x\in N(v_1)-\{u_1\}$. If $x\in S$, then $(S-\{v_1,v_2\})\cup\{u_2\}$ is a dominating set of T of cardinality less than $\gamma(T)$,which is impossible. Hence, $x\in V-S-S'_B=N(S_A)$. Let v be the vertex of S_A adjacent to x. Then, $(S-\{v,v_1,v_2\})\cup\{u_2,x\}$ is a dominating set of T of cardinality less than $\gamma(T)$, which is impossible. Hence, u_1 is a leaf. The desired result follows. □

Claim 6. *If $k\geqslant 2$, then $S_A=\emptyset$.*

Proof. Suppose $k\geqslant 2$ and $S_A\neq\emptyset$. Let F be the bipartite graph with partite sets S_A and $N(S_A)$ and with edge set $E(F)=\{uv\in E(T)\mid v\in S_A, u\in N(S_A)\}$. Since S_A is a packing in T, each vertex of $N(S_A)$ has degree 1 in F. Since each vertex of S has degree at least 2, each vertex of S_A has degree at least 2 in F. Let $u\in N(S_A)$ and let $v\in S_A$ be the neighbor of u in F. Let $w\in N(v)-\{u\}$.

Consider now the sequence $v_1,v_2,\ldots,v_k$ of vertices, where $v_1=w$ and $v_2=u$. Let $S_0,S_1,\ldots,S_k$ be the corresponding sequence of $\gamma(T)$-sets satisfying the hypothesis of Theorem 1. By Theorem 1, there exists a vertex $w'\in S$ adjacent to $v_1=w$ such that $(S-\{w'\})\cup\{w\}$ is a $\gamma(T)$-set. Since $w'\in S_A$, we must have $v=w'$. Thus, $S_1=(S-\{v\})\cup\{w\}$. Since $v_2=u\notin S_1$, there exists a vertex $u_2\in S_1$ adjacent to v_2 such that $S_2=(S_1-\{u_2\})\cup\{v_2\}$. Since, by Claim 3, $N(v_2)\cap S_1\subseteq S_B$, $u_2\in S_B$. Let $u'=\text{epn}(u_2,S)$. By Claim 5, u' is not adjacent to v_1 or v_2. But then u' is not dominated by S_2, a contradiction. Hence, if $k\geqslant 2$, then $S_A=\emptyset$. □

If $S_A=\emptyset$, then by Claim 2, $V(T)=S_B\cup S'_B$. By Claim 5, each vertex of S'_B is a leaf. Thus, T is the corona of the tree $\langle S_B\rangle$ induced by S_B, i.e, $T=\text{cor}(\langle S_B\rangle)$. Hence by Claim 6, it remains only to consider the case when $k=1$ and $S_A\neq\emptyset$. Letting

$N(S_A)=L_A$, it follows readily from Claims 1, 2, 3 and 5, and the fact that each vertex of S has degree at least 2 in the tree T, that $T\in\mathscr{T}$. □

4. Graphs with large *k*-Roman domination number

Our aim in this section is first to show that for any graph G, $\gamma_R^k(G)\leqslant(k+1)\gamma(G)$ and, second, to characterize forests G for which $\gamma_R^k(G)=(k+1)\gamma(G)$.

Lemma 4. *For any graph G and for $k\geqslant 1$,*

$$\gamma_R^k(G)\leqslant(k+1)\gamma(G).$$

Proof. Let $G=(V,E)$ and let $S=\{v_1,\dots,v_\gamma\}$ be a $\gamma(G)$-set. Let $f=(V_0,V_1,\dots,V_{k+1})$ be the function defined by $V_0=V-S$, $V_i=\emptyset$ for $1\leqslant i\leqslant k$ and $V_{k+1}=S$. Since $S\succ V$, we can partition V into sets $W_1,\dots,W_\gamma$, where $v_i\succ W_i$ for each $i=1,\dots,\gamma$. The movement of k legions from v_i to vertices in W_i cannot create an undefended vertex since v_i will still have positive weight and $v_i\succ W_i$. Hence, f is a kRDF, and so $\gamma_R^k(G)\leqslant(k+1)\gamma(G)$. □

