ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 334 (2007) 1–10
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On the existence and uniqueness of solutions of a class of fractional differential equations

Rabha W. Ibrahim [a], Shaher Momani [b,*]

[a] PO Box 14526, Sana'a, Yemen
[b] Department of Mathematics, Mutah University, PO Box 7, Al-Karak, Jordan

Received 5 January 2006
Available online 23 December 2006
Submitted by I. Podlubny

Abstract

In this paper, we investigate the existence and uniqueness of solutions for the following class of multi-order fractional differential equations

$$D_{\beta_1}^{\gamma_1,\delta_1} \cdots D_{\beta_n}^{\gamma_n,\delta_n} u(t) := \prod_{i=1}^{n} D_{\beta_i}^{\gamma_i,\delta_i} u(t) := D_{\beta_i,n}^{\gamma_i,\delta_i} u(t) = f\big(t, u(t)\big), \quad t \in [0,1],$$

$$u(0) = 0, \qquad \sum_{i=1}^{n} \delta_i \leqslant 1, \quad \gamma_i > 0, \quad \beta_i > 0, \quad 1 \leqslant i \leqslant n,$$

where $D_{\beta_i,n}^{\gamma_i,\delta_i}$ denotes the generalized Erdélyi–Kober operator of fractional derivative of order δ_i. Moreover, some properties concerning the positive, maximal, minimal, and continuation of solutions are obtained.

Keywords: Erdélyi–Kober operators; Multi-order fractional differential equations; Existence; Uniqueness; Positive solution; Fixed-point theorem

1. Introduction

Recently, much attention has been paid to the existence and uniqueness of solutions for fractional differential equations of the type

* Corresponding author.
E-mail addresses: rabhaibrahim@yahoo.com (R.W. Ibrahim), shahermm@yahoo.com (S. Momani).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.12.036

$$D^{\delta}u = f\big(t, u(t)\big), \quad u^{(\delta-1)}(t_0) = u_0, \tag{1}$$

where $0 < \delta \leqslant 1$ and D^{δ} denotes Riemann–Liouville fractional derivative of order δ, see [1–4]. Hadid [1] used Schauder fixed-point theorem to obtain local existence, and Tychonov's fixed-point theorem to obtain global existence of solution of the above fractional differential equation. Momani [3] proved local and global uniqueness theorems for (1), by using Bihari's and Gronwall's inequalities. The existence, uniqueness, and structural stability of solutions of the fractional differential equation (1) have been investigated in [4].

In this paper we consider differential equations involving more general operator of fractional differentiation, called Erdélyi–Kober fractional derivatives:

$$D_{\beta_1}^{\gamma_1,\delta_1}\cdots D_{\beta_n}^{\gamma_n,\delta_n}u(t) := \prod_{i=1}^{n} D_{\beta_i}^{\gamma_i,\delta_i}u(t) := D_{\beta_i,n}^{\gamma_i,\delta_i}u(t) = f\big(t,u(t)\big), \quad t \in [0,1],$$

$$u(0) = 0, \qquad \sum_{i=1}^{n}\delta_i \leqslant 1, \quad \gamma_i > 0, \quad \beta_i > 0, \quad 1 \leqslant i \leqslant n, \tag{2}$$

where $D_{\beta_i,n}^{\gamma_i,\delta_i}$ denotes the generalized Erdélyi–Kober operator of fractional derivative of order δ_i, β_j and γ_i, $1 \leqslant i \leqslant n$, are arbitrary constants and δ_i is a parameter describing the order of the fractional derivative. The general response expression contains parameters describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\delta_i = \delta$, $\beta_j = 1$ and $\gamma_i = 0$, the fractional differential equation reduces to the fractional differential equation (1). The additional parameters γ_i, β_i allow more generality and these operators have found a large number of applications in analysis, mathematical physics and other disciplines [5].

In order to proceed, we give some basic definitions and theorems [5–9] which are used further in this paper.

