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The Birman–Schwinger principle in von Neumann algebras of finite type

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Abstract

We introduce a relative index for a pair of dissipative operators in a von Neumann algebra of finite type and prove an analog of the Birman–Schwinger principle in this setting. As an application of this result, revisiting the Birman–Krein formula in the abstract scattering theory, we represent the de la Harpe–Skandalis determinant of the characteristic function of dissipative operators in the algebra in terms of the relative index.

Keywords: Perturbation theory; Dissipative operators; Von Neumann algebras

1. Introduction

In 1961 M.Sh. Birman [3] and J. Schwinger [24] independently introduced a method to control the number of negative eigenvalues of Schrödinger operators. In the abstract operator-theoretic setting, the classical Birman–Schwinger principle (in its simplest form) states (see, e.g., [25]):

Given a self-adjoint strictly positive operator H_0 and a non-negative self-adjoint compact operator V on a Hilbert space $\mathcal{H}$, the number of negative eigenvalues (*counting multiplicity*)

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of the operator $H = H_0 - V$ coincides with the number of eigenvalues greater than one of the Birman–Schwinger operator $V^{1/2} H_0^{-1} V^{1/2}$.

That is,

$$\dim\bigl[E_H(\mathbb{R}_-)\mathcal{H}\bigr] = \dim\bigl[E_{I - V^{1/2} H_0^{-1} V^{1/2}}(\mathbb{R}_-)\mathcal{H}\bigr], \tag{1.1}$$

where $\mathbb{R}_- = (-\infty, 0)$ and $E_T(\cdot)$ is the spectral measure of a self-adjoint operator T.

The sign-definiteness assumptions upon H_0 and V can be relaxed and the principle admits further generalizations. Assume that V is factorized in the form $V = K^* N^{-1} K$, with N a self-adjoint unitary and K a compact operator, and that H_0 and $H = H_0 - V$ have bounded inverses. Then (1.1) can be extended to a more general equality

$$\operatorname{ind}\bigl(E_{H_0}(\mathbb{R}_-), E_H(\mathbb{R}_-)\bigr) = \operatorname{ind}\bigl(E_N(\mathbb{R}_-), E_{N - K H_0^{-1} K^*}(\mathbb{R}_-)\bigr) \tag{1.2}$$

of the Fredholm indices for the associated pairs of the spectral projections (cf. [16] for the proof in the case of trace class perturbations; see also [22,23]). For the concept of the Fredholm index for a pair of orthogonal projections we refer to [1].

The main purpose of this paper is to find an appropriate generalization of the principle (1.2) in the context of perturbation theory in a von Neumann algebra $\mathcal{A}$ of finite type. To accomplish this goal, we introduce the concept of a *relative* index $\xi(M, N)$ associated with a pair (M, N) of dissipative elements in $\mathcal{A}$ via

$$\xi(M, N) = \tau\bigl[\Xi(N)\bigr] - \tau\bigl[\Xi(M)\bigr]. \tag{1.3}$$

Here $\Xi(M)$ denotes the Ξ-operator [16,17] (cf. also [10]) associated with M and τ a normal tracial state on the algebra $\mathcal{A}$.

If both M and N are self-adjoint, the relative index $\xi(M, N)$ can be expressed in terms of the τ-Fredholm indices of the corresponding spectral projections:

$$\xi(M, N) = \operatorname{ind}_\tau\bigl(E_N(\mathbb{R}_-), E_M(\mathbb{R}_-)\bigr) + \frac{1}{2}\operatorname{ind}_\tau\bigl(E_N(\{0\}), E_M(\{0\})\bigr). \tag{1.4}$$

Recall that the notion of the τ-Fredholm index for a pair of orthogonal projections (P, Q) is an analog of the index introduced in [1], where the usual trace has to be replaced by the tracial state τ. In the particular case of von Neumann algebras of finite type, one has $\operatorname{ind}_\tau(P, Q) = \tau(P - Q)$. We refer to [8,9] for the theory of τ-Fredholm operators.

The main result of the present paper (see Theorem 3.3) establishes a generalization of the Birman–Schwinger principle to the case of dissipative operators in a finite von Neumann algebra $\mathcal{A}$. For boundedly invertible dissipative operators M, N, $M - K^* N^{-1} K$, and $N - K M^{-1} K^*$ in $\mathcal{A}$ we prove the relation

$$\xi\bigl(M, M - K^* N^{-1} K\bigr) = \xi\bigl(N, N - K M^{-1} K^*\bigr). \tag{1.5}$$

In the self-adjoint case, this relation together with (1.4) provides an analog of (1.2) for the τ-Fredholm indices.

Relaxing the invertibility assumption on the operators M and/or N, we present an extension of the principle (1.5) (see Theorem 3.7). In particular, if N has a bounded inverse and the family of the operators $N - K(M + i\varepsilon I)^{-1}K^*$, $\varepsilon > 0$, has a limit as $\varepsilon \downarrow 0$ in the norm topology as an invertible (dissipative) operator we show that the relation

$$\xi\big(M, M - K^*N^{-1}K\big) = \xi\big(N, N - K(M + i0I)^{-1}K^*\big) \tag{1.6}$$

holds.

As an application of (1.6) to the self-adjoint case, we study the perturbation problem $H_0 \mapsto H = H_0 - K^*N^{-1}K$, with $N = N^*$ boundedly invertible and $H_0 = H_0^*$. Under mild additional assumptions, (1.6) leads to an equality (see Theorem 4.3) relating the index $\xi(H, H_0)$ to the de la Harpe–Skandalis determinant [14] of the Lifshits characteristic function of the dissipative operator $N - K(H_0 + i0I)^{-1}K^*$. We remark that this result strongly resembles the Birman–Krein formula [4] relating the scattering matrix to the spectral shift function.