Lemma 5. *If $k\geqslant 1$ and G is a graph satisfying $\gamma_R^k(G)=(k+1)\gamma(G)$, then for every $\gamma(G)$-set S and every $v\in S$, $\mathrm{epn}(v,S)$ contains an independent set of $k+1$ vertices.*

Proof. Let $G=(V,E)$ and let $S=\{v_1,\dots,v_\gamma\}$ be a $\gamma(G)$-set. Suppose that $\mathrm{epn}(v_1,S)$ contains no set of $k+1$ independent vertices. Let $G_1=\langle\mathrm{epn}(v_1,S)\cup\{v_1\}\rangle$. Then, $\beta(G_1)\leqslant k$. Since $S\succ V$, we can partition V into sets $W_1,W_2,\dots,W_\gamma$, where $W_1=V(G_1)$ and for $i=2,\dots,\gamma$, $v_i\succ W_i$ for each $i=1,\dots,\gamma$. Let $f=(V_0,V_1,\dots,V_{k+1})$ be the function defined by $V_0=V-S$, $V_k=\{v_1\}$, $V_{k+1}=S-\{v_1\}$ and, if $k\geqslant 2$, $V_i=\emptyset$ for $1\leqslant i\leqslant k-1$.

For $i\geqslant 2$, the movement of k legions from v_i to vertices in W_i cannot create an undefended vertex since after any such movement, v_i will still have positive weight and $v_i\succ W_i$. Hence to prove that f is a kRDF, it suffices to show that the movement of k legions between the vertices of G_1 cannot create an undefended vertex. Suppose, then, that $w_1,\dots,w_k$ is the sequence of (attacked) vertices in G_1 corresponding to such a movement of k legions in G_1.

Initially, we move one legion from v_1 to w_1. If $k=1$, then $\beta(G_1)\leqslant k$ implies that G_1 is a clique, and therefore this movement cannot create an undefended vertex. If $k\geqslant 2$, then for $i=2,\dots,k$, we define our movement as follows: If w_i is adjacent to w_j for some $j<i$, then we move a legion from w_j to w_i (note that after such a movement, v_1 has positive weight and therefore no undefended vertex is created); otherwise, if w_i is not adjacent to w_j for any $j<i$, then we move a legion from v_1 to w_i. After such a movement of k legions, either a legion was moved between two vertices w_j and w_i for some i, j with $1\leqslant j<i\leqslant k$, in which case v_1 has positive weight and therefore no undefended vertex is created, or no legion was moved between two vertices of $\mathrm{epn}(v_1,S)$, in which case $W=\{w_1,w_2,\dots,w_k\}$ is an independent set and each vertex of W has weight 1. By assumption, $\beta(G_1)\leqslant k$ and so W must be a

maximum independent set in G_1. But then $W \succ V(G_1)$, implying that no undefended vertex is created. Hence f is a kRDF, and so $\gamma_R^k(G) \leqslant w(f) = (k+1)\gamma(G) - 1$, contrary to assumption. Therefore, $\mathrm{epn}(v_1, S)$ must contain an independent set of $k+1$ vertices. Similarly, for $i = 2, \ldots, \gamma$, $\beta(\langle \mathrm{epn}(v_i, S) \rangle) \geqslant k+1$. □

The necessary condition in Lemma 5 for a graph G satisfying $\gamma_R^k(G) = (k+1)\gamma(G)$ is not sufficient. For example, for $k \geqslant 1$ and $i = 1, 2$, let F_i be a star $K_{1,k+1}$ with center v_i and with w_i a leaf. Let G be obtained from $F_1 \cup F_2$ by adding the edge $w_1 w_2$. Then, $S = \{v_1, v_2\}$ is the unique $\gamma(G)$-set. For $i = 1, 2$, $\mathrm{epn}(v_i, S) = N(v_i)$ is an independent set of $k+1$ vertices. However the function f defined by $f(v_1) = f(v_2) = k$, $f(w_1) = 1$, and $f(v) = 0$ for all other vertices v of G, is a kRDF of G, and so $\gamma_R^k(G) \leqslant w(f) < 2(k+1) = (k+1)\gamma(G)$. This example is easily extendable to graphs G with arbitrary domination number at least 2.

Gunther et al. [7] presented the following characterization of trees with unique minimum dominating sets.