Definition 1.1. (See [5–9].) The Erdélyi–Kober operator of fractional integration of order δ is defined as

$$I_{\beta}^{\gamma,\delta}f(t) = \frac{t^{-\beta(\gamma+\delta)}}{\Gamma(\delta)}\int_0^t \big(t^{\beta}-\tau^{\beta}\big)^{\delta-1}\tau^{\beta\gamma}f(\tau)\,d\tau^{\beta} = \frac{1}{\Gamma(\delta)}\int_0^1 (1-\sigma)^{\delta-1}\sigma^{\gamma}f\big(t\sigma^{1/\beta}\big)\,d\sigma,$$

and the Erdélyi–Kober operator of fractional derivative is defined as

$$D_{\beta}^{\gamma,\delta}f(t) = \big[\big(t^{-\gamma}D^{\delta}t^{\gamma+\delta}\big)f\big(t^{1/\beta}\big)\big]_{t\to t^{\beta}},$$

where $0 < \delta < 1$, $\gamma \in \mathbb{R}$ and $\beta > 0$.

Definition 1.2. (See [5–9].) The generalized fractional calculus is based on commutative compositions of Erdélyi–Kober operator:

$$I_{(\beta_i),n}^{(\gamma_i),(\delta_i)}f(t) = \left[\prod_{i=1}^{n} I_{\beta_i}^{\gamma_i,\delta_i}\right]f(t)$$

$$= \int_0^1\cdots\int_0^1\left[\prod_{i=1}^{n}\frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]f\big(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\big)\,d\sigma_1\cdots d\sigma_n.$$

Definition 1.3. (See [5–9].) The generalized Erdélyi–Kober operator of fractional derivative is defined as

$$D_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f(t) := D_\eta I_{(\beta_i),n}^{(\gamma_i+\delta_i),(\eta_i-\delta_i)} f(t),$$

where

$$D_\eta = \left[\prod_{i=1}^{n}\prod_{j=1}^{\eta_i}\left(\frac{1}{\beta_j}x\frac{d}{dx}+\gamma_i+j\right)\right],$$

and

$$\eta_i = \begin{cases} [\delta_i]+1, & \text{if } \delta_i \text{ noninteger},\\ \delta_i, & \text{if } \delta_i \text{ integer}, \ i=1,\dots,n. \end{cases}$$

Srivastava et al. [10] generalized this operator in order to include some important functions. For example, the Bessel function and Poisson function can be reduced to this operator [7]. For more details on the mathematical properties of the Erdélyi–Kober fractional derivatives and integrals, see [5–10].

Theorem 1.1. *(See [11].) Let U be a convex subset of Banach space E and $T: U \to U$ be a compact map. Then T has at least one fixed point in U.*

Definition 1.4. A Banach space $\mathcal{B}$ endowed with a closed cone K is an ordered Banach space $(\mathcal{B}, K)$ with a partial order $\leqslant$ in $\mathcal{B}$ as follows: $x \leqslant y$ if $y - x \in K$.

Definition 1.5. For $x, y \in \mathcal{B}$, the order interval $\langle x, y\rangle$ is defined as $\langle x, y\rangle = \{z \in \mathcal{B}\colon\ x \leqslant z \leqslant y\}$.

Theorem 1.2. *(See [12].) Let $(\mathcal{B}, K)$ be an ordered Banach space. Let U_1, U_2 be open subsets of $\mathcal{B}$ with $0 \in U_1$ and $\overline{U}_1 \subset U_2$ and let $F : K \cap (\overline{U}_2 \setminus U_1) \to K$ be completely continuous. Further suppose either*

(i) $\|Fu\| \leqslant \|u\|$ *for* $u \in K \cap \partial U_1$ *and* $\|Fu\| \geqslant \|u\|$ *for* $u \in K \cap \partial U_2$, *or*

(ii) $\|Fu\| \geqslant \|u\|$ *for* $u \in K \cap \partial U_1$ *and* $\|Fu\| \leqslant \|u\|$ *for* $u \in K \cap \partial U_2$.

Then F has a fixed point.