It should be mentioned that in the context of perturbation theory for self-adjoint operators in von Neumann algebras of finite type, the function

$$\mathbb{R} \ni \lambda \mapsto \xi(H - \lambda I, H_0 - \lambda I) \tag{1.7}$$

coincides with the spectral shift function associated with the pair of self-adjoint operators (H, H_0). We recall that the concept of the spectral shift function was introduced by I.M. Lifshits [21] and M.G. Krein [20] for (finite or infinite) factors of type I (see [5,6,27], and references therein) and it has been extended to the case of (semi)finite von Neumann algebras in [2,11] (see also [7]).

Throughout the paper we assume that $\mathcal{A}$ is a von Neumann algebra of finite type and τ a normal tracial state on it. In the case when $\mathcal{A}$ is a factor of type II_1, the symbol $\mathrm{Dim}(\cdot)$ stands for the relative dimension associated with $\mathcal{A}$. The set of the boundedly invertible dissipative operators in $\mathcal{A}$ is given particular consideration and we reserve the symbol $\mathcal{D}_{\mathcal{A}}$ for this set. We use the letter K to refer to an arbitrary operator in $\mathcal{A}$ and M, N to refer to dissipative operators in $\mathcal{A}$. We denote self-adjoint operators in $\mathcal{A}$ by H_0, V, and H while discussing issues of perturbation problems. Auxiliary self-adjoint operators will be denoted by A, B, L and unitary operators by U, S.

2. The Ξ-operator

Suppose M is a dissipative, not necessarily invertible, operator in $\mathcal{A}$ and L its minimal self-adjoint dilation (see [26]) in a Hilbert space $\mathcal{K} \supset \mathcal{H}$. We define the Ξ-operator associated with M by

$$\Xi(M) = P_{\mathcal{H}}\left[E_L(\mathbb{R}_-) + \frac{1}{2}E_L\big(\{0\}\big)\right]\Big|_{\mathcal{H}}, \tag{2.1}$$

where $E_L(\cdot)$ stands for the spectral measure of L and $P_{\mathcal{H}}$ for the orthogonal projection in the space $\mathcal{K}$ onto $\mathcal{H}$.

Theorem 2.1. *If $M \in \mathcal{A}$ is a dissipative operator, then the self-adjoint non-negative contraction $\Xi(M)$ belongs to the algebra $\mathcal{A}$.*

Proof. Suppose first that M has a bounded inverse. Then, by the Langer lemma, the minimal self-adjoint dilation L of M has a trivial kernel, that is,

$$E_L(\{0\}) = 0,$$

and, therefore,

$$\Xi(M) = P_{\mathcal{H}} E_L(\mathbb{R}_-)|_{\mathcal{H}} = \frac{1}{\pi} \operatorname{Im} \log M \tag{2.2}$$

(cf. [17, Lemma 2.7]). Here $\log M$ denotes the principal branch of the operator logarithm of $M \in \mathcal{D}_{\mathcal{A}}$ with the cut along the negative imaginary semi-axis provided by the Riesz functional calculus. Equivalently, the operator logarithm $\log M$ can be understood as the norm-convergent Riemann integral

$$\log M = -\mathrm{i} \int_0^\infty \big((M + \mathrm{i}\lambda I)^{-1} - (1 + \mathrm{i}\lambda)^{-1} I\big)\, d\lambda. \tag{2.3}$$

Representation (2.3) proves that $\Xi(M)$ is an element of $\mathcal{A}$ (under the assumption that M has a bounded inverse).

To prove the claim of the theorem in the general case, it suffices to deduce that

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} \Xi(M + \mathrm{i}\varepsilon I) = \Xi(M) \tag{2.4}$$

whenever M is dissipative. Indeed, given $\varepsilon > 0$, the dissipative operator $M + \mathrm{i}\varepsilon I \in \mathcal{A}$ obviously has a bounded inverse. Hence $\Xi(M + \varepsilon \mathrm{i} I) \in \mathcal{A}$, by the first part of the proof, and the claim follows from (2.4).

In order to prove (2.4), we note that

$$\begin{aligned}
\Xi(M + \mathrm{i}\varepsilon I) &= \frac{1}{\pi} \operatorname{Im} \log(M + \mathrm{i}\varepsilon I) \\
&= -\frac{1}{\pi} \int_0^\infty \operatorname{Re}\big((M + \mathrm{i}\varepsilon I + \mathrm{i}\lambda I)^{-1} - (1 + \mathrm{i}\lambda)^{-1} I_{\mathcal{H}}\big)\, d\lambda \\
&= -\frac{1}{\pi} P_{\mathcal{H}} \int_0^\infty \operatorname{Re}\big((L + \mathrm{i}\varepsilon I + \mathrm{i}\lambda I)^{-1} - (1 + \mathrm{i}\lambda)^{-1} I_{\mathcal{K}}\big)\big|_{\mathcal{H}}\, d\lambda \\
&= \frac{1}{\pi} P_{\mathcal{H}} \operatorname{Im} \log(L + \mathrm{i}\varepsilon I)|_{\mathcal{H}} \\
&= \frac{1}{\pi} P_{\mathcal{H}} \operatorname{Im} \log(L + \mathrm{i}\varepsilon I)\big[E_L(\{0\}) + E_L(\mathbb{R} \setminus \{0\})\big]\big|_{\mathcal{H}}.
\end{aligned} \tag{2.5}$$

Following almost verbatim the arguments in [17], we verify that

$$\operatorname*{s-lim}_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} \log(L + \mathrm{i}\varepsilon I) E_L(\mathbb{R} \setminus \{0\}) = E_L(\mathbb{R}_-), \tag{2.6}$$

with the limit taken in the strong operator topology of the Hilbert space $\mathcal{K}$. Finally, applying the Spectral Theorem to the self-adjoint operator L with the use of (2.5) and (2.6), we conclude that

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} \Xi(M+\mathrm{i}\varepsilon I) = P_{\mathcal{H}}\left[E_L(\mathbb{R}_-) + \frac{1}{2}E_L(\{0\})\right]\Big|_{\mathcal{H}} = \Xi(M). \qquad \square$$

Remark 2.2. As one can see from the proof of Theorem 2.1, the Ξ-operator possesses the continuity property in the sense that

$$\operatorname*{s-lim}_{\varepsilon\downarrow 0} \Xi(M+\mathrm{i}\varepsilon I) = \Xi(M), \tag{2.7}$$

whenever M is a dissipative operator in $\mathcal{A}$. It is also clear that if, in addition, M is self-adjoint, then the Ξ-operator can be expressed in terms of the spectral resolution $E_M(\cdot)$ associated with M via

$$\Xi(M) = E_M(\mathbb{R}_-) + \frac{1}{2}E_M(\{0\}). \tag{2.8}$$

To conclude this section, we link the trace of the Ξ-operator to the phase of the de la Harpe–Skandalis determinant [14]. Basic properties of this determinant can be found in Appendix A.

