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Trichotomies for abstract semilinear differential equations

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Abstract

In this paper we present some results concerning the existence and uniqueness of mild solutions to certain abstract semilinear differential equations and the asymptotic behavior of these solutions. The basic techniques used are the iterative method and the fixed point theory for differential equations in Banach space. However, the most pleasant here is that it can be applied to nonlinear equations without assuming the eigenvalues of the differential operator in the linear parts of the differential equation has non-zero real part.

1. Introduction and notation

The original motivation for this paper is to study sufficient conditions for the existence and uniqueness, as well as the asymptotic behavior of the solutions for the semilinear differential equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \beta u(t,x) + f(t,x,u) \quad \text{on } (0,\infty)\times\Omega, \\ u(t,x) = 0 \quad \text{on } [0,\infty)\times\partial\Omega, \\ u(0,x) = \xi_0(x) \quad \text{on } \Omega. \end{cases} \tag{1.1}$$

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We assume that $\Omega \subset R^n$ is a bounded domain with smooth boundary and that the constant $\beta > 0$ and the function f satisfy certain conditions. If one lets X denote the Hilbert space $L^2(\Omega)$ and if the operator $A : D(A) \to X$ is defined by

$$A\varphi = \Delta\varphi + \beta\varphi \quad \text{for all } \varphi \in D(A),$$

where

$$D(A) = \{\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}) \colon \varphi(x) = 0 \text{ on } \partial\Omega\},$$

then the semilinear differential equation (1.1) can be replaced by the abstract semilinear initial value problem:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + f(t, u(t)) & \text{on } (0, \infty), \\ u(0) = \xi_0 \in X & \text{on } \Omega, \end{cases} \tag{1.2}$$

where $u(\cdot) \in X$. From [6, p. 205], there exists a sequence $\{\varphi_n \colon n \in N\}$ of eigenfunctions correspondent to the eigenvalues $\{\lambda_n \colon n \in N\}$ for A, and $\{\varphi_n \colon n \in N\}$ forms an orthonormal basis for the Hilbert space X. Moreover, the operator A generates a C_0-semigroup $\{T(t) \colon t \geqslant 0\}$ on the Hilbert space X, which is defined by

$$\big(T(t)\varphi\big)(x) = \sum_{k=1}^{\infty} \exp(\lambda_k t)\langle \varphi, \varphi_k\rangle \varphi_k \quad \text{for all } \varphi \in X.$$

Therefore, instead of solving this semilinear differential equation (1.1) directly, we consider more general abstract semilinear initial value problem:

$$\begin{cases} \frac{d}{dt}u(t) = A(t)u(t) + f(t, u(t)) & \text{on } (s, \infty), \\ u(s) = \xi_s \in X & \text{on } \Omega, \end{cases} \tag{1.3}$$

where $s \geqslant 0$ is a fixed real number and $\{A(t) \colon t \geqslant 0\}$ generates a C_0-evolution system $\{U(t, s) \colon 0 \leqslant s \leqslant t < \infty\}$ on the Banach space X.

Under the assumption that $A(t)$ is sectorial for each $t \geqslant 0$, with a dense constant domain $D \equiv D(A(t))$, and $Y \supset D$ is a dense imbedded Banach subspace of X. M.I. Gil' [7, chapter 16] gave a sufficient condition for the zero solution of (1.3) which globally asymptotically Y-stable. In [1], Y.H. Chang and R.H. Martin, Jr. found sufficient conditions for the conditional stability for the zero solution of the abstract semilinear initial value problem (1.3). The basic techniques used in that paper are Lyapunov-like methods and fixed point theory for differential equations in Banach spaces. Y.H. Chang [2] presented some results concerning the existence and its asymptotic behavior of mild solutions for the abstract semilinear initial value problem (1.3). However, if there is an eigenvalue λ_n of A with $\operatorname{Re}\lambda_n = 0$, then the C_0-semigroup $\{T(t) \colon t \geqslant 0\}$ generated by A does not satisfy the conditions in [2]. The existence and the trichotomy properties for vector value ordinary differential equations (where A is a matrix) were extensively studied by Z. Deming and X. Min [4], S. Elaydi and O. Hajek [5] and H. Jialin [8,9]. In this paper, we try to continue Y.H. Chang and R.H. Martin, Jr.'s concept to the generalized semilinear case in which $A(t)$ may be an unbounded operator with some eigenvalues having zero real part.

In the rest of this preliminary section, we will introduce some notation and hypotheses. In Section 2, we apply the iterative approximation method to prove the local existence and uniqueness of the mild solution of (1.3) (see Theorem 2.1) under some assumptions on the forcing term function $f(\cdot, u(\cdot))$. Moreover, with further restrictions on the function f, we approach the global existence and uniqueness of the mild solution for certain types of abstract semilinear equations as well as the conditional asymptotic stability and the conditional stability (see Theorem 2.4 and Theorem 2.7). Practical examples are given in the last section.

Through out this paper, we let X be the Banach space endowed with norm $|\cdot|$, and $\{A(t)\colon t \geqslant 0\}$ generates a C_0-evolution system $\{U(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$ on the Banach space X. $\|\cdot\|$ denotes the norm on the Banach space $B(X)$, where $B(X)$ is the space of bounded linear operators on X. We also let $\|\cdot\|_\infty$ denote the supremum norm on $C([s,\infty); X)$.

Here we assume that there exist nontrivial supplementary projections P_1, P_2 and P_3 on the Banach space X such that $P_i X = X_i$, $i = 1, 2, 3$, and the C_0-evolution system $\{U(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$ satisfies the following conditions:

(A1) $\{U(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$ restricted to X_i, $i = 1, 2$, are total evolution systems (that is, for each $x_1 \in X_1$, $x_2 \in X_2$ and $t < s$, there is a unique $z^i_{t,s} \in X$, $i = 1, 2$, such that $U(s,t) z^i_{t,s} = x_i$, $i = 1, 2$, and in this case we define $U(t,s)x_i = z^i_{t,s}$, $i = 1, 2$, for $t < s$).
(A2) $U(t,s)P_i = P_i U(t,s)$ for all $0 \leqslant s \leqslant t < \infty$ and $i = 1, 2, 3$.
(A3) $\int_s^t \|U(t,\tau)P_3\|\,d\tau + \int_t^\infty \|U(t,\tau)P_1\|\,d\tau \leqslant K$ for all $0 \leqslant s \leqslant t < \infty$.
(A4) $\|U(t,s)P_2\| \leqslant L_2$ for all $0 \leqslant s, t < \infty$.

For the purpose of obtaining the global existence of the mild solution for the differential equation (1.3), we also assume that the function $f : [s,\infty) \times X \to X$ in (1.3) satisfies the following conditions (see Theorem 2.4):

(F1) $f(t,x)$ is continuous in $t \in [s,\infty)$ for any fixed $x \in X$.
(F2) $f(t,x)$ is locally Lipschitz continuous in $x \in X$ for all $t \in [s,\infty)$; that is, for any constant $\alpha > 0$, there exists a constant $\gamma(\alpha)$ such that

$$\bigl|f(t,x) - f(t,y)\bigr| \leqslant \gamma(\alpha)|x-y| \quad \text{for all } s \leqslant t < \infty,\ |x|, |y| \leqslant \alpha.$$

(F3) $f(t,0) \equiv 0$ for all $s \leqslant t < \infty$.
(F4) $\int_s^\infty |P_2 f(\tau,\varphi(\tau)) - P_2 f(\tau,\phi(\tau))|\,d\tau \leqslant \gamma_2 \|\varphi - \phi\|_\infty$ for all $\varphi, \phi \in D$, where

$$D = \bigl\{\varphi \in C([s,\infty); X)\colon \|\varphi\|_\infty \leqslant \alpha\bigr\}$$

and α is the constant given in (F2).

However, if we just wanted to get the local existence of the mild solution for the differential equation (1.3), we only need assume conditions (F1) and (F2) (see Theorem 2.1).