Theorem 1.3. *(See [13].) Let $(\mathcal{B}, K)$ be an ordered Banach space, $[u_0, v_0] \subset \mathcal{B}$, and $T : [u_0, v_0] \to [u_0, v_0]$ an increasing continuous operator. If K is a normal cone and T is completely continuous, then T has a fixed point which lies in $[u_0, v_0]$.*

Theorem 1.4. *(See [13].) Assume that K is a closed subset of a Banach space E. Let F be a contraction mapping with Lipschitz constant $(k < 1)$ from K to itself. Then F has a unique fixed point x^* in K. Moreover, if x_0 is an arbitrary point in K and x_n is defined by $x_{n+1} = Fx_n$ $(n = 0, 1, \dots)$, then $\lim_{n\to\infty} x_n = x^* \in K$ and $d(x_n, x^*) \leqslant (k^n/(1-k))d(x_1, x_0)$.*

2. Existence and uniqueness theorems

In this section, we begin by proving the existence and uniqueness of solution for Eq. (2) using Schauder fixed-point Theorem 1.1 and Banach fixed-point Theorem 1.4, respectively.

Lemma 2.1. *Assume that the continuous function $u(t)$ is in C_p for all $t \in [0,1)$. Then*

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),n} D^{(\gamma_i),(\delta_i)}_{(\beta_i),n} u(t) = u(t),$$
$$D^{(\gamma_i),(\delta_i)}_{(\beta_i),n} I^{(\gamma_i),(\delta_i)}_{(\beta_i),n} u(t) = u(t).$$

Proof. One can verify that the unique solution for the differential equation $D^{(\gamma_i),(\delta_i)}_{(\beta_i),n} u(t) = 0$ is $u(t) = \prod_{i=1}^{n} c_i t^{-\beta_i(\gamma_i+1)}$, $c_i \in \mathbf{R}$ (generalization of Lemma 2.1 in [9]). Then

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),n} D^{(\gamma_i),(\delta_i)}_{(\beta_i),n} u(t) = u(t) + \prod_{i=1}^{n} c_i t^{-\beta_i(\gamma_i+1)}.$$

By continuity of $u(t)$ in $[0,1)$ it implies that $c_i = 0$, for all $i = 1, \dots, n$. Hence we obtain the first law. The second one comes from the assumption of the lemma and in view of [7, Theorem 1.5.5]. □

Let $\mathcal{B} := C[0,1]$ be the Banach space endowed with the max norm, U be a nonempty closed subset of $\mathcal{B}$ defined as $U = \{u \in \mathcal{B}\colon \|u\| \leqslant l,\ l > 0\}$, and $A : U \to U$ be the operator defined as

$$Au(t) = \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{n} \frac{(1-\sigma_i)^{\delta_i - 1} \sigma_i^{\gamma_i}}{\Gamma(\delta_i)} \right] \times f\big(t\sigma_1^{1/\beta_1} \cdots \sigma_n^{1/\beta_n}, u\big(t\sigma_1^{1/\beta_1} \cdots \sigma_n^{1/\beta_n}\big)\big)\, d\sigma_1 \cdots d\sigma_n. \tag{3}$$

To facilitate our discussion, let us first state the following assumption:

Assumption A.

(1) $t \in [0,1]$, and $l > 0$,

(2) $\Omega := \prod_{i=1}^{n} \frac{\Gamma(\gamma_i + 1)}{\Gamma(\gamma_i + \delta_i + 1)}$,

(3) $f : [0,1] \times [-l, l] \to \mathbb{R}$ is a given continuous function such that $\Omega \|f\| < l$.

The properties of the operator A are discussed in the next lemma.