Theorem 2.3. *Assume that $M \in \mathcal{D}_{\mathcal{A}}$. Let $\det_\tau M$ be the de la Harpe–Skandalis determinant associated with the homotopy class of the C^1-paths of invertible operators joining M with the identity I and containing any C^1-path $[0,1] \ni t \mapsto M_t \in \mathcal{D}_{\mathcal{A}}$. Then*

$$\det_\tau M = \exp\bigl(\mathrm{i}\pi\tau\bigl[\Xi(M)\bigr]\bigr)\cdot\Delta(M),$$

with $\Delta(\cdot)$ the Fuglede–Kadison determinant (cf. [15]).

Proof. As any complex number, $\det_\tau M$ can be written in the polar form

$$\det_\tau M = \exp\bigl(\mathrm{i}\,\mathrm{Im}\log[\det_\tau M]\bigr)\cdot|\det_\tau M|. \tag{2.9}$$

Lemma A.2(i) implies that $\det_\tau M = \exp(\tau[\log M])$ and Lemma A.1(ii) that $|\det_\tau M| = \Delta(M)$. Combining the latter representations with (2.9), one gets

$$\det_\tau M = \exp\bigl(\mathrm{i}\pi\ \mathrm{Im}\,\tau[\log M]\bigr)\cdot\Delta(M). \tag{2.10}$$

By positivity of the state τ, one concludes that $\tau\circ\mathrm{Im} = \mathrm{Im}\circ\tau$, and hence the right-hand side of (2.10) equals $\exp(\mathrm{i}\pi\tau[\mathrm{Im}\log M])\cdot\Delta(M)$. Taking into account (2.2) completes the proof. $\square$

Remark 2.4. In the case $\mathcal{A}$ is a finite type factor,

$$\tau\bigl[\Xi(H)\bigr] = \mathrm{Dim}\bigl[E_H\bigl((-\infty,0)\bigr)\mathcal{H}\bigr]$$

whenever H is a self-adjoint invertible element in $\mathcal{A}$. Thus, $\tau[\Xi(M)]$ can be considered a natural generalization of the Morse index of the dissipative element M.

3. The Birman–Schwinger principle

The main aim of this section is to provide an analog of the Birman–Schwinger principle in the context of perturbation theory for dissipative operators in the von Neumann algebra setting.

Definition 3.1. We define the ξ-index associated with the pair (M, N) of dissipative operators M and N in $\mathcal{A}$ by

$$\xi(M, N) = \tau\big[\Xi(N)\big] - \tau\big[\Xi(M)\big]. \tag{3.1}$$

Remark 3.2. The index $\xi(M, N)$ can also be recognized as the argument of the de la Harpe–Skandalis determinant $\Delta(t \mapsto M_t)$ associated with the homotopy class of the nonsingular C^1-paths joining M with N and containing any C^1-path $[0, 1] \ni t \mapsto M_t \in \mathcal{D}_{\mathcal{A}}$ with the endpoints $M_0 = M$ and $M_1 = N$. That is,

$$\Delta(t \mapsto M_t) = \exp\big(\mathrm{i}\pi\xi(M, N)\big) \cdot \Delta\big(NM^{-1}\big),$$

with $\Delta(\cdot)$ the Fuglede–Kadison determinant.

We note that in view of Remark 2.2 the relative index associated with the pair (H_0, H) of self-adjoint operators in $\mathcal{A}$ admits a transparent representation via the τ-Fredholm indices of the corresponding spectral projections

$$\xi(H, H_0) = \operatorname{ind}_\tau\big(E_{H_0}(\mathbb{R}_-), E_H(\mathbb{R}_-)\big) + \frac{1}{2}\operatorname{ind}_\tau\big(E_{H_0}\big(\{0\}\big), E_H\big(\{0\}\big)\big). \tag{3.2}$$

We start with an invariance principle for the ξ-index associated with a pair of boundedly invertible dissipative operators, a natural analog of the *Birman–Schwinger principle* in the perturbation theory for self-adjoint operators in the standard I_∞ setting.

Theorem 3.3. *Let $K \in \mathcal{A}$ and $M, N \in \mathcal{D}_{\mathcal{A}}$. Suppose, in addition, that the dissipative operators $M - K^*N^{-1}K$ and $N - KM^{-1}K^*$ are boundedly invertible. Then*

$$\xi\big(M, M - K^*N^{-1}K\big) = \xi\big(N, N - KM^{-1}K^*\big). \tag{3.3}$$

Before turning to the proof of Theorem 3.3, let us interpret its result in the context of perturbation theory for self-adjoint operators.