2. Main results

The argument of the proof for the Theorem 2.1 is standard. To prove the existence and uniqueness of the mild solution to the differential equation (1.3), one can use a similar argument showed in [3, Theorem 2.3]. On the other hand, one can also obtain the blow-up result by following the proof of Theorem 6-1.4 in Pazy's book [10] with a slight modification. So, we omit the detail here.

Theorem 2.1. *Let linear operators* $\{A(t)\colon t \geqslant 0\}$ *generate a* C_0*-evolution system* $\{U(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$ *on the Banach space* X, *and let* $T > s$ *be any fixed constant. If the function* $f:[s,\infty)\times X \to X$ *in* (1.3) *satisfies conditions* (F1) *and* (F2), *then for any* $\xi_s \in X$, *there is* $t_{\max} \in (s, T]$ *such that the integral equation*

$$u(t) = U(t,s)\xi_s + \int_s^t U(t,\tau) f\bigl(\tau, u(\tau)\bigr)\, d\tau \tag{2.1}$$

has a unique solution u *on the interval* $[s, t_{\max})$. *Moreover, if* $t_{\max} < T$, *then*

$$\lim_{t\to t_{\max}} \bigl|u(t)\bigr| = \infty.$$

To prove the global existence and uniqueness of the solution to the integral equation (2.1) (Theorem 2.4), we need following lemmas as preliminaries.

Lemma 2.2. *Suppose the* C_0*-evolution system* $\{U(t,s)\colon 0 \leqslant s \leqslant t < \infty\}$ *satisfies conditions* (A1)–(A4) *and the function* $f:[s,\infty)\times X \to X$ *satisfies conditions* (F1)–(F4). *If* $\xi_3 \in X_3$ *and the operator* $C : D \to C([s,\infty); X)$ *is defined by*

$$(C\varphi)(t) = U(t,s)\xi_3 + \int_s^t U(t,\tau) P_3 f\bigl(\tau,\varphi(\tau)\bigr)\, d\tau - \int_t^\infty U(t,\tau) P_2 f\bigl(\tau,\varphi(\tau)\bigr)\, d\tau - \int_t^\infty U(t,\tau) P_1 f\bigl(\tau,\varphi(\tau)\bigr)\, d\tau$$

for all $\varphi \in D$, *then* C *is well-defined and*

$$\|C\varphi - C\phi\|_\infty \leqslant (\gamma K + \gamma_2 L_2)\|\varphi - \phi\|_\infty \quad \text{for any } \varphi, \phi \in D,$$

where γ, K, γ_2, L_2 *are the constants in* (A3), (A4), (F3) *and* (F4).

Proof. From the conditions (A1), (A3), (A4), (F3) and (F4), for any $\varepsilon > 0$, $t_1 > t_2 \geqslant s$ and $\varphi \in D$, there exists a constant $T_0 > t_1$ large enough such that

$$\begin{aligned}
&\big|C\varphi(t_1) - C\varphi(t_2)\big| \\
&\quad \leqslant \big|U(t_1,s)\xi_3 - U(t_2,s)\xi_3\big| + \int_s^{t_2} \Big|\big\{U(t_1,\tau) - U(t_2,\tau)\big\} P_3 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau \\
&\qquad + \int_{t_2}^{t_1} \Big|U(t_1,\tau)P_3 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau + \int_{t_2}^{t_1} \Big|U(t_2,\tau)P_2 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau \\
&\qquad + \int_{t_2}^{t_1} \Big|U(t_2,\tau)P_1 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau \\
&\qquad + \int_{t_1}^{T_0} \Big|\big\{U(t_1,\tau) - U(t_2,\tau)\big\}(P_1+P_2) f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau + 2\varepsilon.
\end{aligned}$$

Since the function $t \mapsto U(t,s)\xi_3$ is continuous on $0 \leqslant s \leqslant t < \infty$, and functions $\tau \mapsto |U(t_1,\tau)P_3 f(\tau,\varphi(\tau))|$, $\tau \mapsto |U(t_2,\tau)P_2 f(\tau,\varphi(\tau))|$, $\tau \mapsto |U(t_2,\tau) \times P_1 f(\tau,\varphi(\tau))|$ are uniformly continuous on the compact interval $[t_1,t_2]$, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $|U(t_1,s)\xi_3 - U(t_2,s)\xi_3| < \varepsilon$ for all $|t_1 - t_2| < \delta_1$ and

$$\begin{aligned}
&\int_{t_2}^{t_1} \Big|U(t_1,\tau)P_3 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau + \int_{t_2}^{t_1} \Big|U(t_2,\tau)P_2 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau \\
&\quad + \int_{t_2}^{t_1} \Big|U(t_2,\tau)P_1 f\big(\tau,\varphi(\tau)\big)\Big|\,d\tau < \varepsilon \quad \text{for all } |t_1 - t_2| < \delta_2.
\end{aligned}$$

On the other hand, since the functions

$$(t,\tau) \mapsto U(t,\tau)P_3 f\big(\tau,\varphi(\tau)\big) \quad \text{and} \quad (t,\tau) \mapsto U(t,\tau)(P_1+P_2) f\big(\tau,\varphi(\tau)\big)$$

are uniformly continuous on the compact sets

$$\big\{(t,\tau)\colon\ 0 \leqslant s \leqslant \tau \leqslant t \leqslant T_0\big\} \quad \text{and} \quad \big\{(t,\tau)\colon\ 0 \leqslant s \leqslant t \leqslant \tau \leqslant T_0\big\}$$

respectively, there exists a constant $\delta_3 > 0$ such that

$$\Big|\big\{U(t_1,\tau) - U(t_2,\tau)\big\} P_3 f\big(\tau,\varphi(\tau)\big)\Big| < \varepsilon T_0^{-1}$$

for all $|t_1 - t_2| < \delta_3$, $s \leqslant \tau \leqslant t_2 \leqslant T_0$, and

$$\Big|\big\{U(t_1,\tau) - U(t_2,\tau)\big\}(P_1+P_2) f\big(\tau,\varphi(\tau)\big)\Big| < \varepsilon T_0^{-1}$$

for all $|t_1 - t_2| < \delta_3$, $s \leqslant t_1 \leqslant \tau \leqslant T_0$. Let $\delta \leqslant \min\{\delta_1, \delta_2, \delta_3\}$, then

$$\left|C\varphi(t_1) - C\varphi(t_2)\right| \leqslant \varepsilon + (t_2 - s)\varepsilon T_0^{-1} + (T_0 - t_1)\varepsilon T_0^{-1} + 2\varepsilon \leqslant 6\varepsilon$$

for all $0 \leqslant s \leqslant t_2 \leqslant t_1 \leqslant t_2 + \delta$. Hence, $C\varphi \in C([s, \infty); X)$ for all $\varphi \in D$, and C is well defined. Moreover, for any $\phi, \varphi \in D$,

$$\begin{aligned}\|C\varphi - C\phi\|_\infty \leqslant \sup_{t \geqslant s}\Bigg\{ &\int_s^t \left\|U(t,\tau)P_3\right\|\gamma\left|\varphi(\tau) - \phi(\tau)\right| d\tau \\ &+ \int_t^\infty \left\|U(t,\tau)P_1\right\|\gamma\left|\varphi(\tau) - \phi(\tau)\right| d\tau \Bigg\} \\ &+ \sup_{t \geqslant s}\int_t^\infty \left\|U(t,\tau)P_2\right\|\left|P_2 f\big(\tau, \varphi(\tau)\big) - P_2 f\big(\tau, \phi(\tau)\big)\right| d\tau \\ \leqslant{}& (\gamma K + \gamma_2 L_2)\|\varphi - \phi\|_\infty\end{aligned}$$

and this lemma is proved.