Theorem 6 (Gunther et al. [7]). *Let T be a tree of order at least 3. Then, T has a unique $\gamma(T)$-set if and only if T has a $\gamma(T)$-set S such that $|\mathrm{epn}(v, S)| \geqslant 2$ for every vertex $v \in S$.*

As an immediate consequence of Lemma 5 and Theorem 6, we have the following result.

Corollary 7. *If $k \geqslant 1$ and T is a tree satisfying $\gamma_R^k(T) = (k+1)\gamma(T)$, then T has a unique $\gamma(T)$-set.*

The necessary condition in Corollary 7 for a tree T satisfying $\gamma_R^k(T) = (k+1)\gamma(T)$ is not sufficient as may be seen by considering the tree T constructed in the paragraph following the proof of Lemma 5.

Recall that a support vertex of a tree T is a vertex adjacent to a leaf, while an r-support vertex is adjacent to at least r leaves. Further, the set of all r-support vertices of T is denoted by $S_r(T)$. Note that $S_{r+1}(T) \subseteq S_r(T)$. For $r \geqslant 2$, every r-support vertex of a tree T belongs to every $\gamma(T)$-set. Hence, if $r \geqslant 2$ and a tree T has a unique $\gamma(T)$-set S, then $S_r(T) \subseteq S$. We state this as an observation.

Observation 8. *If $k \geqslant 1$ and T is a tree with a unique $\gamma(T)$-set S, then $S_{k+1}(T) \subseteq S$.*

Lemma 9. *If $k \geqslant 1$ and T is a tree with a unique $\gamma(T)$-set S, and if every vertex of S is a $(k+1)$-support vertex, then $\gamma_R^k(T) = (k+1)\gamma(T)$.*

Proof. By Observation 8, $S_{k+1}(T) \subseteq S$. By assumption, every vertex of S is a $(k+1)$-support vertex, and so $S \subseteq S_{k+1}(T)$. Consequently, $S = S_{k+1}(T)$. Let f be a $\gamma_R^k(T)$-function. We show that $w(f) \geqslant (k+1)\gamma(T)$. For each $v \in S$, let N_v consist of v and every leaf adjacent to v. Since $S = S_{k+1}(T)$, v is adjacent to at least $k+1$

leaves, and so $|N_v| \geqslant k+2$. Suppose $\ell \geqslant 0$ leaves adjacent to v have weight 0 under f. If $\ell \geqslant k+1$, then $f(N_v) \geqslant k+1$. On the other hand, if $\ell \leqslant k$, then $f(v) \geqslant \ell$, and so $f(N_v) \geqslant f(v)+(|N_v|-1-\ell) \geqslant |N_v|-1 \geqslant k+1$. In any event, $f(N_v) \geqslant k+1$. Since the sets $\bigcup_{v\in S} N_v$ are disjoint sets in T, it follows that $\gamma_R^k(T)=w(f) \geqslant \sum_{v\in S} f(N_v) \geqslant (k+1)|S| = (k+1)\gamma(T)$. Consequently, by Lemma 4, $\gamma_R^k(T)=(k+1)\gamma(T)$. □

Lemma 10. *If T is a tree with a unique $\gamma(T)$-set S, and if no vertex of S is a $(k+1)$-support vertex, then $\gamma_R^k(T) < (k+1)\gamma(T)$.*

Proof. We proceed by induction on $\gamma(T)$. Suppose $\gamma(T)=1$. Let $S=\{v\}$. Then, T is a star $K_{1,n}$. Since S is a unique $\gamma(T)$-set, $n \geqslant 2$. On the other hand, since S has no $(k+1)$-support vertex, $n \leqslant k$. Let f be the function that assigns the weight k to v and the weight 0 to every other vertex. Then, f is a kRDF, and so $\gamma_R^k(T) \leqslant w(f) = k < k+1 = (k+1)\gamma(T)$. This establishes the base case.

Suppose, then, that the result of the lemma is true for all trees T' with $\gamma(T') < t$, where $t \geqslant 2$, that satisfy the hypothesis in the statement of the lemma. Let $T=(V,E)$ be a tree with $\gamma(T)=t$ and with a unique $\gamma(T)$-set S such that no vertex of S is a $(k+1)$-support vertex. Let T be rooted at an endvertex r of a longest path. Let w be a vertex at distance $\mathrm{diam}(T)-2$ from r on a longest path starting at r, and let v be the child of w on this path. Let x denote the parent of w, and let y denote the parent of x.