Assume that $H_0 = H_0^*$ is a boundedly invertible element in $\mathcal{A}$, and that the perturbation $V = H - H_0$ can be factored in the form[1] $V = -K^*N^{-1}K$, with $N = N^*$ a boundedly invertible element in $\mathcal{A}$. Then Theorem 3.3 guarantees the coincidence of the τ-Fredholm indices for the dual pairs of the spectral projections

$$\operatorname{ind}_\tau\big(E_{H_0}(\mathbb{R}_-), E_H(\mathbb{R}_-)\big) = \operatorname{ind}_\tau\big(E_N(\mathbb{R}_-), E_{N-KH_0^{-1}K^*}(\mathbb{R}_-)\big). \tag{3.4}$$

[1] Such a factorization is available for any $V = V^*$; for instance, one can take $K = \sqrt{|V|}$ and $N = -\operatorname{sgn}(V)$, with $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ -1 & \text{if } x < 0. \end{cases}$

In particular, if $\mathcal{A}$ is a factor of finite type, H_0 and V are positive, and both H_0 and H have bounded inverses, principle (3.4) acquires the traditional "counting dimensions" flavor (cf. (1.1)):

$$\operatorname{Dim}\big[E_{H_0-V}(\mathbb{R}_-)\mathcal{H}\big]=\operatorname{Dim}\big[E_{V^{1/2}H_0^{-1}V^{1/2}}\big((1,\infty)\big)\mathcal{H}\big]. \tag{3.5}$$

Proof of Theorem 3.3. Introduce an auxiliary Herglotz operator-valued function

$$z\mapsto \mathfrak{M}(z)=\begin{pmatrix} M+zI & K^* \\ K & N+zI\end{pmatrix},\quad z\in\mathbb{C}_+,$$

with values in the von Neumann algebra $\mathcal{A}\,\overline{\otimes}\,M_2$, where M_2 is the space of 2×2 (scalar) matrices. Note that $\mathfrak{M}(z)$, $z\in\mathbb{C}_+$, are boundedly invertible operators in $\mathcal{A}\,\overline{\otimes}\,M_2$ and the diagonal entries of $\mathfrak{M}^{-1}(z)$ are the inverses of the operators

$$\begin{aligned}\mathcal{M}(z)&=M+zI-K^*(N+zI)^{-1}K,\\ \mathcal{N}(z)&=N+zI-K(M+zI)^{-1}K^*,\end{aligned} \tag{3.6}$$

the Schur complements of $\mathfrak{M}(z)$.

Taking into account that

$$\frac{d}{dz}\mathfrak{M}(z)=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}$$

and using the Dixmier–Fuglede–Kadison differentiation formula (cf. [13,15]) yields

$$\frac{d}{dz}\tau^{(2)}\big[\log\mathfrak{M}(z)\big]=\tau^{(2)}\big[\mathfrak{M}^{-1}(z)\big]=\frac{1}{2}\tau\big[\mathcal{M}^{-1}(z)\big]+\frac{1}{2}\tau\big[\mathcal{N}^{-1}(z)\big]. \tag{3.7}$$

Here $\tau^{(2)}$ denotes the normal tracial state on the von Neumann algebra $\mathcal{A}\,\overline{\otimes}\,M_2$ given by

$$\tau^{(2)}\left[\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right]=\frac{\tau(A)+\tau(D)}{2},\quad A,B,C,D\in\mathcal{A}. \tag{3.8}$$

By direct computations, we get

$$\mathcal{N}^{-1}(z)=(N+zI)^{-1}+(N+zI)^{-1}K\mathcal{M}^{-1}(z)K^*(N+zI)^{-1}. \tag{3.9}$$

Employing the additivity and cyclicity of the state τ and representation (3.9), we derive

$$\begin{aligned}&\tau\big[\mathcal{M}^{-1}(z)\big]+\tau\big[\mathcal{N}^{-1}(z)\big]\\ &\quad=\tau\big[\mathcal{M}^{-1}(z)\big]+\tau\big[\mathcal{M}^{-1}(z)K^*(N+zI)^{-2}K\big]+\tau\big[(N+zI)^{-1}\big]\\ &\quad=\tau\big[\mathcal{M}^{-1}(z)\big(I+K^*(N+zI)^{-2}K\big)\big]+\tau\big[(N+zI)^{-1}\big]\\ &\quad=\frac{d}{dz}\big(\tau\big[\log\mathcal{M}(z)\big]+\tau\big[\log(N+zI)\big]\big).\end{aligned} \tag{3.10}$$

Comparing (3.7) and (3.10) gives

$$\frac{d}{dz}\big(2\tau^{(2)}\big[\log\mathfrak{M}(z)\big]\big) = \frac{d}{dz}\big(\tau\big[\log\mathcal{M}(z)\big] + \tau\big[\log(N+zI)\big]\big).$$

From this, we conclude that

$$2\tau^{(2)}\big[\log\mathfrak{M}(z)\big] = \tau\big[\log\mathcal{M}(z)\big] + \tau\big[\log(N+zI)\big] + C, \tag{3.11}$$

with C a constant. Combining the asymptotic expansions

$$\tau^{(2)}\big[\log\mathfrak{M}(\mathrm{i}y)\big] = \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right),$$
$$\tau\big[\log\mathcal{M}(\mathrm{i}y)\big] = \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right),$$
$$\tau\big[\log(N+\mathrm{i}yI)\big] = \log(\mathrm{i}y) + \mathcal{O}\left(\frac{1}{y}\right)$$

as $y \to +\infty$, we infer that the constant C in (3.11) equals zero and, hence,

$$2\tau^{(2)}\big[\log\mathfrak{M}(z)\big] = \tau\big[\log\mathcal{M}(z)\big] + \tau\big[\log(N+zI)\big], \quad \operatorname{Im} z > 0. \tag{3.12}$$

Computing the normal boundary values as $z \downarrow 0$ in (3.12) ensures the equality

$$\tau^{(2)}[\log\mathbf{M}] = \frac{1}{2}\big(\tau\big[\log\big(M - K^*N^{-1}K\big)\big] + \tau[\log N]\big), \tag{3.13}$$

where

$$\mathbf{M} = \mathfrak{M}(0).$$

Next, we note that

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} M & K^* \\ K & N \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} N & K \\ K^* & M \end{pmatrix}.$$

It is straightforward to check that, for any unitary operator $U \in \mathcal{A}\,\overline{\otimes}\,M_2$ and any $H \in \mathcal{D}_{\mathcal{A}\overline{\otimes}M_2}$,

$$\log\big(UHU^{-1}\big) = U(\log H)U^{-1}, \tag{3.14}$$

which along with the invariance of the state $\tau^{(2)}$ with respect to unitary transformations yields

$$\tau^{(2)}\big[\log\big(UHU^{-1}\big)\big] = \tau^{(2)}[\log H].$$

Hence, (3.13) implies the equality

$$\tau^{(2)}[\log\mathbf{M}] = \frac{1}{2}\big(\tau\big[\log\big(N - KM^{-1}K^*\big)\big] + \tau[\log M]\big). \tag{3.15}$$