Lemma 2.3. *Suppose the C_0-evolution system $\{U(t,s)\colon\ 0 \leqslant s \leqslant t < \infty\}$ satisfies conditions* (A1)–(A4). *Then* $\lim_{t\to\infty} \|U(t,\tau)P_3\| = 0$ *for all $\tau \geqslant s$, and there is a constant $L_3 = L_3(s) > 0$ such that $\|U(t,\tau)P_3\| \leqslant L_3$ for all $0 \leqslant s \leqslant t < \infty$. Furthermore, if the function $f : [s,\infty) \times X \to X$ satisfies conditions* (F1)–(F4) *and the constants γ, K, γ_2, L_2 in* (A3), (A4), (F3) *and* (F4) *satisfy $\gamma K + \gamma_2 L_2 < 1$, then for any $\xi_3 \in P_3X$ with $|\xi_3| < (1 - \gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, the operator C is a contraction mapping from D into D.*

Proof. From the condition (A3), $\int_s^t \|U(t,\tau)P_3\|\, d\tau \leqslant K$ for all $t \geqslant s$. Set $\varphi(t) = \|U(t,s)P_3\|^{-1}$ for all $t \geqslant s$, then for any fixed $\xi \in X$ and $t \geqslant s$,

$$\begin{aligned}\left|\left(\int_s^t \varphi(\tau)\, d\tau\right) U(t,s)P_3\xi\right| &\leqslant \int_s^t \varphi(\tau)\left\|U(t,\tau)P_3\right\|\left\|U(\tau,s)P_3\right\||\xi|\, d\tau \\ &\leqslant K|\xi|.\end{aligned}$$

Thus, for all $t \geqslant s$, $\varphi(t)^{-1}\int_s^t \varphi(\tau)\, d\tau \leqslant K$. Let $\Psi(t) = \int_s^t \varphi(\tau)\, d\tau$ for all $t \geqslant s$. Then

$$\Psi'(t) = \varphi(t) \geqslant \frac{1}{K}\int_s^t \varphi(\tau)\, d\tau = \frac{1}{K}\Psi(t),$$

and hence $\Psi(t) \geqslant \Psi(t_0)\exp\{K^{-1}(t - t_0)\}$ for all $t \geqslant t_0 > s$. This implies

$$\lim_{t\to\infty}\left\|U(t,\tau)P_3\right\| = 0 \quad \text{for all } \tau \geqslant s$$

and

$$\begin{aligned}\|U(t,s)P_3\| &\leqslant K\Psi(t)^{-1}\\ &\leqslant \bigl\{K\Psi(s+1)^{-1}\exp\bigl(K^{-1}(s+1)\bigr)\bigr\}\exp(-K^{-1}t)\end{aligned}$$

for all $t \geqslant s+1$, since the function $t \mapsto U(t,s)P_3\xi$ is uniformly continuous on the compact interval $[s, s+1]$. It follows from Theorem 2.1 that there exists a constant $M_1(s) > 0$ such that $\|U(t,s)P_3\| \leqslant M_1(s)$, for all $t \in [s, s+1]$. Let

$$L_3 = \max\bigl\{M_1(s),\, K\Psi(s+1)^{-1}\exp\bigl(K^{-1}(s+1)\bigr)\bigr\}.$$

Then $\|U(t,s)P_3\| \leqslant L_3$ for $t \geqslant s$. If $\gamma K + \gamma_2 L_2 < 1$ and $\xi_3 \in P_3X$ with $|\xi| < (1-\gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, then for any $\varphi \in D$

$$\begin{aligned}\|C\varphi\|_\infty &\leqslant \sup_{t\geqslant s}\bigl|U(t,s)P_3\xi_3\bigr| + \sup_{t\geqslant s}\int_t^\infty \bigl\|U(t,\tau)P_2\bigr\|\bigl|P_2 f\bigl(\tau,\varphi(\tau)\bigr)\bigr|\,d\tau\\ &\quad+\sup_{t\geqslant s}\Biggl(\int_s^t \bigl\|U(t,\tau)P_3\bigr\|\bigl|f\bigl(\tau,\varphi(\tau)\bigr)\bigr|\,d\tau\\ &\quad+\int_t^\infty \bigl\|U(t,\tau)P_1\bigr\|\bigl|f\bigl(\tau,\varphi(\tau)\bigr)\bigr|\,d\tau\Biggr) \leqslant \alpha.\end{aligned}$$

Hence, $C\varphi \in D$ for any $\varphi \in D$, and $C(D) \subset D$. Moreover, from Lemma 2.2,

$$\|C\varphi - C\phi\|_\infty \leqslant (\gamma K + \gamma_2 L_2)\|\varphi - \phi\|_\infty \quad \text{for any } \varphi, \phi \in D.$$

Hence, $C : D \to D$ is a contraction mapping on D with a contraction constant $\gamma K + \gamma_2 L_2$. The assertion of this lemma is established now.

Theorem 2.4. *Suppose the C_0-evolution system $\{U(t,s)\colon\ 0 \leqslant s \leqslant t < \infty\}$ satisfies conditions* (A1)–(A4) *and the function $f : [s,\infty) \times X \to X$ satisfies conditions* (F1)–(F4). *If the constants γ, K, γ_2, and L_2 in* (A3), (A4), (F3) *and* (F4) *satisfy $\gamma K + \gamma_2 L_2 < 1$, then for any $\xi_3 \in X_3$ with $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, there exists $\xi_s \in X$ such that $P_3\xi_s = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem* (1.3) *is bounded on* $[s,\infty)$. *Furthermore,* $\lim_{t\to\infty}|u(t)| = 0$.

Proof. From Lemma 2.3, $C : D \to D$ is a contraction mapping on D with contraction constant $\gamma K + \gamma_2 L_2$. Then there exists a unique u in D such that $Cu = u$. Hence $u(t)$ is bounded on $[s,\infty)$, and

$$u(t) = U(t,s)\xi_3 + \int_s^t U(t,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$

$$-\int_t^\infty U(t,\tau)P_2 f\big(\tau,u(\tau)\big)\,d\tau - \int_t^\infty U(t,\tau)P_1 f\big(\tau,u(\tau)\big)\,d\tau.$$

Thus

$$u(s) = \xi_3 - \int_s^\infty U(s,\tau)P_2 f\big(\tau,u(\tau)\big)\,d\tau - \int_s^\infty U(s,\tau)P_1 f\big(\tau,u(\tau)\big)\,d\tau.$$

Let $\xi_s = u(s) \in X$. Following from the facts $P_3U(t,s) = U(t,s)P_3$ and $P_jP_3 = 0$, for each $j = 1, 2$, we have $P_3\xi_s = P_3u(s) = \xi_3$. On the other hand,

$$\begin{aligned}
u(t) &= U(t,s)\xi_s + U(t,s)\int_s^\infty U(s,\tau)P_2 f\big(\tau,u(\tau)\big)\,d\tau \\
&\quad + U(t,s)\int_s^\infty U(s,\tau)P_1 f\big(\tau,u(\tau)\big)\,d\tau + \int_s^t U(t,\tau)P_3 f\big(\tau,u(\tau)\big)\,d\tau \\
&\quad - \int_t^\infty U(t,\tau)P_2 f\big(\tau,u(\tau)\big)\,d\tau - \int_t^\infty U(t,\tau)P_1 f\big(\tau,u(\tau)\big)\,d\tau \\
&= U(t,s)\xi_s + \int_s^t U(t,\tau)(P_1+P_2+P_3) f\big(\tau,u(\tau)\big)\,d\tau \\
&= U(t,s)\xi_s + \int_s^t U(t,\tau) f\big(\tau,u(\tau)\big)\,d\tau
\end{aligned}$$

for any $t \geqslant s$. This shows that $u(t)$ is a bounded mild solution to the abstract semilinear initial value problem (1.3) with initial value ξ_s on $[s,\infty)$ which satisfies $P_3\xi_s = \xi_3$. From Theorem 2.1, the solution $u(t)$ is unique on $[s,\infty)$.