By Observation 8, $S_{k+1}(T) \subseteq S$. Hence, since S is the unique $\gamma(T)$-set and no vertex of S is a $(k+1)$-support vertex, $S_{k+1}(T)=\emptyset$. By Theorem 6, no leaf belongs to S, and so $v\in S$. Therefore v is adjacent to at most k leaves. If $\deg v \leqslant k$, then $\mathrm{epn}(v,S)$ does not contain an independent set of $k+1$ vertices, and so, by Lemma 5, $\gamma_R^k(T) < (k+1)\gamma(T)$. Hence we may assume that $\deg v = k+1$ and that $\mathrm{epn}(v,S)=N(v)$. It follows that $\deg w = 2$ and that $w,x \notin S$. Thus, x cannot be a support vertex.

Suppose x has a child w' that is a support vertex. Then it follows from Theorem 6 that $w'\in S$. If w' has a child v' that is a support vertex, then (as with the vertex v) $\deg v' = k+1$, $\mathrm{epn}(v',S)=N(v')$ and $w' \notin S$, a contradiction. Hence every child of w' is a leaf. But then $\deg w' = k+1$ and $\mathrm{epn}(w',S)=N(w')$. Let $f=(V_0,V_1,\ldots,V_{k+1})$ be the function defined by $V_0 = V-S$, $V_{k+1} = S-\{v,w'\}$, $V_1=\{v,w',x\}$ if $k=1$ while $V_1=\{x\}$ and $V_k=\{v,w'\}$ if $k \geqslant 2$, and if $2 \leqslant i \leqslant k-1$, let $V_i=\emptyset$. Then f is a kRDF, and so $\gamma_R^k(T) \leqslant w(f)=(k+1)|S|-1=(k+1)\gamma(T)-1$, contrary to assumption. Hence, no child of x is a support vertex.

Suppose $\deg x \geqslant 3$. Let $w' \in C(x)-\{w\}$. Then, w' is neither a leaf nor a support vertex. Let v' be a child of w' and let u' be a child of v'. As shown earlier (with the vertex v), $\deg v' = k+1$, $v'\in S$ and $\deg w' = 2$. Let $g=(V_0,V_1,\ldots,V_{k+1})$ be the function defined by $V_0=V-S$, $V_{k+1}=S-\{v,v'\}$, $V_1=\{x,v,v'\}$ if $k=1$ while $V_1=\{x\}$ and $V_k=\{v,v'\}$ if $k \geqslant 2$, and if $2 \leqslant i \leqslant k-1$, let $V_i=\emptyset$. Then g is a kRDF, and so $\gamma_R^k(T) \leqslant w(g)=(k+1)|S|-1=(k+1)\gamma(T)-1$, contrary to assumption. Hence, $\deg x = 2$. Since $w,x \notin S$, we must therefore have $y\in S$.

Let $T'=T-C(v)-\{v,w,x\}$. Since $y\in S$, $S-\{v\}$ is a dominating set of T', and so $\gamma(T') \leqslant |S|-1=\gamma(T)-1$. Let h' be a $\gamma_R^k(T')$-function and let $h: V \to \{0,1,\ldots,k+1\}$ be the function defined by $h(z)=h'(z)$ if $z\in V(T')$, $h(x)=1$, $h(v)=k$, $h(w)=0$ and

$h(u)=0$ for each child u of v. Then, h is a kRDF of T, and so

$$\gamma_R^k(T) \leqslant w(h) = w(h') + k + 1 \leqslant (k+1)\gamma(T') + k + 1 \leqslant (k+1)\gamma(T). \tag{1}$$

Suppose we have equality throughout the inequality chain (1). In particular, $\gamma_R^k(T')=(k+1)\gamma(T')$ and $\gamma(T')=\gamma(T)-1$. By Corollary 7, T' has a unique $\gamma(T')$-set, namely $S'=S-\{v\}$. In particular, since $y\in S'$, y is not a leaf in T'. Hence, every leaf in T' is also a leaf in T. Therefore since T has no $(k+1)$-support vertex, neither does T'. Consequently, T' is a tree with $\gamma(T')<t$ and with a unique $\gamma(T')$-set S' such that no vertex of S' is a $(k+1)$-support vertex. Applying the inductive hypothesis to T', $\gamma_R^k(T')<(k+1)\gamma(T')$, a contradiction. Hence, we cannot have equality throughout the inequality chain (1), i.e., $\gamma_R^k(T)<(k+1)\gamma(T)$ as desired. □

As an immediate consequence of Lemma 10 we have the following result.