Combining (3.13) and (3.15) we get

$$\tau\bigl[\log\bigl(M-K^*N^{-1}K\bigr)\bigr]+\tau[\log N]=\tau\bigl[\log\bigl(N-KM^{-1}K^*\bigr)\bigr]+\tau[\log M].$$

By (2.2) and (3.1), this completes the proof of the theorem. □

Remark 3.4. The requirement that both the Shur complements $M-K^*N^{-1}K$ and $N-KN^{-1}K^*$ of $\mathbf{M}$ have bounded inverses is redundant. It is sufficient to require that at least one of the Shur complements is nonsingular since $M-K^*N^{-1}K\in\mathcal{D}_{\mathcal{A}}$ implies $N-KN^{-1}K^*\in\mathcal{D}_{\mathcal{A}}$ and vice versa (cf. representation (3.9)).

The following consequence suggests a recipe for the computation of the (relative) Morse index of a 2×2 operator matrix (cf. Remark 2.4). It also provides a representation for the ξ-index associated with an off-diagonal perturbation problem.

Corollary 3.5. *Assume hypothesis of Theorem* 3.3 *and let* $\mathbf{M}$ *be the operator matrix*

$$\mathbf{M}=\begin{pmatrix} M & K^* \\ K & N\end{pmatrix}.$$

Let U and W be isometries from $\mathcal{H}$ into $\mathcal{H}\oplus\mathcal{H}$ such that $U^*\mathbf{M}U=M$ and $W^*\mathbf{M}W=N$. Then

$$\begin{aligned}2\tau^{(2)}\bigl[\Xi(\mathbf{M})\bigr]&=\tau\bigl[\Xi\bigl(\bigl(W^*\mathbf{M}^{-1}W\bigr)^{-1}\bigr)\bigr]+\tau\bigl[\Xi(U^*\mathbf{M}U)\bigr]\\&=\tau\bigl[\Xi\bigl(\bigl(U^*\mathbf{M}^{-1}U\bigr)^{-1}\bigr)\bigr]+\tau\bigl[\Xi(W^*\mathbf{M}W)\bigr].\end{aligned}\tag{3.16}$$

In particular,

$$\begin{aligned}2\xi(\mathbf{M}_0,\mathbf{M})&=\xi\bigl(U^*\mathbf{M}U,\bigl(U^*\mathbf{M}^{-1}U\bigr)^{-1}\bigr)\\&=\xi\bigl(W^*\mathbf{M}W,\bigl(W^*\mathbf{M}^{-1}W\bigr)^{-1}\bigr),\end{aligned}$$

where

$$\mathbf{M}_0=\begin{pmatrix} M & 0 \\ 0 & N\end{pmatrix}.$$

Proof. We notice that equalities (3.13) and (3.15) yield

$$\begin{aligned}2\tau^{(2)}\bigl[\Xi(\mathbf{M})\bigr]&=\tau\bigl[\Xi\bigl(N-KM^{-1}K^*\bigr)\bigr]+\tau\bigl[\Xi(M)\bigr]\\&=\tau\bigl[\Xi\bigl(M-K^*N^{-1}K\bigr)\bigr]+\tau\bigl[\Xi(N)\bigr].\end{aligned}\tag{3.17}$$

According to the definition of the Shur complements of $\mathbf{M}$ (cf. (3.6)), we get

$$U^*\mathbf{M}^{-1}U=\bigl(M-K^*N^{-1}K\bigr)^{-1},\qquad W^*\mathbf{M}^{-1}W=\bigl(N-KM^{-1}K^*\bigr)^{-1}$$

and hence (3.16) follows from (3.17). □

Remark 3.6. In the I_∞ setting, a relation similar to (3.16) has been recently derived in [12].

Our next goal is to obtain an extension of the basic invariance principle stated in Theorem 3.3 by relaxing the invertibility hypotheses.

Theorem 3.7. *Let $M, N \in \mathcal{A}$ be dissipative and K an arbitrary operator in $\mathcal{A}$. Then the following assertions hold.*

(i)
$$\lim_{\varepsilon\downarrow 0} \xi\big(M + \mathrm{i}\varepsilon I, M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big) = \lim_{\varepsilon\downarrow 0} \xi\big(N + \mathrm{i}\varepsilon I, N + \mathrm{i}\varepsilon I - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\big).$$

(ii) *Assume that N has a bounded inverse. Then*

$$\xi\big(M, M - K^*N^{-1}K\big) = \lim_{\varepsilon\downarrow 0} \xi\big(N, N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\big). \tag{3.18}$$

(iii) *If, in addition, the limit*

$$K(M + \mathrm{i}0I)^{-1}K^* = \operatorname*{n-lim}_{\varepsilon\downarrow 0} K(M + \mathrm{i}\varepsilon I)^{-1}K^*$$

exists and $N - K(M + \mathrm{i}0I)^{-1}K^*$ has a bounded inverse, then

$$\xi\big(M, M - K^*N^{-1}K\big) = \xi\big(N, N - K(M + \mathrm{i}0I)^{-1}K^*\big). \tag{3.19}$$

Proof. (i) Theorem 3.3 guarantees that

$$\xi\big(M + \mathrm{i}\varepsilon I, M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big) = \xi\big(N + \mathrm{i}\varepsilon I, N + \mathrm{i}\varepsilon I - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\big), \quad \varepsilon > 0.$$

Therefore, to prove the claim it is sufficient to establish the existence of the limit

$$\lim_{\varepsilon\downarrow 0} \xi\big(M + \mathrm{i}\varepsilon I, M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big) = \lim_{\varepsilon\downarrow 0}\big(\tau\big[\Xi\big(M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big)\big] - \tau\big[\Xi(M + \mathrm{i}\varepsilon I)\big]\big).$$

By Remark 2.2, the limit $\lim_{\varepsilon\downarrow 0} \tau[\Xi(M + \mathrm{i}\varepsilon I)]$ exists. Next, by Remark 3.6,

$$\tau\big[\Xi\big(M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big)\big] = 2\tau^{(2)}\big[\Xi(\mathbf{M} + \mathrm{i}\varepsilon\mathbf{I})\big] - \tau\big[\Xi(N + \mathrm{i}\varepsilon I)\big], \tag{3.20}$$

where $\mathbf{M} = \left(\begin{smallmatrix} M & K^* \\ K & N \end{smallmatrix}\right)$ is a 2×2 operator matrix in $\mathcal{A}\,\overline{\otimes}\, M_2$. Applying Remark 2.2 to the dissipative elements $\mathbf{M}$ and N in the algebras $\mathcal{A}\,\overline{\otimes}\, M_2$ and $\mathcal{A}$, respectively, insures the existence of the limit of the left-hand side of (3.20) as $\varepsilon \downarrow 0$, completing the proof.