Since $|u(t)| \leqslant \alpha$ for all $t \geqslant s$, there exists $\mu \in [0,\infty)$ such that $\mu = \limsup_{t\to\infty} |u(t)|$. If $\mu > 0$, then there is a constant $\theta \in (0,1)$ and $t_1 \geqslant s$ such that $\theta > \gamma K + \gamma_2 L_2$ and $|u(t)| \leqslant \theta^{-1}\mu$, for all $t \geqslant t_1$. From Lemma 2.3, one may have

$$\lim_{t\to\infty} \big\|U(t,s)P_3\big\| = \lim_{t\to\infty} \big\|U(t,t_1)P_3\big\| = 0.$$

For any $t \geqslant t_1 \geqslant s$, with t large enough,

$$\big|u(t)\big| \leqslant \big\|U(t,s)P_3\big\||\xi_3| + \big\|U(t,t_1)P_3\big\| \int_s^{t_1} \big|U(t_1,\tau)P_3 f\big(\tau,u(\tau)\big)\big|\,d\tau$$

$$+\int_{t_1}^{t}\bigl|U(s,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\bigr|\,d\tau+\int_{t}^{\infty}\bigl|U(t,\tau)P_1 f\bigl(\tau,u(\tau)\bigr)\bigr|\,d\tau$$
$$+\int_{t}^{\infty}\bigl|U(t,\tau)P_2 f\bigl(\tau,u(\tau)\bigr)\bigr|\,d\tau$$
$$\leqslant \bigl\|U(t,s)P_3\bigr\||\xi_3|+\bigl\|U(t,t_1)P_3\bigr\|\int_{s}^{t_1}\bigl|U(t_1,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\bigr|\,d\tau$$
$$+K\gamma\theta^{-1}\mu+L_2\int_{t}^{\infty}\bigl|P_2 f\bigl(\tau,u(\tau)\bigr)\bigr|\,d\tau.$$

Thus $\mu=\limsup_{t\to\infty}|u(t)|\leqslant(\gamma K+\gamma_2L_2)\theta^{-1}\mu<\mu$. This is impossible, and hence $\mu=0$. This shows that $\lim_{t\to\infty}|u(t)|=0$, and this theorem is completely proved now.

With the same processes as in the proofs of Lemma 2.2 and Lemma 2.3, one may easily obtain the following Lemma 2.5 and Lemma 2.6.

Lemma 2.5. *Suppose the C_0-evolution system $\{U(t,s)\colon\ 0\leqslant s\leqslant t<\infty\}$ satisfies conditions* (A1)–(A4) *and the function $f:[s,\infty)\times X\to X$ satisfies conditions* (F1)–(F4). *For any fixed $\xi_2\in X_2$, $\xi_3\in X_3$, let the operator $B:D\to C([s,\infty);X)$ be defined by*

$$(B\varphi)(t)=U(t,s)\xi_2+U(t,s)\xi_3+\int_{s}^{t}U(t,\tau)P_2 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$
$$+\int_{s}^{t}U(t,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\,d\tau-\int_{t}^{\infty}U(t,\tau)P_1 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$

for all $\varphi\in D$, then B is well-defined and

$$\|B\varphi-B\phi\|_\infty\leqslant(\gamma K+\gamma_2L_2)\|\varphi-\phi\|_\infty\quad\text{for any }\varphi,\phi\in D,$$

where γ, K, γ_2, L_2 are the constants in (A3), (A4), (F3) *and* (F4).

Lemma 2.6. *Suppose the C_0-evolution system $\{U(t,s)\colon\ 0\leqslant s\leqslant t<\infty\}$ satisfies conditions* (A1)–(A4) *and the function $f:[s,\infty)\times X\to X$ satisfies conditions* (F1)–(F4). *If the constants γ, K, γ_2, L_2 in* (A3), (A4), (F3) *and* (F4) *satisfy $\gamma K+\gamma_2L_2<1$, then for any $\xi_2\in P_2X$, $\xi_3\in P_3X$ with both $|\xi_2|$ and $|\xi_3|$ strictly less than $(1-\gamma K-\gamma_2L_2)\alpha(L_2+L_3)^{-1}$, where L_3 is as in Lemma* 2.3, *the operator B is a contraction mapping from D into D.*

Theorem 2.7. *Suppose the* C_0*-evolution system* $\{U(t,s)\colon\ 0 \leqslant s \leqslant t < \infty\}$ *satisfies conditions* (A1)–(A4) *and the function* $f:[s,\infty)\times X \to X$ *satisfies conditions* (F1)–(F4). *If the constants* γ, K, γ_2, *and* L_2 *in* (A3), (A4), (F3) *and* (F4) *satisfy* $\gamma K + \gamma_2 L_2 < 1$. *Then for any fixed* $\xi_2 \in P_2X$, $\xi_3 \in P_3X$ *with* $|\xi_2|$, $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha(L_2+L_3)^{-1}$, *there exists* $\xi_s \in X$ *such that* $P_3\xi_s = \xi_3$, $P_2\xi_s = \xi_2$ *and the corresponding unique mild solution* $u(t)$ *to the abstract semilinear initial value problem* (1.3) *is bounded on* $[s,\infty)$. *Furthermore,*

$$\|u\|_\infty \leqslant \frac{L_2}{1-\gamma K - \gamma_2 L_2}|\xi_2| + \frac{L_3}{1-\gamma K - \gamma_2 L_2}|\xi_3|.$$

Proof. From Lemma 2.6, $B:D\to D$ is a contraction mapping on D with a contraction constant $\gamma K + \gamma_2 L_2$. Hence there exists a unique $u \in D$ such that $Bu = u$, $u(t)$ is bounded on $[s,\infty)$, and

$$u(t) = U(t,s)(\xi_2+\xi_3) + \int_s^t U(t,\tau)P_2 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$
$$+ \int_s^t U(t,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\,d\tau - \int_t^\infty U(t,\tau)P_1 f\bigl(\tau,u(\tau)\bigr)\,d\tau,$$
$$u(s) = \xi_2 + \xi_3 - \int_s^\infty U(t,\tau)P_1 f\bigl(\tau,u(\tau)\bigr)\,d\tau.$$

Let $\xi_s = u(s) \in X$. Since $P_jU(t,s) = U(t,s)P_j$ and $P_jP_i = 0$, $i \neq j$ for all i, $j \in \{1,2,3\}$, this implies that $P_2\xi_s = P_2u(s) = \xi_2$, $P_3\xi_s = P_3u(s) = \xi_3$. On the other hand,

$$u(t) = U(t,s)\xi_s + U(t,s)\int_s^\infty U(s,\tau) f\bigl(\tau,u(\tau)\bigr)\,d\tau$$
$$+ \int_s^t U(t,\tau)P_2 f\bigl(\tau,u(\tau)\bigr)\,d\tau + \int_s^t U(t,\tau)P_3 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$
$$- \int_t^\infty U(t,\tau)P_1 f\bigl(\tau,u(\tau)\bigr)\,d\tau$$
$$= U(t,s)\xi_s + + \int_s^t U(t,\tau) f\bigl(\tau,u(\tau)\bigr)\,d\tau$$

for any $t \geqslant s$. Thus $u(t)$ is a bounded mild solution to the abstract semilinear initial value problem (1.3) on $[s,\infty)$ with initial value ξ_s which satisfies $P_2\xi_s =$

ξ_2, $P_3\xi_s = \xi_3$. The uniqueness of the solution can be obtained as in the proof of Theorem 2.1 immediately. Furthermore,

$$\|u\|_\infty = L_2|\xi_2| + L_3|\xi_3| + L_2 \int_s^t \left|P_2 f\bigl(\tau, u(\tau)\bigr)\right| d\tau$$
$$+ \sup_{t \geqslant s} \left(\int_s^t \|U(t,\tau)P_3\| \gamma \left|u(\tau)\right| d\tau - \int_t^\infty \|U(t,\tau)P_1\| \gamma \left|u(\tau)\right| d\tau \right)$$
$$\leqslant L_2|\xi_2| + L_3|\xi_3| + (\gamma K + \gamma_2 L_2)\|u\|_\infty.$$

Thus $\|u\|_\infty \leqslant (1 - \lambda K + -\gamma_2 L_2)^{-1}(L_2|\xi_2| + L_3|\xi_3|)$. The proof of this theorem is completed now.