Corollary 11. *If F is a forest with a unique $\gamma(F)$-set S, and if F has a component with no $(k+1)$-support vertex, then $\gamma_R^k(F)<(k+1)\gamma(F)$.*

In order to characterize the trees T for which $\gamma_R^k(T)=(k+1)\gamma(T)$, we construct a family $\mathscr{F}$ of forests as follows.

Let F be a forest with a unique $\gamma(F)$-set S such that each component of F contains a $(k+1)$-support vertex. It follows from Observation 8 that $S_{k+1}(F)\subseteq S$. If $S_{k+1}(F)=S$, then we let $\tilde{F}=F$. Otherwise, if $S_{k+1}(F)\neq S$, then we define the subforest $\tilde{F}$ of F recursively by means of a sequence of subforests $F_0,F_1,\ldots,F_t$ of F, where $F_0=F$, as follows: For $i=0,\ldots,t-1$, let $S_i=S\cap V(F_i)$. If every component of F_i has a $(k+1)$-support vertex and if $S_i-S_{k+1}(F_i)\neq\emptyset$, then let

$$F_{i+1}=F_i-\left(\bigcup_{v\in S_{k+1}(F_i)} N[v]-(S_i-S_{k+1}(F_i))\right).$$

Hence, F_{i+1} is obtained from F_i by deleting all vertices, except for possibly any vertices of $S_i-S_{k+1}(F_i)$, in the closed neighborhoods of every $(k+1)$-support vertex in F_i. Since $\gamma(F)$ is finite, there exists an integer $t\geqslant 1$ such that F_t has a component with no $(k+1)$-support vertex or $S_t=S_{k+1}(F_t)$. Then, $\tilde{F}=F_t$. For $i=1,\ldots,t$, we call F_{i+1} the *pruning of F_i* and we define t to be the *number of prunings of the forest F*. Note that, if $t>i\geqslant 0$, then $S_{i+1}=S_i-S_{k+1}(F_i)$. The three prunings F_0, F_1 and F_2 of a forest $F=F_0$ with $k=2$ are illustrated in Fig. 2, where for $i=0,1,2$, the unique $\gamma(F_i)$-set S_i is indicated by the large darkened vertices.

Observation 12. *For $i=0,\ldots,t$, the set S_i is the unique $\gamma(F_i)$-set.*

Proof. We proceed by induction on i. If $i=0$, then $S_0=S$ and $F_0=F$, and so S_0 is the unique $\gamma(F_0)$-set. Thus the statement is true for $i=0$. Suppose that the set S_m is the unique $\gamma(F_m)$-set, where $0\leqslant m<t$. By construction, S_{m+1} is a dominating set of F_{m+1}, and so $\gamma(F_{m+1})\leqslant|S_{m+1}|$. If $\gamma(F_{m+1})<|S_{m+1}|$, then adding the set $S_{k+1}(F_m)$ to any $\gamma(F_{m+1})$-set produces a dominating set of F_m of cardinality $|S_{k+1}(F_m)|+\gamma(F_{m+1})<$

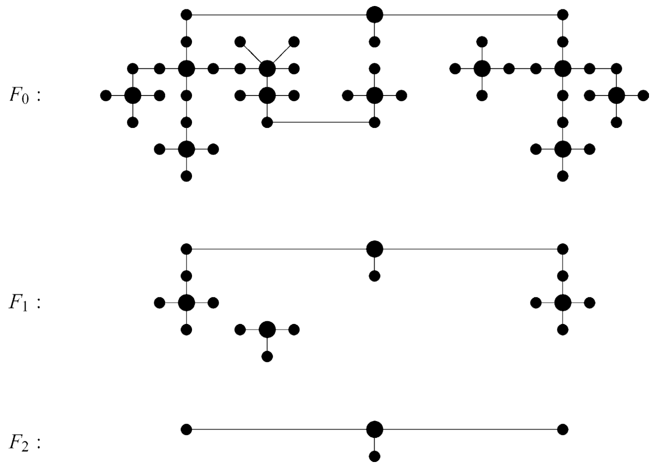

Fig. 2. Prunings of the forest $F = F_0$ with $k = 2$.