Lemma 2.2. *Let Assumption* A *hold. Then the operator A is completely continuous.*

Proof. For $u \in U$, we find

$$\begin{aligned}
\big|Au(t)\big| &\leqslant \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{n} \frac{(1-\sigma_i)^{\delta_i - 1} \sigma_i^{\gamma_i}}{\Gamma(\delta_i)} \right] \times \big|f\big(t\sigma_1^{1/\beta_1} \cdots \sigma_n^{1/\beta_n}, u\big(t\sigma_1^{1/\beta_1} \cdots \sigma_n^{1/\beta_n}\big)\big)\big|\, d\sigma_1 \cdots d\sigma_n \\
&\leqslant \|f\| \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{n} \frac{(1-\sigma_i)^{\delta_i - 1} \sigma_i^{\gamma_i}}{\Gamma(\delta_i)} \right] d\sigma_1 \cdots d\sigma_n = \Omega \|f\| < l.
\end{aligned}$$

Therefore A maps U to itself. Moreover, $A(U)$ is bounded operator.

Now, we prove that A is continuous. Since f is continuous function in a compact set $[0,1]\times[-l,l]$, then it is uniformly continuous there. Thus given $\epsilon>0$, we can find $\mu>0$ such that $\|f(t,u)-f(t,v)\|<\frac{\epsilon}{\Omega}$ when $\|u-v\|<\mu$. Then

$$\begin{aligned}\bigl|Au(t)-Av(t)\bigr| &\leqslant \int_0^1\cdots\int_0^1\left[\prod_{i=1}^n \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]\\ &\quad\times\bigl|f\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},u\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\\ &\quad-f\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},v\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\bigr|\,d\sigma_1\cdots d\sigma_n\\ &\leqslant\bigl\|f(t,u)-f(t,v)\bigr\|\int_0^1\cdots\int_0^1\left[\prod_{i=1}^n \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]d\sigma_1\cdots d\sigma_n\\ &=\Omega\bigl\|f(t,u)-f(t,v)\bigr\|<\epsilon.\end{aligned}$$

Now, we shall prove that A is equicontinuous. Let $u\in U$ and $t_1,t_2\in[0,1]$. Then

$$\begin{aligned}\bigl|Au(t_1)-Au(t_2)\bigr| &\leqslant \int_0^1\cdots\int_0^1\left[\prod_{i=1}^n \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]\\ &\quad\times\bigl|f\bigl(t_1\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},u\bigl(t_1\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\\ &\quad-f\bigl(t_2\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},u\bigl(t_2\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\bigr|\,d\sigma_1\cdots d\sigma_n\leqslant 2\Omega\|f\|\end{aligned}$$

which is independent of u i.e. A is relatively compact. The Arzela–Ascoli Theorem implies that A is completely continuous. The proof is therefore complete. □

Now, we give the main results of this section.

Theorem 2.1. *Let Assumption* A *hold. Then* (2) *has at least one solution.*

Proof. We need only to prove that the operator A has a fixed point. Since A is completely continuous (Lemma 2.2), that is, A is compact on the set U. Hence, in view of Theorem 1.1, A has a fixed point, which is a solution for Eq. (2). The proof is complete. □

Theorem 2.2. *Let Assumption* A *be satisfied and* $\|f(t,u)-f(t,v)\|<L\|u-v\|$, *where* L *is a constant such that* $\Omega L<1$. *Then Eq.* (2) *has a unique solution.*

Proof. We need only to prove that the operator A has a unique fixed point.

$$\begin{aligned}\bigl|Au(t)-Av(t)\bigr| &\leqslant \int_0^1\cdots\int_0^1\left[\prod_{i=1}^n \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]\\ &\quad\times\bigl|f\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},u\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\\ &\quad-f\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n},v\bigl(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\bigr)\bigr)\bigr|\,d\sigma_1\cdots d\sigma_n\\ &\leqslant\bigl\|f(t,u)-f(t,v)\bigr\|\int_0^1\cdots\int_0^1\left[\prod_{i=1}^n \frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]d\sigma_1\cdots d\sigma_n\\ &<\Omega L\|u-v\|.\end{aligned}$$

Then in view of Theorem 1.4, A has a unique fixed point which is corresponding to the unique solution for Eq. (2). □

3. Positive solution theorems

Here we use Theorem 1.2 to study the existence of positive, continuous solution for Eq. (2). For this purpose, we shall illustrate the following assumption:

Assumption B. For $t \in [0,1]$,

(1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a given continuous function.