3. Applications

Example 3.1. We first consider the semilinear initial-boundary value problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) + \beta u(t,x) + f(t,x,u) \quad \text{on } (0,\infty) \times \Omega, \\ u(t,x) = 0 \quad \text{on } [0,\infty) \times \partial\Omega, \\ u(0,x) = \xi_0(x) \quad \text{on } \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary, $\beta > 0$ is a constant, the function f satisfy conditions (F1)–(F4), and $\xi_0(\cdot)$ is in $L^2(\Omega)$. Let X be the Hilbert space $L^2(\Omega)$, and let the operator $A : D(A) \to X$ be defined by $A\varphi = \Delta\varphi + \beta\varphi$ for all $\varphi \in D(A)$ with $D(A) = \{\varphi \in C^2(\Omega)\colon \varphi(x) = 0 \text{ on } \partial\Omega\}$. Then the semilinear differential equation (3.1) can be replaced by the abstract semilinear initial value problem (1.2). It can be shown that there exists a sequence of eigenfunctions $\{\varphi_n\colon n \in N\}$ corresponding to the sequence of eigenvalues $\{\lambda_n\colon n \in N\}$ for A, and $\{\varphi_n\colon n \in N\}$ is an orthonormal basis for the Hilbert space X [6, p. 205]. This implies that $\varphi = \sum_{k=1}^\infty \langle \varphi, \varphi_k \rangle \varphi_k$ for all $\varphi \in X$ and the C_0-semigroup $\{T(t)\colon t \geqslant 0\}$ generated by A on X is defined by

$$\bigl(T(t)\varphi\bigr)(x) = \sum_{k=1}^\infty \exp(\lambda_k t)\langle \varphi, \varphi_k \rangle \varphi_k$$

for all $\varphi \in X$. Suppose $\beta > 0$ be a constant such that the eigenvalues of A satisfies $\operatorname{Re}\lambda_1 \geqslant \cdots \geqslant \operatorname{Re}\lambda_n > 0$, $\operatorname{Re}\lambda_{n+1} = \cdots = \operatorname{Re}\lambda_m = 0$ and $0 > \operatorname{Re}\lambda_{m+1} \geqslant \operatorname{Re}\lambda_{m+2} \geqslant \cdots$. We may define linear operators P_1, P_2 and P_3 on X by

$$P_1\varphi = \sum_{k=1}^n \langle \varphi, \varphi_k \rangle \varphi_k, \quad P_2\varphi = \sum_{k=n+1}^m \langle \varphi, \varphi_k \rangle \varphi_k, \quad P_3\varphi = \sum_{k=m+1}^\infty \langle \varphi, \varphi_k \rangle \varphi_k$$

for all $\varphi \in X$. Then operators P_1, P_2 and P_3 are projections on the Hilbert space X. Let X_i be the range of a projection P_i for each $i = 1, 2, 3$. Then the dimensions of X_1 and X_2 are n and $m-n$, respectively. Let $U(t,s)=T(t-s)$ for all $t \geqslant s \geqslant 0$, then $\{U(t,s)\colon 0 \leqslant s \leqslant t\}$ is a C_0-evolution system with the infinitesimal generator $A(t) \equiv A$. It is easy to see that the conditions (A1) and (A2) are satisfied.

For any $t \in R$ and $\varphi \in X$,

$$\left|T(t)P_1\varphi\right| = \left|\sum_{k=1}^{n} \langle \varphi, \varphi_k\rangle \exp(\lambda_k t)\varphi_k\right| \leqslant |\varphi| \sum_{k=1}^{n} \exp(t \operatorname{Re}\lambda_k),$$

and hence

$$\left\|U(t,\tau)P_1\right\| = \left\|T(t-\tau)P_1\right\| \leqslant \sum_{k=1}^{n} \exp\bigl(\operatorname{Re}\lambda_k(t-\tau)\bigr)$$

for all $t,\tau \in R$. Let $\varpi = 2^{-1}\min\{\operatorname{Re}\lambda_n, -\operatorname{Re}\lambda_{m+1}\}$.Then $\operatorname{Re}\lambda_k(t-\tau) \leqslant \varpi(t-\tau) \leqslant 0$ for all $t \leqslant \tau < \infty$ and for all $k=1,2,\ldots,n$. Hence $\|U(t,\tau)P_1\| \leqslant n\exp(\varpi(t-\tau))$ for all $t \leqslant \tau < \infty$. We define a function $V:[0,\infty)\times X \to R$ by

$$V(t,\varphi) = \left|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi\right|$$

for all $t \in [0,\infty)$ and for all $\varphi \in X$. Then for all $t \geqslant 0$ and $\varphi \in D(A)$,

$$\begin{aligned}
\frac{d}{dt}V(t,\varphi) = {} & \frac{1}{V(t,\varphi)} \operatorname{Re}\Bigl\langle \frac{d}{dt}\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi, \\
& \qquad \exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi\Bigr\rangle \\
= {} & -\varpi V(t,\varphi) + \frac{|e^{-(\varpi+\lambda_{m+1})t}|^2}{V(t,\varphi)} \\
& \times \operatorname{Re}\Bigl\langle \sum_{k=m+1}^{\infty} (\lambda_k - \lambda_{m+1})\langle\varphi,\varphi_k\rangle T(t)\varphi_k, T(t)P_3\varphi \Bigr\rangle \\
= {} & -\varpi V(t,\varphi) + \frac{|\exp(-(\varpi+\lambda_{m+1})t)|^2}{V(t,\varphi)} \\
& \times \operatorname{Re} \sum_{k=m+1}^{\infty} \left|\langle\varphi,\varphi_k\rangle\right|^2 \left|e^{\lambda_k t}\right|^2 (\lambda_k - \lambda_{m+1}) \\
= {} & -\varpi V(t,\varphi) + \frac{|e^{-(\varpi+\lambda_{m+1})t}|^2}{V(t,\varphi)} \\
& \times \sum_{k=m+1}^{\infty} \left|\langle\varphi,\varphi_k\rangle\right|^2 \left|e^{\lambda_k t}\right|^2 (\operatorname{Re}\lambda_k - \operatorname{Re}\lambda_{m+1}) \\
\leqslant {} & -\varpi V(t,\varphi).
\end{aligned}$$

This implies that

$$V(t,\varphi) \leqslant V(0,\varphi)\exp(-\varpi t) = \exp(-\varpi t)|P_3\varphi| \leqslant \exp(-\varpi t)|\varphi|,$$

and hence

$$\left|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi\right| \leqslant \exp(-\varpi t)|\varphi|$$

for all $t \geqslant 0$ and $\varphi \in D(A)$. Since $D(A)$ is dense in X, for any $\varphi \in X$, there is a sequence $\{\varphi_j\colon j \in N\}$ in $D(A)$ such that $\lim_{j\to\infty} \varphi_j = \varphi$. This implies that

$$\begin{aligned}\left|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi\right| &= \lim_{j\to\infty}\left|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\varphi_j\right| \\ &\leqslant \lim_{j\to\infty} \exp(-\varpi t)|\varphi_j| \\ &= \exp(-\varpi t)|\varphi|\end{aligned}$$

for all $t \geqslant 0$ and $\varphi \in X$. Hence for all $t \geqslant 0$,

$$\begin{aligned}\left|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)\right|\left\|T(t)P_3\right\| &= \left\|\exp\bigl(-(\varpi+\lambda_{m+1})t\bigr)T(t)P_3\right\| \\ &\leqslant \exp(-\varpi t).\end{aligned}$$

This shows that $\|T(t)P_3\| \leqslant \exp(\operatorname{Re}\lambda_{m+1}t) \leqslant \exp(-\varpi t)$ for all $t \geqslant 0$, and for all $t \geqslant \tau \geqslant 0$,

$$\left\|U(t,\tau)P_3\right\| = \left\|T(t-\tau)P_3\right\| \leqslant \exp\bigl(-\varpi(t-\tau)\bigr).$$