$|S_{k+1}(F_m)| + |S_{m+1}| = |S_{k+1}(F_m)| + |S_m - S_{k+1}(F_m)| = |S_m| = \gamma(F_m)$, which is impossible. Hence, $\gamma(F_{m+1}) = |S_{m+1}|$. If F_{m+1} has two distinct $\gamma(F_{m+1})$-sets X and Y, then $X \cup S_{k+1}(F_m)$ and $Y \cup S_{k+1}(F_m)$ are both $\gamma(F_m)$-sets, contradicting the inductive hypothesis that S_m is the unique $\gamma(F_m)$-set. Hence, S_{m+1} is the unique $\gamma(F_{m+1})$-set. □

We define the family $\mathscr{F}$ to consist of all forests F, every component of which contains a $(k+1)$-support vertex, that have a unique $\gamma(F)$-set S such that $\tilde{F} = F_t$ and $S_t = S_{k+1}(F_t)$. Note that if $F \in \mathscr{F}$ and $\tilde{F} = F_t$, then each of the subgraphs $F_0, \ldots, F_t$ belong to the family $\mathscr{F}$.

Lemma 13. *If $F \in \mathscr{F}$, then $\gamma_R^k(F) = (k+1)\gamma(F)$.*

Proof. We proceed by induction on the number t of prunings of the forest F. Let S be the unique $\gamma(F)$-set. We shall adopt the notation introduced in constructing the family $\mathscr{F}$. Suppose $t = 0$. Then, $\tilde{F} = F$ and $S = S_{k+1}(F)$. Thus every vertex of S is a $(k+1)$-support vertex. Hence it follows from Lemma 9 that $\gamma_R^k(F) = (k+1)\gamma(F)$. Therefore the base case when $t = 0$ is true.

Suppose that all forests $F \in \mathscr{F}$ with $\tilde{F} = F_m$ where $0 \leqslant m < t$ satisfy $\gamma_R^k(F) = (k+1)\gamma(F)$. Let $F \in \mathscr{F}$ satisfy $\tilde{F} = F_t$. Then, $S_t = S_{k+1}(F_t)$. Since $t \geqslant 1$, $S - S_{k+1}(F) \neq \emptyset$. We consider the forest $F_1 = F - (\bigcup_{v \in S_{k+1}(F)} N[v] - S_1)$. By Observation 12, S_1 is the unique $\gamma(F_1)$-set. Since $F \in \mathscr{F}$, every component of F_1 has a $(k+1)$-support vertex (possibly, $S_1 = S_{k+1}(F_1)$). Now, $F_1 \in \mathscr{F}$ and $t-1$ prunings of the forest F_1 are needed to construct the forest $\widetilde{F}_1$. Applying the inductive hypothesis to F_1, $\gamma_R^k(F_1) = (k+1)\gamma(F_1)$.

Let f_1 be a $\gamma_R^k(F_1)$-function, and let $f : V(F) \to \{0, 1, \ldots, k+1\}$ be defined by $f(v) = f_1(v)$ if $v \in V(F_1)$, $f(v) = k+1$ if $v \in S_{k+1}(F)$, and $f(v) = 0$ otherwise. Then, f is a kRDF of F, and so $\gamma_R^k(F) \leqslant w(f) = w(f_1) + (k+1)|S_{k+1}(F)| = \gamma_R^k(F_1) + (k+1)|S_{k+1}(F)|$. On the other hand, let g be a $\gamma_R^k(F)$-function. Suppose $v \in S_{k+1}(F)$ and u is a leaf adjacent to v. If $g(u) = 1$, then we can reassign to v the value $g(v) + 1$ and to u the value 0. Hence we may assume that $g(v)=k+1$ for each $v\in S_{k+1}(F)$ and $g(u)=0$ for each leaf u adjacent to v. Furthermore, if $u\in N[S_{k+1}(F)]-S$ and if u is not a leaf, then we may assume that $g(u)=0$ for otherwise we can shift the positive weight on u to a vertex of F_1. Let g' be the restriction of g to F_1. Then, g' is a kRDF of F_1, and so $\gamma_R^k(F_1)\leqslant w(g')=w(g)-(k+1)|S_{k+1}(F)|=\gamma_R^k(F)-(k+1)|S_{k+1}(F)|$. Consequently, $\gamma_R^k(F)=\gamma_R^k(F_1)+(k+1)|S_{k+1}(F)|$.