(2) there exist two distinct positive constants m and M such that $m \leqslant f \leqslant M$.

Let $K \subset \mathcal{B}$ be a cone defined by $K = \{u \in \mathcal{B}\colon\ u(t) \geqslant 0,\ 0 \leqslant t \leqslant 1\}$. Then $(\mathcal{B}, K)$ forms an ordered Banach space. Let $A:K \to K$ be the operator defined as in Eq. (3), then we have the following lemma.

Lemma 3.1. *Let Assumption* B *be satisfied. Then A is completely continuous.*

Proof. The operator A is a bounded mapping (see proof of Lemma 2.2). We proceed to prove that $A:K \to K$ is continuous. Let $u \in K$, where $\|u\| \leqslant l$. Let $S = \{v \in K\colon\ \|u - v\| < r_1\}$. Then $\|v\| < l + r_1 := r$, $\forall v \in S$. Since f is continuous on $[0,1]\times[0,r]$, then it is uniformly continuous there. Hence, given $\epsilon > 0$, $\exists \mu > 0$ $(\mu < r_1)$ such that $\|f(t,u) - f(t,v)\| < \epsilon/\Omega$, for $\|u - v\| < \mu$, $0 \leqslant t \leqslant 1$. If $\|u - v\| < \mu$ then $v \in S$ and $\|v\| \leqslant r$. As $v \in S \subset K$, $\|v\| \leqslant r$, similarly $\|u\| \leqslant r$. So we have $\|Au - Av\| < \epsilon$, hence A is continuous. Then, A has a fixed point (see Lemma 2.2). □

Then we have the following results.

Theorem 3.1. *Let Assumption* B *hold. Then* (2) *has at least one positive solution.*

Proof. Let $U_1 = \{u \in \mathcal{B}\colon\ \|u\| \leqslant \Omega m\}$ and $U_2 = \{u \in \mathcal{B}\colon\ \|u\| \leqslant \Omega M\}$. For $u \in K \cap \partial U_2$, we have $0 \leqslant u(t) \leqslant \Omega M$, $t \in [0,1]$. Since $f(t,u) \leqslant M$, we have

$$\begin{aligned} Au(t) &= \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{n} \frac{(1-\sigma_i)^{\delta_i - 1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right] \\ &\quad \times f\left(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}, u\left(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\right)\right) d\sigma_1\cdots d\sigma_n \\ &\leqslant M \int_0^1 \cdots \int_0^1 \left[\prod_{i=1}^{n} \frac{(1-\sigma_i)^{\delta_i - 1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right] d\sigma_1\cdots d\sigma_n = \Omega M. \end{aligned}$$

Hence $\|Au\| \leqslant \|u\|$. On the other hand, for $u \in K \cap \partial U_1$, we have $0 \leqslant u(t) \leqslant \Omega m$, $t \in [0,1]$. Since $m \leqslant f(t,u)$, we have $Au(t) \geqslant \Omega m$. Thus $\|Au\| \geqslant \Omega m = \|u\|$, and in view of Theorem 1.2, A has a fixed point in $K \cap (\overline{U}_2 \setminus U_1)$, which corresponds to the positive solution for Eq. (2). Hence the proof is complete. □

Theorem 3.2. *Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous and $f(t,.)$ increasing for each $t\in[0,1]$. Let there exist u_o, v_o satisfying $D^{\gamma_i,\delta_i}_{\beta_i,n}u_o\leqslant u_o$, $D^{\gamma_i,\delta_i}_{\beta_i,n}v_o\geqslant v_o$ and $0\leqslant u_o\leqslant v_o$, $0\leqslant t\leqslant 1$. Then* (2) *has a positive solution.*