Therefore, for all $0 \leqslant t < \infty$,

$$\begin{aligned}&\int_0^t \left\|U(t,\tau)P_3\right\| d\tau + \int_t^\infty \left\|U(t,\tau)P_1\right\| d\tau \\ &\quad\leqslant \int_0^t \exp\bigl(-\varpi(t-\tau)\bigr)\, d\tau + \int_t^\infty n\exp\bigl(\varpi(t-\tau)\bigr)\, d\tau \\ &\quad\leqslant 1 + n\varpi^{-1}.\end{aligned}$$

This implies that the condition (A3) holds with $K = 1 + n\varpi^{-1}$ and $L_3 = 1$. For all $t \in R$ and $\varphi \in X$, we have

$$\left|T(t)P_2\varphi\right| = \left|\sum_{k=n+1}^{m} \langle\varphi,\varphi_k\rangle \exp(\lambda_k t)\varphi_k\right| \leqslant (m-n)|\varphi|.$$

Thus $\|U(t,\tau)P_2\| = \|T(t-\tau)P_2\| \leqslant (m-n)$ for all $0 \leqslant s, t < \infty$, and hence the condition (A4) holds with $L_2 = m - n$.

Suppose the forcing term function $f(t,\varphi)$ satisfies conditions (F1)–(F4) with constants γ, γ_2, α. If $(n+\varpi)\varpi^{-1}\gamma + (m-n)\gamma_2 < 1$, then from Theorem 2.4, for any $\xi_3 \in X_3$ with $|\xi_3| < (1 - (n+\varpi)\varpi^{-1}\gamma - (m-n)\gamma_2)\alpha$, there exists a $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the semilinear initial value problem (1.2) satisfies $\lim_{t\to\infty} |u(t)| = 0$. Furthermore, from Theorem 2.7, for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with $|\xi_2|, |\xi_3| <$

$\varpi^{-1}(m-n+1)^{-1}\{\varpi-(n+\varpi)\gamma-\varpi(m-n)\gamma_2\}\alpha$, there exists a $\xi_0 \in X$ such that $P_2\xi_0 = \xi_2$, $P_3\xi_0 = \xi_3$ and the corresponding unique bounded mild solution $u(t)$ to the semilinear initial value problem (1.2) satisfies

$$\|u\|_\infty \leqslant (1-\gamma K-\gamma_2 L_2)^{-1}\big\{(m-n)|\xi_2|+|\xi_3|\big\}.$$

Example 3.2. Let $s \geqslant 0$, $\varpi_1 > 0$, $\varpi_3 > 0$, ξ_1, ξ_2 and ξ_3 are given real constants. Consider the differential system:

$$\begin{cases} u_1'(t) = \varpi_1 u_1(t) + a_{11}(t)u_1^2(t) + a_{12}(t)u_2^2(t) + a_{13}(t)u_3^2(t), \\ u_2'(t) = a_{21}(t)u_1^2(t) + a_{22}(t)u_2^2(t) + a_{23}(t)u_3^2(t), \\ u_3'(t) = -\varpi_3 u_3(t) + a_{31}(t)u_1^2(t) + a_{32}(t)u_2^2(t) + a_{33}(t)u_3^2(t), \\ u_1(s) = \xi_1,\ u_2(s) = \xi_2,\ \text{and } u_3(s) = \xi_3, \end{cases} \quad (t > s) \qquad (3.2)$$

where $a_{ij} \in C([0,\infty); R)$ satisfies $\|a_{ij}\|_\infty \leqslant M$ and $\int_s^\infty |a_{2j}(\tau)|\,d\tau \leqslant L$ for some constants M, L, for all $i, j = 1, 2, 3$. Let X be the Banach space R^3 with the Euclidean norm, and let projections $P_1, P_2, P_3 : X \to X$ be defined by $P_1 x = (x_1, 0, 0)$, $P_2 x = (0, x_2, 0)$, and $P_3 x = (0, 0, x_3)$ for all $x = (x_1, x_2, x_3) \in X$. Suppose the operator $A : X \to X$ and the function $f : [s,\infty) \times X \to X$ are defined by

$$Ax = (\varpi_1 x_1, 0, -\varpi_3 x_3),$$

$$f(t,x)^T = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$$

for all $x = (x_1, x_2, x_3) \in X$. Then the differential equations (3.2) can be written as the abstract semilinear differential equation (1.2) with the initial value $u(s) = (\xi_1, \xi_2, \xi_3)$. Moreover, since A is a bounded operator, the C_0-semigroup $\{T(t)\colon t \geqslant 0\}$ generated by A is a C_0-group, and it can be represented as

$$T(t)x = \big(\exp(\varpi_1 t)x_1, x_2, \exp(-\varpi_3 t)x_3\big)$$

for all $x = (x_1, x_2, x_3) \in X$.

Let $X_j = P_j X$, $j = 1, 2, 3$. Then $\dim X_j = 1$, for all $j = 1, 2, 3$, since

$$T(t)P_1 x = \big(\exp(\varpi_1 t)x_1, 0, 0\big),$$

$$T(t)P_2 x = (0, x_2, 0),$$

and

$$T(t)P_3 x = \big(0, 0, \exp(-\varpi_3 t)x_3\big),$$

for all $x = (x_1, x_2, x_3) \in X$. Let $U(t,s) = T(t-s)$ for all $t, s \in R$. Then $\{U(t,s)\colon t, s \in R\}$ is a C_0-evolution system with the infinitesimal generator $A(t) \equiv A$. The conditions (A1) and (A2) obviously hold, since for all $t \in R$,

$$\begin{aligned}\|T(t)P_1\| &= \sup_{|x|=1} \left|\exp(\varpi_1 t)x_1\right| \\ &= \exp(\varpi_1 t) \sup_{|x|=1} |x_1| \leqslant \exp(\varpi_1 t) \sup_{|x|=1} |x| = \exp(\varpi_1 t), \\ \|T(t)P_2\| &= \sup_{|x|=1} |x_2| \leqslant \sup_{|x|=1} |x| = 1,\end{aligned}$$

and

$$\|T(t)P_3\| = \sup_{|x|=1} \left|\exp(-\varpi_3 t)x_3\right| \leqslant \exp(-\varpi_3 t) \sup_{|x|=1} |x| = \exp(-\varpi_3 t).$$

This implies that

$$\begin{aligned}\|U(t,\tau)P_1\| &= \|T(t-\tau)P_1\| \leqslant e^{\varpi_1(t-\tau)}, \\ \|U(t,\tau)P_2\| &= \|T(t-\tau)P_2\| \leqslant 1,\end{aligned}$$

and

$$\|U(t,\tau)P_3\| = \|T(t-\tau)P_3\| \leqslant e^{-\varpi_3(t-\tau)},$$

for all $t, \tau \in R$. Thus

$$\begin{aligned}&\int_s^t \|U(t,\tau)P_3\|\,d\tau + \int_t^\infty \|U(t,\tau)P_1\|\,d\tau \\ &\quad \leqslant (\varpi_1\varpi_3)^{-1}(\varpi_1+\varpi_3) \quad \text{for all } t \geqslant s.\end{aligned}$$

Therefore, the conditions (A3) and (A4) hold with the constants $K = (\varpi_1\varpi_3)^{-1} \times (\varpi_1 + \varpi_3)$, $L_2 = 1$ and $L_3 = 1$.