Since S_1 is the unique $\gamma(F_1)$-set, $\gamma(F_1)=|S_1|=|S|-|S_{k+1}(F)|=\gamma(F)-|S_{k+1}(F)|$. Thus, since $\gamma_R^k(F_1)=(k+1)\gamma(F_1)$, it follows that $\gamma_R^k(F)=\gamma_R^k(F_1)+(k+1)|S_{k+1}(F)|=(k+1)(\gamma(F_1)+|S_{k+1}(F)|)=(k+1)\gamma(F)$. □

Lemma 14. *Let F be a forest. If $F\notin\mathscr{F}$, then $\gamma_R^k(F)<(k+1)\gamma(F)$.*

Proof. Suppose $F\notin\mathscr{F}$. If the forest F does not have a unique $\gamma(F)$-set, then it follows from Corollary 7 that $\gamma_R^k(F)<(k+1)\gamma(F)$. Hence we may assume that F has a unique $\gamma(F)$-set S. If F has a component with no $(k+1)$-support vertex, then, by Corollary 11, $\gamma_R^k(F)<(k+1)\gamma(F)$. Hence we may assume that each component of F contains a $(k+1)$-support vertex. Now since $F\notin\mathscr{F}$, it follows that $\tilde{F}=F_t$ where F_t has a component with no $(k+1)$-support vertex. Let g be a $\gamma_R^k(F_t)$-function. Then, by Corollary 11, $w(g)=\gamma_R^k(F_t)<(k+1)\gamma(F_t)$. By Observation 12, S_t is the unique $\gamma(F_t)$-set and, by construction, $S-S_t$ is a dominating set of $F-V(F_t)$. Let $f:V(F)\to\{0,1,\ldots,k+1\}$ be defined by $f(v)=g(v)$ if $v\in V(F_t)$, $f(v)=k+1$ if $v\in S-S_t$, and $f(v)=0$ otherwise. Then, f is a kRDF of F, and so $\gamma_R^k(F)\leqslant w(f)=w(g)+(k+1)|S-S_t|<(k+1)\gamma(F_t)+(k+1)(|S|-|S_t|)=(k+1)|S_t|+(k+1)(|S|-|S_t|)=(k+1)|S|=(k+1)\gamma(F)$. □

As an immediate consequence of Lemmas 13 and 14, we have the following characterization of forests F that satisfy $\gamma_R^k(F)=(k+1)\gamma(F)$.

Theorem 15. *Let F be a forest. Then $\gamma_R^k(F)=(k+1)\gamma(F)$ if and only if $F\in\mathscr{F}$.*

5. Complexity

The following decision problem for the domination number of a graph is known to be NP-complete, even when restricted to bipartite graphs (see Dewdney [5]) or chordal graphs (see Booth [1] and Booth and Johnson [2]).

DOMINATING SET (DM)

INSTANCE: A graph G and a positive integer $\ell\leqslant|V(G)|$.

QUESTION: Does G have a dominating set of cardinality ℓ or less?

We will demonstrate a polynomial time reduction of this problem to our k-Roman dominating function problem.

k-ROMAN DOMINATING FUNCTION (kRDF)

INSTANCE: A graph H and a positive integer $j\leqslant(k+1)|V(H)|$.

QUESTION: Does H have a kRDF of weight j or less?