Proof. Let $u, v\in K$ such that $u<v$, then we have

$$\begin{aligned}
Au(t) &= \int_0^1\cdots\int_0^1\left[\prod_{i=1}^{n}\frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]\\
&\quad\times f\big(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}, u\big(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\big)\big)\,d\sigma_1\cdots d\sigma_n\\
&\leqslant \int_0^1\cdots\int_0^1\left[\prod_{i=1}^{n}\frac{(1-\sigma_i)^{\delta_i-1}\sigma_i^{\gamma_i}}{\Gamma(\delta_i)}\right]\\
&\quad\times f\big(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}, v\big(t\sigma_1^{1/\beta_1}\cdots\sigma_n^{1/\beta_n}\big)\big)\,d\sigma_1\cdots d\sigma_n\\
&= Av(t).
\end{aligned}$$

Therefore $Au(t)\leqslant Av(t)$, $\forall t$, then $Au\leqslant Av$. As $\exists u_o, v_o$ such that $0\leqslant u_o\leqslant v_o$ with $Au_o\leqslant u_o$, $Av_o\geqslant v_o$, in view of Theorem 1.3, A is compact and has a fixed point in $\langle u,v\rangle$. Hence $A:\langle u_o,v_o\rangle\to\langle u_o,v_o\rangle$ is compact, by Theorem 1.3, A has a fixed point $w\in\langle u,v\rangle$, which is the positive solution. This proves the theorem. □

In the following theorems, let the function f be continuous, increasing and have finite limit as $u\to\infty$, then in view of Theorem 3.2, Eq. (2) has a positive solution.

Theorem 3.3. *Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous and $f(t,.)$ increasing for each $t\in[0,1]$. If $0<\lim_{u\to\infty}f(t,u)<\infty$, $\forall t\in[0,1]$, then* (2) *has a positive solution.*

As a consequence of Theorem 3.3, the following theorem holds.

Theorem 3.4. *Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous and $f(t,.)$ increasing for each $t\in[0,1]$. If $0\leqslant\lim_{\|u\|\to\infty}\max_{0\leqslant t\leqslant 1}\frac{f(t,u)}{\|u\|}<\infty$, then* (2) *has a positive solution.*

In general we have the following theorem.

Theorem 3.5. *Let $f(t,u(t))=c+Mu(t)$, where c and M are positive constants. Then* (2) *has a positive solution.*

Theorem 3.6. *Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be continuous and $\|f(t,u)-f(t,v)\|<L\|u-v\|$, $\forall u,v\in[0,\infty)$ such that $\Omega L<1$. Then* (2) *has unique solution which is positive.*

Proof. Let $u, v\in K$, so we have

$$\big|Au(t)-Av(t)\big|\leqslant\Omega\big\|f(t,u)-f(t,v)\big\|<\Omega L\|u-v\|.$$

Then by Theorem 1.4, A has a unique fixed point (positive solution) in K. □

4. Maximal and minimal solutions theorem

In this section, we consider the existence of maximal and minimal solutions for Eq. (2).

Definition 4.1. Let m be a solution of Eq. (2) in $[0, 1]$, then m is said to be a maximal solution of (2), if for every solution u of (2) existing on $[0, 1]$, the inequality $u(t) \leqslant m(t)$, $t \in [0, 1]$, holds. A minimal solution may be define similarly by reversing the last inequality.

Theorem 4.1. *Let $f : [0, 1] \times [0, \infty) \to [0, \infty)$ be a given continuous and non-decreasing function in u. Assume that there exist two positive constants μ, ν $(\mu < \nu)$ such that*

$$\frac{\mu}{\Omega f(t, \mu)} < 1 < \frac{\nu}{\Omega f(t, \nu)}.$$

Then there exists a maximal and minimal solution of Eq. (2) *on* $[0, 1]$.