From the definition of the function f and the assumptions of a_{ij}, for all $i, j = 1, 2, 3$, f is continuous in t and $f(t, 0) \equiv 0$. Thus conditions (F1) and (F3) hold. Furthermore, for all $t \geqslant s$ and $x, y \in X$, with $|x|, |y| \leqslant \alpha$,

$$\begin{aligned}&\left|f(t,x) - f(t,y)\right| \\ &\quad \leqslant \sum_{i=1}^3 \left|a_{i1}(t)\left(x_1^2 - y_1^2\right) + a_{i2}(t)\left(x_2^2 - y_2^2\right) + a_{i3}(t)\left(x_3^2 - y_3^2\right)\right| \\ &\quad \leqslant M \sum_{i=1}^3 \sum_{j=1}^3 \left(|x| + |y|\right)|x-y| \\ &\quad \leqslant 18\alpha M|x-y|.\end{aligned}$$

Thus the condition (F2) holds with $\gamma = 18\alpha M$. If $\varphi, \phi \in C([s,\infty); X)$ satisfy $\|\varphi\|_\infty, \|\phi\|_\infty \leqslant \alpha$, then

$$\begin{aligned}
&\int_s^\infty \left|P_2 f\big(\tau,\varphi(\tau)\big) - P_2 f\big(\tau,\phi(\tau)\big)\right| d\tau \\
&\quad = \int_s^\infty \left| \sum_{j=1}^3 a_{2j}(\tau)\big(\varphi_j^2(\tau) - \phi_j^2(\tau)\big)\right| d\tau \\
&\quad \leqslant \sum_{j=1}^3 \int_s^\infty \left|a_{2j}(\tau)\right|\big(\|\varphi\|_\infty + \|\phi\|_\infty\big)\|\varphi-\phi\|_\infty \, d\tau \\
&\quad \leqslant 2\alpha\|\varphi-\phi\|_\infty \sum_{j=1}^3 \int_s^\infty \left|a_{2j}(\tau)\right| d\tau \\
&\quad \leqslant 6L\alpha\|\varphi-\phi\|_\infty
\end{aligned}$$

and hence the condition (F4) holds with $\gamma_2 = 6L\alpha$.

If the constant $\alpha > 0$ satisfies

$$\alpha < \frac{\varpi_1\varpi_3}{6(L\varpi_1\varpi_3 + 3M\varpi_1 + 3M\varpi_3)},$$

then $\gamma K + \gamma_2 L_2 < 1$. From Theorem 2.4, for any $\xi_3 \in X_3$ with

$$|\xi_3| < \alpha(\varpi_1\varpi_3)^{-1}\big\{(1-6L\alpha)\varpi_1\varpi_3 - 18M\alpha(\varpi_1+\varpi_3)\big\},$$

there exists a $\xi_s \in X$ such that $P_3\xi_s = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (1.2) satisfies $\lim_{t\to\infty}|u(t)| = 0$. This implies $\lim_{t\to\infty} u(t) = 0$, where $u(t) = (u_1(t), u_2(t), u_3(t))$ is the mild solution to the semilinear differential equation (3.2). From Theorem 2.7, for any fixed $\xi_2 \in X_2$, $\xi_3 \in X_3$ with

$$|\xi_2| < \alpha(2\varpi_1\varpi_3)^{-1}\big\{(1-6L\alpha)\varpi_1\varpi_3 - 18M\alpha(\varpi_1+\varpi_3)\big\},$$

and

$$|\xi_3| < \alpha(2\varpi_1\varpi_3)^{-1}\big\{(1-6L\alpha)\varpi_1\varpi_3 - 18M\alpha(\varpi_1+\varpi_3)\big\},$$

there exists a $\xi_s \in X$ such that $P_2\xi_s = \xi_2$, $P_3\xi_s = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (1.2) satisfies

$$\|u\|_\infty \leqslant (1-\gamma K - \gamma_2 L_2)^{-1}\big(|\xi_2| + |\xi_3|\big).$$

Example 3.3. Consider the semilinear initial-boundary value problem:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \beta(t)u(t,x) + f(t,x) & \text{on } (0,\infty)\times(0,\pi), \\ u(t,0) = u(t,\pi) = 0 & \text{on } [0,\infty), \\ u(0,x) = \xi_0(x) & \text{on } (0,\pi), \end{cases} \tag{3.3}$$

where β is a continuous function on the interval $[0,\infty)$ satisfying the following conditions:

(1) There is a positive integer n such that $n^2 = \inf_{t\geqslant 0}\beta(t) \leqslant \sup_{t\geqslant 0}\beta(t) < (n+1)^2$.
(2) β is a constant function on $[T,\infty)$ for some $T \geqslant 0$,
(3) $\int_0^\infty (\beta(\tau) - n^2)\,d\tau$ is finite.

Let $X = L^2[0,\pi]$,

$$D = \bigl\{\varphi \in C^2(0,\pi) \cap C^1[0,\pi]:\ \varphi(0) = \varphi(\pi) = 0\bigr\},$$

and the operator $A(t) : D \to X$ is defined by

$$\bigl(A(t)\varphi\bigr)(x) = \frac{\partial^2}{\partial x^2}\varphi(x) + \beta(t)\varphi(x)$$

for all $\varphi \in D$ and for all $t \in [0,\infty)$. Then initial-boundary value problem (3.3) can be replaced by the abstract semilinear initial value problem (1.3) with the initial value $u(0) = \xi_0 \in X$. From the definition of $A(t)$, $\varphi_k(x) = \sqrt{2\pi^{-1}}\,\sin(kx)$ is an eigenfunction of $A(t)$ corresponding to the eigenvalue $\lambda_k(t) = \beta(t) - k^2$ of $A(t)$ for each fixed $t \geqslant 0$ and for all $k \in N$. On the other hand, the sequence of functions $\{\varphi_k:\ k \in N\}$ forms an orthonormal basis for the Hilbert space X [6, p. 231] and each φ in X can be represented as $\varphi = \sum_{k=1}^\infty \langle\varphi,\varphi_k\rangle\varphi_k$ [11, pp. 137–139]. Moreover, the operator $A(t)$ generates a C_0-evolution $\{U(t,s):\ 0 \leqslant s \leqslant t < \infty\}$ on the Hilbert space X which satisfies

$$U(t,s)\varphi = \sum_{k=1}^{\infty} \exp\left(\int_s^t \bigl(\beta(\tau) - k^2\bigr)\,d\tau\right)\langle\varphi,\varphi_k\rangle\varphi_k \tag{3.4}$$

for all $\varphi \in X$ and $0 \leqslant s \leqslant t < \infty$. Since $\lambda_k(t) = \beta(t) - k^2$ and $\lambda_1(t) > \lambda_2(t) > \cdots$ for each $t \in [0,\infty)$. This implies that

$$\inf_{t\geqslant 0}\lambda_1(t) > \cdots > \inf_{t\geqslant 0}\lambda_n(t) = 0 > \inf_{t\geqslant 0}\lambda_{n+1}(t) > \cdots$$

and

$$\sup_{t\geqslant 0}\lambda_1(t) > \cdots > \sup_{t\geqslant 0}\lambda_n(t) \geqslant 0 > \sup_{t\geqslant 0}\lambda_{n+1}(t) > \cdots$$

for all $t \in [0,\infty)$. Let P_1, P_2 and P_3 are projections on X which are defined by

$$P_1\varphi = \sum_{k=1}^{n-1}\langle\varphi,\varphi_k\rangle\varphi_k, \quad P_2\varphi = \langle\varphi,\varphi_n\rangle\varphi_n, \quad \text{and} \quad P_3\varphi = \sum_{k=n+1}^{\infty}\langle\varphi,\varphi_k\rangle\varphi_k$$

for all $\varphi \in X$. Let X_i be the range of a projection P_i, for each $i = 1, 2, 3$. Then X_1 and X_2 are finite dimensional spaces, and hence the conditions (A1) and (A2) hold. Since

$$P_1U(t,s)\varphi = U(t,s)P_1\varphi = \sum_{k=1}^{n-1}\langle\varphi,\varphi_k\rangle \exp\left(\int_s^t \bigl(\beta(\tau) - k^2\bigr)\,d\tau\right)\varphi_k$$

for all $0 \leqslant s, t < \infty$ and $\varphi \in X$. If we set $\varpi = 2^{-1} \inf_{t \geqslant 0} \lambda_{n-1}(t)$, then

$$\lambda_1(t) > \cdots > \lambda_{n-1}(t) > \varpi > 0 \quad \text{for all } t \geqslant 0.$$

This derives

$$\left\| U(t,s)P_1 \right\| \leqslant \sum_{k=1}^{n-1} \exp\left(\int_s^t \left(\beta(\tau) - k^2\right) d\tau \right) \leqslant (n-1) \exp\bigl(\varpi(t-s)\bigr)$$

for all $0 \leqslant t \leqslant s < \infty$ and $\int_t^\infty \|U(t,\tau)P_1\|\, d\tau \leqslant (n-1)\varpi^{-1}$ is finite for all $t \geqslant 0$. As long as we can show that there is constants $L_3 > 0$ and $\eta > 0$ such that

$$\left\| U(t,s)P_3 \right\| \leqslant L_3 \exp\bigl(-\eta(t-s)\bigr) \quad \text{for all } 0 \leqslant s \leqslant t < \infty.$$

Then for all $t \geqslant 0$,

$$\int_0^t \left\| U(t,\tau)P_3 \right\| d\tau \leqslant L_3 \eta^{-1} \exp(-\eta)$$

is finite and this shows the condition (A3) to be true with the constant K which equals to $L_3\eta^{-1}\exp(-\eta) + (n-1)\varpi^{-1}$.