(ii) From Theorem 3.3, we obtain that the equality

$$\xi\big(M + \mathrm{i}\varepsilon I, M - K^*N^{-1}K + \mathrm{i}\varepsilon I\big) = \xi\big(N, N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\big) \tag{3.21}$$

holds for all $\varepsilon > 0$. Observe that invertibility of N implies that of the operator $N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*$ for any $\varepsilon > 0$. Indeed, the Herglotz operator-valued function

$$z \mapsto N - K(M + zI)^{-1}K^*$$

in the upper-half plane is invertible for $|\mathrm{Im}\, z|$ large enough and, therefore, it is invertible for all $z \in \mathbb{C}_+$ (cf. [17, Lemma 2.3]). Passing to the limit $\varepsilon \downarrow 0$ in (3.21) and making use of Remark 2.2 implies (3.18).

Since by hypothesis $N - K(M + \mathrm{i}0I)^{-1}K^*$ has a bounded inverse, using continuity of the operator logarithm (cf. (2.3)) and that of the state τ, we attain

$$\lim_{\varepsilon\downarrow 0} \tau\big[\Xi\big(N - K(M + \mathrm{i}\varepsilon I)^{-1}K^*\big)\big] = \tau\big[\Xi\big(N - K(M + \mathrm{i}0I)^{-1}K^*\big)\big].$$

Now the claim follows from (ii). □

Remark 3.8. As one can see from the proof, for any dissipative elements M and N in $\mathcal{A}$ and any $K \in \mathcal{A}$, the limit

$$\lim_{\varepsilon\downarrow 0} \tau\big[\Xi\big(M + \mathrm{i}\varepsilon I - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big)\big]$$

exists. If, in addition, the dissipative operator M has a bounded inverse, claim (ii) infers the existence of the limit

$$\lim_{\varepsilon\downarrow 0} \tau\big[\Xi\big(M - K^*(N + \mathrm{i}\varepsilon I)^{-1}K\big)\big].$$

4. The Birman–Krein formula revisited

As an application of Theorem 3.3, first we state a result regarding the computation of the relative index associated with purely imaginary dissipative perturbations $A \mapsto A + \mathrm{i}B$, $B \geqslant 0$, of a self-adjoint operator A. The following theorem sheds some light on the role of the characteristic function of a dissipative operator in the relative index theory. We recall that the Lifshits characteristic function $\mathbf{S}$ of the dissipative operator $A + \mathrm{i}B$ calculated at the spectral point $\lambda = 0$ (see, e.g., [18, Section IV.6]) is given by

$$\mathbf{S} = I - 2\mathrm{i}B^{1/2}(A + \mathrm{i}B)^{-1}B^{1/2}, \tag{4.1}$$

provided that $A + \mathrm{i}B$ has a bounded inverse.

Theorem 4.1. *Let $A = A^*$ and $B = B^* \geqslant 0$ be elements in $\mathcal{A}$. Suppose both A and $A + \mathrm{i}B$ have bounded inverses. Then*

$$\xi(A, A + \mathrm{i}B) = \frac{1}{\pi}\tau\big[\arctan\big(B^{1/2}A^{-1}B^{1/2}\big)\big] = \frac{1}{2\pi}\tau[\arg \mathbf{S}]. \tag{4.2}$$

Here **S** *is as in* (4.1) *and the argument of* **S** *is defined by the Spectral Theorem*

$$\arg \mathbf{S} = \int\limits_{|z|=1} \arg z \, dE_{\mathbf{S}}(z), \quad \arg z \in (-\pi, \pi], \; z \in \mathbb{C} \setminus \{0\},$$

with the cut along the negative semi-axis.

Proof. Introduce a self-adjoint operator $H = B^{1/2}A^{-1}B^{1/2}$. Theorem 3.3 implies that

$$\begin{aligned} \xi(A, A + \mathrm{i}B) = \xi(\mathrm{i}I, \mathrm{i}I - H) &= \tau\big[\Xi(\mathrm{i}I - H)\big] - \frac{1}{2} \\ &= \frac{1}{\pi}\tau\big[\operatorname{Im}\log(\mathrm{i}I - H)\big] - \frac{1}{2}. \end{aligned} \tag{4.3}$$

By the Spectral Theorem applied to H, we obtain

$$\begin{aligned} \tau\big[\operatorname{Im}\log(\mathrm{i}I - H)\big] - \frac{\pi}{2} &= \int\limits_{\mathbb{R}} \left(\operatorname{Im}\log(\mathrm{i} - \lambda) - \frac{\pi}{2}\right) d\tau\big[E_H(\lambda)\big] \\ &= \int\limits_{\mathbb{R}} \operatorname{Im}\log(1 + \mathrm{i}\lambda)\, d\tau\big[E_H(\lambda)\big] = \tau[\arctan H]. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), completes the proof of the first equality in (4.2).