Proof. The integral equation of Eq. (2) is

$$u(t) = I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t, u(t)\big). \tag{4}$$

Consider the fractional order integral equation

$$u(t) = \epsilon + I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t, u(t)\big), \quad t \in [0, 1],\ \epsilon > 0. \tag{5}$$

Then by Lemma 2.1, Eq. (5) is a solution of Eq. (2) in (μ, ν), $t \in [0, 1]$, for some positive constants μ, ν such that

$$\frac{\mu}{\epsilon + \Omega f(t, \mu)} < 1 < \frac{\nu}{\epsilon + \Omega f(t, \nu)}.$$

Now, let $0 < \epsilon_2 < \epsilon_1 \leqslant \epsilon$. Then we have $u_{\epsilon_2}(0) < u_{\epsilon_1}(0)$. Thus we can prove that

$$u_{\epsilon_2}(t) < u_{\epsilon_1}(t), \quad \forall t \in [0, 1]. \tag{6}$$

Assume that it is false. Then there exist a t_1 such that

$$u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1) \quad \text{and} \quad u_{\epsilon_2}(t) < u_{\epsilon_1}(t), \quad \forall t \in [0, t_1).$$

Since f is monotonic non-decreasing in u, it follows that $f(t, u_{\epsilon_2}(t)) \leqslant f(t, u_{\epsilon_1}(t))$. Consequently, using Eq. (5), we get

$$u_{\epsilon_2}(t_1) = \epsilon_2 + I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t_1, u_{\epsilon_2}(t_1)\big) < \epsilon_1 + I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t_1, u_{\epsilon_1}(t_1)\big) = u_{u_{\epsilon_1}}(t_1),$$

which contradicts the fact that $u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1)$. Hence the inequality (6) is true. That is, there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly in $[0, 1]$. We denote this limiting value by $m(t)$. Obviously, by the uniform continuity of f (see Lemma 3.1), the equation

$$u_{\epsilon_n}(t) = I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t, u_{\epsilon_n}(t)\big),$$

yields that m is a solution of Eq. (2). To show that m is a maximal solution of Eq. (2), let u be any solution of Eq. (2) in $[0, 1]$. Then

$$u(t) < \epsilon + I_{(\beta_i),n}^{(\gamma_i),(\delta_i)} f\big(t, u(t)\big) = u_\epsilon(t).$$

Since the maximal solution is unique (see [14] and [15]), it is clear that $u_\epsilon(t)$ tends to $m(t)$ uniformly in $[0, 1]$ as $\epsilon \to 0$, which proves the existence of maximal solution for Eq. (2). A similar argument holds for the minimal solution. □

5. The continuation theorem

In this section, we study the continuation of solution of Eq. (2) when $0 < \delta_i \leqslant 1$, and for the case $(\beta_i = 1, \gamma_i = 0)$, $\forall i = 1, \dots, n$, then $D^{(\gamma_i),(\delta_i)}_{(\beta_i),n}$ reduces to the multi-order Riemann–Liouville fractional derivative operator D^{δ_i} (see [5–9]). Hence we have the equation

$$D^{\delta_i} u(t) = f\bigl(t, u(t)\bigr). \tag{7}$$

The evolution equation corresponding to Eq. (7) is

$$Du(t) = f\bigl(t, u(t)\bigr), \quad D = \frac{d}{dt}. \tag{8}$$

Then we have the following properties.

Lemma 5.1. *Let $f(t, u(t))$ be continuous function, then*

$$\lim_{\delta_i \to p} I^{(\delta_i)}_{1,n} f\bigl(t, u(t)\bigr) = I^p f\bigl(t, u(t)\bigr),$$

Proof. Without the loss of generality, let $n = 1$. We have

$$\begin{aligned} \bigl|I^\delta f\bigl(t, u(t)\bigr) - I^p f\bigl(t, u(t)\bigr)\bigr| &= \left|\int_0^1 \left(\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} - \frac{(1-\sigma)^{p-1}}{\Gamma(p)}\right) f(t\sigma)\, d\sigma\right| \\ &\leqslant \|f\| \int_0^1 \left|\left(\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} - \frac{(1-\sigma)^{p-1}}{\Gamma(p)}\right)\right| d\sigma \end{aligned}$$