Since

$$\left\| U(t,s)P_3 \right\| \leqslant \sum_{k=1}^{\infty} \exp\left(\int_s^t \beta(\tau)\, d\tau - (n+k)^2 (t-s) \right)$$

for all $t \geqslant s \geqslant 0$ and there $0 \leqslant \varepsilon < 1$ is such that $\sup_{t \geqslant 0} \beta(t) \leqslant (n+\varepsilon)^2 < (n+1)^2$.

$$\begin{aligned} \left\| U(t,s)P_3 \right\| &\leqslant \sum_{k=1}^{\infty} \frac{\exp(\int_s^t (n+\varepsilon)^2\, d\tau)}{\exp(((n+\varepsilon)+(k-\varepsilon))^2 (t-s))} \\ &= \sum_{k=1}^{\infty} \frac{1}{\exp((2(n+\varepsilon)(k-\varepsilon) + (k-\varepsilon)^2)(t-s))} \\ &\leqslant \exp\bigl(-(1-\varepsilon)^2(t-s)\bigr) \sum_{k=1}^{\infty} \bigl(\exp\bigl(-2n(t-s)\bigr)\bigr)^{k-1} \\ &= \frac{\exp(2n(t-s))}{\exp(2n(t-s))-1} \exp\bigl(-(1-\varepsilon)^2(t-s)\bigr) \end{aligned} \tag{3.5}$$

for all $t > s \geqslant 0$. The constant $(\exp(2n(t-s))-1)^{-1}\exp(2n(t-s))$ is dependent on t, s, and

$$\lim_{s \to t} \frac{\exp(2n(t-s))}{\exp(2n(t-s))-1} = \infty.$$

So, we can not directly estimate $\|U(t,s)P_3\|$ from (3.5). To overcome this difficulty, we need to consider the parameters of the C_0-evolution system $U(t,s)$ in the following cases:

(1) The first parameter t is in the interval $[0, T+1]$ and the second parameter s satisfies $0 \leqslant s \leqslant t \leqslant T+1$.
(2) The first parameter t is in the interval $(T+1, \infty)$ and the second parameter s satisfies $T < t-1 \leqslant s \leqslant t < \infty$.
(3) The first parameter t is in the interval $(T+1, \infty)$ and the second parameter s satisfies $0 \leqslant s < t-1$.

Case 1: By using the same technique as used in the proof of Theorem 2.1, one may have $M_1 = \{\|U(t,s)P_3\|\colon 0 \leqslant s \leqslant t \leqslant T+1\}$ which is a finite constant. So, we obtain the estimation

$$\big\|U(t,s)P_3\big\| \leqslant M_1 \exp\big((1-\varepsilon)^2(T+1)\big)\exp\big(-(1-\varepsilon)^2(t-s)\big)$$

for all $0 \leqslant s \leqslant t \leqslant T+1$.

Case 2: From the assumption (2) of the function β and (3.4), it is easy to see that $U(t,s) = U(t-s+T, T)$ for all $T < t-1 \leqslant s \leqslant t < \infty$. Therefore,

$$M_2 = \big\{\big\|U(t,s)P_3\big\|\colon t-1 \leqslant s \leqslant t\big\} = \big\{\big\|U(T+h,T)P_3\big\|\colon 0 \leqslant h \leqslant 1\big\}$$

is finite for all $T+1 < t < \infty$ and hence

$$\big\|U(t,s)P_3\big\| \leqslant M_2 \exp\big((1-\varepsilon)^2\big)\exp\big(-(1-\varepsilon)^2(t-s)\big)$$

for all $T < t-1 \leqslant s \leqslant t$.

Case 3: From (3.5), it is easy to see that

$$\big\|U(t,s)P_3\big\| \leqslant \frac{\exp(2n)}{\exp(2n)-1}\exp\big(-(1-\varepsilon)^2(t-s)\big).$$

Finally, let η and L_3 are the constants $\eta = (1-\varepsilon)^2$ and

$$L_3 = \max\left\{M_1 \exp\big((1-\varepsilon)^2(T+1)\big),\, M_2\exp\big((1-\varepsilon)^2\big),\, \frac{\exp(2n)}{\exp(2n)-1}\right\},$$

then we may get the estimation of $\|U(t,s)P_3\|$ as $\|U(t,s)P_3\| \leqslant L_3 \exp(-\eta \times (t-s))$ for all $0 \leqslant s \leqslant t < \infty$. This shows the condition (A3) to be true with the constant

$$K = L_3(1-\varepsilon)^{-2}\exp\big(-(1-\varepsilon)^2\big) + (n-1)\varpi^{-1}.$$

On the other hand, since for all $0 \leqslant s, t < \infty$,

$$P_2U(t,s)\varphi = U(t,s)P_2\varphi = \langle\varphi, \varphi_n\rangle \exp\left(\int_s^t \big(\beta(\tau) - n^2\big)\,d\tau\right)\varphi_n$$

and

$$\|U(t,s)P_2\| \leqslant \exp\left(\int_s^t \left(\beta(\tau)-n^2\right)d\tau\right).$$

From the assumption (3) of the function β,

$$\|U(t,s)P_2\| \leqslant \max\left\{\exp\left(\int_0^\infty \left(\beta(\tau)-n^2\right)d\tau\right), \exp\left(\int_0^\infty \left(n^2-\beta(\tau)\right)d\tau\right)\right\}$$

for all $0 \leqslant s, t < \infty$. This implies the condition (A4) holds, and

$$L_2 = \max\left\{\exp\left(\int_0^\infty \left(\beta(\tau)-n^2\right)d\tau\right), \exp\left(\int_0^\infty \left(n^2-\beta(\tau)\right)d\tau\right)\right\}.$$

Suppose the function $f(t,\varphi)$ satisfies conditions (F1)–(F4). If $\gamma K + \gamma_2 L_2 < 1$, then followed from Theorem 2.4, for any $\xi_3 \in X_3$ which satisfies $|\xi_3| < (1-\gamma K - \gamma_2 L_2)\alpha L_3^{-1}$, there exists a $\xi_0 \in X$ such that $P_3\xi_0 = \xi_3$ and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (1.3) satisfies $\lim_{t\to\infty} |u(t)| = 0$. On the other hand, according to Theorem 2.7, for any $\xi_2 \in X_2$, $\xi_3 \in X_3$ with both $|\xi_2|$ and $|\xi_3|$ less than $(1-\gamma K - \gamma_2 L_2)\alpha(L_2+L_3)^{-1}$, there exists a $\xi_0 \in X$ such that $P_2\xi_0 = \xi_2$, $P_3\xi_0 = \xi_3$, and the corresponding unique mild solution $u(t)$ to the abstract semilinear initial value problem (1.3) satisfies

$$\|u\|_\infty \leqslant (1-\gamma K - \gamma_2 L_2)^{-1}\left(L_2|\xi_2| + L_3|\xi_3|\right).$$

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