It is straightforward to verify that

$$\mathbf{S} = (\mathrm{i}I - H)(\mathrm{i}I + H)^{-1}, \tag{4.5}$$

which, in particular, implies that **S** is unitary. For a smooth path of unitaries

$$[0, 1] \ni t \mapsto U_t = (\mathrm{i}I - tH)(\mathrm{i}I + tH)^{-1} \tag{4.6}$$

linking the identity $I = U_0$ with $\mathbf{S} = U_1$, we derive that

$$\tau[\arg \mathbf{S}] = \operatorname{Im}\tau(\log \mathbf{S}) = \operatorname{Im}\int\limits_0^1 \frac{d}{dt}\tau(\log U_t)\, dt = \operatorname{Im}\int\limits_0^1 \tau\big[\dot{U}_t U_t^{-1}\big]\, dt. \tag{4.7}$$

Observing that

$$\begin{aligned} \tau\big[\dot{U}_t U_t^{-1}\big] &= -\tau\big[\big(H(\mathrm{i}I + tH)^{-1} + (\mathrm{i}I - tH)(\mathrm{i}I + tH)^{-2}H\big)(\mathrm{i}I + tH)(\mathrm{i}I - tH)^{-1}\big] \\ &= -\tau\big[H(\mathrm{i}I + tH)^{-1}\big] - \tau\big[(\mathrm{i}I - tH)^{-1}H\big] \\ &= -\frac{d}{dt}\tau\big[\log(\mathrm{i}I + tH)\big] + \frac{d}{dt}\tau\big[\log(\mathrm{i}I - tH)\big], \end{aligned} \tag{4.8}$$

we arrive at the equality

$$\operatorname{Im}\int_0^1 \tau\big[\dot U_t U_t^{-1}\big]\,dt = \operatorname{Im}\big(\tau\big[\log(\mathrm{i}I - H)\big] - \tau\big[\log(\mathrm{i}I)\big]\big)$$
$$- \operatorname{Im}\big(\tau\big[\log(\mathrm{i}I + H)\big] - \tau\big[\log(\mathrm{i}I)\big]\big).$$

By the Spectral Theorem this equality implies

$$\begin{aligned}\operatorname{Im}\int_0^1 \tau\big[\dot U_t U_t^{-1}\big]\,dt &= \int_{\mathbb{R}} \left(\operatorname{Im}\log\left(\frac{1-\mathrm{i}\lambda}{\mathrm{i}}\right) - \operatorname{Im}\log\left(\frac{1+\mathrm{i}\lambda}{\mathrm{i}}\right)\right) d\tau\big[E_H(\lambda)\big] \\ &= \int_{\mathbb{R}} \big(\arg(1+\mathrm{i}\lambda) - \arg(1-\mathrm{i}\lambda)\big)\,d\tau\big[E_H(\lambda)\big] \\ &= 2\int \arg(1+\mathrm{i}\lambda)\,d\tau\big[E_H(\lambda)\big] = 2\tau[\arctan H]. \qquad (4.9)\end{aligned}$$

Comparing (4.7) and (4.9) proves the second equality in (4.2). □

Before turning back to the context of perturbation theory for self-adjoint operators, it is convenient to collect basic assumptions and related notation in the form of a hypothesis.

Hypothesis 4.2. Suppose that $H_0 = H_0^*$ and $V = V^*$ are elements in $\mathcal{A}$ and $H = H_0 - V$. Assume that V is factored in the form $V = -K^*N^{-1}K$, where $K \in \mathcal{A}$ and $N = N^*$ an element with a bounded inverse in $\mathcal{A}$. Assume that the norm-limit

$$K^*(H_0 + \mathrm{i}0I)^{-1}K = \operatorname*{n-lim}_{\varepsilon\downarrow 0} K^*(H_0 + \mathrm{i}\varepsilon I)^{-1}K$$

exists and both the operators $\mathcal{N} = N - K^*(H_0 + \mathrm{i}0I)^{-1}K$ and $\operatorname{Re}\mathcal{N}$ have bounded inverses.

Assume, in addition, that $\mathbf{S}$ is the characteristic function of the dissipative operator $\mathcal{N}$ at the zero value of the spectral parameter, that is,

$$\mathbf{S} = I - 2\mathrm{i}(\operatorname{Im}\mathcal{N})^{1/2}\mathcal{N}^{-1}(\operatorname{Im}\mathcal{N})^{1/2}. \qquad (4.10)$$

We conclude (under Hypothesis 4.2) with a result relating the relative index $\xi(H, H_0)$ to the de la Harpe–Skandalis determinant of the characteristic function $\mathbf{S}$ of the dissipative operator $\mathcal{N}$.

Theorem 4.3. *Assume Hypothesis* 4.2. *Let* $\det_\tau \mathbf{S}$ *be the de la Harpe–Skandalis determinant associated with the homotopy class of the* C^1*-paths of invertible operators joining* $\mathbf{S}$ *and* I *and containing the path*

$$[0,1] \ni t \mapsto t\mathbf{S} + (1-t)I. \qquad (4.11)$$

Then

$$\det_\tau \mathbf{S} = \Theta \exp\big(-2\pi\mathrm{i}\xi(H, H_0)\big), \qquad (4.12)$$

where

$$\Theta = \exp\bigl(-2\pi \mathrm{i}\xi(N, \operatorname{Re}\mathcal{N})\bigr). \tag{4.13}$$

Proof. Applying Theorem 3.7 yields

$$\xi(H_0, H) = \xi(N, \mathcal{N}) = \xi(N, \operatorname{Re}\mathcal{N}) + \xi(\operatorname{Re}\mathcal{N}, \mathcal{N}). \tag{4.14}$$

By Theorem 4.1 one has

$$\xi(\operatorname{Re}\mathcal{N}, \mathcal{N}) = \frac{1}{2\pi}\tau[\arg \mathbf{S}]$$

and, hence,

$$\xi(H_0, H) = \xi(N, \operatorname{Re}\mathcal{N}) + \frac{1}{2\pi}\tau[\arg \mathbf{S}] \tag{4.15}$$

holds. Multiplying by $2\pi\mathrm{i}$ on both sides of (4.15) and then exponentiating ensures, by Lemma A.2(ii), that

$$\Delta(t \mapsto U_t) = \Theta \exp\bigl(-2\pi \mathrm{i}\xi(H, H_0)\bigr),$$

where the nonsingular path $[0,1] \ni t \mapsto U_t$ is given by

$$\begin{aligned} U_t = {} & \bigl(\mathrm{i}I - t(\operatorname{Im}\mathcal{N})^{1/2}(\operatorname{Re}\mathcal{N})^{-1}(\operatorname{Im}\mathcal{N})^{1/2}\bigr) \\ & \times \bigl(\mathrm{i}I + t(\operatorname{Im}\mathcal{N})^{1/2}(\operatorname{Re}\mathcal{N})^{-1}(\operatorname{Im}\mathcal{N})^{1/2}\bigr)^{-1}, \quad t \in [0,1]. \end{aligned}$$