but since

$$\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} \to \frac{(1-\sigma)^{p-1}}{\Gamma(p)}, \quad \text{as } \delta \to p,\ p = 1, 2, 3, \dots,$$

we get the result. □

Theorem 5.1. *If the solution u_1 of Eq.* (8) *exists, and if u_δ is the solution of Eq.* (2)*, then*

$$\lim_{\delta_i \to 1} u_{\delta_i}(t) = u_1(t).$$

Proof. Since $u_\delta(t) = I^{(\delta_i)}_{1,n} f(t, u_\delta(t))$ and $u_1(t) = If(t, u_1(t))$, then

$$\begin{aligned} \bigl|u_\delta(t) - u_1(t)\bigr| &= \bigl|I^{(\delta_i)}_{1,n} f\bigl(t, u_\delta(t)\bigr) - I^{(\delta_i)}_{1,n} f\bigl(t, u_1(t)\bigr) + I^{(\delta_i)}_{1,n} f\bigl(t, u_1(t)\bigr) - If\bigl(t, u_1(t)\bigr)\bigr| \\ &\leqslant \bigl|I^{(\delta_i)}_{1,n} f\bigl(t, u_\delta(t)\bigr) - I^{(\delta_i)}_{1,n} f\bigl(t, u_1(t)\bigr)\bigr| + \bigl|I^{(\delta_i)}_{1,n} f\bigl(t, u_1(t)\bigr) - If\bigl(t, u_1(t)\bigr)\bigr| \\ &\leqslant \Omega L \|u_\delta - u_1\| + \bigl|I^{(\delta_i)}_{1,n} f\bigl(t, u_1(t)\bigr) - If\bigl(t, u_1(t)\bigr)\bigr|. \end{aligned}$$

Thus

$$\|u_\delta - u_1\| \leqslant \frac{|I^{(\delta_i)}_{1,n} f(t, u_1(t)) - If(t, u_1(t))|}{1 - \Omega L}$$

where $\Omega L < 1$ (Uniqueness Theorem). Then in view of Lemma 5.1, we have $\|u_\delta - u_1\| \to 0$ as $\delta_i \to 1$, and hence the proof is complete. □

References

[1] S.B. Hadid, Local and global existence theorems on differential equations of non-integer order, J. Fract. Calc. 7 (1995) 101–105.

[2] S. Hadid, B. Masaedeh, S. Momani, On the existence of maximal and minimal solutions of differential equations of non-integer order, J. Fract. Calc. 9 (1996) 41–44.

[3] S.M. Momani, Local and global uniqueness theorems on differential equations of non-integer order via Bihari's and Gronwall's inequalities, Rev. Tecn. J. 23 (2000) 66–69.

[4] K. Diethelm, N. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002) 229–248.

[5] I.N. Sneddon, The use in mathematical analysis of Erdélyi–Kober operators and some of their applications, in: B. Ross (Ed.), Fractional Calculus and Applications, in: Lecture Notes in Math., vol. 457, Springer, New York, 1975, pp. 37–79.

[6] I. Ali, V. Kiryakova, S.L. Kalla, Solutions of fractional multi-order integral and differential equations using a Poisson-type transform, J. Math. Anal. Appl. 269 (2002) 172–199.

[7] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Res. Notes Math. Ser., vol. 301, Longman/Wiley, New York, 1994.

[8] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives (Theory and Applications), Gorden and Breach, New York, 1993.

[9] S. Zahang, Existence of positive solution for some of nonlinear fractional differential equations, J. Math. Anal. Appl. 278 (2003) 136–148.

[10] H.M. Srivastava, S.L. Kala, L. Galue, Further results on an H-function generalized fractional calculus, J. Fract. Calc. 4 (1993) 89–102.

[11] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.

[12] M.A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.

[13] M.C. Joshi, R.K. Bose, Some Topics in Nonlinear Functional Analysis, Wiley Eastern, New Delhi, 1985.

[14] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, vol. 1, Academic Press, New York, 1969.

[15] M.R. Rao, Ordinary Differential Equations, East–West Press, 1980.