Theorem 16. *kRDF is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that kRDF is a member of NP since we can, in polynomial time, guess at a function $f: V(H) \to \{0, 1, \dots, k+1\}$ and verify that f has weight at most j and is a kRDF. We next show how a polynomial time algorithm for kRDF could be used to solve DM in polynomial time. Given a graph G and a positive integer ℓ construct the graph H by adding for each vertex v of G a star $K_{1,k+1}$, joining v to an endvertex of this star and then subdividing the resulting edge once. Note that if $k = 1$, then we have added a path of length 4 to v. It is easy to see that the construction of the graph H can be accomplished in polynomial time. Note that if G is a bipartite or chordal graph, then so too is H.

Lemma 17. $\gamma_R^k(H) = \gamma(G) + (k+1)|V(G)|$.

Proof. Let $f = (V_0, V_1, \dots, V_{k+1})$ be a $\gamma_R^k(H)$-function. Let $v \in V(G)$ and let F_v be the component of $H - (V(G) - \{v\})$ containing v. Then, F_v is obtainable from a star $K_{1,k+1}$ by subdividing one edge twice. Let v, w, x, y denote the path from v to the center y of the star that was added to v to produce H. We may assume that $f(z) = 0$ for each of the k leaves z adjacent to y (for otherwise we can simply shift any positive weight from z to its neighbor y). Since f is a kRDF, it follows that $f(y) \geqslant k$ and $f(N[w]) \geqslant 1$. Thus, $f(V(F_v)) \geqslant k+1$. Let $S = (\bigcup_{i=1}^{k+1} V_i) \cap V(G)$.

If $f(V(F_v)) \geqslant k+2$, then we may assume that $f(v) \geqslant 1$, $f(w) = 1$, $f(x) = 0$, $f(y) = k$, and $f(z) = 0$ for each leaf z adjacent to y (for otherwise we can simply shift any additional positive weight in the subgraph F_v to v). Hence, if $f(V(F_v)) \geqslant k+2$, then $v \in S$.

Suppose that $f(V(F_v)) = k+1$. Then, $f(N[w]) = 1$ and $f(y) = k$. If $f(x) = 1$, then $f(v) = f(w) = 0$. In particular, $v \in V_0$, and so v must be adjacent to a vertex u of positive weight in f. Since $w \in V_0$, $u \in V(G)$. Hence, v is adjacent to a vertex of S. On the other hand, suppose $f(x) = 0$. If $f(w) = 0$, then the movement of k legions from y to its k leaves will create an undefended vertex, namely x. Hence, $f(w) = 1$ and so $f(v) = 0$. Consider now the sequence of k vertices consisting of $k-1$ leaves of y followed by the vertex x. The movement of k legions from y to these k vertices will create an undefended vertex, namely a leaf of y. Thus the movement of a legion from w to x cannot create an undefended vertex. But this implies that the vertex v must be adjacent to a vertex of S. Hence, if $f(V(F_v)) = k+1$, then v is dominated by S.

Thus, S is a dominating set of G, and so $\gamma(G) \leqslant |S|$. Furthermore, if $v \in S$, then $f(V(F_v)) \geqslant k+2$, while if $v \notin S$, then $f(V(F_v)) = k+1$. Hence, $\gamma_R^k(H) = w(f) \geqslant (k+2)|S| + (k+1)(|V(G)| - |S|) = |S| + (k+1)|V(G)| \geqslant \gamma(G) + (k+1)|V(G)|$.

On the other hand, let D be a $\gamma(G)$-set. Let $g: V(H) \to \{0, 1, \dots, k+1\}$ be the function defined as follows: if $v \in D$, then let $g(v) = g(w) = 1$, $g(x) = 0$, $g(y) = k$, and $g(z) = 0$ for each leaf z adjacent to y, while if $v \notin D$, then let $g(v) = 0$, $g(w) = 1$, $g(x) = 0$, $g(y) = k$, and $g(z) = 0$ for each leaf z adjacent to y. Then, g is a kRDF of

H, and so $\gamma_R^k(H) \leqslant w(g) = (k+2)|D| + (k+1)(|V(G)| - |D|) = |D| + (k+1)|V(G)| = \gamma(G) + (k+1)|V(G)|$. Consequently, $\gamma_R^k(H) = \gamma(G) + (k+1)|V(G)|$, as desired. □

Lemma 17 implies that if we let $j = \ell + (k+1)|V(G)|$, then $\gamma(G) \leqslant \ell$ if and only if $\gamma_R^k(H) \leqslant j$. This completes the proof of Theorem 16. □

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