Since the path of unitary operators $t \mapsto U_t$ with endpoints $\mathbf{S}$ and I is homotopically equivalent to the path of invertible operators $[0,1] \ni t \mapsto t\mathbf{S} + (1-t)I$ (the point -1 does not belong to the spectrum of $\mathbf{S}$), the result follows upon applying Lemma A.1(i). □

Remark 4.4. Note that the characteristic function $\mathbf{S}$ of the dissipative operator $\mathcal{N} = N - K^*(H_0 + \mathrm{i}0I)^{-1}K$ given by (4.10) can also be understood as the abstract scattering operator associated with the pair (H_0, H) (cf. [27]). As distinct from the classical Birman–Krein formula [4] where the argument of the determinant of the scattering matrix is directly related to the spectral shift function (mod $\mathbb{Z}$), representation (4.12) for $\det_\tau \mathbf{S}$ via the ξ-index contains a unimodular factor Θ (4.13). Presence of the additional factor Θ in (4.12) can be explained by the non-integer nature of the τ-Fredholm index for the pair of orthogonal projections

$$\xi(N, \operatorname{Re}\mathcal{N}) = \operatorname{ind}_\tau\bigl(E_N(\mathbb{R}_-), E_{\operatorname{Re}\mathcal{N}}(\mathbb{R}_-)\bigr) \in [-1, 1].$$

Appendix A

In this appendix, we recall the concept of a determinant introduced by P. de la Harpe and G. Skandalis in [14].

Let $GL^0(\mathcal{A})$ be the set of boundedly invertible elements of $\mathcal{A}$. Given a nonsingular C^1-path of operators $[0,1] \ni t \mapsto H_t \in GL^0(\mathcal{A})$, the de la Harpe–Skandalis determinant associated with the path $t \mapsto H_t$ is defined by

$$\Delta(t \mapsto H_t) = \exp\left(\int_0^1 \tau\left[\dot{H}_t H_t^{-1}\right] dt\right). \tag{A.1}$$

Some important properties of the de la Harpe–Skandalis determinant are listed in the lemma below. The proofs of these facts can be found in [14, Lemma 1 and Proposition 2].

Lemma A.1. *Suppose that* $[0,1] \ni t \mapsto H_t$ *is a* C^1*-path of operators in* $GL^0(\mathcal{A})$.

(i) *The determinant* $\Delta(t \mapsto H_t)$ *is invariant under fixed endpoint homotopies.*

(ii) *The absolute value of the perturbation determinant* $\Delta(t \mapsto H_t)$ *is path-independent. Moreover,*

$$\left|\Delta(t \mapsto H_t)\right| = \Delta\left(H_1 H_0^{-1}\right),$$

where $\Delta(A) = \exp(\tau[\log\sqrt{A^*A}\,])$ *denotes the Fuglede–Kadison determinant of a boundedly invertible operator* $A \in \mathcal{A}$.

(iii) *If* $\|H_t - I\| < 1$ *for all* $t \in [0,1]$, *then*

$$\Delta(t \mapsto H_t) = \exp\left(\tau[\log H_1] - \tau[\log H_0]\right), \tag{A.2}$$

where the operator logarithm $\log H_j$, $j = 1, 2$, *in* (A.2) *is understood as the norm convergent series*

$$\log H_j = -\sum_{k=1}^{\infty} \frac{(I - H_j)^k}{k}, \quad j = 0, 1.$$

(iv) *Let* $H_t^{(j)}\colon [0,1] \to GL^0(\mathcal{A})$, $j = 1, 2$, *be* C^1*-paths. Then*

$$\Delta\left(t \mapsto H_t^{(1)} H_t^{(2)}\right) = \Delta\left(t \mapsto H_t^{(1)}\right)\Delta\left(t \mapsto H_t^{(2)}\right).$$

The following result reduces the computation of the determinant for paths of operators in either $\mathcal{D}_\mathcal{A}$ or

$$\mathcal{U}_\mathcal{A} = \left\{U\colon U = (\mathrm{i}I - H)(\mathrm{i}I + H)^{-1} \text{ for some } H = H^* \in \mathcal{A}\right\}$$

to that of the state τ of the operator logarithm.

Lemma A.2.

(i) *For a C^1-path of operators* $[0,1] \ni t \mapsto H_t \in \mathcal{D}_{\mathcal{A}}$ *with* $H_0 = I$,

$$\Delta(t \mapsto H_t) = \exp\bigl(\tau[\log H_1]\bigr), \tag{A.3}$$

where $\log(\cdot)$ *is the principal branch of the operator logarithm of* H_1 *with the cut along the negative imaginary semi-axis provided by the Riesz functional calculus.*

(ii) *For a C^1-path of operators* $[0,1] \ni t \mapsto U_t \in \mathcal{U}_{\mathcal{A}}$ *with* $U_0 = I$,

$$\Delta(t \mapsto U_t) = \exp\bigl(\tau[\widetilde{\log}\, U_1]\bigr),$$

where $\widetilde{\log}(\cdot)$ *is the principal branch of the operator logarithm of* U_1 *with the cut along the negative real semi-axis provided by the Spectral Theorem.*

Proof. (i) One notices that $\tau[\dot{H}_t H_t^{-1}] = \frac{d}{dt}\tau[\log H_t]$. Integrating the latter expression from 0 to 1 and comparing the result with (A.1) implies (A.3). The proof of (ii) goes along the same lines as that of (i). □

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